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CERTAIN STRUCTURE OF SUBDIAGONAL ALGEBRAS

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Abstract. Let $\mathfrak{A}$ be a maximal subdiagonal algebra of a von Neumann algebra $\mathcal{M}$ with respect to a faithful normal expectation Φ. Then we show that if φ is a faithful normal state of $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$, then $\mathfrak{A}$ is σ_t^φ-invariant, where $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ is the modular automorphism group associated with φ. As an application, we prove that every σ-weakly closed subdiagonal algebra of $\mathcal{B}(\mathcal{H})$ is a nest algebra with an atomic nest.

Keywords: *von Neumann algebra, subdiagonal algebra.*

MSC (2000): Primary 46L10; Secondary 47D25.

1. INTRODUCTION AND PRELIMINARIES

In [1], Arveson introduced the notion of subdiagonal algebras to study the analyticity in operator algebras. At first, we start by given the definition of subdiagonal algebras. Let $\mathcal{M}$ be a von Neumann algebra on a separable complex Hilbert space $\mathcal{H}$, and let Φ be a faithful normal positive idempotent linear map from $\mathcal{M}$ onto a von Neumann subalgebra $\mathfrak{D}$ of $\mathcal{M}$. A σ-weakly closed subalgebra $\mathfrak{A}$ of $\mathcal{M}$, containing $\mathfrak{D}$, is called subdiagonal algebra in $\mathcal{M}$ with respect to Φ if

(i) $\mathfrak{A} \cap \mathfrak{A}^* = \mathfrak{D}$,
(ii) Φ is multiplicative on $\mathfrak{A}$, and
(iii) $\mathfrak{A} + \mathfrak{A}^*$ is σ-weakly dense in $\mathcal{M}$.

The algebra $\mathfrak{D}$ is called the diagonal of $\mathfrak{A}$. We say that $\mathfrak{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to Φ in case $\mathfrak{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to Φ.

Although subdiagonal algebras are not assumed to be σ-weakly closed in [1], the σ-weak closure of a subdiagonal algebra is again a subdiagonal algebra ([1], Remark 2.1.2). Thus we assume that our subdiagonal algebras are always σ-weakly closed.

In [1], Arveson asked whether all (σ-weakly closed) subdiagonal algebras are maximal subdiagonal algebras. In [3], Exel gave an affirmative answer to this problem in finite case. That is, Exel showed that if $\mathfrak{A}$ is finite in the sense that there exists a faithful normal finite trace τ on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$, then $\mathfrak{A}$ is automatically maximal subdiagonal.

In general, let $\mathfrak{A}$ be a subdiagonal algebra of $\mathcal{M}$ with respect to Φ. Then there exists a faithful normal state φ of $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$ by the separability of $\mathcal{H}$.

First, we shall prove that, if $\mathfrak{A}$ is maximal, then $\mathfrak{A}$ is invariant under the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ of φ (cf. [5]). Further, as an application, we shall show that every subdiagonal algebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is a nest algebra with an atomic nest.

2. σ_t^φ-INVARIANCE OF SUBDIAGONAL ALGEBRAS

Let $\mathcal{M}$ be a von Neumann algebra, acting on a separable Hilbert space $\mathcal{H}$, and let $\mathfrak{A}$ be a subdiagonal algebra of $\mathcal{M}$ with respect to Φ. Put $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{A}^*$. Then there exists a faithful normal state φ on $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$. Without loss of generality, we may assume that $\mathcal{M}$ has a cyclic and separating vector ξ_0 in $\mathcal{H}$ such that $\varphi(T) = (T\xi_0, \xi_0)$ for any $T \in \mathcal{M}$.

Put $\mathfrak{A}_0 = \{X \in \mathfrak{A} : \Phi(X) = 0\}$, and we define the closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ by $\mathcal{H}_1 = [\mathfrak{A}_0\xi_0]$, $\mathcal{H}_2 = [\mathfrak{D}\xi_0]$ and $\mathcal{H}_3 = [\mathfrak{A}_0^*\xi_0]$ respectively, where $[\mathcal{S}]$ is the closed linear span of a subset $\mathcal{S}$ of $\mathcal{H}$. Let P_i be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_i$ for every $i = 1, 2, 3$. Let $\mathfrak{A}_m$ be the set of all $A \in \mathcal{M}$ such that $\Phi(\mathfrak{A}A\mathfrak{A}_0) = \Phi(\mathfrak{A}_0A\mathfrak{A}) = 0$. By [1], Theorem 2.2.1, we recall that $\mathfrak{A}_m$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to Φ containing $\mathfrak{A}$. Then we easily have the following lemma.

