J. OPERATOR THEORY
39(1998), 297–308

A DECOMPOSITION THEOREM IN CLIFFORD ANALYSIS

JOHN RYAN

Communicated by Norberto Salinas

Abstract. A modified Cauchy integral formula is used to show that each monogenic function f defined on a sector domain and satisfying $\|f(x)\| \leqslant C\|x\|^{-n+1}$ can be expressed as $f = f_1 + f_2$. Here f_1 is monogenic on the sector domain and monogenically extends to upper half space, while f_2 monogenically extends to lower half space. Moreover, $\|f_j(x)\| \leqslant C\|x\|^{-n+1}$ on these extended domains, for $j = 1$ or 2. Similar decompositions are obtained over more general unbounded domains, and for more general types of monogenic functions.

Keywords: *Cauchy kernels, monogenic functions.*

MSC (2000): Primary 30G35; Secondary 53A50, 42B20.

INTRODUCTION

A general result describing monogenic decompositions of monogenic functions is given in [1]. In [8], [9], [10] an elementary decomposition is explicitly provided for special types of monogenic functions defined over a sector domain in $\mathbb{R}^n$. These monogenic functions satisfy two basic conditions which enable these functions to be utilised as convolution operators of Calderon-Zygmund type acting on the L^p spaces of Lipschitz surfaces in $\mathbb{R}^n$, for $1 < p < \infty$. By adding on extra terms to the Cauchy kernel of Clifford analysis we are able to produce a number of Cauchy integral formulae for a wide class of monogenic functions defined on many unbounded domains, including sector domains. Using these integral formulae we are also able to provide explicit decompositions for these monogenic functions. The method of obtaining these decompositions is different from the method given in [8], [9], [10]. The decomposition works well for mongenic functions satisfying the first of the two criteria satisfied by the monogenic functions lying on sector domains, and used in [8], [9], [10].

The intention of this paper is to provide suitable groundwork to the type of operator theory needed to solve boundary value problems for unbounded domains, but where modified versions of the Clifford analysis versions of the Plemelj formulae used in [6], [8], [9] are needed. Some of these applications have already been developed in [4].

1. PRELIMINARIES

We shall consider the complex Clifford algebra, $\mathrm{Cl}_n(\mathbb{C})$, generated from $\mathbb{R}^n$. So that if $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$ then

$$1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n$$

is a basis for $\mathrm{Cl}_n(\mathbb{C})$, where $1 \leqslant r \leqslant n$ and $e_{j_1} < \cdots < e_{j_r}$. Moreover, the basis vectors, $e_1, \ldots, e_n$, satisfy the anticommutation relationship

$$e_i e_j + e_j e_i = -2\delta_{i,j},$$

where $\delta_{i,j}$ is the Kroneker delta function.

It may be observed that each non-zero vector $x \in \mathbb{R}^n \subset \mathrm{Cl}_n(\mathbb{C})$ has a multiplicative inverse, $x^{-1} = -x\|x\|^{-2}$. Up to the sign this inverse corresponds to the usual Kelvin inverse of a non-zero vector in $\mathbb{R}^n$.

For an element $A = a_0 + \cdots + a_{1\cdots n} e_1 \cdots e_n \in \mathrm{Cl}_n(\mathbb{C})$ its norm is defined to be $\|A\| = (a_0^2 + \cdots + a_{1\cdots n}^2)^{\frac{1}{2}}$.

Suppose now that U is a domain in $\mathbb{R}^n$. We shall be interested in differentiable functions defined on U and taking their values in the Clifford algebra $\mathrm{Cl}_n(\mathbb{C})$. In particular we shall be interested in such a function, f, which satisfies the equation $Df = 0$, where $D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$. Such a function is called a left monogenic function. We are also interested in right monogenic functions. A right monogenic function satisfies the equation $gD = 0$. An example of a function which is both left and right monogenic is $G(x) = \frac{1}{\omega_n} \frac{x}{\|x\|^n}$, where ω_n is the surface area of the unit sphere lying in $\mathbb{R}^n$. Using this function we may easily establish the following analogue of Cauchy's integral formula:

Theorem 1.1. *Suppose $f : U \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function, and U is a bounded domain with a Lipschitz continuous boundary. Suppose also that f has a continuous extension to the boundary of U, and we denote this extension also by f. Then for each $y \in U$ we have that $f(y) = \int\limits_{\partial U} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)$, where $n(x)$ is the outward pointing normal vector to U at x, and σ is the Lebesgue measure on the surface ∂U.*

