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UNIQUE EXTENSION OF PURE STATES OF C^*-ALGEBRAS

L.J. BUNCE and C.-H. CHU

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Abstract. Let A be a C^*-subalgebra of a C^*-algebra B. We say that A has the *pure extension property* in B if every pure state of A has a unique pure state extension to B.

We show that A has the pure extension property in B if and only if there is a weak expectation on B for the atomic representation of A, among several equivalent conditions, including the unique extension of type I factor states. If A is separable and B is a von Neumann algebra, we show that the pure extension property is equivalent to that every factor state of A extends to a unique factor state of B which is in turn equivalent to that A is dual and the minimal projections of A are minimal in B. If A has the pure extension property in B, then there is a natural map $\widehat{\alpha}$ between their spectra $\widehat{A}$ and $\widehat{B}$. We study the relationship of $\widehat{A}$ and $\widehat{B}$ under $\widehat{\alpha}$ as well as the unique extension of atomic states.

Keywords: *C^*-algebra, pure extension property, atomic extension property, weak expectation, hereditary subalgebra.*

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1. INTRODUCTION

Let A be a C^*-subalgebra of a C^*-algebra B. The set of extensions, in the state space of B, of a pure state φ of A is a weak*-closed face so that by the Krein-Milman Theorem φ extends to at least one pure state of B which, if unique, is also the unique extension of φ in the state space of B ([25], 4.1.17).

We say that A has the *pure extension property* (PEP) *in* B if every pure state of A extends uniquely to a pure state of B.

When B is abelian, the pure extension property of A in B is in outcome a minor variation of the Stone-Weierstrass Theorem. But deep investigations of

Kadison and Singer ([21]), Anderson ([8]) and Archbold ([9]) reveal the subtlety of the pure extension property when A is abelian but B is not abelian. A strong form of pure extension property institutionalised in the theory of *perfect* C^*-algebras has been investigated in penetrating detail by Akemann and Shultz ([5]).

In this paper we investigate the pure extension property for arbitrary C^*-algebras A and B. Contrary to the special case in which A is abelian ([17]), the PEP of A in B need not be implemented by a conditional expectation from B onto A in general. Instead, as we will show, PEP is equivalent to the existence of certain *weak expectations* as well as to other conditions, including unique extension of type I factor states (cf. Theorem 2.8). A consequence of the classical Stone-Weierstrass Theorem is that if A is *abelian* and has the PEP in B, then A is an ideal of a maximal abelian subalgebra of B. With the assistance of a non-commutative version (Theorem 3.3) of this result for arbitrary A, we give several characterisations (Theorem 3.8) of the PEP for a separable C^*-subalgebra of a von Neumann algebra. When A has the PEP in B, there is a natural map $\widehat{\alpha} : \widehat{A} \to \widehat{B}$, between the corresponding spectra $\widehat{A}$ and $\widehat{B}$, which we exploit in Section 4 to study the extensions of atomic states.

Generally we shall use standard notation to be found in [25]. If A is a C^*-algebra, $S(A)$ will denote the state space of A and $P(A)$ the set of pure states of A. Given a normal state φ of a von Neumann algebra M, we denote by $\mathrm{s}(\varphi)$ and $\mathrm{c}(\varphi)$, respectively, the support projection and the central support projection of φ in M. For a C^*-algebra A, here habitually identified with its canonical image in A^{**}, the states of A identify with the normal states of A^{**}. Given a state φ of A, we define the homomorphism $\tau_\varphi : A \to A^{**}\mathrm{c}(\varphi)$ by $\tau_\varphi(a) = a\mathrm{c}(\varphi)$. We let $(\pi_\varphi, H_\varphi, h_\varphi)$ denote the GNS-representation associated with φ. The normal extension of π_φ to A^{**} restricts to an isomorphism from $A^{**}\mathrm{c}(\varphi)$ onto $\overline{\pi_\varphi(A)}$, the weak*-closure of $\pi_\varphi(A)$ in the algebra $B(H_\varphi)$ of bounded linear operators on H_φ. We call φ a *factor state* of A if $A^{**}\mathrm{c}(\varphi)$ is a factor. Further, φ is called a *factor state of type* I if $A^{**}\mathrm{c}(\varphi)$ is a factor of type I. The set of factor states and the set of factor states of type I will be denoted by $F(A)$ and $F_{\mathrm{I}}(A)$ respectively. Let z_A be the supremum of all minimal central projections in A^{**}. Then $A^{**}z_A$ is the atomic part of A^{**} and we refer to $\tau_a : a \in A \mapsto az_A \in A^{**}z_A$ as the *atomic representation* of A. The canonical inclusion $A \hookrightarrow A^{**}$ is the *universal representation* of A.

It follows from [25], 3.1.6, 4.1.7 that A has the PEP in B if and only if A has the PEP in $H(A)$, the hereditary C^*-subalgebra generated by A in B. If e is the identity of A^{**} (identified with the weak*-closure of A in B^{**}), an increasing net (a_λ) in A, with $0 \leqslant a_\lambda \leqslant e$ for all λ, is an approximate unit of A if and only if

$a_\lambda \to e$ strongly. This follows from [4], 3.1, as does the fact that the following are equivalent (see also [5], 2.32):

(a) $H(A) = B$;
(b) A and B have a common approximate unit;
(c) no pure state of B vanishes on A;
(d) every pure state of B restricts to a state of A.

Further, (b) implies that *every* approximate unit of A is an approximate unit of B; and "pure state" in (c) and (d) may be replaced by "state".

We also note that A is hereditary in B (i.e. $A = H(A)$) if (and only if) every state of A has unique extension to a state of B ([22]). We have been informed by the referee that this result has also appeared in [20].

A C^*-algebra B is called *scattered* if B^{**} is a direct sum of type I factors, or equivalently, if B has a composition series in which each successive quotient is isomorphic to the C^*-algebra $K(H)$ of compact operators on some Hilbert space H ([16], [23]). A C^*-algebra B is called *dual* if it is isomorphic to a C^*-subalgebra of some $K(H)$, or equivalently, every maximal abelian subalgebra of A is generated by minimal projections [19], 4.7.20.

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2. UNIQUE EXTENSIONS AND WEAK EXPECTATIONS

Let A be a C^*-subalgebra of a C^*-algebra B and let M be a von Neumann algebra. A *weak expectation* for a *-homomorphism $\pi : A \to M$ is a linear contraction $\mathcal{P} : B \to \overline{\pi(A)}$ satisfying $\mathcal{P}|A = \pi$, where $\overline{\pi(A)}$ is the *weak**-closure of $\pi(A)$ in M. Given a factor state φ of A with GNS representation π_φ, Tsui ([31]) has shown that φ can be extended to a factor state of B if there is a weak expectation for π_φ. Tsui's proof is based on Sakai's lecture in the 1973 Wabash Conference. The connection between weak expectations and factor state extensions has been developed in fine detail by Archbold and Batty ([12]). We note that the normal extension $\overline{\pi} : A^{**} \to M$ of $\pi : A \to M$ factors through the isomorphism $A^{**}\mathrm{c}(\pi) = A^{**}/\overline{\pi}^{-1}(0) \to \overline{\pi(A)}$ where $\mathrm{c}(\pi) \in A^{**}$ is the central support of π. Therefore the existence of a weak expectation $\mathcal{P}$ for π amounts to the existence of a contractive projection $\mathcal{Q} : B^{**} \to A^{**}\mathrm{c}(\pi)$ such that $\overline{\pi} \circ \mathcal{Q}|A^{**} = \overline{\pi}$, where $\mathcal{P}$ and $\mathcal{Q}$ are related by $\mathcal{P} = \overline{\pi} \circ \mathcal{Q}|B$. It follows that $\mathcal{P}$ is completely positive and satisfies

$$\mathcal{P}(aba') = \pi(a)\mathcal{P}(b)\pi(a')$$

for $a, a' \in A$ and $b \in B$ (cf. [12], Proposition 2.1). In particular, the latter property implies that, if $\varphi \in S(A)$ has the GNS representation $(\pi_\varphi, H_\varphi, h_\varphi)$ and if $\mathcal{P}$ is a weak expectation for π_φ, then the map $\mathcal{P} \mapsto \varphi \circ \mathcal{P} = \langle \mathcal{P}(\cdot) h_\varphi, h_\varphi \rangle$ is injective ([12], Corollary 2.2). We will sometimes substitute for the GNS representation π_φ its equivalent $\tau_\varphi : A \to A^{**}\mathrm{c}(\varphi)$ where $\tau_\varphi(a) = a\mathrm{c}(\varphi)$. The normal extension of τ_φ to A^{**} will also be denoted by τ_φ.

