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# A NOTE ON THE SPECTRA OF CERTAIN COMPOSITION OPERATORS ON HARDY SPACES

GEORG HOEVER

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ABSTRACT. H. Kamowitz claimed that the spectra of certain composition operators on Hardy spaces, induced by analytic selfmappings $\varphi$ of the open unit disk, which have Denjoy-Wolff point of modulus one with angular derivative less than one, are discs. C.C. Cowen discovered a gap in the proof and gave a new suggestion, applying different methods for different areas of points asserted to lie in the spectrum. In one of these he uses a continuity assertion of his 'model of iteration', which seems not to be proved completly. This paper fills the gap in Kamowitz's proof precisely in this case and gives little generalizations.

KEYWORDS: *Composition operator, spectrum.*

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## 1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and, for $1 \leqslant p < \infty$, let $H^p$ denote the usual Hardy spaces (cf. [5]):

$$H^p = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic}, \ \|f\|_p := \sup_{r\in(0,1)} \left\{ \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \, \mathrm{d}\theta \right]^{\frac{1}{p}} \right\} < \infty \right\}.$$

For an analytic selfmapping $\varphi$ of $\mathbb{D}$, we consider the composition operator

$$C_\varphi : H^p \to H^p, \quad f \mapsto f \circ \varphi.$$

This operator has been studied intensively during the last three decades; for a survey compare [3], [9] and [4]. In particular, it is a basic fact that $C_\varphi$ is well-defined and bounded for all analytic selfmappings $\varphi$ (see e.g. [4], Corollary 3.7). Suppose that $\varphi$ is not an elliptic automorphism. Then $\sigma(C_\varphi)$, the spectrum of $C_\varphi$, depends essentially on the position of the Denjoy-Wolff point $a$ of $\varphi$, that is the uniquely determined limit point of all iteration sequences $(z_n)_{n=0}^{\infty}$, $z_{n+1} = \varphi(z_n)$ (cf. [3], p. 137, and, for the definition of the Denjoy-Wolff point, [4], Theorem 2.51 and Definition 2.52).

If $\varphi$ is analytic in a neighborhood of $\overline{\mathbb{D}}$ and not a Möbius transformation, $|a| = 1$ and $\varphi'(a) < 1$, H. Kamowitz attempted to show

$$\sigma(C_\varphi) = \left\{\lambda \in \mathbb{C} : |\lambda| \leqslant (\varphi'(a))^{-\frac{1}{p}}\right\}$$

([7], Theorem 3.1). There is a gap in the proof, which C.C. Cowen mentioned in [2], p. 79: It has to be shown that for $g(z) = (z-a)^\alpha h(z)$ with a certain complex $\alpha$ and bounded analytic function $h$ and for certain complex $\lambda$, the sum $\sum\limits_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k}$ converges in $H^p$, where $\varphi_k$ denotes the $k$-th iterate of $\varphi$ ($\varphi_0 := \mathrm{id}$, the identity map, and $\varphi_{k+1} := \varphi_k \circ \varphi$).

C. C. Cowen did not fill this gap but suggested a different proof. He restricted himself to $p = 2$ and, in the case that $\varphi$ is an inner function, extensively used the Hilbert space structure of $H^2$ ([2], Theorem 5.1; here and below cf. [4], too) so that an extension to $H^p$ is not obvious. If $\varphi$ is not an inner function, he used methods which can be transferred to $H^p$, $1 < p < \infty$ (but not to $p = 1$): For a certain radius $r$ and all $s \in (0, r)$ he shows that $\sigma(C_\varphi) \cap \{\lambda : |\lambda| = s\} \neq \emptyset$, applying special iteration sequences ([2], Corollary 3.6). By using the so-called model of iteration, that is the fact that there is a nonconstant analytic selfmapping $\sigma$ of $\mathbb{D}$ and a Möbius transformation $\psi$ such that $\psi \circ \sigma = \sigma \circ \varphi$ (this is a deep result developed in [1]), he finds that the spectrum is invariant under rotations ([2], Theorem 4.3), hence $\{\lambda \in \mathbb{C} : |\lambda| < r\} \subseteq \sigma(C_\varphi)$; for a generalization to $H^p$, $1 < p < \infty$, see [6], Korollar 6.2.2. In order to show that $\{\lambda \in \mathbb{C} : r < |\lambda| < (\varphi'(a))^{-\frac{1}{2}}\} \subseteq \sigma(C_\varphi)$ ([2], Theorem 4.6) he uses the assertion $\sigma(z) \to 1$ as $z \to a$ in his model of iteration, which is mentioned in the remark in [1], p. 80. Unfortunately we do not see how this follows from the construction. (It is obvious that $\sigma(z_n) \to 1$ for all iteration sequences, but how does continuity in $a$ follow?) Precisely in this case (considering non-inner functions and $\lambda$, $r < |\lambda| < (\varphi'(a))^{-\frac{1}{p}}$) we will fill the gap in Kamowitz's proof and hence obtain a complete proof of Theorem 1.1.

THEOREM 1.1. *Let $1 < p < \infty$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic in a neighborhood of $\overline{\mathbb{D}}$ and not being an inner function. Suppose that $a \in \partial\mathbb{D}$ is the Denjoy-Wolff point of $\varphi$ and $\varphi'(a) < 1$. Then the spectrum of $C_\varphi : H^p \to H^p$ is given by*

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leqslant (\varphi'(a))^{-\frac{1}{p}}\}.$$

In the next section we develop the crucial estimates to prove the convergence of the above-mentioned sum in a simplified case. In the third section we give the details to fill the gap. We will require weaker assumptions than in the application above, hence Theorem 3.1 below is interesting in itself (e.g., $p = 1$ is allowed and $\varphi$ need not to be analytic in a neighborhood of $\overline{\mathbb{D}}$.).