This theorem was first established in a modified form in 3 dimensions in [2]. The proof follows the same basic lines as the classical case in complex analysis of one variable. Many basic properties of monogenic functions and the Clifford-Cauchy integral formula are given in [1], [14] and elsewhere, see for instance the work of Moisil and Theodorescu ([11]) and Iftimie ([6]). It should also be mentioned that in [5], [7], Cauchy integral formulae have been introduced for Dirac operators more closely related to the Helmholtz equation, than the one used here. However, the ideas presented here may be adapted to fit that context too, though some of the estimates used will need to be modified.

In [3] the following modification of Theorem 1.1 is established:

Theorem 1.2. *Suppose that U is a domain lying in a half space of $\mathbb{R}^n$, and that f is a bounded left monogenic function defined on U. Suppose also that U has a Lipschitz continuous boundary and that f continuously extends to the boundary of U. Then for each $y \in U$ we have that*

$$f(y) = \int\limits_{\partial U} M(x,y)n(x)f(x)\,\mathrm{d}\sigma(x),$$

where $M(x,y) = G(x-y) - G(x+y)$.

In [3] we also show that Theorem 1.2 is also true if we assume that $\|f(x)\| \leqslant C\|x\|^s$ for some constant $C \in \mathbb{R}^+$ and some $s \in [0,1)$.

Adapting arguments worked out in the four dimensional setting in [14], see for instance [13], we have:

Proposition 1.3. *Suppose that f is left monogenic in the variable $y = x^{-1}$ on the domain U^{-1}, where U^{-1} is the Kelvin inverse of the domain U. Then the function $G(x)f(x^{-1})$ is left monogenic on the domain U.*

By combining the remark following Theorem 1.1 with Proposition 1.3, and noting that $x^{-1} - y^{-1} = y^{-1}(y-x)x^{-1}$, in [3] we deduce:

THEOREM 1.4. *Suppose that U is a domain lying in a half space and that 0 lies on the boundary of U. Suppose also that f is a left monogenic function defined in a neighbourhood of $U \setminus \{0\}$, and that there is a real, positive constant C such that $\|f(x)\| \leqslant C\|x\|^{-n+1-s}$ on U, where $s \in [0,1)$. Then for each $y \in U$ we have that*

$$f(y) = \mathrm{P.V.} \int_{\partial U} L(x,y)n(x)f(x)\,\mathrm{d}\sigma(x),$$

where $L(x,y) = G(x-y) + G(x+y)$.

2. THE DECOMPOSITION

We begin this section by introducing a Cauchy integral formula for special types of monogenic functions defined on special types of domains.

THEOREM 2.1. *Suppose that $f : V \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function which satisfies $\|f(x)\| < C\|x\|^{-n+1-s}$ for some $C \in \mathbb{R}^+$ and $s \in (0,1)$. Suppose also that the domain U has a piecewise C^1 boundary, ∂U, that $0 \in \partial U$, and that V is a neighbouhood of $\mathrm{cl}(U) \setminus \{0\}$. Then for each $y \in U$ we have*

$$f(y) = \mathrm{P.V.} \int_{\partial U} (G(x-y) + G(y))n(x)f(x)\,\mathrm{d}\sigma(x). \tag{2.1}$$

Proof. First we choose $r, R \in \mathbb{R}^+$ such that $r < \|y\| < R$. Then from Cauchy's integral formula and Cauchy's theorem we have that

$$f(y) = \int_{\partial U(r,R)} (G(x-y) + G(y))n(x)f(x)\,\mathrm{d}\sigma(x),$$

where $U(r,R) = (U \cap B(0,R)) \setminus B(0,r)$, and $B(0,r) = \{x \in \mathbb{R}^n : \|x\| < r\}$. As $(x-y)^{-1} + y^{-1} = y^{-1}x(y-x)^{-1}$, it may be determined that

$$\|G(x-y) + G(y)\| < C_1\|x\| \sum_{j=0}^{n-2} \|y\|^{-j-1}\|x-y\|^{-n+1+j},$$

for some $C_1 \in \mathbb{R}^+$. Consequently,

$$\int_{S^{n-1}(0,r)\cap U} \|(G(x-y) + G(y))n(x)f(x)\|\,\mathrm{d}\sigma(x) < C(y)r^{n-1-s}$$

for some $C(y) \in \mathbb{R}^+$, provided $\|y\| > 2\|x\|$. So

$$\lim_{r\to 0} \int\limits_{S^{n-1}(0,r)\cap U} (G(x-y)+G(y))n(x)f(x)\,\mathrm{d}\sigma(x) = 0.$$

