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SPECTRAL DECOMPOSITION OF SOME NONSELFADJOINT BLOCK OPERATOR MATRICES

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Abstract. In this note we study spectral properties of a block operator matrix $\widetilde{A}$ (see (1.1) below), where A and $-D$ are m-accretive, and B, D are bounded operators. Under an additional assumption, the spectrum of $\widetilde{A}$ consists of one part in the extended right and one part in the left half plane, and the corresponding spectral subspaces allow representations by means of angular operators. If the part of the spectrum of $\widetilde{A}$ in the right half plane is discrete, a half range completeness statement follows. As an essential tool the quadratic numerical range of a block operator matrix is introduced.

Keywords: *Block operator matrices, spectral subspaces, half range completeness, numerical range.*

MSC (2000): Primary 47A15; Secondary 47A12.

1. INTRODUCTION

In the paper [1] selfadjoint operators $\widetilde{A}$ in a Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \times \widehat{\mathcal{H}}$, given by a block operator matrix

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \tag{1.1}$$

were considered under the assumption that the spectra of A and D are separated and B, D are bounded operators. In [2] and in [6] these investigations were extended to the case that all entries are unbounded.

In this note the results of [1] are generalized to a nonselfadjoint situation. We consider an operator $\widetilde{A}$ given by a block operator matrix (1.1) such that A and $-D$

are m-sectorial (see [4]) and their numerical ranges have a positive distance from the imaginary axis, and B is bounded. We prove in Section 3 that the imaginary axis belongs to the resolvent set $\rho(\widetilde{A})$. If the spaces $\mathcal{H}$, $\widehat{\mathcal{H}}$ are not trivial, then the spectrum $\sigma(\widetilde{A})$ consists of some part $\sigma_-(\widetilde{A})$ in the left half plane and another part $\sigma_+(\widetilde{A})$ in the right half plane (if A or D are unbounded, then $\sigma_+(\widetilde{A})$ or $\sigma_-(\widetilde{A})$ may be empty, which implies that ∞ belongs to the extended spectrum of $\widetilde{A}$).

As the main result of this note, it is shown in Section 4 that, if the operator D is bounded, then the spectral subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ corresponding to $\sigma_+(\widetilde{A})$ and $\sigma_-(\widetilde{A})$ are supported on $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively; that is, e.g., $\mathcal{L}_+$ admits a representation

$$\mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ K_+ x \end{pmatrix} : x \in \mathcal{H} \right\}$$

with some bounded linear operator K_+ from $\mathcal{H}$ into $\widehat{\mathcal{H}}$. The invariance of $\mathcal{L}_+$ and $\mathcal{L}_-$ under $\widetilde{A}$ implies that K_+ and the corresponding operator K_- for $\mathcal{L}_-$ satisfy certain Riccati equations.

As in [1], this result is used in Section 5 in order to prove a half range completeness statement: If $\sigma_+(\widetilde{A})$ is discrete, under certain assumptions, the first components of a system of root vectors of $\widetilde{A}$ corresponding to $\sigma_+(\widetilde{A})$ ("half" of the spectrum of $\widetilde{A}$) form a complete system in $\mathcal{H}$. An example of an eigenvalue problem of Sturm-Liouville type where this result can be applied is given at the end of Section 5.

In order to locate the spectrum $\sigma(\widetilde{A})$ of the block operator matrix $\widetilde{A}$ in Section 3, we introduce the notion of the quadratic numerical range of a general block operator matrix

$$\widetilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with closed operators A, D and bounded operators B, C in Section 2. This quadratic numerical range is the set of all eigenvalues of the matrices

$$\begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} & \frac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\[2ex] \frac{(Cx,\hat{x})}{\|x\|\|\hat{x}\|} & \frac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} \end{pmatrix},$$

where $x \in \mathcal{D}(A), \hat{x} \in \mathcal{D}(D)$, $x, \hat{x} \neq 0$. It has some properties analogous to those of the numerical range. We mention that the quadratic numerical range of a block operator matrix $\widetilde{A}$ turns out to be especially useful in the particular case of a selfadjoint operator $\widetilde{A}$. This question will be considered elsewhere.