## 2. THE CRUCIAL CONVERGENCE THEOREM

Let $H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$ and, for $h \in H^\infty(\mathbb{D})$, $\|h\|_\infty := \sup\{|h(z)| : z \in \mathbb{D}\}$.

THEOREM 2.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic with Denjoy-Wolff point $a \in \partial\mathbb{D}$ and $1 \leqslant p < \infty$. Suppose that $\varphi$ and $\varphi'$ are continuously continuable on $\overline{\mathbb{D}}$ (the continuation is denoted by $\varphi$ or $\varphi'$, resp.) and $d := \varphi'(a) < 1$. Suppose $S := \{b \in \partial\mathbb{D} : |\varphi(b)| = 1\}$ is finite and $\tilde{S} := \{\varphi(b) : b \in S\}$ consists only of fixed points of $\varphi$.*

*If $\tilde{S} = \{a\}$, let $r := 0$, otherwise $r := \max\{(\varphi'(\tilde{b}))^{-1} : \tilde{b} \in \tilde{S},\ \tilde{b} \neq a\}$. Suppose $-\frac{1}{p} < \alpha < \frac{1}{p}\frac{\log r}{\log d}$ ($\alpha \in (-\frac{1}{p}, \infty)$ in the case $r = 0$), $h \in H^\infty(\mathbb{D})$, and $g(z) := (z-a)^\alpha h(z)$ using an analytic branch of $(z-a)^\alpha$. Then for $\lambda \in \mathbb{C}$, $|\lambda| > d^{\,\alpha}$, the sum*

$$\sum_{k=0}^{\infty} \lambda^{-k} g \circ \varphi_k$$

*converges in $H^p$.*

REMARK 2.2. (i) Since $\varphi(a) = a$ we have $a \in S$ and $a \in \tilde{S}$.

(ii) The Julia-Carathéodory Theorem (see e.g. [9], p. 57) asserts that $\varphi'(\tilde{b})$ is real and positive for all $\tilde{b} \in \tilde{S}$, and the Grand Iteration Theorem ([9], p. 78) insures $\varphi'(\tilde{b}) > 1$ for $\tilde{b} \in \tilde{S}$, $\tilde{b} \neq a$, hence $r < 1$.

(iii) In our modification of H. Kamowitz's proof we may restrict ourselves to real $\alpha$. Recall that $(z-a)^\alpha \in H^p$ for $\alpha > -\frac{1}{p}$, as can be seen by an elementary calculation similar to that in [5], Lemma, p. 65. In [7], Lemma 1.3 and Lemma 1.4, it is shown that the assumptions on $S$ and $\tilde{S}$ are fulfilled at least for an iterate of a selfmapping that is analytic in a neighborhood of $\overline{\mathbb{D}}$ and is not an inner function.

(iv) There is a possibility to weaken the assumption, namely to allow $\varphi'(b) = \infty$ in some sense. We use the continuity of $\varphi'$ in $b \in S$, $\varphi(b) \neq a$ to deduce the inequality (2.6) below and its consequences. It is easy to see that one can obtain similar estimates in some cases where $\varphi'(b) = \infty$, e.g., for $\varphi(z) = 2\sqrt{\frac{z+1}{2}} - 1$ ($\sqrt{1} := 1$), where $S = \tilde{S} = \{-1, 1\}$, $\varphi'(-1) = \infty$ and $\varphi'(1) = \frac{1}{2}$ (hence 1 is the Denjoy-Wolff point); then the assertion of Theorem 2.1. holds, using $\frac{1}{\infty} = 0$, hence $r = 0$ in the example. The details of this case are left to the reader.

*Proof of Theorem* 2.1. The proof is divided into three parts. First we choose a constant $\delta$ such that we can control the behavior of $\varphi$ in $B_\delta(b) := \{z \in \overline{\mathbb{D}} : |z - b| < \delta\}$, $b \in S$, and we deduce some properties for the iterates $\varphi_k$ of $\varphi$. In the second and third part we examine the behavior of $\varphi_k$ and the consequences for the growth of $|g \circ \varphi_k|$ more accurately in the cases $\alpha > 0$, $|\lambda| \leqslant 1$ (I) and $\alpha < 0$ (II), which leads to norm estimates in each case. After that we are done, because the other cases are clear: For $\alpha = 0$ the assertion is true because then $|\lambda| > 1$ and $g$ is bounded in $\mathbb{D}$, hence $\|g \circ \varphi_k\|_p \leqslant \|g\|_\infty$ for all $k$, similarly in the case $\alpha > 0$, $|\lambda| > 1$.

We fix a constant $\varepsilon > 0$ so small that

— in case I: $d + \varepsilon < |\lambda|^{\frac{1}{\alpha}}$ and, unless $\tilde{S} = \{a\}$, $\varphi'(\tilde{b}) - \varepsilon > d^{-\alpha p}$ (in particular $> 1$) for all $\tilde{b} \in \tilde{S}$, $\tilde{b} \neq a$; note that due to our assumptions we have $\varphi'(\tilde{b})^{-\frac{1}{p}} \leqslant r^{\frac{1}{p}} < d^\alpha$ for $\tilde{b} \in \tilde{S}$, $\tilde{b} \neq a$; further, since the Julia-Carathéodory Theorem ([9], p. 57) asserts that the angular derivatives $\varphi'(b)$ are different from zero, we may assume that $\varepsilon$ is so small that $|\varphi'(b)| > \varepsilon$ for all $b \in S$;
— in case II: $(d - \varepsilon)^\alpha < |\lambda|$ and $\varepsilon < \min\{d, 1 - d\}$.