Similarly for $\|x\| > 2\|y\|$

$$\int\limits_{S^{n-1}(0,R)\cap U} \|(G(x-y)+G(y))n(x)f(x)\|\,\mathrm{d}\sigma(x) < C_2 R^{-s}.$$

So

$$\lim_{R\to\infty} \int\limits_{S^{n-1}(0,R)\cap U} (G(x-y)+G(y))n(x)f(x)\,\mathrm{d}\sigma(x) = 0.$$

The result follows. ∎

An interesting case to consider is when the domain, U, is a sector domain. Such domains were considered in [8], [9], [10] and elsewhere. For $0 < \mu < \frac{\pi}{2}$, we shall consider the sector domain

$$S_\mu = \{x \in \mathbb{R}^n : |x_1| < \|x_2 e_2 + \cdots + x_n e_n\| \tan\mu\}.$$

This type of sector domains is not as general as the type appearing in [8], [9], [10]. However, one can simply rotate the sector domains introduced here to obtain those more general sector domains. The results that we obtain here holds equally well on those more general sector domains.

In [8], [9], [10] one considers right monogenic functions $f : S_\mu \to \mathrm{Cl}_n(\mathbb{C})$ which satisfy $\|f(x)\| < C\|x\|^{-n+1}$ for some constant $C \in \mathbb{R}^+$. The integral formula appearing in the previous theorem does not work for such functions. However, one can replace the kernel $G(x-y)+G(y)$ appearing in Theorem 2.1 by the kernel $G(x-y)-G(x-w)$, where $w \in \mathbb{R}^n \setminus \mathrm{cl}(U)$. In this case

$$\|G(x-y)-G(x-w)\| < C\|y-w\| \sum_{j=1}^{n-1} \|x-y\|^j \|x-w\|^{n-j}.$$

Now using this kernel, and the previous inequality, one may readily adapt arguments appearing in [3] to obtain:

THEOREM 2.2. *Suppose that U is a domain in $\mathbb{R}^n$, and that U has a piecewise smooth, or Lipschitz, boundary and that 0 lies on the boundary of U. Suppose also that $w \in \mathbb{R}^n \setminus \mathrm{cl}(U)$. Then for each left monogenic function $f : V \to \mathrm{Cl}_n(\mathbb{C})$ satisfying $\|f(x)\| < C\|x\|^s$ for some $s \in [0,1)$, where V is a neighbourhood of $\mathrm{cl}(U) \setminus \{0\}$, then*

$$f(y) = \mathrm{P.V.} \int_{\partial U} (G(x-y) - G(x-w))n(x)f(x)\,\mathrm{d}\sigma(x), \tag{2.2}$$

for each $y \in U$.

When $s = 0$ Theorem 2.2 gives a Cauchy integral formula for a bounded monogenic function on U. To get a Cauchy integral formula for the type of monogenic functions described in [8], [9], [10] we need to use Kelvin inversion. Using the identity $u^{-1} - v^{-1} = u^{-1}(v-u)v^{-1}$ one may determine that

$$G(u^{-1} - v^{-1}) = \omega_n^2 G(u)^{-1} G(v-u) G(v)^{-1},$$

see for instance [12]. On placing $x = u^{-1}$ and $y = v^{-1}$ and substituting into (2.2) we have from Proposition 1.3, on noting that $n(x)\mathrm{d}\sigma(x) = \omega_n^2 G(u)n(u)G(u)\mathrm{d}\sigma(u)$ for $x = u^{-1}$:

THEOREM 2.3. *Suppose that $g : W \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function defined on the neighbourhood W of $\mathrm{cl}(U)^{-1}$, the Kelvin inverse of U. Suppose also that $\|g(v)\| < C\|v\|^{-n+1-s}$ for some $C \in \mathbb{R}^+$, some $s \in [0,1)$ and for each $v \in W$. Then*

$$g(v) = \mathrm{P.V.} \int_{\partial U^{-1}} K(u,v)n(u)g(u)\,\mathrm{d}\sigma(u)$$

for each $v \in U^{-1}$, where $K(u,v) = G(v)G(q)G(u-q) + G(u-v)$ and $q = w^{-1}$, and $w \in \mathbb{R}^n \setminus \mathrm{cl}(U)$.