Given $\varphi \in S(A)$, the (possibly empty) convex set $\mathcal{E}_\varphi$ of weak expectations for $\tau_\varphi : A \to A^{**}\mathrm{c}(\varphi)$ is compact in the point-weak topology. Let $S_\varphi = \{\varphi \circ \mathcal{P} : \mathcal{P} \in \mathcal{E}_\varphi\}$. Then $S_\varphi \subset E_\varphi \equiv \{\psi \in S(B) : \psi | A = \varphi\}$. If $\mathcal{E}_\varphi$ is nonempty, we define a map $\alpha_\varphi : \mathcal{E}_\varphi \to S_\varphi$ by $\alpha_\varphi(\mathcal{P}) = \varphi \circ \mathcal{P}$. This notation is retained in the following two results of which we shall make frequent use.

Theorem 2.1. ([31]) *Let A be a C^*-subalgebra of a C^*-algebra B. If $\varphi \in F(A)$ and $\mathcal{E}_\varphi$ is nonempty, then the extreme points of S_φ are factor states of B (extending φ).*

Theorem 2.2. ([12]) *Let A be a C^*-subalgebra of a C^*-algebra B. We have:*

(i) *$\alpha_\varphi : \mathcal{E}_\varphi \to S_\varphi$ is an affine homeomorphism for every $\varphi \in S(A)$ with $\mathcal{E}_\varphi \neq \emptyset$;*

(ii) *for every $\varphi \in P(A)$, $\mathcal{E}_\varphi \neq \emptyset$ and $E_\varphi = S_\varphi$.*

Proof. (i) This is [12], Corollary 2.2 as remarked above.

(ii) This is implicit in [12], Theorem 2.3. Indeed, given $\varphi \in P(A)$ with extension $\psi \in E_\varphi$, employing the usual GNS notation as above, the following identification (cf. [19], 2.10.2) may be made: $h_\varphi = h_\psi = h$ say, $H_\varphi = [\pi_\psi(A)h]$ and $\pi_\varphi(a) = \pi_\psi(a)|H_\varphi$ for $a \in A$. Let E be the orthogonal projection of H_ψ onto H_φ. Then the map $\mathcal{Q} : B \to \overline{\pi_\varphi(A)} = B(H_\varphi)$ given by

$$\mathcal{Q}(b) = E\pi_\psi(b)E$$

is a weak expectation for π_φ with $\psi(b) = \langle \mathcal{Q}(b)h, h\rangle$. Hence $\mathcal{P} : B \xrightarrow{\mathcal{Q}} \overline{\pi_\varphi(A)} \xrightarrow{\text{iso}} A^{**}\mathrm{c}(\varphi)$ is a weak expectation for τ_φ and $\psi = \varphi \circ \mathcal{P}$. ∎

Lemma 2.3. *Let M be a von Neumann subalgebra of a von Neumann algebra N where M is a factor containing the identity of N. Suppose each normal state of M extends to a unique normal factor state of N. Then N contains a unique minimal central projection z, and also $Mz = Nz$.*

Proof. By assumption, N has normal factor states and hence at least one minimal central projection z say.

We show that every normal state of Mz extends to a unique normal state of Nz which will yield $Mz = Nz$.

Let φ be a normal state of Mz and let $\overline{\varphi}$ be a normal state of Nz extending φ. Then $\overline{\varphi}$ extends to a unique normal state ψ of N given by $\psi(\cdot) = \overline{\varphi}(\cdot\ z)$. As $\psi(z) = 1$, ψ is a factor state of N. On the other hand, the normal state $\varphi(\cdot\ z)$ of M extends to a unique normal factor state ω of N, and as $\psi|M = \varphi(\cdot\ z)$, we have $\psi = \omega$ which gives $\overline{\varphi} = \omega|Nz$.

Finally, fix any normal state ψ of M with unique normal factor state extension ω on N. As M is a factor, it is isomorphic to Mz via the isomorphism $x \mapsto xz$. So there is a normal state φ on Mz such that $\psi(\cdot) = \varphi(\cdot\ z)$. By the above arguments, we have $\omega|Nz = \omega|Mz = \varphi$. So $\omega(z) = 1$ and $z = \mathrm{c}(\omega)$. This shows that z is unique. ∎

REMARK 2.4. We note that in Lemma 2.3, the algebra N need not be a factor. For example, let $N = (L^\infty(0,1)\overline{\otimes}A) \oplus (1 \otimes A)$ where A is a factor, and let $M = \{(x,x) : x \in 1 \otimes A\}$. Then $1_N \in M \subset N$ and $Me = Ne$ where $e = (0, f)$ with f being the identity of $1 \otimes A$. We have that e is the unique minimal central projection of N and each normal state of M has a unique extension to a normal factor state of N.

THEOREM 2.5. *Let A be a C^*-subalgebra of a C^*-algebra B. Then we have* (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) *in the following conditions:*

(i) *each factor state of A has a unique extension to a state of B;*

(ii) *each factor state of A has a unique extension to a factor state of B;*

(iii) *there is a weak expectation $\mathcal{P} : B \longrightarrow A^{**}$ for $A \hookrightarrow A^{**}$ such that for each $\varphi \in F(A)$ with an extension $\overline{\varphi} \in F(B)$, we have $\overline{\varphi} = \varphi \circ \mathcal{P}$.*

Proof. (ii) $\Rightarrow$ (iii) Let $\varphi \in F(A)$ and let $\overline{\varphi} \in F(B)$ extend φ. Consider the inclusion

$$A^{**}\mathrm{c}(\varphi) \hookrightarrow \mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi).$$

We show that every normal state ψ of $A^{**}\mathrm{c}(\varphi)$ extends uniquely to a normal factor state of $\mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$. Such a state ψ may be identified with $\psi \in F(A)$ satisfying $\mathrm{c}(\psi) = \mathrm{c}(\varphi)$. Let $\overline{\psi} \in F(B)$ extend ψ. Then $\overline{\psi}$ acts on $\mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$ as a normal factor state (as $\overline{\psi}(\mathrm{c}(\varphi)\mathrm{c}(\overline{\psi})) = 1$) extending ψ on $A^{**}\mathrm{c}(\varphi)$. Now let ω be any normal factor state of $\mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$ extending ψ. Then $\omega(z\mathrm{c}(\varphi)) = 1$ for some minimal central projection z in B^{**}. Hence the unique normal extension of ω to B^{**} is supported by z and therefore is a factor state extending ψ and must equal $\overline{\psi}$ by (ii). So ψ extends uniquely to a normal factor state of $\mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$. Hence, by Lemma 2.3, we have $A^{**}\mathrm{c}(\varphi)\mathrm{c}(\overline{\varphi}) = \mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)\mathrm{c}(\overline{\varphi})$.

Consider the maps

$$\sigma : b \in B \longmapsto \mathrm{c}(\varphi)b\mathrm{c}(\varphi)\mathrm{c}(\overline{\varphi}) \in \mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)\mathrm{c}(\overline{\varphi});$$

$$\tau : x \in A^{**}\mathrm{c}(\varphi) \longmapsto x\mathrm{c}(\overline{\varphi}) \in A^{**}\mathrm{c}(\varphi)\mathrm{c}(\overline{\varphi})$$

and note that σ is a linear contraction, that τ is a $*$-isomorphism since $A^{**}\mathrm{c}(\varphi)$ is a factor, and that $\tau^{-1} \circ \sigma|A = \tau_\varphi$. Hence $\mathcal{P}_\varphi = \tau^{-1} \circ \sigma : B \to A^{**}\mathrm{c}(\varphi)$ is a weak expectation for τ_φ which, by (ii), Theorem 2.1 and Theorem 2.2 (i), is the unique such weak expectation. It now follows from [12], Theorem 2.6 that there is a weak expectation $\mathcal{P} : B \to A^{**}$ for $A \hookrightarrow A^{**}$. Hence $\tau_\varphi \circ \mathcal{P} = \mathcal{P}_\varphi$ by uniqueness, so that

$$\varphi \circ \mathcal{P} = \varphi \circ \tau_\varphi \circ \mathcal{P} = \varphi \circ \mathcal{P}_\varphi = \overline{\varphi}$$

where the final equality comes from Theorem 2.1.