LEMMA 2.1. *Keep the notation as above. Then*
(i) $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$;
(ii) $\mathfrak{D}\mathcal{H}_i \subseteq \mathcal{H}_i$ $(i = 1, 2, 3)$;
(iii) $\mathcal{H}_1 = [(\mathfrak{A}_m)_0\xi_0]$ *and* $\mathcal{H}_3 = [(\mathfrak{A}_m)_0^*\xi_0]$;
(iv) $(\mathfrak{A}_m)_0(\mathcal{H}_1 \oplus \mathcal{H}_2) \subseteq \mathcal{H}_1$ *and* $(\mathfrak{A}_m)_0^*(\mathcal{H}_2 \oplus \mathcal{H}_3) \subseteq \mathcal{H}_3$.

Considering the Hilbert space decomposition in (i), we have the following lemma.

LEMMA 2.2. *Keep the assumptions and the notation as above. Then*

$$\mathfrak{D} = \left\{ D \in \mathcal{M} : D = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \right\}$$

and

$$(\mathfrak{A}_m)_0 = \left\{ X \in \mathcal{M} : X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} \right\}.$$

Proof. Put

$$\mathfrak{B} = \left\{ D \in \mathcal{M} : D = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \right\}$$

and

$$\mathfrak{C} = \left\{ X \in \mathcal{M} : X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} \right\}$$

respectively. Then it is clear that $\mathfrak{D} \subseteq \mathfrak{B}$ and $(\mathfrak{A}_m)_0 \subseteq \mathfrak{C}$.

If $D \in \mathfrak{B}$, then $\Phi(D) \in \mathfrak{D} \subseteq \mathfrak{B}$ and so $\Phi(D)$ has the matrix form as follows:

$$\Phi(D) = \begin{pmatrix} V_{11} & 0 & 0 \\ 0 & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix}.$$

Since $\mathfrak{A}_0 + \mathfrak{D} + \mathfrak{A}_0^*$ is σ-weakly dense in $\mathcal{M}$, $\mathfrak{A}_0 + \mathfrak{A}_0^*$ is σ-weakly dense in $\mathrm{Ker}(\Phi)$. Let P_2 be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_2$. Then it is clear that $P_2(X\xi_0) = \Phi(X)\xi_0$, $X \in \mathcal{M}$. Thus $P_2XP_2 = 0$ for every $X \in \mathrm{Ker}(\Phi)$. Since $D - \Phi(D) \in \mathrm{Ker}(\Phi)$, we have $D_{22} - V_{22} = P_2(D - \Phi(D))P_2 = 0$. Hence we have

$$(D - \Phi(D))\xi_0 = (D_{22} - V_{22})\xi_0 = 0.$$

Since ξ_0 is a separating vector for $\mathcal{M}$ and $\xi_0 \in \mathcal{H}_2$, we have $D = \Phi(D) \in \mathfrak{D}$ and so $\mathfrak{D} = \mathfrak{B}$.

On the other hand, take any $X \in \mathfrak{C}$. Since $\Phi(X) \in \mathfrak{D}$, $\Phi(X)$ is of the form

$$\Phi(X) = \begin{pmatrix} V_{11} & 0 & 0 \\ 0 & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix}.$$

Then we similarly have $P_2(\Phi(X) - X)P_2 = V_{22} = 0$. Thus $\Phi(X) = 0$.