On noting that $S_\mu^{-1} = S_\mu$ for each sector domain, S_μ, it can be seen that the previous theorem gives a Cauchy integral formula for the type of monogenic functions used in [8], [9], [10].

Each sector domain, S_μ, is bounded by two cones

$$C_\mu^+ = \{x \in \mathbb{R}^n : x_1 = \|x_2 e_2 + \cdots + x_n e_n\| \tan\mu\}$$

and

$$C_\mu^- = \{x \in \mathbb{R}^n : -x_1 = \|x_2 e_2 + \cdots + x_n e_n\| \tan\mu\}.$$

Also, $S_\mu = S_\mu^+ \cap S_\mu^-$, where $S_\mu^+ = S_\mu \cup \mathbb{R}^{n,+}$ and $S_\mu^- = S_\mu \cup \mathbb{R}^{n,-}$, with $\mathbb{R}^{n,+} = \{x \in \mathbb{R}^n : x_1 > 0\}$, and $\mathbb{R}^{n,-} = \{x \in \mathbb{R}^n : x_1 < 0\}$.

From Theorem 2.3 it follows that if $f : S_\mu \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function, and $\|f(x)\| < C\|x\|^{-n+1-s}$ for some $s \in [0,1)$ and some $C \in \mathbb{R}^+$, then $f = f_1 + f_2$, where $f_1 : S_\mu^+ \to \mathrm{Cl}_n(\mathbb{C})$, and $f_2 : S_\mu^- \to \mathrm{Cl}_n(\mathbb{C})$ are left monogenic functions. Moreover,

$$f_1(y) = \mathrm{P.V.} \int\limits_{C_\mu^+} K(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

for each $y \in S_\mu^+$, and

$$f_2(y) = \mathrm{P.V.} \int\limits_{C_\mu^-} K(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

for each $y \in S_\mu^-$.

The decomposition described in the previous paragraph occurs not only over sector domains, but over many other types of domains too, including the rotated paraboloid $\{x \in \mathbb{R}^n : |x_1| < (x_2^2 + \cdots + x_n^2)\}$. In fact the type of decomposition described in the previous paragraph happens over any domain U which satisfies

(i) $0 \in U$,

(ii) $U = U_1 \cap U_2$, where $U_1 \cup U_2 = \mathbb{R}^n \setminus \{0\}$ and $\partial U = \partial U_1 \cap \partial U_2$ with $\partial U_1 \cap \partial U_2 = \{0\}$,

(iii) ∂U is Lipschitz continuous,

(iv) the surfaces ∂U_1 and ∂U_2 have no boundary.

We shall call such a domain a decomposition domain. So we have:

PROPOSITION 2.4. *Suppose that $f : V \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function defined in a neighbourhood V of a decomposition domain U. Suppose also that $\|f(x)\| < C\|x\|^{-n+1-s}$ for some $C \in \mathbb{R}^+$ and $s \in [0,1)$. Then $f = f_1 + f_2$ where the left monogenic function f_j is defined by*

$$f_j(y) = \mathrm{P.V.} \int\limits_{\partial U_j} K(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

for each $y \in U_j$ and $j = 1,2$.

A general decomposition result for monogenic functions is obtained in [1] over general domains, and using different techniques. The methods employed here give explicit computations for a wide class of domains, and monogenic functions. The results obtained here also improve on Theorem 1.4 as we no longer need to restrict ourselves to domains lying in a half plane.

Again using Kelvin inversion and Propositions 1.3 and 2.4 we have:

PROPOSITION 2.5. *Suppose that $f : V \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function, and that V is a neighbourhood of $\mathrm{cl}(U) \setminus \{0\}$. Suppose also that $\|f(x)\| < C\|x\|^s$ for some $C \in \mathbb{R}^+$ and $s \in [0,1)$, and U is a decomposition domain. Then $f = f_1 + f_2$, where*

$$f_j(y) = \int_{\partial U_j} (G(x-y) - G(x-w))n(x)f(x)\,\mathrm{d}\sigma(x)$$

for $j = 1,2$, $y \in U_j$, and $w \in \mathbb{R}^n \setminus \mathrm{cl}(U)$.