(iii) $\Rightarrow$ (ii) Given (iii), by [12], Theorem 2.6 (4) $\Rightarrow$ (3), together with Theorem 2.1 (see also [12], Proposition 2.5) each $\varphi \in F(A)$ *has* an extension in $F(B)$ from which (ii) is now immediate.

(i) $\Rightarrow$ (ii) Take $\varphi \in F(A)$ with extension $\overline{\varphi} \in S(B)$. By an argument similar to that in the proof of (ii) $\Rightarrow$ (iii), every normal state of $A^{**}\mathrm{c}(\varphi)$ extends to a unique normal factor state of $\mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$ and therefore, as before, the map $\mathcal{P}_\varphi : B \to A^{**}\mathrm{c}(\varphi)$ is a weak expectation for τ_φ. Hence $\overline{\varphi} \in F(B)$ by Theorem 2.1. ∎

Corollary 2.6. *Let A be a C^*-subalgebra of a C^*-algebra B. Suppose that A is a von Neumann algebra. The following conditions are equivalent:*

(i) *every $\varphi \in F(A)$ has a unique extension to some $\overline{\varphi} \in F(B)$;*

(ii) *there is a contractive projection $\mathcal{P} : B \to A$ such that for each $\varphi \in F(A)$ with an extension $\overline{\varphi} \in F(B)$, we have $\overline{\varphi} = \varphi \circ \mathcal{P}$.*

Proof. Let $\pi : A \to \pi(A) \subset B(H)$ be a faithful normal representation. If each factor state of A extends to a unique factor state of B, then by Theorem 2.5 (ii) $\Rightarrow$ (iii) and [12], Theorem 2.6 (4) $\Rightarrow$ (3), there is a weak expectation $\mathcal{Q} : B \to \overline{\pi(A)} = \pi(A)$ for π. Then $\mathcal{P} = \pi^{-1} \circ \mathcal{Q}$ is a contractive projection from B onto A. The proof is concluded as in Theorem 2.5. ∎

Proposition 2.7. *Let A be a C^*-subalgebra of a C^*-algebra B. Let $\varphi \in P(A)$ with an extension $\overline{\varphi} \in P(B)$. The following conditions are equivalent:*

(i) *$\overline{\varphi}$ is the unique extension of φ;*

(ii) $\mathrm{s}(\varphi) = \mathrm{s}(\overline{\varphi})$*;*

(iii) *$A^{**}\mathrm{c}(\varphi)$ is an hereditary subalgebra of $B^{**}\mathrm{c}(\overline{\varphi})$;*

(iv) *there is a unique weak expectation $\mathcal{P}_\varphi : B \to A^{**}\mathrm{c}(\varphi)$ for $\tau_\varphi : A \to A^{**}\mathrm{c}(\varphi)$ where $\mathcal{P}_\varphi(b) = \mathrm{c}(\varphi)b\mathrm{c}(\varphi)$.*

Proof. (i) $\Rightarrow$ (ii) As $\overline{\varphi}(\mathrm{s}(\varphi)) = 1$, we have $\mathrm{s}(\overline{\varphi}) \leqslant \mathrm{s}(\varphi)$ in B^{**}. If $\mathrm{s}(\overline{\varphi}) \neq \mathrm{s}(\varphi)$, we can choose $\tau \in S(B)$ such that $\tau(\mathrm{s}(\varphi) - \mathrm{s}(\overline{\varphi})) = 1$. Then $\tau(\mathrm{s}(\varphi)) = 1$ and

$\tau(\mathrm{s}(\overline{\varphi})) = 0$. In particular, $\mathrm{s}(\tau|A) = \mathrm{s}(\varphi)$, by minimality, so that $\tau|A = \varphi$ and hence that $\tau = \overline{\varphi}$ contradicting $\tau(\mathrm{s}(\overline{\varphi})) = 0$. Hence $\mathrm{s}(\overline{\varphi}) = \mathrm{s}(\varphi)$.

(ii) $\Rightarrow$ (iii) Let $\mathrm{s}(\varphi) = \mathrm{s}(\overline{\varphi})$. Then $\mathrm{s}(\varphi)$ lies in the weak*-closed ideal $A^{**} \cap B^{**}\mathrm{c}(\overline{\varphi})$ as therefore does $\mathrm{c}(\varphi)$. Hence $\mathrm{c}(\varphi) \leqslant \mathrm{c}(\overline{\varphi})$. Therefore $\mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$ is a type I factor which we may identify with some $B(H)$ and $A^{**}\mathrm{c}(\varphi)$ accordingly with a type I subfactor M containing 1_H and a minimal projection e of $B(H)$. Let $z \in M'$ and let f be any minimal projection of M. Then f is minimal in $B(H)$ as it is equivalent to e in M. Therefore we have $zf = fzf \in \mathbb{C} \cdot f \subset M$. It follows that $z \in M$ and hence $M' = \mathbb{C} \cdot 1$ which gives $M = M'' = B(H)$, proving (iii).

(iii) $\Rightarrow$ (i) and (iv). Given any extension τ of φ, we have $\tau(\mathrm{c}(\varphi)) = 1$ so that

$$\tau(b) = \tau(\mathrm{c}(\varphi)b\mathrm{c}(\varphi)) = \varphi(\mathrm{c}(\varphi)b\mathrm{c}(\varphi)) = \overline{\varphi}(b)$$

for $b \in B$, proving (i) which implies that the map $\mathcal{P}_\varphi$ in (iv) is a weak expectation. Its uniqueness then follows from Theorem 2.2.

(iv) $\Rightarrow$ (i) Since (iv) implies that $\mathrm{c}(\varphi)B\mathrm{c}(\varphi) \subset A^{**}\mathrm{c}(\varphi)$, we see (i) follows as in the proof of (iii) $\Rightarrow$ (i). ∎

THEOREM 2.8. *Let A be a C^*-subalgebra of a C^*-algebra B. The following conditions are equivalent:*

(i) *each $\varphi \in P(A)$ has a unique extension in $P(B)$;*

(ii) *Each $\varphi \in F_{\mathrm{I}}(A)$ has a unique extension in $S(B)$;*

(iii) *each $\varphi \in F_{\mathrm{I}}(A)$ has a unique extension in $F(B)$;*

(iv) *each $\varphi \in F_{\mathrm{I}}(A)$ has a unique extension in $F_{\mathrm{I}}(B)$;*

(v) *$A^{**}z_A$ is an l^∞-sum of hereditary subalgebras of $B^{**}z_B$;*

(vi) *there is a unique weak expectation $\mathcal{P} : B \to A^{**}z_A$ for the atomic representation τ_{a} of A and it is given by $\mathcal{P}(b) = \sum e_j b e_j$ for $b \in B$, where e_j are minimal central projections in $A^{**}z_A$ with $z_A = \sum e_j$;*

(vii) *there is a contractive projection $\mathcal{Q} : B^{**} \to A^{**}z_A$ such that $\varphi \circ \mathcal{Q}$ is the unique extension in $S(B)$ of each $\varphi \in P(A)$.*

Proof. (i) $\Rightarrow$ (v) Let (φ_j) be a family of mutually inequivalent pure states of A such that $z_A = \sum \mathrm{c}(\varphi_j)$. By Proposition 2.7, we have

$$A^{**}z_A = \sum A^{**}\mathrm{c}(\varphi_j) = \sum \mathrm{c}(\varphi_j)B^{**}\mathrm{c}(\varphi_j) \subset B^{**}z_B.$$

(v) $\Rightarrow$ (ii) Let $\varphi \in F_{\mathrm{I}}(A)$. Then $\mathrm{c}(\varphi) = \mathrm{c}(\psi)$ for some $\psi \in P(B)$ and (*v*) implies that $A^{**}\mathrm{c}(\varphi) = A^{**}\mathrm{c}(\psi) = \mathrm{c}(\psi)B^{**}\mathrm{c}(\psi) = \mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi)$ which in turn implies (ii).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Given $\varphi \in F_{\mathrm{I}}(A)$, then $A^{**}\mathrm{c}(\varphi)$ is a type I factor and hence injective. So there is a weak expectation $\mathcal{P} : B \to A^{**}\mathrm{c}(\varphi)$ for $A \to A^{**}\mathrm{c}(\varphi)$

implying that φ has a factor state extension on B, by Theorem 2.1, some of which must be type I ([12], Theorem 3.2).