Since $\varphi'$ is continuous, there is a constant $\delta_0 \in (0, 1)$ such that

$$|\varphi'(z) - \varphi'(b)| < \varepsilon \quad \text{for all } b \in S,\ z \in B_{\delta_0}(b). \tag{2.1}$$

Moreover, because of the continuity of $\varphi$ and the fact that all $\tilde{b} \in \tilde{S}$ are fixpoints of $\varphi$ and $S$ is finite, it is easy to see that $\delta_0$ can be chosen so small that

$$\varphi(B_{\delta_0}(\tilde{b})) \cap B_{\delta_0}(b) = \emptyset \quad \text{for all } \tilde{b} \in \tilde{S},\ b \in S,\ b \neq \tilde{b} \tag{2.2}$$

and $B_{\delta_0}(b_1) \cap B_{\delta_0}(b_2) = \emptyset$ for all $b_1$, $b_2 \in S$, $b_1 \neq b_2$.

By the very definition of $S$ we obtain that $\varphi\Big(\overline{\mathbb{D}} \setminus \bigcup_{b \in S} B_{\delta_0}(b)\Big)$ is a closed set lying in $\mathbb{D}$. Hence there is a constant $\delta$, which we may assume to be smaller than $\delta_0$, satisfying

$$B_\delta(b) \cap \varphi\left(\overline{\mathbb{D}} \setminus \bigcup_{b \in S} B_{\delta_0}(b)\right) = \emptyset \quad \text{for all } b \in S,$$

and further that, for all $b \in S \setminus \tilde{S}$, there is a point $\tilde{b} \in \tilde{S}$ with $\varphi(B_\delta(b)) \subset B_{\delta_0}(\tilde{b})$.

Let $D := \overline{\mathbb{D}} \setminus \bigcup_{b\in S} B_\delta(b)$. Then $\varphi(D)$ is a closed set lying in $\mathbb{D}$. Hence $\varphi_k(D)$ converges uniformly to the Denjoy-Wolff point $a$ ([9], Grand Iteration Theorem, p. 78). In particular, there is an integer $k_0$ satisfying $\varphi_{k_0}(D) \subset B_\delta(a)$.

Finally we want to introduce integers dependent on $z$ which control the rate of convergence of $\varphi_k(z)$ to $a$: By (2.2), for all $z \in B_{\delta_0}(\tilde b) \cap \mathbb{D}$, $\tilde b \in \tilde S$, $\tilde b \neq a$, we have $\varphi(z) \in B_{\delta_0}(\tilde b)$ or $\varphi(z) \in D$ or both. Since $\varphi_k(z) \to a$, in particular $\varphi_k(z) \notin B_{\delta_0}(\tilde b)$ for large $k$, there is an integer $k$ with $\varphi_k(z) \in D$. We denote the smallest such integer by $k(z)$ ($k(z) \geqslant 0$). Then $\varphi_k(z) \in B_\delta(\tilde b)$ for all $0 \leqslant k < k(z)$.

For $z \in B_\delta(b) \cap \mathbb{D}$, $b \in S \setminus \tilde S$, $\varphi(b) \neq a$, our construction yields $\varphi(z) \in B_{\delta_0}(\tilde b) \cap \mathbb{D}$ for $\tilde b = \varphi(b) \in \tilde S$, $\tilde b \neq a$. Hence for this $z$, we can define $k(z) := 1 + k(\varphi(z))$, which is the smallest integer satisfying $\varphi_{k(z)}(z) \in D$.

Next we consider the two cases mentioned above in more detail:

**Case I.** $\alpha > 0$, $|\lambda| \leqslant 1$. We shall estimate $|\varphi_k(z) - a|$. The estimates we find depend on the position of $z$:

— $z \in B_{\delta_0}(a)$. Using (2.1), we obtain

$$|\varphi(z) - a| = \left| \int_a^z \varphi'(\xi)\,\mathrm{d}\xi \right| \leqslant |z-a|(d+\varepsilon). \tag{2.3}$$

Because of $d + \varepsilon < |\lambda|^{\frac{1}{\alpha}} \leqslant 1$ it follows that $\varphi(z) \in B_{\delta_0}(a)$ and, inductively, $|\varphi_k(z) - a| \leqslant |z-a|(d+\varepsilon)^k$, hence

$$|\varphi_k(z) - a| \leqslant \delta_0 (d+\varepsilon)^k \quad (z \in B_{\delta_0}(a),\ k \in \{0,1,\ldots\}). \tag{2.4}$$

— $z \in D$. The definition of $k_0$ in the first part of the proof implies that $\varphi_{k_0}(z) \in B_\delta(a)$, hence the result above insures that, for $k \geqslant k_0$, $|\varphi_k(z) - a| \leqslant (d+\varepsilon)^{k-k_0} |\varphi_{k_0}(z) - a|$. Thus there is a constant $c_0$, independent of $z$ and $k$, such that

$$|\varphi_k(z) - a| \leqslant c_0 (d+\varepsilon)^k \quad (z \in D,\ k \in \{0,1,\ldots\}). \tag{2.5}$$

— $z \in B_{\delta_0}(\tilde b) \cap \mathbb{D}$, $\tilde b \in \tilde S$, $\tilde b \neq a$. Another application of (2.1) yields

$$|\varphi(z) - \tilde b| = \left| (z-\tilde b)\varphi'(\tilde b) + \int_{\tilde b}^z \big(\varphi'(\xi) - \varphi'(\tilde b)\big)\,\mathrm{d}\xi \right| \geqslant |z - \tilde b|(\varphi'(\tilde b) - \varepsilon). \tag{2.6}$$