2. QUADRATIC NUMERICAL RANGE

Let $\mathcal{H}$ and $\widehat{\mathcal{H}}$ be Hilbert spaces. We consider an operator $\widetilde{A}$ in the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \times \widehat{\mathcal{H}}$ given by a block operator matrix

$$\widetilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

where A and D are densely defined closed operators in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, and the operators $B \in L(\widehat{\mathcal{H}}, \mathcal{H}), C \in L(\mathcal{H}, \widehat{\mathcal{H}})$ are bounded.

For the operator $\widetilde{A}$ we call the set

$$\begin{aligned} W^2_{\widetilde{A}} := \Bigg\{ \lambda \in \mathbb{C} : \det \begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} - \lambda & \frac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\ \frac{(Cx,\hat{x})}{\|x\|\|\hat{x}\|} & \frac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} - \lambda \end{pmatrix} = 0, \\ x \in \mathcal{D}(A),\, \hat{x} \in \mathcal{D}(D),\, x, \hat{x} \neq 0 \Bigg\} \end{aligned} \tag{2.2}$$

the *quadratic numerical range* (with respect to the block operator representation (2.1)).

Thus, for each element $\widetilde{x} = (x, \hat{x})^{\mathrm{t}} \in \widetilde{\mathcal{H}}$ such that $x \in \mathcal{D}(A)$, $\hat{x} \in \mathcal{D}(D)$, $x, \hat{x} \neq 0$, two complex numbers λ_1, λ_2 are defined as the solutions of the quadratic equation in (2.2), which are the eigenvalues of the matrix

$$\begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} & \frac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\ \frac{(Cx,\hat{x})}{\|x\|\|\hat{x}\|} & \frac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} \end{pmatrix},$$

and $W^2_{\widetilde{A}}$ is the set of all these solutions or of all these eigenvalues.

It is easy to see that, for a bounded operator $\widetilde{A}$, the quadratic numerical range is a bounded subset of $\mathbb{C}$. It is also not difficult to see that the quadratic numerical range consists of at most two connected sets. If $B = 0$ or $C = 0$, then $W^2_{\widetilde{A}} = W_A \cup W_D$, where for a closed operator T in a Hilbert space, W_T denotes its numerical range,

$$W_T := \left\{ \frac{(Tx, x)}{\|x\|^2} : x \in \mathcal{D}(T),\, x \neq 0 \right\}.$$

Let $\mathrm{r}(\widetilde{A})$ be the set of points of regular type of $\widetilde{A}$, that is, $\lambda \in \mathrm{r}(\widetilde{A})$ is equivalent to

$$\|(\widetilde{A} - \lambda)\widetilde{x}\| \geqslant \gamma_\lambda \|\widetilde{x}\|, \quad \widetilde{x} \in \mathcal{D}(\widetilde{A}),$$

for some $\gamma_\lambda > 0$.

THEOREM 2.1. *For the quadratic numerical range $W^2_{\widetilde{A}}$ the following inclusions hold:*

$$\sigma_{\mathrm{p}}(\widetilde{A}) \subset W^2_{\widetilde{A}}, \quad \mathbb{C} \setminus \mathrm{r}(\widetilde{A}) \subset \overline{W^2_{\widetilde{A}}}.$$

Proof. Let $\lambda \in \sigma_{\mathrm{p}}(\widetilde{A})$. Then there exists a nontrivial vector $(x, \hat{x})^{\mathrm{t}} \in \widetilde{\mathcal{H}}$, $x \in \mathcal{D}(A)$, $\hat{x} \in \mathcal{D}(D)$, such that

$$\begin{aligned} (A-\lambda)x + B\hat{x} &= 0, \\ Cx + (D-\lambda)\hat{x} &= 0. \end{aligned} \tag{2.3}$$