We also have the following Cauchy integral formula:

THEOREM 2.6. *Suppose that $f : W \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function on a neighbourhood W of $\mathrm{cl}(U) \setminus \{0\}$, and that $0 \in \partial U$, and $\|f(x)\| < C\|x\|^s$ for some $C \in \mathbb{R}^+$ and $s \in (-n+1, 0)$. Then*

$$f(y) = \mathrm{P.V.} \int_{\partial U} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x).$$

Proof. First we choose r and $R \in \mathbb{R}^+$ such that $r < \|y\| < R$. Then from Cauchy's integral formula we have that

$$f(y) = \int_{\partial U(r,R)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x),$$

where $U(r,R)$ is as in the proof of Theorem 2.1.

Now

$$\Big\| \int_{S^{n-1}(0,R)\cap U} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < CR^s$$

for some $C \in \mathbb{R}^+$, provided that R is sufficently large.

Also,

$$\Big\| \int_{S^{n-1}(0,r)\cap U} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < C(y)r^{n-1-s}$$

for some $C(y) \in \mathbb{R}^+$.

Consequently,

$$f(y) = \lim_{r\to 0}\lim_{R\to\infty} \int_{\partial U(r,R)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x).$$

The result follows. ∎

One can also deduce that if U is a decomposition domain then $f = f_1 + f_2$, where

$$f_j(y) = \int\limits_{\partial U_j} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x),$$

for $j = 1, 2$.

Using this observation and Propositions 2.4 and 2.5 we have the following decomposition result:

Theorem 2.7. *Suppose that* $f : W \to \mathrm{Cl}_n(\mathbb{C})$ *is a left monogenic function defined on the neighbourhood,* W *of* $\mathrm{cl}(U)\backslash\{0\}$*, where* U *is a decomposition domain. Suppose also that* $\|f(x)\| < C\|x\|^s$ *for some* $C \in \mathbb{R}^+$ *and* $s \in (-n, 1)$*. Then* $f = f_1 + f_2$*, where* f_j *is a left monogenic function defined on* U_j *for* $j = 1, 2$*.*

The previous result is a special case of a result in [1]. However, the following result does not automatically follow from that result in [1].

Theorem 2.8. *Suppose that* S_μ *is a sector domain and* $f : W \to \mathrm{Cl}_n(\mathbb{C})$ *is a left monogenic function defined on the neighbourhood* W *of* $\mathrm{cl}(S_\mu) \setminus \{0\}$*. Suppose also that* $\|f(x)\| < C\|x\|^s$ *for some* $C \in \mathbb{R}^+$ *and some* $s \in (-n+1, 0)$*. Then* $\|f_j(x)\| < C_{j,\nu}\|x\|^s$*, where* $C_{j,\nu} \in \mathbb{R}^+$*,* $x \in S_\nu^\pm$*, with* $0 < \nu < \mu$ *and* f_j *is as in Theorem* 2.7*, for* $j = 1, 2$*.*

Proof. First let us place $C_\mu^\pm = V_{\pm,a}(y) \cup V_{\pm,b}(y)$, where $V_{\pm,a}(y) = \{x \in C_\mu^\pm : \|x\| > N\|y\|\}$ and $V_{\pm,b}(y) = C_\mu^\pm \setminus V_{\pm,a}(y)$. Then

$$\Big\| \int\limits_{V_{\pm,a}(y)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < C \int\limits_{N\|y\|}^{\infty} r^{s-1}\,\mathrm{d}r,$$

for some $C \in \mathbb{R}^+$, provided that the integer N is chosen to be large enough. Consequently,

$$\Big\| \int\limits_{V_{\pm,a}(y)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < C_1\|y\|^s.$$

Also, elementary inequalities and trigonometry give

$$\Big\| \int\limits_{V_{\pm,b}(y)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < C(\nu)\|y\|^{-n+1} \int\limits_{0}^{N\|y\|} r^{n-2+s}\,\mathrm{d}r,$$

for some $C(\nu) \in \mathbb{R}^+$, provided $y \in S_\nu^\pm$. The result follows. ∎