It is trivial that $\mathfrak{C}$ is a $\mathfrak{D}$-bimodule and $(\mathfrak{A}_m)_0 \subseteq \mathfrak{C}$. Hence it is easy to check that $\mathfrak{D} + \mathfrak{C}$ is a subdiagonal algebra of $\mathcal{M}$ with respect to Φ, containing $\mathfrak{A}_m$. By the maximality of $\mathfrak{A}_m$, we have $\mathfrak{A}_m = \mathfrak{D} + \mathfrak{C}$. This implies that $(\mathfrak{A}_m)_0 = \mathfrak{C}$. This completes the proof. ∎

From Tomita-Takesaki theory, we define conjugate-linear operators S_0 and F_0, with dense domains $\{\mathcal{M}\xi_0\}$ and $\{\mathcal{M}'\xi_0\}$ respectively by

$$S_0 A\xi_0 = A^*\xi_0 \quad \text{and} \quad F_0 B\xi_0 = B^*\xi_0 \quad (A \in \mathcal{M},\ B \in \mathcal{M}').$$

By [5], Lemma 9.2.1, the operator S_0 is preclosed, and the adjoint F is an extension of F_0. Further, if S is the closure of S_0, then $S^* = F$. Let $\mathcal{D}(S)$ be the domain of S, and let $G(S)$ be the graph of S.

LEMMA 2.3. *Keep the notation as above. Then the closed operator S has the following matrix decomposition with respect to Lemma* 2.1 (i)*:*

$$S = \begin{pmatrix} 0 & 0 & S_3 \\ 0 & S_2 & 0 \\ S_1 & 0 & 0 \end{pmatrix}$$

where for $i = 1, 2, 3$, S_i is a closed operator with domain $\mathfrak{F}_i$ in $\mathcal{H}_i$ such that $S_1\mathfrak{F}_1 = \mathfrak{F}_3$, $S_2\mathfrak{F}_2 = \mathfrak{F}_2$ and $S_3\mathfrak{F}_3 = \mathfrak{F}_1$.

Proof. Since $\{\mathfrak{A}_0\xi_0\} \oplus \{\mathfrak{D}\xi_0\} \oplus \{\mathfrak{A}_0^*\xi_0\} \subseteq \mathcal{D}(S)$, we can define a preclosed operator V_0 by

$$V_0(A + D + B^*)\xi_0 = (A^* + D^* + B)\xi_0$$

for $A,\ B \in \mathfrak{A}_0$ and $D \in \mathfrak{D}$. Since S_0 is an extension of V_0, S is an extension of the closure V of V_0. Since $G(S)$ is the norm closure of $\{X\xi_0 \oplus X^*\xi_0 : X \in \mathcal{M}\}$ and $\mathfrak{A}_0 + \mathfrak{D} + \mathfrak{A}_0^*$ is σ-weakly dense in $\mathcal{M}$, it is easy to prove that $S = V$.

Let $\zeta \oplus S\zeta \in G(S)$. Since S is the closure of V_0, there exist $\{A_n, B_n\}_{n=1}^{\infty}$ in $\mathfrak{A}_0$ and $\{D_n\}_{n=1}^{\infty}$ in $\mathfrak{D}$ such that

$$\lim_{n\to\infty} (\|(A_n + D_n + B_n^*)\xi_0 - \zeta\|^2 + \|(A_n^* + D_n^* + B_n)\xi_0 - S\zeta\|^2) = 0.$$

Then, we have

$$\begin{aligned}\lim_{n\to\infty} (&\|A_n\xi_0 - P_1\zeta\|^2 + \|D_n\xi_0 - P_2\zeta\|^2 + \|B_n^*\xi_0 - P_3\zeta\|^2 \\ &+ \|A_n^*\xi_0 - P_3S\zeta\|^2 + \|D_n^*\xi_0 - P_2S\zeta\|^2 + \|B_n\xi_0 - P_1S\zeta\|^2) = 0,\end{aligned}$$

where P_i is the projection from $\mathcal{H}$ onto $\mathcal{H}_i$ for $i = 1, 2, 3$. This implies that $P_i\zeta \oplus SP_i\zeta \in G(S)$ and $P_i\mathcal{D}(S) \subset \mathcal{D}(S)$ for $i = 1, 2, 3$. Put $\mathfrak{F}_i = P_i\mathcal{D}(S)$. Since $S\mathfrak{F}_1 = \mathfrak{F}_3, S\mathfrak{F}_2 = \mathfrak{F}_2$ and $S\mathfrak{F}_3 = \mathfrak{F}_1$, we have the desired matrix form of S with respect to Lemma 2.1 (i). This completes the proof. ∎

Put $\Delta = S^*S$. We recall that the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ of $\mathcal{M}$ associated with φ has the following form:

$$\sigma_t^\varphi(X) = \Delta^{\mathrm{i}t} X \Delta^{-\mathrm{i}t} \quad (\forall\, t\in\mathbb{R},\ X\in\mathcal{M}).$$

Then we have the following theorem.