Suppose first that $x, \hat{x} \neq 0$. Taking the inner product of the first (second, respectively) equation in (2.3) with x ($\hat{x}$, respectively), we find that the system

$$\begin{pmatrix} (Ax,x) - \lambda\|x\|^2 & (B\hat{x},x) \\ (Cx,\hat{x}) & (D\hat{x},\hat{x}) - \lambda\|\hat{x}\|^2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = 0$$

has the solution $\zeta_1 = 1$, $\zeta_2 = 1$, hence

$$\det \begin{pmatrix} (Ax,x) - \lambda\|x\|^2 & (B\hat{x},x) \\ (Cx,\hat{x}) & (D\hat{x},\hat{x}) - \lambda\|\hat{x}\|^2 \end{pmatrix} = 0.$$

But this equation is equivalent to the equation in the definition (2.2) of $W^2_{\widetilde{A}}$, and hence $\lambda \in W^2_{\widetilde{A}}$. Now let $\hat{x} = 0$. Then $(A-\lambda)x = 0$, $Cx = 0$, and hence

$$\det \begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} - \lambda & \frac{(B\hat{x}',x)}{\|\hat{x}'\|\|x\|} \\ \frac{(Cx,\hat{x}')}{\|x\|\|\hat{x}'\|} & \frac{(D\hat{x}',\hat{x}')}{\|\hat{x}'\|^2} - \lambda \end{pmatrix} = \det \begin{pmatrix} 0 & \frac{(B\hat{x}',x)}{\|\hat{x}'\|\|x\|} \\ 0 & \frac{(D\hat{x}',\hat{x}')}{\|\hat{x}'\|^2} - \lambda \end{pmatrix} = 0$$

for all $\hat{x}' \in \mathcal{D}(D), \hat{x}' \neq 0$, which implies $\lambda \in W^2_{\widetilde{A}}$. The case $x = 0$ is analogous.

More generally, if $\lambda_0 \notin \mathrm{r}(\widetilde{A})$, then there exists a sequence $(\widetilde{x}_n)_1^\infty \subset \widetilde{\mathcal{H}}$, $\widetilde{x}_n = (x_n, \hat{x}_n)^{\mathrm{t}}$, $x_n \in \mathcal{D}(A)$, $\hat{x}_n \in \mathcal{D}(D)$, $\|\widetilde{x}_n\| = 1$, such that $\|(\widetilde{A} - \lambda_0)\widetilde{x}_n\| \to 0$ for $n \to \infty$, that is,

$$\begin{aligned} (A-\lambda_0)x_n + B\hat{x}_n &= f_n, \\ Cx_n + (D-\lambda_0)\hat{x}_n &= \hat{f}_n, \end{aligned}$$

where $(f_n)_1^\infty \subset \mathcal{H}, (\hat{f}_n)_1^\infty \subset \widehat{\mathcal{H}}$, $\|f_n\| \to 0, \|\hat{f}_n\| \to 0$ for $n \to \infty$. Suppose first that $\liminf\limits_{n\to\infty} \|x_n\| > 0$, $\liminf\limits_{n\to\infty} \|\hat{x}_n\| > 0$. Without loss of generality we can assume $\|x_n\| > 0, \|\hat{x}_n\| > 0$ for $n = 1, 2, \dots$. Then

$$\frac{(Ax_n, x_n)}{\|x_n\|^2} - \lambda_0 + \frac{(B\hat{x}_n, x_n)}{\|x_n\|^2} = \frac{(f_n, x_n)}{\|x_n\|^2},$$

$$\frac{(Cx_n, \hat{x}_n)}{\|\hat{x}_n\|^2} + \frac{(D\hat{x}_n, \hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda_0 = \frac{(\hat{f}_n, \hat{x}_n)}{\|\hat{x}_n\|^2}.$$