As long as $\varphi_k(z) \in B_{\delta_0}(\tilde{b}) \cap \mathbb{D}$ we obtain from (2.6): $|\varphi_k(z) - \tilde{b}| \geqslant |z - \tilde{b}|(\varphi'(\tilde{b}) - \varepsilon)^k$. Thus $k(z)$, defined in the first part of the proof, is smaller than the smallest integer $k$ satisfying $|z - \tilde{b}|(\varphi'(\tilde{b}) - \varepsilon)^k \geqslant \delta_0$ (recall $\varphi'(\tilde{b}) - \varepsilon > d^{-\alpha p} > 1$), hence

$$k(z) \leqslant \frac{\log \delta_0 - \log |z - \tilde{b}|}{\log(\varphi'(\tilde{b}) - \varepsilon)} + 1 \leqslant \frac{\log \delta_0 - \log |z - \tilde{b}|}{\log d^{-\alpha p}} + 1 =: K(z)$$

(as smaller $|z - \tilde{b}|$, as greater $K(z)$, i.e. as slower the convergence of $\varphi_k(z)$ to $a$ could be).

Combining this with (2.5), we obtain, for $k \geqslant K(z)$,

$$|\varphi_k(z) - a| = |\varphi_{k-k(z)}(\varphi_{k(z)}(z)) - a| \leqslant c_0(d + \varepsilon)^{k-k(z)} \leqslant c_0(d + \varepsilon)^{k-K(z)}. \tag{2.7}$$

– $z \in B_\delta(b) \cap \mathbb{D}$, $b \in S \setminus \tilde{S}$, $\varphi(b) \neq a$. We can apply (2.7) with $\varphi(z)$ replacing $z$ because $\varphi(z) \in B_{\delta_0}(\varphi(b)) \cap \mathbb{D}$:

$$|\varphi_k(z) - a| = |\varphi_{k-1}(\varphi(z)) - a| \leqslant c_0(d + \varepsilon)^{k-1-K(\varphi(z))} \quad \text{for } k - 1 \geqslant K(\varphi(z)).$$

Similar to (2.6) we have $|\varphi(z) - \varphi(b)| \geqslant |z - b|(|\varphi'(b)| - \varepsilon)$, and so

$$\begin{aligned} 1 + K(\varphi(z)) &= 1 + \frac{\log \delta_0 - \log |\varphi(z) - \varphi(b)|}{\log d^{-\alpha p}} + 1 \\ &\leqslant 2 + \frac{\log \delta_0 - \log(|z - b|(|\varphi'(b)| - \varepsilon))}{\log d^{-\alpha p}} =: K(z). \end{aligned}$$

We can summarize the last two cases: For all $b \in S$, $\varphi(b) \neq a$, there is a constant $c_b$, such that, for all $z \in B_\delta(b) \cap \mathbb{D}$,

$$K(z) = c_b - \frac{\log |z - b|}{\log d^{-\alpha p}} \tag{2.8}$$

and

$$|\varphi_k(z) - a| \leqslant c_0(d + \varepsilon)^{k-K(z)} \quad (k \geqslant K(z)). \tag{2.9}$$

Now we are able to estimate $\|g \circ \varphi_k\|_p^p = \sup\limits_{s \in (0,1)} \frac{1}{2\pi} \int\limits_0^{2\pi} |g \circ \varphi_k(s\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta$ for a large fixed integer $k$. We fix $s \in (0, 1)$ and divide the integral into several parts:

$$\begin{aligned} M_0 &:= \{\theta \in [0, 2\pi) : s\mathrm{e}^{\mathrm{i}\theta} \in D\} \\ M(b) &:= \{\theta \in [0, 2\pi) : s\mathrm{e}^{\mathrm{i}\theta} \in B_\delta(b)\} \quad (b \in S). \end{aligned}$$

Then $M_0 \cup \bigcup\limits_{b \in S} M(b) = [0, 2\pi)$.

From (2.4) and the definition of $g$ we obtain

$$\frac{1}{2\pi}\int\limits_{M(a)} |g\circ\varphi_k(se^{i\theta})|^p\,d\theta \leqslant \frac{1}{2\pi}\int\limits_{M(a)} \Big[\big(\delta_0(d+\varepsilon)^k\big)^\alpha\ \|h\|_\infty\Big]^p\,d\theta$$
$$\leqslant \delta_0^{\alpha p}\|h\|_\infty^p(d+\varepsilon)^{k\alpha p}.$$

Similarly (2.5) yields

$$\frac{1}{2\pi}\int\limits_{M_0} |g\circ\varphi_k(se^{i\theta})|^p\,d\theta \leqslant c_0^{\alpha p}\|h\|_\infty^p(d+\varepsilon)^{k\alpha p}.$$

For $b\in S$, $b\neq a$ but $\varphi(b)=a$, our construction insures $\varphi(se^{i\theta})\in B_{\delta_0}(a)$ for all $\theta\in M(b)$. Hence (2.4) asserts, for $k\geqslant 1$,

$$\frac{1}{2\pi}\int\limits_{M(b)} |g\circ\varphi_k(se^{i\theta})|^p\,d\theta \leqslant \delta_0^{\alpha p}\|h\|_\infty^p(d+\varepsilon)^{(k-1)\alpha p}.$$

To compensate the possibly slow convergence of $\varphi_k(z)$ to $a$ for $z$ near $b\in S$, $\varphi(b)\neq a$, we use the fact that the sets where the convergence could be very slow are very small; more accurate: For $b\in S$, $\varphi(b)\neq a$, we divide $M(b)$ again:

$$M_l(b) := \Big\{\theta\in[0,2\pi) : \delta d^{\alpha p l}\leqslant |se^{i\theta}-b| < \delta d^{\alpha p(l-1)}\Big\}\quad (l\in\{1,2,\dots\}).$$