(iv) $\Rightarrow$ (i) This is obvious.

(i) $\Rightarrow$ (vi) By injectivity of $A^{**}z_A$, there is a weak expectation $\mathcal{P} : B \to A^{**}z_A$ for the atomic representation $\tau_{\rm a} : A \to A^{**}z_A$. Let $\mathcal{Q}$ be another weak expectation for $\tau_{\rm a}$. Then (i) implies that $\varphi \circ \mathcal{P} = \varphi \circ \mathcal{Q}$ for all $\varphi \in P(A)$ which gives $\mathcal{P} = \mathcal{Q}$ since $P(A)$ separates points of $A^{**}z_A$. Now the formula for $\mathcal{P}$ is seen from the proof of (i) $\Rightarrow$ (v).

(vi) $\Rightarrow$ (i), (vii) For each $\varphi \in P(A)$, the condition (vi) implies that $A^{**}{\rm c}(\varphi) = {\rm c}(\varphi)B^{**}{\rm c}(\varphi)$ from which (i) follows as in the proof of Proposition 2.7 (iv) $\Rightarrow$ (i). In addition, we see that $\mathcal{Q}(b) = \sum e_i b e_i$ where the e_i are minimal central projections in A^{**} with $\sum e_i = z_A$ and $b \in B^{**}$, defines a contractive projection $\mathcal{Q} : B^{**} \to A^{**}z_A$ satisfying (vii). ∎

By Theorem 2.5 and Theorem 2.8, we have

Corollary 2.9. *Let A be a type* I *C^*-subalgebra of a C^*-algebra B. The following conditions are equivalent:*

(i) *A has the* PEP *in B;*

(ii) *each $\varphi \in F(A)$ has a unique extension to $\overline{\varphi} \in F(B)$;*

(iii) *each $\varphi \in F(A)$ has a unique extension to $\overline{\varphi} \in S(B)$;*

(iv) *there is a weak expectation $\mathcal{P} : B \to A^{**}$ for $A \hookrightarrow A^{**}$ such that $\varphi \circ \mathcal{P} \in F(B)$ is the unique extension of each $\varphi \in F(A)$.*

Remark 2.10. In (iv) of the above corollary, the weak expectation $\mathcal{P} : B \to A^{**}$ for $A \hookrightarrow A^{**}$ cannot be replaced by a contractive projection $B \to A$ even when A is liminal and separable, as may be seen by combining [5], Proposition 3.14 and the proof of [11], Theorem 2.1.

3. RELATIVE COMMUTANTS

Given a C^*-subalgebra A of a C^*-algebra B, we let

$$A^{\rm c} = \{b \in B : ab = ba \ \forall\, a \in A\}$$

$$A^{\perp} = \{b \in B : bA = Ab = 0\}$$

denote respectively the *relative commutant* and the *annihilator* of A in B. The centre of A is denoted by $Z(A)$. For subsets S and T of B, we let $S \cdot T = \{st : s \in S,\ t \in T\}$ and let $[S]$ denote the norm closed linear span of S while $C^*(S)$ denotes the C^*-subalgebra generated by S. If $S = \{x_1, \ldots, x_n\}$, we will also write $C^*(S) = C^*(x_1, \ldots, x_n)$.

LEMMA 3.1. *Let A be a prime C^*-subalgebra of a unital C^*-algebra B and let A have the* PEP *in B. Then we have $A^c = A^\perp + \mathbb{C} \cdot 1$.*

Proof. We show that the quotient $A^c/A^\perp$ has no zero divisor and hence is one-dimensional.

For self-adjoint $x \in A^c$, we let $C_x = [A \cdot C^*(1, x)]$ and $J_x = [A \cdot C^*(x)]$. Then A has PEP in C_x.

Note that the pure states of C_x restrict to pure states of A. Indeed, given $\varphi \in P(C_x)$ with the GNS-representation $\pi_\varphi : C_x \to B(H_\varphi)$ and normal extension $\tilde{\pi}_\varphi$, then $\tilde{\pi}_\varphi(x) \in \mathbb{C} \cdot 1$ so that $\overline{\pi_\varphi(A)} = B(H_\varphi)$. But A and C_x have common approximate unit so that $\varphi|A$ is a state, hence $\pi_\varphi|A$ is irreducible and so $\varphi|A \in P(A)$. By PEP, A separates points of $P(C_x) \cup \{0\}$. As J_x is a two-sided ideal of C_x, by [19], 11.1.3 and 11.1.7 the irreducible representations of J_x restrict to those of $A \cap J_x$. So $J_x = 0$ if $A \cap J_x = 0$.

Now take self-adjoint elements x and y in A^c such that $xy \in A^\perp$. Then $C^*(x) \cdot C^*(y) \cdot A = 0$. Hence $J_x \cdot J_y = 0$ and therefore $(A \cap J_x) \cap (A \cap J_y) = A \cap J_x \cap J_y = 0$ which implies either $A \cap J_x = 0$ or $A \cap J_y = 0$ because A is prime. It follows that $J_x = 0$ or $J_y = 0$, that is, $x \in A^\perp$ or $y \in A^\perp$. This shows that $A^c/A^\perp$ has no zero divisor. ∎

Given a proper C^*-subalgebra A of a C^*-algebra B, consider the weak*-compact convex set

$$S = \{f \in B^* : f = f^*,\ \|f\| \leqslant 1,\ f(A) = 0\}.$$

Let $g \in \partial S$ and let $g = g_1 - g_2$ be its orthogonal decomposition with $g_1, g_2 \geqslant 0$. Put $\tau = g_1 + g_2$. The following lemma is taken from Sakai's book ([28]).

LEMMA 3.2. *Let A, B and τ be as above and suppose that:*
(i) *A and B have a common approximate unit;*
(ii) $Z(\overline{\pi_\tau(A)}) \subset Z(\overline{\pi_\tau(B)})$.
Then $\overline{\pi_\tau(A)}$ is a nonzero factor.

Proof. The required argument is the same as in [28], 4.1.9 with lines 10–14 of that proof omitted. ∎

THEOREM 3.3. *Let A be a C^*-subalgebra with* PEP *in a C^*-algebra B. Then:*
(i) $A \cdot A^c \subset A$;
(ii) $A^c = Z(A)$ *if A and B have a common approximate unit.*

Proof. (i) We may suppose that B has a unit. Let x be a self-adjoint element of A^c. We show that the C^*-algebra $E = [A \cdot C^*(1, x)]$ is equal to A. Suppose that $A \neq E$. Let g, g_1, g_2 and τ be chosen as in the remarks preceding Lemma 3.2. We claim that $\overline{\pi_\tau(A)}$ is a nonzero factor. Indeed, it is evident that Condition (i) of Lemma 3.2 is satisfied. To see that Condition (ii) of Lemma 3.2 holds, we note that E is a two-sided ideal of $D = C^*(A \cup \{x\})$ so that $\pi_\tau : E \to B(H_\tau)$ extends to $\overline{\pi_\tau} : D \to B(H_\tau)$ with $\overline{\overline{\pi_\tau}(D)} = \overline{\pi_\tau(E)}$. But $\overline{\overline{\pi_\tau}(D)}$ is generated by $\overline{\pi_\tau}(x)$ and $\overline{\pi_\tau(A)}$, and the former lies in the commutant of the latter. Hence we have $Z(\overline{\pi_\tau(A)}) \subset Z(\overline{\overline{\pi_\tau}(D)}) = Z(\overline{\pi_\tau(E)})$.

Therefore $\pi_\tau(A)$ is a prime C^*-algebra. But $\pi_\tau(A) = \overline{\pi_\tau}(A)$ has the PEP in $\overline{\pi_\tau}(D)$. Therefore $\pi_\tau(E) = \overline{\pi_\tau}(E) = [\overline{\pi_\tau}(A) \cdot C^*(1, \overline{\pi_\tau}(x))] = \pi_\tau(A)$ where the final equality comes from Lemma 3.1. Hence we have $E = A + \ker \pi_\tau$. As $g(\ker \pi_\tau) = 0$, this implies that $g(E) = 0$ which is a contradiction proving (i).