Theorem 2.4. *Let $\mathfrak{A}$ be a maximal subdiagonal algebra of $\mathcal{M}$ with respect to Φ and let φ is a faithful normal state of $\mathcal{M}$ such that $\varphi\circ\Phi = \varphi$. Then $\mathfrak{A}$ is σ_t^φ-invariant, that is $\sigma_t^\varphi(\mathfrak{A}) = \mathfrak{A}$.*

Proof. From Lemma 2.3, the adjoint S^* of S has the matrix form

$$S^* = \begin{pmatrix} 0 & 0 & S_1^* \\ 0 & S_2^* & 0 \\ S_3^* & 0 & 0 \end{pmatrix}$$

where S_i^* is the adjoint operator of S_i with domain $\mathfrak{F}_i^*$ $(i = 1,2,3)$. Then the modular operator Δ has the matrix form

$$\Delta = \begin{pmatrix} S_1^*S_1 & 0 & 0 \\ 0 & S_2^*S_2 & 0 \\ 0 & 0 & S_3^*S_3 \end{pmatrix}.$$

By Lemmas 2.2 and 2.3, it is easy to prove that, for every $t\in\mathbb{R}$,

$$\sigma_t^\varphi(\mathfrak{D}) = \mathfrak{D} \quad \text{and} \quad \sigma_t^\varphi(\mathfrak{A}_0) = \mathfrak{A}_0 \quad (t\in\mathbb{R}).$$

Thus we have the theorem. ∎

Let $\mathcal{M}^\varphi$ be the centralizer of $\mathcal{M}$ associated with φ, that is, $\mathcal{M}^\varphi = \{A\in\mathcal{M} : \varphi(AB) = \varphi(BA),\ \forall\, B\in\mathcal{M}\}$. Recall that $\mathcal{M}^\varphi$ is the fixed point algebra of $\mathcal{M}$ with respect to $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ and there exists a faithful normal expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{M}^\varphi$ (cf. [6], Theorem 1.2). We remark that $\mathcal{M}^\varphi$ is a finite von Neumann algebra.

Corollary 2.5. *Keep the notation and assumptions as above. Then $\mathcal{E}(\mathfrak{A})$ is a maximal finite subdiagonal algebra of $\mathcal{M}^\varphi$ with respect to $\Phi|_{\mathcal{M}^\varphi}$.*

Proof. Let $X\in\mathcal{M}$. By [6], Theorem 1.2, $\mathcal{E}(X)$ is in the σ-weak closure of convex hull of $\{\sigma_t^\varphi(X) : t\in\mathbb{R}\}$. Since $\mathfrak{A}$ is σ_t^φ-invariant by Theorem 2.4, we have $\mathcal{E}(\mathfrak{A}_0)\subset\mathfrak{A}_0$ and $\mathcal{E}(\mathfrak{D})\subset\mathfrak{D}$. Then we easily prove that $\mathcal{E}(\mathfrak{A})$ is a subdiagonal algebra of $\mathcal{M}^\varphi$ with respect to $\Phi|_{\mathcal{M}^\varphi}$. By [5], Proposition 9.2.14, $\varphi|_{\mathcal{M}^\varphi}$ is a faithful normal trace on $\mathcal{M}^\varphi$. Thus $\mathcal{E}(\mathfrak{A})$ is a finite subdiagonal algebra of $\mathcal{M}^\varphi$. By [3], Theorem 7, $\mathcal{E}(\mathfrak{A})$ is a maximal finite subdiagonal algebra of $\mathcal{M}^\varphi$. This completes the proof. ∎

Finally, we remark about the question in [1], Remark 2.2.3, whether every subdiagonal algebra must nessesarily be maximal subdiagonal. In [3], Exel showed that every finite subdiagonal algebra is maximal subdiagonal. By Theorem 2.4, we have following two questions.