By (2.8) we have, for $\theta\in M_l(b)$ and $z=se^{i\theta}$,

$$K(z)\leqslant c_b-\frac{\log(\delta d^{\alpha p l})}{\log d^{-\alpha p}} = c_b+\frac{\log\delta}{\log d^{\alpha p}}+l.$$

Hence, using $c_b' := c_b+\frac{\log\delta}{\log d^{\alpha p}}$, it follows from (2.9) that, if $k\geqslant c_b'+l$, i.e., $l\leqslant k-c_b'$,

$$\frac{1}{2\pi}\int\limits_{M_l(b)} |g\circ\varphi_k(se^{i\theta})|^p\,d\theta \leqslant \frac{|M_l(b)|}{2\pi}\Big[\big(c_0(d+\varepsilon)^{k-c_b'-l}\big)^\alpha\|h\|_\infty\Big]^p,$$

where $|M_l(b)|$ denotes the Lebesgue measure of $M_l(b)$. It is easy to see that $|M_l(b)|\leqslant 2\pi\delta d^{\alpha p(l-1)}\leqslant 2\pi\delta(d+\varepsilon)^{\alpha p(l-1)}$, hence

$$\frac{1}{2\pi}\int\limits_{M_l(b)} |g\circ\varphi_k(se^{i\theta})|^p\,d\theta \leqslant \delta c_0^{\alpha p}\|h\|_\infty^p(d+\varepsilon)^{(k-c_b'-1)\alpha p}\quad (l\leqslant k-c_b).$$

Let $k_b$ denote the largest integer smaller than $k-c_b'$, in particular $k_b\geqslant k-c_b'-1$. We only look for large $k$, hence we may assume $k\geqslant c_b'+1$. For $R(b) := \{\theta\in$

$[0,2\pi) : |se^{\mathrm{i}\theta} - b| < \delta d^{\alpha p k_b}\}$, we have $|R(b)| \leqslant 2\pi\delta d^{\alpha p k_b} \leqslant 2\pi\delta(d+\varepsilon)^{\alpha p(k-c'_b-1)}$, hence, using $|\varphi_k(z) - a| \leqslant 2$,

$$\frac{1}{2\pi}\int\limits_{R(b)} |g\circ\varphi_k(se^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta \leqslant \delta 2^{\alpha p}\|h\|_\infty^p\,(d+\varepsilon)^{(k-c'_b-1)\alpha p}.$$

Since $M(b) = R(b) \cup \bigcup\limits_{l=1}^{k_b} M_l(b)$ we obtain

$$\begin{aligned}\frac{1}{2\pi}\int\limits_{M(b)} |g\circ\varphi_k(se^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta &\leqslant (2^{\alpha p} + k_b c_0^{\alpha p})\delta\|h\|_\infty^p(d+\varepsilon)^{(k-c'_b-1)\alpha p}\\ &\leqslant (2^{\alpha p} + (k-c'_b)c_0^{\alpha p})\delta\|h\|_\infty^p(d+\varepsilon)^{(-c'_b-1)\alpha p}(d+\varepsilon)^{k\alpha p}.\end{aligned}$$

Putting all the information together we see that there is a constant $C$, independent of $s$ such that, for all $k \geqslant 1 + \max\{c'_b : b \in S, \varphi(b) \neq a\} \cup \{0\}$,

$$\frac{1}{2\pi}\int\limits_0^{2\pi} |g\circ\varphi_k(se^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta \leqslant Ck(d+\varepsilon)^{\alpha p k},$$

hence

$$\|\lambda^{-k} g\circ\varphi_k\|_p \leqslant |\lambda|^{-k}\left(Ck(d+\varepsilon)^{\alpha p k}\right)^{\frac{1}{p}} = C^{\frac{1}{p}}\left(k^{\frac{1}{pk}}\frac{(d+\varepsilon)^\alpha}{|\lambda|}\right)^k.$$

Since $(d+\varepsilon)^\alpha < |\lambda|$ and $k^{\frac{1}{pk}} \to 1$ for $k\to\infty$, there is a constant $0<q<1$ such that $k^{\frac{1}{pk}}\frac{(d+\varepsilon)^\alpha}{|\lambda|} < q$ for all large $k$. Hence the sum $\sum\limits_{k=0}^{\infty}\lambda^{-k} g\circ\varphi_k$ converges in $H^p$.

**Case II.** $\alpha < 0$. To estimate $\frac{1}{2\pi}\int\limits_0^{2\pi}|g\circ\varphi_k(se^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta$ we consider again the different positions of $z = se^{\mathrm{i}\theta}$:

— $z \in B_{\delta_0}(a)$: Similar to (2.3) and (2.6) we obtain

$$|z-a|(d+\varepsilon) \geqslant |\varphi(z) - a| \geqslant |z-a|(d-\varepsilon) \quad (z \in B_{\delta_0}(a)),$$

in particular $\varphi(z) \in B_{\delta_0}(a)$, and it follows inductively that

$$|g\circ\varphi_k(z)| \leqslant \left((d-\varepsilon)^k|z-a|\right)^\alpha \|h\|_\infty \quad (z\in B_{\delta_0}(a)) \tag{2.10}$$

(be always aware of the consequences of $\alpha < 0$ for the inequalities).