(ii) Let (a_λ) be a common approximate unit of A and B, and let $x \in A^{\mathrm{c}}$. Then by (i), we have $x = \lim a_\lambda x \in A$. ∎

A Banach space X is called a *Grothendieck space* ([18]) if each $\sigma(X^*, X)$-convergent sequence in X^* is $\sigma(X^*, X^{**})$-convergent. The quotient of a Grothendieck space is also a Grothendieck space. Pfitzner ([26]) has shown that every von Neumann algebra is a Grothendieck space. In Proposition 3.6 and Theorem 3.8 below, one can actually replace the von Neumann algebra M by a C^*-algebra A which is a Grothendieck space. The proofs, however, only make use of a weaker property that $\sigma(A^*, A)$-convergent sequence of *positive* functionals is $\sigma(A^*, A^{**})$-convergent. This fact has been proved by Akemann, Dodd and Gamlen ([3]) for von Neumann algebras.

Lemma 3.4. *Let A be a dual C^*-subalgebra of a C^*-algebra B. Then the following conditions are equivalent:*

(i) *A has the* PEP *in B;*
(ii) *the minimal projections of A are minimal in B;*
(iii) *A is a c_0-sum of hereditary C^*-subalgebras of B.*

Proof. (i) ⇒ (ii) Let $p \in A$ be a minimal projection. Then $\mathbb{C} \cdot p$ has PEP in A. By Condition (i), $\mathbb{C} \cdot p$ has PEP in pBp and hence pBp can not have two distinct pure states. So $pBp = \mathbb{C} \cdot p$, that is, p is minimal in B.

(ii) ⇒ (iii) By [19], 4.7.20, we may suppose that A is simple dual. In this case, it follows from Proposition 2.7 (ii) ⇒ (iii) that the type I factor $A^{**} = A^{**} z_A$ is an hereditary subalgebra of $B^{**} z_B$. Hence $A = A^{**} \cap B$ is an hereditary subalgebra of B.

(iii) ⇒ (i) Each pure state of A is supported by a hereditary subalgebra of B and hence has unique extension in $S(B)$. ∎

LEMMA 3.5. *Let A be a separable C^*-subalgebra of a C^*-algebra B and let B be a Grothendieck space. If A has the* PEP *in B, then A is scattered.*

Proof. By [17], Theorem 7 together with Theorem 2.7, it suffices to show that A is of type I. To this end let $\varphi, \psi \in P(A)$ be such that $\ker \pi_\varphi = \ker \pi_\psi$. By a theorem of Glimm (see [19], p. 190), it is sufficient to show that π_φ and π_ψ are equivalent. By [19], 3.4.2 (ii) and separability, φ is the w*-limit of a sequence of pure states (ψ_n) associated with π_ψ. By [17], Lemma 1, we have $\overline{\varphi} = \text{w*-}\lim \overline{\psi}_n$ where $\overline{\tau} \in P(B)$ denotes the unique extension of $\tau \in P(A)$. As B is a Grothendieck space, this implies $\overline{\varphi} = \sigma(B^*, B^{**})\text{-}\lim \overline{\psi}_n$ which gives $\varphi(c_\psi) = \overline{\varphi}(c_\psi) = \lim \overline{\psi_n}(c_\psi) = 1$. Hence $\mathrm{c}(\varphi) = \mathrm{c}(\psi)$ proving that π_φ and π_ψ are equivalent. ∎

PROPOSITION 3.6. *Let A be a nonzero separable C^*-subalgebra of M/I where M is a von Neumann algebra and I is the norm-closed ideal of M such that M/I is antiliminal. Then A does not have* PEP *in M/I.*

In particular, no nonzero separable C^*-subalgebra of the Calkin algebra $B(H)/K(H)$ has the PEP *in $B(H)/K(H)$.*

Proof. Suppose otherwise, then by Lemma 3.5, A must contain a nonzero simple dual ideal which necessarily has the PEP in M/I. Now Lemma 3.4 contradicts the fact that M/I is antiliminal. ∎

LEMMA 3.7. *Let A be a separable C^*-algebra acting irreducibly on a Hilbert space H. If A has the* PEP *in $B(H)$, then $A = K(H)$.*

Proof. By Lemma 3.5, A is type I and so contains $K(H)$ which implies that $A/K(H)$ has the PEP in $B(H)/K(H)$. Hence $A = K(H)$ by Proposition 3.6. ∎

The following extends Theorem 6 and Theorem 7 of [17].

THEOREM 3.8. *Let A be a separable C^*-subalgebra of a von Neumann algebra M. The following conditions are equivalent:*

(i) *A has the* PEP *in M;*

(ii) *every $\varphi \in F(A)$ has unique extension in $S(M)$;*

(iii) *every $\varphi \in F(A)$ has unique extension in $\overline{F(M)}$;*

(iv) *every $\varphi \in F(A)$ has unique extension in $F(M)$;*

(v) *A is a dual C^*-algebra and each minimal projection of A is minimal in M;*

(vi) *A is a c_o-sum of hereditary subalgebras of M.*

Proof. In view of Corollary 2.9 and Lemma 3.4, it is sufficient to show that (i) implies that A is dual.

By Lemma 3.5, Condition (i) implies that there is a sequence (z_n) of orthogonal central projections in the weak*-closure $\overline{A}$ with $\overline{A} = \Big(\bigoplus_n \overline{A}z_n\Big)_{l_\infty}$ where each $\overline{A}z_n$ is a type I factor. As A has the PEP in $\overline{A}$, each Az_n has the PEP in $\overline{A}z_n$ implying that Az_n is simple dual, by Lemma 3.7, contained in A by Theorem 3.3 (i). Let $D = \Big(\bigoplus_n Az_n\Big)_{c_o}$ which is a dual C^*-subalgebra of A. We show that $A = D$. It is evident if (z_n) is finite. Suppose (z_n) is infinite. Let $a \in A$. If $\|az_n\| \not\to 0$, then passing to a subsequence and scaling, we may suppose that $\|az_n\| \geqslant 1$ for all n. Given any subset $\alpha \subset \mathbb{N}$, let $a_\alpha = \Big(\bigoplus_{n\in\alpha} az_n\Big)_{l_\infty} = a\Big(\bigoplus_{n\in\alpha} z_n\Big)_{l_\infty}$ which is in A by Theorem 3.3 (i). But for $\alpha, \beta \subset \mathbb{N}$ with $\alpha \neq \beta$, we have $\|a_\alpha - a_\beta\| \geqslant 1$. This contradicts separability. Therefore $\|az_n\| \to 0$ and so $a = \Big(\bigoplus_n az_n\Big)_{c_0} \in D$. Hence $A = D$ and the proof is complete. ∎

4. ATOMIC EXTENSIONS

Let A be a C^*-algebra and let $K \subset S(A)$. We define the *σ-convex hull* of K to be the following set in which the sum is norm-convergent:

$$\sigma(K) = \Big\{ \sum \lambda_n \varphi_n : \varphi \in K,\ \lambda_n \geqslant 0,\ \sum \lambda_n = 1 \Big\}.$$

We have $\sigma(P(A)) = \{\varphi \in S(A) : \varphi(z_A) = 1\}$, the set of *atomic states* of A which identifies with the normal state space of $A^{**}z_A$. It is more generally true that the normal state space of an atomic von Neumann algebra is the σ-convex hull of its pure normal states. We have $F_{\mathrm{I}}(A) \subset \sigma(P(A))$. In fact, a state lies in $F_{\mathrm{I}}(A)$ if and only if it is a σ-convex sum of equivalent pure states ([13]). In particular, $F_{\mathrm{I}}(A)$ consists precisely of the atomic factor states of A. As a natural development of previous sections, we shall consider the general question of unique extension of atomic states.

Let A be a C^*-subalgebra of a C^*-algebra B. We say that A has the *atomic extension property* (AEP) in B if each atomic state of A has unique extension to an atomic state of B. Note that AEP implies PEP.

It is evident that every atomic state of A extends to an atomic state of B. In particular, if A is a hereditary subalgebra of B, then A has the AEP in B. However, this may not be true if A is the sum of two orthogonal hereditary subalgebras of B because, for instance, when A is finite-dimensional, the AEP in B implies unique extension of states of A.