QUESTION 2.6. Is there a subdiagonal algebra which is not σ_t^φ-invariant for some faithful normal state φ on $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$?

QUESTION 2.7. If $\mathfrak{A}$ is a σ_t^φ-invariant subdiagonal algebra of $\mathcal{M}$ for every faithful normal state φ on $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$, is $\mathfrak{A}$ maximal subdiagonal ?

3. SUBDIAGONAL ALGEBRAS OF $\mathcal{B}(\mathcal{H})$

Let $\mathcal{N}$ be a nest, that is, $\mathcal{N}$ is a chain of closed subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is closed under intersection and closed span. Then the nest algebra $\operatorname{alg}\mathcal{N}$ is the set of all operators T in $\mathcal{B}(\mathcal{H})$ such that $TN \subseteq N$ for every $N \in \mathcal{N}$. The intersection $\mathfrak{D} = \operatorname{alg}\mathcal{N} \cap (\operatorname{alg}\mathcal{N})^*$ is the diagonal of $\operatorname{alg}\mathcal{N}$. We recall that there exists a faithful normal expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathfrak{D}$ if and only if $\mathcal{N}$ is atomic (cf. [2], Theorem 8.6). In this case, $\operatorname{alg}\mathcal{N}$ is a subdiagonal algebra of $\mathcal{B}(\mathcal{H})$ with respect to the expectation. In this section, we consider the converse, that is, if $\mathfrak{A}$ is a subdiagonal algebra of $\mathcal{B}(\mathcal{H})$ with respect to a faithful normal expectation Φ, then $\mathfrak{A}$ is a nest algebra with an atomic nest.

Our main theorem in this section is the following.

THEOREM 3.1. *Let $\mathfrak{A}$ be a subdiagonal algebra of $\mathcal{B}(\mathcal{H})$ with respect to a faithful normal expectation Φ. Then there exists an atomic nest $\mathcal{N}$ such that $\mathfrak{A} = \operatorname{alg}\mathcal{N}$.*

The proof of Theorem 3.1 requires a few preliminary results.

Let $\mathfrak{A}$ be a subdiagonal algebra of $\mathcal{B}(\mathcal{H})$ with respect to a faithful normal expectation Φ. Since $\mathcal{H}$ is separable, there exists a faithful normal state φ on $\mathcal{B}(\mathcal{H})$ such that $\varphi \circ \Phi = \varphi$. Let ρ be the canonical trace of $\mathcal{B}(\mathcal{H})$. By [5], Lemma 9.2.19, there is a positive contraction K in $\mathcal{B}(\mathcal{H})$ such that $I - K$ is a trace-class operator and

$$\rho((I-K)A) = \varphi(KA) = \varphi(AK) \quad (A \in \mathcal{B}(\mathcal{H})).$$

Moreover, both K and $I-K$ are injective. Put $F = K^{-1}(I-K)$. Then by [5], Lemma 9.2.20, the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ of $\mathcal{B}(\mathcal{H})$ associated with φ is written as

$$\sigma_t^\varphi(X) = F^{\mathrm{i}t} X F^{-\mathrm{i}t} \quad (X \in \mathcal{B}(\mathcal{H}),\ t \in \mathbb{R}).$$

Let $\mathcal{B}(\mathcal{H})^\varphi$ be the centralizer of φ of $\mathcal{B}(\mathcal{H})$. Since the centralizer $\mathcal{B}(\mathcal{H})^\varphi$ is the fixed point algebra of $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ (cf. [5], p. 697), we can show that $\mathcal{B}(\mathcal{H})^\varphi$ is the commutant of $\{F\}$. Since $I-K$ is a trace class positive operator, we can write $I-K = \sum_{n=1}^{\infty}{}^{\oplus}\lambda_n P_n$, where $\{\lambda_n\}_{n=1}^\infty$ are the distinct eigenvalues of $I-K$ and for every n, P_n is the spectral projection of $I-K$ corresponding to λ_n. Since $F = K^{-1}(I-K) = \sum_{n=1}^{\infty}{}^{\oplus}\lambda_n(1-\lambda_n)^{-1}P_n$, we can decompose

$$\mathcal{B}(\mathcal{H})^\varphi = \sum_{n=1}^{\infty}{}^{\oplus} M_{k_n},$$

where k_n is the dimension of $P_n\mathcal{H}$, and M_{k_n} is the set of all $k_n \times k_n$ matrices on $P_n\mathcal{H}$.