— $z \in D$: By the very definition of $D$ and since $\varphi(D), \dots, \varphi_{k_0}(D)$ are compact sets lying in $\mathbb{D}$, there is a constant $\delta_1$ satisfying

$$|\varphi_m(z) - a| > \delta_1 \quad \text{for all } z \in D,\ 0 \leqslant m \leqslant k_0. \tag{2.11}$$

This, together with the definition of $k_0$ in the first part of the proof and (2.10) yields, for $k \geqslant k_0$

$$\begin{aligned} |g \circ \varphi_k(z)| &\leqslant \big((d-\varepsilon)^{k-k_0}|\varphi_{k_0}(z) - a|\big)^{\alpha} \|h\|_\infty \\ &< \delta_1^\alpha \|h\|_\infty (d-\varepsilon)^{(k-k_0)\alpha} \quad (z \in D). \end{aligned} \tag{2.12}$$

— $z \in B_\delta(b)$, $b \in S$, $\varphi(b) \neq a$. Let $k(z)$ be the integer defined in the first part of the proof. If $k < k(z)$ then $\varphi_k(z) \in B_{\delta_0}(b)$ for some $b \in S$, $b \neq a$, hence $|\varphi_k(z) - a| \geqslant \delta_0$; if $k(z) \leqslant k < k(z) + k_0$ then $\varphi_k(z)$ belongs to $\varphi_{k-k(z)}(D)$ and by (2.11): $|\varphi_k(z) - a| > \delta_1$. In both cases we have

$$|g \circ \varphi_k(z)| \leqslant (\min\{\delta_0, \delta_1\})^\alpha \|h\|_\infty \leqslant (\min\{\delta_0, \delta_1\})^\alpha \|h\|_\infty (d-\varepsilon)^{k\alpha}.$$

If $k \geqslant k(z) + k_0$, using (2.12) we can estimate

$$|g \circ \varphi_k(z)| \leqslant {\delta_1}^\alpha \|h\|_\infty (d-\varepsilon)^{(k-k(z)-k_0)\alpha} \leqslant {\delta_1}^\alpha \|h\|_\infty (d-\varepsilon)^{k\alpha}.$$

— $z = s\mathrm{e}^{\mathrm{i}\theta} \in B_\delta(b)$, $b \in S$, $\varphi(b) = a$: Then $\theta \in M := \{\theta \in [0, 2\pi) : \varphi(s\mathrm{e}^{\mathrm{i}\theta}) \in B_{\delta_0}(a)\}$ and (2.10) yields

$$\begin{aligned} \frac{1}{2\pi}\int\limits_M |g \circ \varphi_k(s\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta &\leqslant \frac{1}{2\pi}\int\limits_M (d-\varepsilon)^{(k-1)\alpha p}|\varphi(s\mathrm{e}^{\mathrm{i}\theta}) - a|^{\alpha p}\|h\|_\infty^p \,\mathrm{d}\theta \\ &\leqslant (d-\varepsilon)^{-\alpha p}\|h\|_\infty^p \|G\|_p^p (d-\varepsilon)^{k\alpha p}, \end{aligned}$$

with $G(z) = (\varphi(z) - a)^\alpha = C_\varphi(z \mapsto (z-a)^\alpha) \in H^p$.

Thus we can find a constant $C$, independent of $k$, satisfying

$$\|\lambda^{-k} g \circ \varphi_k\|_p \leqslant |\lambda|^{-k} \left(C(d-\varepsilon)^{\alpha p k}\right)^{\frac{1}{p}} = C^{\frac{1}{p}} \left(\frac{(d-\varepsilon)^\alpha}{|\lambda|}\right)^k,$$

and since $(d-\varepsilon)^\alpha < |\lambda|$, the sum $\sum\limits_{k=0}^{\infty} \lambda^{-k} g \circ \varphi_k$ converges in $H^p$. ∎

## 3. THE DETAILS TO FILL THE GAP

As mentioned in the introduction we cannot fill the gap for all $\lambda$ Kamowitz's proof uses. Let us recall the situation of Theorem 1.1: A selfmapping $\varphi$ of $\mathbb{D}$, analytic in a neighborhood of $\overline{\mathbb{D}}$ and not being an inner function is given, which has Denjoy-Wolff point $a \in \partial\mathbb{D}$ with angular derivative $d := \varphi'(a) < 1$. Then there is a positive integer $N$ such that $S := \{b \in \partial\mathbb{D} : |\varphi_N(b)| = 1\}$ is finite and $\tilde{S} := \{\varphi_N(b) : b \in S\}$ consists only of fixed points of $\varphi_N$ ([7], Lemma 1.3 and 1.4). Fix such an integer $N$. (In the proof of Corollary 4.8 in [2], and in [4], Exercise 7.5.2, it is claimed that $S$ consists only of fixed points of $\varphi$ for some $N$, which one can see to be false by considering $z \mapsto \frac{z^2+1}{2}$. But after slight modifications Corollary 4.8 will be proved if the continuity in $a$ of the mapping $\sigma$, mentioned in the introduction, is guaranteed.)

If $\tilde{S} \neq \{a\}$ let $r := \max\{(\varphi_N'(\tilde{b}))^{-1} : \tilde{b} \in \tilde{S}, \tilde{b} \neq a\}$. Then $\sigma(C_{\varphi_N})$ intersects every circle with radius $\rho < r^{\frac{1}{p}}$ (see [2], Corollary 3.6, for the case $p = 2$ and [6], Satz 6.2.1, for the extension to arbitrary $1 < p < \infty$). Since $C_{\varphi_N} = (C_\varphi)^N$ it follows from the spectral mapping theorem (see e.g. [8], Theorem 3.4 in Section V) that $\sigma(C_\varphi)$intersects every circle with radius $\rho < r^{\frac{1}{Np}}$. In our case the spectrum is invariant under rotations ([2], Theorem 4.3, and [6], Satz 3.3.1, both using the model of iteration but not the continuity of $\sigma$ in $a$), hence $\{\lambda \in \mathbb{C} : |\lambda| < r^{\frac{1}{Np}}\} \subseteq \sigma(C_\varphi)$.