Let $\widehat{A}$ and $\operatorname{Prim} A$ denote the space of equivalence classes of irreducible representations of A and the primitive ideal space of A. In notation, we shall not distinguish between an irreducible representation of A and its equivalence class. Recall that the canonical surjections ([26], 4.2.12, 4.3.3)

$$\varphi \in P(A) \to \pi_\varphi \in \widehat{A}, \quad \pi \in \widehat{A} \to \ker \pi \in \operatorname{Prim} A$$

are open and continuous.

Let A have the PEP in B and let

$$\alpha : \varphi \in P(A) \to \overline{\varphi} \in P(B)$$

denote the unique extension map.

Let $\varphi_1, \varphi_2 \in P(A)$.

(a) If φ_1 and φ_2 are equivalent, then $\varphi_1(\cdot) = \varphi_2(a \cdot a^*)$ for some $a \in A$. By PEP, we have $\overline{\varphi}_1(\cdot) = \overline{\varphi}_2(a \cdot a^*)$. Hence $\overline{\varphi}_1$ and $\overline{\varphi}_2$ are equivalent. This gives rise to the mapping

$$\widehat{\alpha} : \pi_\varphi \in \widehat{A} \to \pi_{\overline{\varphi}} \in \widehat{B} \quad (\varphi \in P(A)).$$

(b) If $\ker \pi_{\varphi_1} = \ker \pi_{\varphi_2}$, then φ_2 is a weak*-limit of pure states equivalent to φ_1 by [19], 3.4.3. By (a), together with the continuity of α ([17], Lemma 1), $\overline{\varphi_2}$ is a weak*-limit of pure states equivalent to $\overline{\varphi_1}$ from which it follows that $\ker \pi_{\overline{\varphi}_1} \subset \ker \overline{\varphi}_2$ and hence that $\ker \pi_{\overline{\varphi}_1} \subset \ker \pi_{\overline{\varphi}_2}$ ([19], 2.4.11). Therefore $\ker \pi_{\overline{\varphi}_1} = \ker \pi_{\overline{\varphi}_2}$. Thus the mapping

$$\check{\alpha} : \ker \pi_\varphi \in \operatorname{Prim} A \to \ker \pi_{\overline{\varphi}} \in \operatorname{Prim} B \quad (\varphi \in P(A))$$

is well-defined.

Retaining the above notation, we have

PROPOSITION 4.1. *If A has the* PEP *in B, then the following*

$$\begin{array}{ccccc} P(A) & \longrightarrow & \widehat{A} & \longrightarrow & \operatorname{Prim} A \\ \alpha\big\downarrow & & \widehat{\alpha}\big\downarrow & & \check{\alpha}\big\downarrow \\ P(B) & \longrightarrow & \widehat{B} & \longrightarrow & \operatorname{Prim} B \end{array}$$

is a commutative diagram of continuous maps, where the horizontal maps are the canonical ones.

Proof. The maps $\widehat{\alpha}$ and $\check{\alpha}$ are continuous because α is continuous and the horizontal maps are open and continuous surjections. ∎

If A has PEP in B, we shall write

$$P_A(B) = \{\psi \in P(B) : \psi|A \in P(A)\}.$$

A subset $K \subset P(B)$ is said to be *saturated* if K is a union of equivalence classes of pure states in $P(B)$.

Theorem 4.2. *Let A have the* PEP *in B. Then the following conditions are equivalent:*

(i) *A has the atomic extension property in B;*
(ii) *each atomic state of A has unique extension in $S(B)$;*
(iii) *$A^{**}z_A$ is a hereditary subalgebra of $B^{**}z_B$;*
(iv) *$\mathrm{c}(\varphi) = \mathrm{c}(\overline{\varphi})z_A$ for all $\varphi \in P(A)$ with extension $\overline{\varphi} \in P(B)$;*
(v) *$\widehat{\alpha} : \widehat{A} \to \widehat{B}$ is injective;*
(vi) *$\sigma(P_A(B))$ is a norm-closed face of $S(B)$.*

Proof. Given $\varphi \in P(A)$, let $\overline{\varphi} \in P(B)$ be its unique extension.

(i) $\Rightarrow$ (iii) $A^{**}z_A$ and $z_AB^{**}z_A$ have the same predual and hence are equal.

(iii) $\Rightarrow$ (iv) By the proof of Proposition 2.7, condition (iii) implies that for $\varphi \in P(A)$, $\mathrm{c}(\overline{\varphi})z_A$ is a minimal central projection of $z_AB^{**}z_A = A^{**}z_A$ majorising, and hence being equal to, $\mathrm{c}(\varphi)$.

(iv) $\Rightarrow$ (v) This is obvious.

(v) $\Rightarrow$ (iii) $\Rightarrow$ (vi) Let $\widehat{\alpha}$ be injective. Write $z_A = \sum \mathrm{c}(\varphi_i)$ where (φ_i) is a family of mutually inequivalent pure states of A. By assumption, the $\mathrm{c}(\overline{\varphi}_i)$ are mutually orthogonal, and each $\mathrm{c}(\varphi_i) \leqslant \mathrm{c}(\overline{\varphi}_i)$ by the proof of Proposition 2.7. It follows from this and Proposition 2.7 (i) $\Rightarrow$ (iv) that

$$z_AB^{**}z_A = \sum \mathrm{c}(\varphi_i)B^{**}\mathrm{c}(\varphi_i) = \sum A^{**}\mathrm{c}(\varphi_i) = A^{**}z_A,$$

giving (iii). In turn, this identifies $P_A(B)$ with the set of all pure normal states of $z_AB^{**}z_A$. Hence we have $\sigma(P_A(B)) = \{\psi \in S(B) : \psi(z_A) = 1\}$.

(vi) $\Rightarrow$ (ii) Put $K = P_A(B)$. We have $\psi(z_A) = 1$ for all $\psi \in \sigma(K)$. Assuming (v), by [15], p. 245, we have $\sigma(K) = \{\psi \in S(B) : \psi(e) = 1\}$ for some projection $e \leqslant z_A$ in B^{**}. But Proposition 2.7 implies that $\mathrm{s}(\varphi) = \mathrm{s}(\overline{\varphi}) \leqslant e$ for all $\varphi \in P(A)$, from which we deduce that $e = z_A$. Hence $\psi \in S(B)$ lies in $\sigma(K)$ if and only if $\psi|A$ is an atomic state.

Let φ be an atomic state of A and let $\psi \in S(B)$ be an extension of φ. We may write φ as a σ-convex sum of a sequence of pure states of A. Partitioning these pure states by equivalence classes, we may organise φ as a sum $\varphi = \sum\limits_n \alpha_n\varphi_n$ where (α_n) is a finite or infinite sequence of positive real numbers and (φ_n) a mutually disjoint sequence in $F_{\mathrm{I}}(A)$ (cf. [13]). By Theorem 2.8, each φ_n has a unique extension to $\overline{\varphi}_n \in S(B)$. We claim that $\psi = \sum_n \alpha_n \overline{\varphi}_n$.

By earlier argument, we have $\psi = \sum_1^\infty \lambda_n \overline{\tau}_n$ for some $\lambda_n \geqslant 0$ with $\sum \lambda_n = 1$ and $\tau_n \in P(A)$. Hence

$$\sum_n \alpha_n \varphi_n = \sum_n \lambda_n \tau_n.$$

For each m and n, we have $\mathrm{c}(\tau_n) = \mathrm{c}(\varphi_m)$ or $\mathrm{c}(\tau_n)\mathrm{c}(\varphi_m) = 0$. Thus, putting for each m, $S_m = \{n : \mathrm{c}(\tau_n) = \mathrm{c}(\varphi_m)\}$, we have, for $x \in A$,

$$\alpha_m \varphi_m(x) = \sum_n \alpha_n \varphi_n(x\mathrm{c}(\varphi_m)) = \sum_n \lambda_n \tau_n(x\mathrm{c}(\varphi_m)) = \sum_{n \in S_m} \lambda_n \tau_n(x),$$

so that $\alpha_m^{-1} \sum_{n \in S_m} \lambda_n \overline{\tau}_n \in S(B)$ extends φ_m and therefore equals $\overline{\varphi}_m$. It follows that $\psi = \sum_n \alpha_n \overline{\varphi}_n$ as required. ∎

Remark 4.3. We are grateful to the referee for indicated to us condition (ii) in Theorem 4.2.