The proof of Theorem 2.8 can easily be adapted to establish the existence of left monogenic functions which satisfy the inequality $\|f(x)\| < C\|x\|^s$ for some $s \in (-n+1, 0)$ on any sector domain S_μ or associated domain $S_\mu^{\pm}$. A minor adaptation of the proof of Theorem 2.8 gives:

PROPOSITION 2.9. *Suppose that $g : C_\mu^{\pm} \to \mathrm{Cl}_n(\mathbb{C})$ is a measurable function which satisfies the inequality $\|g(x)\| < C\|x\|^s$ for some $C \in \mathbb{R}^+$, some $s \in (-n+1, 0)$ and for almost all $x \in C_\mu^{\pm}$. Then the left monogenic function*

$$f(y) = \int\limits_{\partial S_\mu^{\pm}} G(x-y)n(x)g(x)\,\mathrm{d}\sigma(x)$$

satisfies $\|f(x)\| < C_\nu\|x\|^s$ on $S_\nu^{\pm}$ for some $C_\nu \in \mathbb{R}^+$, and each ν with $0 < \nu < \mu$.

We now proceed to find analogues for the previous two results for the two extremal cases, $s = 0$, and $s = -n+1$. We start with $s = 0$.

THEOREM 2.10. *Suppose that $f : W \to \mathrm{Cl}_n(\mathbb{C})$ is a bounded left mongenic function defined on a neighbourhood W of $\mathrm{cl}(S_\mu) \setminus \{0\}$, for some sector domain S_μ. Then $\|f_j(x)\| < C(\nu)$ for some $C(\nu) \in \mathbb{R}^+$ with $0 < \nu < \mu$, and f_j is as in Theorem 8.*

Proof. We shall use the kernel $G(x-y) - G(x+e_1)$. Now

$$\|G(x-y) - G(x+e_1)\| < C\|y-e_1\| \sum_{j=1}^{n-1} \|x-e_1\|^{-j}\|x-y\|^{-n+j} \tag{2.3}$$

for some real number C. We shall first consider the case where $\|y\| > 1$. In this case $\|y+e_1\| < 2\|y\|$. It follows from (3) that

$$\|G(x-y) - G(x-e_1)\| < C(1)\|y\| \sum_{1}^{n-1} \|x-e_1\|^{-j}\|x-y\|^{-n+j},$$

for some $C(1) \in \mathbb{R}^+$. As in the proof of Theorem 2.8 we shall place $C_\mu^{\pm} = V_{\pm,a}(y) \cup V_{\pm,b}(y)$. It follows that

$$\Big\| \int\limits_{V_{\pm,a}(y)} (G(x-y) - G(X+e_1))n(x)f(x)\,\mathrm{d}\sigma(x) \Big\| < C(2)\|y\| \int\limits_{N\|y\|}^{\infty} r^{-2}\,\mathrm{d}r,$$

for some $C(2) \in \mathbb{R}^+$. Consequently,

$$\Big\| \int\limits_{V_{\pm,a}(y)} (G(x-y) - G(x+e_1))n(x)f(x)\,\mathrm{d}\sigma(x) \Big\| < C(3),$$

for some $C(3) \in \mathbb{R}^+$.

Now let us consider $\int_{V_{\pm,b}(y)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)$. The same elementary trigonometry and inequalities used in the proof of Theorem 2.8 show that

$$\Big\| \int_{V_{\pm,b}(y)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < C_\nu \|y\|^{n-1} \int_0^{N\|y\|} r^{n-2}\,\mathrm{d}r,$$

for $y \in S_\nu^{\pm}$. So $\int_{V_{\pm,b}(y)} G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)\| < C_\nu(1)$ for some $C_\nu(1) \in \mathbb{R}^+$.

Let us now consider $\int_{V_{\pm,b}(y)} G(x+e_1)n(x)f(x)\,\mathrm{d}\sigma(x)$. Now $\|x+e_1\| > |\sin\mu|\,\|x\|$. Elementary trigonometry now reveals that

$$\Big\| \int_{V_{\pm,b}(y)} G(x-e_1)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < C(3)|\operatorname{cosec}\mu|^{-n+2}\|y\|^{-1} \int_0^{N\|y\|} \mathrm{d}r,$$

for some $C(3) \in \mathbb{R}^+$.