It is well-known that $\sigma(C_\varphi) \subseteq \{\lambda : |\lambda| \leqslant d^{-\frac{1}{p}}\}$ (cf., for example, [4], Theorem 3.9). Hence in order to prove Theorem 1.1 it is sufficient to show that $\{\lambda \in \mathbb{C} : r^{\frac{1}{Np}} \leqslant |\lambda| \leqslant d^{-\frac{1}{p}}\} \subseteq \sigma(C_\varphi)$ with $r = 0$ if $\tilde{S} = \{a\}$ and $r$ like above otherwise. We do this under somewhat weaker assumptions.

THEOREM 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic with Denjoy-Wolff point $a \in \partial\mathbb{D}$ and $1 \leqslant p < \infty$. Suppose that $\varphi$ and $\varphi'$ are continuously continuable on $\overline{\mathbb{D}}$ (the continuation is denoted by $\varphi$ or $\varphi'$, resp.), $\varphi''$ is bounded near $a$ and $d := \varphi'(a) < 1$. Suppose there is an integer $N$ such that $S := \{b \in \partial\mathbb{D} : |\varphi_N(b)| = 1\}$ is finite and $\tilde{S} := \{\varphi_N(b) : b \in S\}$ consists only of fixed points of $\varphi_N$. If $\tilde{S} = \{a\}$ let $r := 0$, otherwise $r := \max\{(\varphi_N'(\tilde{b}))^{-1} : \tilde{b} \in \tilde{S}, \tilde{b} \neq a\}$. Then for the spectrum of $C_\varphi : H^p \to H^p$ we have*

$$\{\lambda \in \mathbb{C} : r^{\frac{1}{Np}} \leqslant |\lambda| \leqslant d^{-\frac{1}{p}}\} \subseteq \sigma(C_\varphi).$$

REMARK 3.2. As in Theorem 2.1, in some cases we may allow continuity in the extended sense ($\varphi'(b) = \infty$ for some $b \in S$) and then obtain the result for a greater class of selfmappings. E.g., for $\varphi(z) = 2\sqrt{\frac{z+1}{2}} - 1$ (cf. Remark 2.2 (iv)) we have $r = 0$, $d = \frac{1}{2}$, hence $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \left(\frac{1}{2}\right)^{-\frac{1}{p}}\} = \sigma(C_\varphi)$ (equality holds because $d^{\,-\frac{1}{p}}$ is the spectral radius of $\varphi$, see [4], Theorem 3.9).

Apart from slight modifications we will follow the proof of Theorem 3.1 in [7] and hence only sketch the main steps. For some estimates Kamowitz used his Theorem 2.5, which requires analyticity in a neighborhood of $a$. Thus we shall first prove similar assertions under the weaker assumptions.

Lemma 3.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic with Denjoy-Wolff point $a \in \partial\mathbb{D}$. Suppose that $\varphi$ and $\varphi'$ are continuously continuable on $\overline{\mathbb{D}}$ (the continuation is denoted by $\varphi$ or $\varphi'$, resp.), $\varphi''$ is bounded near $a$ and $d := \varphi'(a) < 1$. Then $(\varphi_k(0))_{k=0}^{\infty}$ is an interpolating sequence and there are constants $0 < c_1 < c_2 < \infty$ such that for all $k = 0, 1, \dots$*

$$c_1 \leqslant \frac{d^k}{\varphi_k(0) - a} \leqslant c_2.$$

*Proof.* For the first assertion see [4], Theorem 2.65. To prove the second assertion let $\varphi''$ be bounded by $M$ near $a$ and $b_k := |\varphi_k(0) - a|$. Then $b_k \overset{k\to\infty}{\longrightarrow} 0$ (hence $b_k M \leqslant \frac{1-d}{2}$ for large $k$) and, for large $k$ and $\xi$ between $\varphi_k(0)$ and $a$,

$$|\varphi'(\xi) - d| = \left| \int_a^{\xi} \varphi''(\eta)\, \mathrm{d}\eta \right| \leqslant |\xi - a| \cdot M \leqslant |\varphi_k(0) - a| \cdot M = b_k M,$$

hence for large $k$, say $k \geqslant k_0$,

$$\begin{aligned} b_{k+1} &= |(\varphi_k(0) - a)\, d + \int_a^{\varphi_k(0)} (\varphi'(\xi) - d)\, \mathrm{d}\xi| \\ &\leqslant b_k\, d + b_k \cdot b_k M \leqslant b_k\, d + b_k \cdot \frac{1-d}{2} = b_k \frac{1+d}{2}. \end{aligned} \tag{3.1}$$

Thus, by induction $\sum\limits_{k=k_0}^{\infty} b_k \leqslant b_{k_0} \sum\limits_{l=0}^{\infty} \left(\frac{1+d}{2}\right)^l < \infty$ and so $\delta := \prod\limits_{k=k_0}^{\infty} \left(1 - \frac{M}{d}\, b_k\right) > 0$ (without loss of generality we can assume that $\frac{M}{d}\, b_k < 1$ for $k \geqslant k_0$) and $D := \prod\limits_{k=k_0}^{\infty} \left(1 + \frac{M}{d}\, b_k\right) < \infty$.