Corollary 4.4. *Let A have the* PEP *in B. The following conditions are equivalent:*

(i) $P_A(B)$ *is saturated;*

(ii) $\mathrm{c}(\varphi) = \mathrm{c}(\overline{\varphi})$ *for every* $\varphi \in P(A)$*;*

(iii) $\sigma(P_A(B))$ *is a split face of* $S(B)$.

Proof. (i) $\Rightarrow$ (ii) Let $\varphi \in P(A)$ with extension $\overline{\varphi} \in P(B)$. By Proposition 2.7 (iii), we have $A^{**}\mathrm{c}(\varphi) = \mathrm{c}(\varphi)B^{**}\mathrm{c}(\varphi) \subset B^{**}\mathrm{c}(\overline{\varphi})$. Let e be a minimal projection of $B^{**}\mathrm{c}(\overline{\varphi})$. Then $e = \mathrm{s}(\psi)$ for some $\psi \in P(B)$ and $\mathrm{c}(\overline{\varphi}) = \mathrm{c}(\psi)$. As $P_A(B)$ is saturated, we have $\psi|A \in P(A)$ so that $e \in A^{**}$ by Proposition 2.7 (ii). Therefore $B^{**}\mathrm{c}(\overline{\varphi}) \subset A^{**}$ and $\mathrm{c}(\overline{\varphi})$ must be a minimal central projection in A^{**} implying that $\mathrm{c}(\overline{\varphi}) = \mathrm{c}(\varphi)$. It follows that z_A is in the centre of B^{**}.

(ii) $\Rightarrow$ (iii) We have $\mathrm{c}(\varphi) = \mathrm{c}(\overline{\varphi})$ for every $\varphi \in P(A)$ and hence the proof of Theorem 4.2 gives

$$\sigma(P_A(B)) = \{\psi \in S(B) : \psi(z_A) = 1\}$$

which is a split face of $S(B)$ ([15], p. 245).

(iii) $\Rightarrow$ (i) Let $\sigma(P_A(B))$ be a split face of $S(B)$. Then Theorem 4.2 together with [15], p. 245 implies that A has the atomic extension property in B and that z_A is a central projection in $B^{**}z_B$. Therefore, if $\psi \in P(B)$ is equivalent to a state in $P_A(B)$, then $\psi|A$ must be atomic and ψ its unique extension, forcing $\psi|A \in P(A)$ so that $\psi \in P_A(B)$. So $P_A(B)$ is saturated. ∎

We remark that the map $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ may be injective without being an *embedding* (i.e. $\widehat{\alpha}$ may not be a homeomorphism onto $\widehat{\alpha}(\widehat{A})$). In fact, it is possible for $\widehat{\alpha}$ to be a bijection without being a homeomorphism even when B is separable type I and A is abelian, as is shown by the following example.

Example 4.5. Let H be an infinite dimensional separable Hilbert space. Let e be a minimal projection in $B(H)$ and let M be a maximal abelian von Neumann subalgebra of $(1-e)B(H)(1-e)$ without atomic part. Choose (as we may) a separable weak*-dense C^*-subalgebra D of M containing $1-e$. Put $A = D + \mathbb{C}\cdot e$ and $B = D + K$ where $K = K(H)$. Then $1 \in A \subset B$ where A is abelian and B is separable of type I, and A has the PEP in B. To see the latter, let $\varphi \in P(A)$ and let $\overline{\varphi} \in P(B)$ be an extension of φ. If $\varphi(e) = 1$, then $\overline{\varphi}$ is concentrated on K and is clearly the *unique* extension of φ. Otherwise $\varphi(e) = 0$ in which case, to show that $\overline{\varphi}$ is unique, it is enough to show that $\overline{\varphi}(K) = 0$. But, if $\overline{\varphi}(K) \neq 0$, then $\overline{\varphi}$ has unique extension to a pure normal state ψ of $B(H)$. This leads to the contradiction that $\psi|M$ is a pure normal state of M. Indeed, given $\psi|M = 1/2(\psi_1 + \psi_2)$ with $\psi_1, \psi_2 \in S(M)$, then ψ_1, ψ_2 are normal and $\psi_1|A = \psi_2|A = \varphi$, so that $\psi_1 = \psi_2$, as required, since D is weak*-dense in M. The map $\check{\alpha} : \mathrm{Prim}(A) \to \mathrm{Prim}(B)$ is a bijection given by $\check{\alpha}(Q + \mathbb{C}\cdot e) = Q + K$ for each $Q \in \mathrm{Prim}(D)$ and $\check{\alpha}(D) = \{0\}$. However, as B is not liminal, $\mathrm{Prim}(B)$ is not Hausdorff so $\widehat{\alpha}$ $(= \check{\alpha})$ is not a homeomorphism.

On the other hand, we have the following result.

Proposition 4.6. *Let A have the* PEP *in B with $P_A(B)$ saturated. Then $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ is an embedding.*

Proof. We note that $\widehat{\alpha}$ is injective by Corollary 4.4 (i) $\Rightarrow$ (iii) and Theo rem 4.2 (vi) $\Rightarrow$ (v). Let $\mathcal{I}$ be a closed two-sided ideal of A and put $\mathcal{J} = \cap\{\ker \pi_{\overline{\varphi}} : \varphi \in P(A), \varphi(\mathcal{I}) = 0\}$, where $\overline{\varphi} \in P(B)$ denotes the unique extension of $\varphi \in P(A)$. We have $\mathrm{c}(\varphi) = \mathrm{c}(\overline{\varphi})$ for each $\varphi \in P(A)$ by the proof of Corollary 4.4. It follows that $\mathcal{I} = A \cap \mathcal{J}$. Thus, for $\varphi \in P(A)$, we have $\pi_\varphi(\mathcal{I}) \neq \{0\}$ if and only if $\pi_{\overline{\varphi}}(\mathcal{J}) \neq \{0\}$. Hence $\widehat{\alpha}(\widehat{\mathcal{I}}) = \widehat{\mathcal{J}} \cap \widehat{\alpha}(\widehat{A})$ which, together with Proposition 4.1, proves that $\widehat{\alpha} : \widehat{A} \to \widehat{\alpha}(\widehat{A})$ is a homeomorphism. ∎

Given a C^*-algebra A, let $\mathrm{Ideal}(A)$ denote the set of all norm-closed two-sided ideals of A.

PROPOSITION 4.7. *Let A have* PEP *in B and let $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ be a homeomorphism. Then the map $\beta : \mathcal{I} \in \mathrm{Ideal}(A) \mapsto \mathcal{I}_B \in \mathrm{Ideal}(B)$ is a bijection with inverse $\mathcal{J} \in \mathrm{Ideal}(B) \mapsto \mathcal{J} \cap A \in \mathrm{Ideal}(A)$ where $\mathcal{I}_B$ is the norm-closed two-sided ideal in B generated by $\mathcal{I}$. Moreover, $\beta|\mathrm{Prim}(A) = \check{\alpha} : \mathrm{Prim}(A) \to \mathrm{Prim}(B)$, which is also a homeomorphism.*

Proof. Let $\mathcal{I} \in \mathrm{Ideal}(A)$. By assumption, $\widehat{\alpha}(\widehat{\mathcal{I}}) = \widehat{\mathcal{J}}$ for some $\mathcal{J} \in \mathrm{Ideal}(B)$. For $\varphi \in P(A)$, with unique extension $\overline{\varphi} \in P(B)$, we have $\varphi(\mathcal{I}) = 0$ if and only if $\overline{\varphi}(\mathcal{J}) = 0$; but $\overline{\varphi}(\mathcal{J}) = 0$ if and only if $\varphi(A \cap \mathcal{J}) = 0$. Hence $A \cap \mathcal{J} = \mathcal{I}$. In particular $\mathcal{I}_B \subset \mathcal{J}$. Let $\pi \in \widehat{B}$ with $\pi(\mathcal{I}_B) = 0$. By assumption, π is equivalent to $\pi_{\overline{\varphi}}$ for some $\varphi \in P(A)$. We have $\overline{\varphi}(\mathcal{I}_B) = 0$ so that $\varphi(\mathcal{I}) = 0$ and hence $\overline{\varphi}(\mathcal{J}) = 0$ which implies $\pi(\mathcal{J}) = 0$. Hence $\mathcal{I}_B = \mathcal{J}$. Given $K \in \mathrm{Ideal}(B)$, a simple argument gives $K = (K \cap A)_B$, proving the first statement.