Therefore,

$$\Big\| \int_{V_{\pm,b}(y)} G(x-e_1)n(x)f(x)\,\mathrm{d}\sigma(x)\Big\| < NC(3).$$

To consider the cases where $\|y\| \leqslant 1$ we repeat the previous arguments but in these cases we integrate over $V_{\pm,a}(e_1)$ and $V_{\pm,b}(e_1)$. In these cases we obtain similar estimates to those already obtained in this proof. This completes the proof. ∎

The Kelvin inverse of S_μ^+ is S_μ^-, while the Kelvin inverse of S_μ^- is S_μ^+. Consequently, on applying Proposition 1.3 to Theorem 2.10 we obtain

THEOREM 2.11. *Suppose that $f : W \to \mathrm{Cl}_n(\mathbb{C})$ is a left monogenic function and that W is a neighbourhood of $\mathrm{cl}(S_\mu)\backslash\{0\}$, for some sector domain S_μ. Suppose also that $\|f(x)\| < C\|x\|^{-n+1}$ for some $C \in \mathbb{R}^+$. Then $\|f_j(x)\| < C(j,\nu)\|x\|^{-n+1}$ for some $C(j,\nu) \in \mathbb{R}^+$, where f_j is as in Theorem 2.7, and $0 < \nu < \mu$.*

The proof of Theorem 2.10 and the argument given to establish Theorem 2.11 can be adapted to give the apropriate analogues of Proposition 2.9 for the extreme cases $s = 0$ and $s = -n+1$.

Acknowledgements. This paper was initiated while the author was a recipient of an Arkansas Science and Technology Authority grant, and was completed while the author was the recipient of a von Humboldt Research Fellowship, visiting the Bergakademie in Freiberg, Saxony, Germany. The central idea for this paper arose while the author was a visitor to the Analysis Research Group at Macquarie University, Sydney, Australia. The author is grateful to Alan McIntosh for that invitation, and to Edwin Franks for coversations that lead to the basic ideas behind this paper.

REFERENCES

1. F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Res. Notes Math. Ser., vol. 76, London 1982.
2. A.C. Dixon, On the Newtonian potential, *Quart. J. Math. Oxford Ser. (2)* **35**(1904), 283–296.
3. E. Franks, J. Ryan, Bounded monogenic functions on unbounded domains, *Contemp. Math.*, to appear.
4. K. Guerlebeck, U. Kaehler, J. Ryan, W. Sproessig, Clifford analysis over unbounded domains, *Adv. Appl. Math.* **19**(1997), 216–239.
5. K. Guerlebeck, W. Sproessig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Birkhauser, Boston 1990.
6. V. Iftimie, Fonctions hypercomplexes, *Bull. Math. Soc. Sci. Math. R.S. Roumanie* **9**(1965), 279–332.
7. V. Kravchenko, M. Shapiro, *Integral Representations for Spatial Models of Mathematical Physics*, Pitman, Res. Notes Math. Ser., vol. 351, London 1996.
8. C. Li, A. McIntosh, T. Qian, Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces, *Rev. Mat. Iberoamericana* **10**(1994), 665–721.
9. C. Li, A. McIntosh, S. Semmes, Convolution singular integrals on Lipschitz surfaces, *J. Amer. Math. Soc.* **5**(1992), 455–481.
10. A. McIntosh, Clifford algebras, Fourier theory, singular integrals, and hermonic functions on Lipschitz domains, in *Clifford Algebras in Analysis and Related Topics*, Stud. Adv. Math., CRC Press, Boca Raton, 1995, pp. 33–87.
11. Gr. Moisil, N. Theodorescu, Fonctions holomorphes dans l'espace, *Mathematica (Cluj)* **5**(1931), 142–159.
12. J. Peetre, T. Qian, Moebius covariance of iterated Dirac operators, *J. Austral. Math. Soc. Ser. A* **56**(1994), 403–414.
13. J. Ryan, Conformally covariant operators in Clifford analysis, *Z. Anal. Anwendungen* **14**(1995), 677–704.
14. A. Sudbery, Quaternionic analysis, *Math. Proc. Cambridge Philos. Soc.* **86**(1979), 199–225.

JOHN RYAN
Department of Mathematics
University of Arkansas
Fayetteville, AR 72701
U.S.A.

Received September 21, 1996; revised January 6, 1997.