As in Section 2, there exists a faithful normal expectation $\mathcal{E}$ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})^\varphi$. Then we have the following proposition.

Proposition 3.2. *Keep the notation and the assumptions as above. Then there exists a family $\{Q_n\}_{n=1}^\infty$ of mutually orthogonal rank one projections in the diagonal $\mathfrak{D}$ of $\mathfrak{A}$ such that $\sum_{n=1}^{\infty} Q_n = I$.*

Proof. As in Section 2, let $\mathfrak{A}_m$ be the maximal subdiagonal algebra with respect to Φ of $B(H)$ containing $\mathfrak{A}$. Let P_n be as above. At first, we shall show that $P_n \in \mathcal{E}(\mathfrak{D}) \subset \mathfrak{D}$. We consider an invertible operator $X = 2P_n \oplus (I-P_n)$ on $\mathcal{H} = P_n\mathcal{H} \oplus P_n^\perp\mathcal{H}$. By Corollary 2.5, $\mathcal{E}(\mathfrak{A}_m)$ is a finite maximal subdiagonal algebra of $\mathcal{B}(\mathcal{H})^\varphi$ with diagonal $\mathcal{E}(\mathfrak{D})$. Hence, by [1], Theorem 4.4.1, there exist a unitary operator U in $\mathcal{B}(\mathcal{H})^\varphi$ and an invertible operator A in $\mathcal{E}(\mathfrak{A}_m)\cap\mathcal{E}(\mathfrak{A}_m)^{-1}$ such that $X = UA$. Since P_n is a central projection of $\mathcal{B}(\mathcal{H})^\varphi$, we can decompose $U = U_1 \oplus U_2$ and $A = A_1 \oplus A_2$. It is clear that $A_1 = 2U_1^*$ and $A_2 = U_2^*$, so we have that $\sigma(A) = 2\sigma(U_1^*) \cup \sigma(U_2^*)$. It is also clear that $\sigma(U_1^*)(\subset \mathbb{T})$ is a finite subset of $\mathbb{T}$ and $\sigma(U_2^*) \subseteq \mathbb{T}$, where $\mathbb{T}$ is the unit circle. If $\sigma(U_2^*) \neq \mathbb{T}$, then we know that $\mathbb{C}\setminus\sigma(A)$ is connected. If $\sigma(U_2^*) = \mathbb{T}$, then the only bounded component of $\mathbb{C}\setminus\sigma(A)$ is the open disc $\mathbb{D}$. Thus we can choose a neighborhood Ω of $\sigma(A)$ such that

(1) $\Omega = \Omega_1 \cup \Omega_2$, $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$,

(2) $\sigma(A_1) \subset \Omega_1$, $\sigma(A_2) \subset \Omega_2$, and

(3) the only bounded component of $\mathbb{C} \setminus \Omega$ is a closed disc $\{\lambda \in \mathbb{C} : |\lambda| \leqslant r\}$ with radius r less than one.

Define the holomorphic function f on Ω by

$$f(z) = \begin{cases} 1, & z \in \Omega_1 \\ 0, & z \in \Omega_2. \end{cases}$$

By [8], Theorem 13.9 there is a sequence $\{R_i\}$ of rational functions with poles at 0, such that $R_i \to f$ uniformly on compact subsets of Ω. Since both A and A^{-1} are in $\mathcal{E}(\mathfrak{A}_m)$, we have that $P_n(= f(A)) \in \mathcal{E}(\mathfrak{A}_m)$, and so $P_n \in \mathcal{E}(\mathfrak{D})$. By Corollary 2.5, $P_n \in \mathfrak{D}$. Since $P_n\mathcal{E}(\mathfrak{A}_m)P_n$ is a subdiagonal algebra of $\mathcal{M}_{k_n}$, by [4], Theorem 2.1, $P_n\mathcal{E}(\mathfrak{A}_m)P_n$ is a nest subalgebra of $\mathcal{M}_{k_n}$ with a finite nest. So $P_n\mathcal{E}(\mathfrak{A}_m)P_n$ contains a family of mutually orthogonal rank one projections whose sum is P_n. This completes the proof. ∎

LEMMA 3.3. *Keep the assumptions as above and also assume that* $\mathfrak{A} \neq \mathcal{B}(\mathcal{H})$. *Then there exists a nontrivial* $\mathfrak{A}$*-invariant subspace of* $\mathcal{H}$.