For $\varphi \in P(A)$, we have $A \cap \ker \pi_{\overline{\varphi}} \subset \ker \pi_\varphi = \mathcal{I}$, say, as π_φ is equivalent to a subrepresentation of $\pi_{\overline{\varphi}}|A$. By the first part of the proof, $\varphi(\mathcal{I}) = 0$ implies $\overline{\varphi}(\mathcal{I}_B) = 0$. So

$$\mathcal{I} = A \cap \mathcal{I}_B \subset A \cap \ker \pi_{\overline{\varphi}} \subset \mathcal{I}$$

which gives $\mathcal{I}_B = \ker \pi_{\overline{\varphi}}$ since β^{-1} is injective. ∎

REMARK 4.8. Let A have the PEP in B.

(a) It follows from Proposition 2.7, Theorem 4.2 and Corollary 4.4 (cf. [5], Proposition 2.24) that the following are equivalent:

(i) $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ is a homeomorphism and z_A is a central projection in B^{**};

(ii) $A^{**} z_A = B^{**} z_B$;

(iii) A separates $P(B) \cup \{0\}$;

(b) $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ may be a homeomorphism without $P_A(B)$ being saturated. For example, if B is nonabelian, choose $\psi \in P(B)$ which is not a homomorphism and put $A = L_\psi \cap L_\psi^*$ where $L_\psi = \{x \in B : \psi(x^*x) = 0\}$. Restriction induces a homeomorphism $\widehat{B} \to \widehat{A}$ ([25], 4.1.0), the inverse of which is $\widehat{\alpha}$. As $z_A = z_B - \mathrm{s}(\psi)$ is not central in B^{**}, $P_A(B)$ is not saturated by Corollary 4.4. In Example 4.10 we will give another example in which $1 \in A \subset B$.

Given a C^*-algebra A, let

$$A_{\mathrm{c}} = \{x \in A^{**} z_A : x, x^*x \text{ and } xx^* \text{ are continuous on } P(A) \cup \{0\}\}.$$

Then A_{c} is a C^*-algebra with an approximate unit in common with A and, when A is identified with $A z_A$, satisfies the following conditions ([5], 2.9):

(i) A has the PEP in A_{c}.

(ii) $P_A(A_{\mathrm{c}})$ is saturated and dense in $P(A_{\mathrm{c}})$.

Proposition 4.9. *Let A have the atomic extension property in B.*

(i) *If all primitive quotients of B are scattered, then A is a hereditary subalgebra of B.*

(ii) *If all primitive quotients of A are scattered and if $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ is a homeomorphism, then A is a hereditary subalgebra of B.*

Proof. (i) Let all primitive quotients of B be scattered and note that this condition is inherited by $H(A)$, the hereditary C^*-subalgebra of B generated by A. Note also that A has the AEP in $H(A)$. Thus, cutting down to $H(A)$, we may suppose that A and B have common approximate unit. Let $\psi \in P(B)$. Then $A\mathrm{c}(\psi)$ has the AEP in $B\mathrm{c}(\psi)$ and they have common approximate unit. But $B\mathrm{c}(\psi)$ is scattered, as therefore is $A\mathrm{c}(\psi)$, and so every state of $A\mathrm{c}(\psi)$ has unique extension to a state of $B\mathrm{c}(\psi)$. Hence $A\mathrm{c}(\psi) = B\mathrm{c}(\psi)$ by [22]. So $A^{**}\mathrm{c}(\psi) = B^{**}\mathrm{c}(\psi)$ is a type I factor implying that $\mathrm{c}(\psi)z_A \neq 0$. It follows that $\mathrm{c}(\psi) = \mathrm{c}(\overline{\varphi}) \geqslant \mathrm{c}(\varphi)$ for some $\varphi \in P(A)$ with extension $\overline{\varphi} \in P(B)$ (using the proof of Proposition 2.7) so that $A^{**}\mathrm{c}(\varphi) = B^{**}\mathrm{c}(\varphi)$ and we deduce that $\mathrm{c}(\varphi)$ is central in B^{**} and in turn, that $\mathrm{c}(\psi) = \mathrm{c}(\varphi)$. Therefore $A^{**}z_A = B^{**}z_B$ and so A separates $P(B) \cup \{0\}$ (cf. Remark 4.8). Hence $A = B$ by Kaplansky's theorem ([19], 11.1.8), as B is type I.

(ii) The inclusions $A \hookrightarrow H(A)$ and $H(A) \hookrightarrow B$ exhibit the AEP. Let $\widehat{\alpha}_1 : \widehat{A} \to \widehat{H(A)}$ and $\widehat{\alpha}_2 : \widehat{H(A)} \to \widehat{B}$ be the corresponding continuous maps given by Proposition 4.1, both of which are injective, by Theorem 4.2, and hence bijective as $\widehat{\alpha} = \widehat{\alpha}_2 \circ \widehat{\alpha}_1$. Therefore $\widehat{\alpha}_1$ is a homeomorphism. Consequently we may suppose that $H(A) = B$.

Let $\psi \in P(B)$. Then $\mathrm{c}(\psi) = \mathrm{c}(\overline{\varphi})$ for some $\overline{\varphi} \in P(B)$ with $\varphi = \overline{\varphi}|A \in P(A)$, by assumption. As in (i), $A\mathrm{c}(\psi), B\mathrm{c}(\psi)$ have common approximate unit and $A\mathrm{c}(\psi)$ has AEP in $B\mathrm{c}(\psi)$. But $A\mathrm{c}(\psi)$ is scattered as $\ker \pi_\psi \cap A = \ker \pi_{\overline{\varphi}} \cap A = \ker \pi_\varphi$ by Proposition 4.7, and so $A\mathrm{c}(\psi) = B\mathrm{c}(\psi)$. Hence all the primitive quotients of B are scattered and the result follows from (i). ∎

Regarding Proposition 4.9, if either A or B is liminal, then its primitive quotients are automatically scattered. We conclude with an example which shows that a slight relaxation in the conditions can render both parts of Proposition 4.9 false. To this end we exhibit below two unequal primitive type I C^*-algebras A and B such that $1 \in A \subset B$, A has the AEP in B and $\widehat{\alpha} : \widehat{A} \to \widehat{B}$ is a homeomorphism.

Example 4.10. Let $D = C[0,1]$ and embed D^{**} as a von Neumann subalgebra of some $B(H)$, with the same identity, such that all minimal projections of D^{**} are properly infinite in $B(H)$. Put $K = K(H)$ and let zD^{**} ($\neq D^{**}$) be the atomic part of D^{**}. We note that zKz is a hereditary C^*-subalgebra of $B = D + K$ and is an ideal of the C^*-algebra $A = D + zKz$. Further, as $D^{**} \cap K = \{0\}$, the map

$A \to zB(H)z : a \mapsto az$ is faithful, inducing a faithful irreducible representation $A \to B(zH)$. In particular, A and B are primitive type I with $1 \in A \subsetneq B$. We claim that A has the PEP in B.

Indeed, let $\varphi \in P(A)$ with extension $\overline{\varphi} \in P(B)$. As zKz is hereditary in B, we may suppose to establish uniqueness of $\overline{\varphi}$ that $\varphi(zKz) = 0$. If $\overline{\varphi}(K) \neq 0$, then $\overline{\varphi}$ extends to a vector state $\omega_h = \langle \cdot h, h \rangle$ on $B(H)$, in which case, $\omega_h | D^{**}$ is a normal extension of $\varphi | D \in P(D)$ which implies that $\omega_h(z) = 1$ and hence that $\omega_h(K) = \omega_h(zKz) = \varphi(zKz) = 0$. This contradiction proves that $\overline{\varphi}(K) = 0$ and in turn that A has the PEP in B, as claimed. Finally, by Theorem 4.2 (v) $\Rightarrow$ (i), A has the AEP in B because the map $\widehat{\alpha} = \check{\alpha} : \operatorname{Prim} A \to \operatorname{Prim} B$ is given by $\check{\alpha}(0) = 0$ and $\check{\alpha}(Q + zKz) = Q + K$ for each $Q \in \operatorname{Prim} D$, which is easily seen to be a homeomorphism.

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L.J. BUNCE
Department of Mathematics
University of Reading
Reading RG6 2AX
ENGLAND

C.-H. CHU
Goldsmiths College
University of London
London SE14 6NW
ENGLAND

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