Now let $c_k := \frac{b_k}{d^k}$. Then (3.1) yields

$$c_{k+1} \leqslant c_k + \frac{M}{d} b_k\, c_k,$$

hence, by induction

$$c_{k+1} \leqslant c_{k_0} \cdot \prod_{k=k_0}^{k} \left(1 + \frac{M}{d} b_k\right) \leqslant c_{k_0}\, D.$$

Similar to (3.1), for $k \geqslant k_0$,

$$b_{k+1} \geqslant |(\varphi_k(0) - a)\, d| - \left| \int\limits_a^{\varphi_k(0)} (\varphi'(\xi) - d)\, \mathrm{d}\xi \right| \geqslant b_k\, d - b_k \cdot b_k M,$$

which yields $c_{k+1} \geqslant c_k - \frac{M}{d} b_k\, c_k$, hence, by induction

$$c_{k+1} \geqslant c_{k_0} \cdot \prod_{k=k_0}^{k} \left(1 - \frac{M}{d} b_k\right) \geqslant c_{k_0}\, \delta.$$

Thus, for $k \geqslant k_0$, we have

$$\frac{d^k}{|\varphi_k(0) - a|} = \frac{1}{c_k} \leqslant \frac{1}{c_{k_0}\, \delta} \quad \text{and} \quad \frac{d^k}{|\varphi_k(0) - a|} \geqslant \frac{1}{c_{k_0}\, D}$$

and the second assertion follows. ∎

*Proof of Theorem* 3.1. Since the spectrum is closed we may restrict ourselves to the interior of the ring asserted to lie in the spectrum. (The interior is nonempty, since $r < 1$ — cf. Remark 2.2 (ii) — and $d^{-\frac{1}{p}} > 1$.)

Fix $\lambda_0$ with $r^{\frac{1}{Np}} < |\lambda_0| < d^{-\frac{1}{p}}$ and suppose $\lambda_0 \notin \sigma(C_\varphi)$. Let $|\lambda_0| = d^\alpha$, i.e., $\alpha = \frac{\log|\lambda_0|}{\log\, d}$, hence by the restriction for $|\lambda_0|$:

$$\frac{\log\, d^{-\frac{1}{p}}}{\log\, d} < \alpha < \frac{\log\, r^{\frac{1}{Np}}}{\log\, d}, \quad \text{i.e.} \quad -\frac{1}{p} < \alpha < \frac{1}{p}\, \frac{\log\, r}{\log d^{\,N}} \tag{3.2}$$

($\alpha \in (-\frac{1}{p}, \infty)$ in the case $r = 0$).

Let $\lambda(s) = s\lambda_0$ for $s > 1$. Then, for $s$ near 1, the operator $\lambda(s) - C_\varphi$ is invertible and $|\lambda(s)| > |\lambda_0|$. Choose an analytic branch of $(z - a)^\alpha$ for $z \in \mathbb{D}$ and then define

$$\gamma_k := \frac{{\lambda_0}^k}{(\varphi_k(0) - a)^\alpha} = \left(\frac{d^k}{\varphi_k(0) - a}\right)^\alpha.$$

(Kamowitz uses all $\lambda$ near $\lambda_0$, $|\lambda| > |\lambda_0|$. His definition of $\gamma_k$ depends on $\lambda$, which should not be. It is sufficient to consider $s\lambda_0$, which leads to our definition of $\gamma_k$.) Applying Lemma 3.3 we see that $(\gamma_k)_{k=0}^\infty$ is bounded and that $(\varphi_k(0))_{k=0}^\infty$ is an interpolating sequence. Hence there is a function $h \in H^\infty(\mathbb{D})$ satisfying $h(\varphi_k(0)) = \gamma_k$.

Let $g(z) = (z - a)^\alpha h(z)$. To fill the gap in Kamowitz's proof we have to show that $\sum\limits_{k=0}^{\infty} \lambda(s)^{-k} g \circ \varphi_k$ converges in $H^p$. Therefore we apply Theorem 2.1 for $\hat{\varphi} := \varphi_N$. It is easy to see that $a$ is the Denjoy-Wolff point of $\hat{\varphi}$ and $\hat{\varphi}'(a) = d^N$.

The definitions of $S$, $\tilde{S}$ and $r$ above and those given in Theorem 2.1 applied for $\hat{\varphi}$ coincide and the condition concerning $\alpha$ is fulfilled (see (3.2)). Since $|\lambda(s)^N| > |\lambda_0|^N = (d^{\,N})^\alpha$, Theorem 2.1 asserts that

$$h_0 := \sum_{k=0}^{\infty}(\lambda(s)^N)^{-k} g\circ\hat{\varphi}_k = \sum_{k=0}^{\infty}\lambda(s)^{-kN} g\circ\varphi_{kN}$$

converges in $H^p$. Thus

$$h_j := \lambda(s)^{-j} C_\varphi^j h_0 = \sum_{k=0}^{\infty}\lambda(s)^{-kN-j} g\circ\varphi_{kN+j} \quad (j=1,\dots,N-1)$$

converges in $H^p$, too. Considering $h_0+\dots+h_{N-1}$ we obtain the convergence of $\sum\limits_{k=0}^{\infty}\lambda(s)^{-k} g\circ\varphi_k$ in $H^p$.

Now an easy calculation yields $g = [\lambda(s)-C_\varphi](\lambda(s)^{-1}\sum\limits_{k=0}^{\infty}\lambda(s)^{-k} g\circ\varphi_k)$, thus

$$\begin{aligned}\left[(\lambda(s)-C_\varphi)^{-1} g\right](0) &= \lambda(s)^{-1}\sum_{k=0}^{\infty}\lambda(s)^{-k} g(\varphi_k(0))\\ &= \lambda(s)^{-1}\sum_{k=0}^{\infty}(s\lambda_0)^{-k}(\varphi_k(0)-a)^\alpha\frac{\lambda_0{}^k}{(\varphi_k(0)-a)^\alpha}\\ &= \lambda(s)^{-1}\sum_{k=0}^{\infty}s^{-k} = \frac{1}{s\lambda_0}\,\frac{1}{1-s},\end{aligned}$$

hence

$$\left[(\lambda_0-C_\varphi)^{-1} g\right](0) = \lim_{s\to 1+}\left[(\lambda(s)-C_\varphi)^{-1} g\right](0) = \infty,$$

a contradiction. ∎

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*GEORG HOEVER*
*Mathematisches Institut*
*Universität München*
*Theresienstr. 39*
*80333 München*
*GERMANY*

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