Proof. By Proposition 3.2, we know that $\mathfrak{D}'$ is abelian. Hence, by Theorems 6.2.1 and 6.2.2 in [1], we have $\rho \circ \Phi = \rho$, where ρ is the canonical trace of $\mathcal{B}(\mathcal{H})$. By Proposition 3.2, there is a rank one projection $e \otimes e \in \mathfrak{D}$ such that $\{\mathfrak{A}_0 e\} \neq \{0\}$, where $e \otimes e(x) = (x, e)e$, $\forall x \in \mathcal{H}$. Put $\mathfrak{M} = [\mathfrak{A}_0 e]$, then $\mathfrak{M}$ is $\mathfrak{A}$-invariant. If $T \in \mathfrak{A}_0$, then

$$(Te, e) = \rho(T(e \otimes e)) = \rho \circ \Phi(T(e \otimes e)) = 0.$$

Hence, $\mathfrak{M}$ is nontrivial. This completes the proof. ∎

Proof of Theorem 3.1. By Lemma 3.3 and Zorn's lemma, there exists a maximal nest $\mathcal{N}$ of $\mathfrak{A}$-invariant subspaces of $\mathcal{H}$. Since $\mathcal{N} \subset \mathfrak{D}'$ and $\mathfrak{D}'$ is atomic, the nest $\mathcal{N}$ is atomic. Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be the set of all atoms of $\mathcal{N}$. By the maximality of $\mathcal{N}$ and Lemma 3.3, we have $E_\lambda\mathcal{B}(\mathcal{H})E_\lambda \subset \mathfrak{D}$. This implies that $\mathcal{N}' = \mathfrak{D}$ and so $\Phi(T) = \sum_{\lambda\in\Lambda} E_\lambda T E_\lambda$ for every T in $\mathcal{B}(\mathcal{H})$. It is clear that $\mathfrak{A} \subset \operatorname{alg}\mathcal{N}$.

Conversely, if $T \in \operatorname{alg}\mathcal{N}$, then there exist nets $A_\alpha,\ B_\alpha \in \mathfrak{A}$ such that $A_\alpha + B_\alpha^* \to T$ σ-weakly. For $\lambda,\ \mu \in \Lambda$, there exist $P_\lambda,\ P_\mu \in \mathcal{N}$ such that $E_\lambda = P_\lambda \ominus P_{\lambda-}$ and $E_\mu = P_\mu \ominus P_{\mu-}$ respectively. We define an order on Λ by

$$\lambda < \mu \Longleftrightarrow P_\lambda < P_\mu.$$

If $\lambda > \mu$, then $E_\lambda T E_\mu = 0$ because $T \in \operatorname{alg}\mathcal{N}$. If $\lambda = \mu$, then $E_\lambda T E_\lambda \in \mathfrak{D} \subset \mathfrak{A}$. Since $E_\lambda B_\alpha^* E_\mu = 0$ in case that $\lambda < \mu$, we have $E_\lambda T E_\mu \in \mathfrak{A}$. Therefore $T = \sum_{\lambda\leqslant\mu} E_\lambda T E_\mu \in \mathfrak{A}$. This completes the proof. ∎

In [9], Theorem 2, the third author and Watatani proved that if $\mathfrak{D}$ is a subfactor of a finite dimensional factor $\mathcal{M}$, then there exist no maximal subdiagonal algebras of $\mathcal{M}$ with respect to the canonical conditional expectation from $\mathcal{M}$ onto $\mathfrak{D}$ unless $\mathfrak{D} = \mathcal{M}$. The following corollary is in case that $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

COROLLARY 3.4. *If $\mathfrak{D}$ is a subfactor of $\mathcal{B}(\mathcal{H})$, then there exist no maximal subdiagonal algebras with diagonal $\mathfrak{D}$ unless $\mathfrak{D} = \mathcal{B}(\mathcal{H})$.*

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