We introduce the polynomials

$$d_n(\lambda) := \det\begin{pmatrix} \frac{(Ax_n,x_n)}{\|x_n\|^2} - \lambda & \frac{(B\hat{x}_n,x_n)}{\|x_n\|^2} \\ \frac{(Cx_n,\hat{x}_n)}{\|\hat{x}_n\|^2} & \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda \end{pmatrix} = \det\begin{pmatrix} \frac{(Ax_n,x_n)}{\|x_n\|^2} - \lambda & \frac{(B\hat{x}_n,x_n)}{\|\hat{x}_n\|\|x_n\|} \\ \frac{(Cx_n,\hat{x}_n)}{\|x_n\|\|\hat{x}_n\|} & \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda \end{pmatrix}.$$

Then $f_n, \hat{f}_n \to 0$ for $n \to \infty$ implies

$$d_n(\lambda_0) = \det\begin{pmatrix} \frac{(f_n,x_n)}{\|x_n\|^2} & \frac{(B\hat{x}_n,x_n)}{\|x_n\|^2} \\ \frac{(\hat{f}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} & \frac{(D\hat{x}_n,\hat{x}_n)}{\|\hat{x}_n\|^2} - \lambda_0 \end{pmatrix} \to 0, \quad n \to \infty.$$

For each n, d_n is a monic quadratic polynomial in λ. If λ_n^1, λ_n^2 are the zeros of d_n, then $\lambda_n^1, \lambda_n^2 \in W^2_{\widetilde{A}}$ and $d_n(\lambda) = (\lambda - \lambda_n^1)(\lambda - \lambda_n^2)$. As $d_n(\lambda_0) \to 0$ for $n \to \infty$, we have $\lambda_n^1 \to \lambda_0$ or $\lambda_n^2 \to \lambda_0$ for $n \to \infty$ and thus $\lambda_0 \in \overline{W^2_{\widetilde{A}}}$.

Let $\liminf\limits_{n\to\infty} \|\hat{x}_n\| = 0$, without loss of generality $\hat{x}_n \to 0$ for $n \to \infty$, $\|x_n\| > 0$ for $n = 1, 2, \dots$. If we define $\lambda_n := \frac{(Ax_n,x_n)}{\|x_n\|^2}$ and choose $\hat{x}'_n \in \widehat{\mathcal{H}}$, $\hat{x}'_n \neq 0$, such that $(Cx_n, \hat{x}'_n) = 0$ for $n = 1, 2, \dots$, then

$$\det\begin{pmatrix} \frac{(Ax_n,x_n)}{\|x_n\|^2} - \lambda_n & \frac{(B\hat{x}'_n,x_n)}{\|\hat{x}'_n\|\|x_n\|)} \\ \frac{(Cx_n,\hat{x}'_n)}{\|x_n\|\|\hat{x}'_n\|} & \frac{(D\hat{x}'_n,\hat{x}'_n)}{\|\hat{x}'_n\|^2} - \lambda_n \end{pmatrix} = \det\begin{pmatrix} 0 & \frac{(B\hat{x}'_n,x_n)}{\|\hat{x}'_n\|\|x_n\|} \\ 0 & \frac{(D\hat{x}'_n,\hat{x}'_n)}{\|\hat{x}'_n\|^2} - \lambda_n \end{pmatrix} = 0,$$

that is, $\lambda_n \in W^2_{\widetilde{A}}$. As $\hat{x}_n \to 0$ and $f_n \to 0$ for $n \to \infty$, the relation

$$((A - \lambda_0)x_n, x_n) - (B\hat{x}_n, x_n) = (f_n, x_n)$$

implies $\lambda_n \to \lambda_0$. The case $\liminf\limits_{n\to\infty} \|x_n\| = 0$ is analogous. This proves the theorem. ∎

Later we will need the following well-known result connected with the numerical range of a closed operator. Here and throughout this note we assume that $\arg z \in (-\pi, \pi]$ for complex numbers $z \in \mathbb{C}$.

LEMMA 2.2. *Let T be a closed operator in a Hilbert space. Assume that its numerical range W_T is contained in a sector $\Delta_T = \{z \in \mathbb{C} : |\arg z| \leqslant \theta_T\}$ for some θ_T, $0 < \theta_T < \frac{\pi}{2}$. Then there exists a constant $C > 0$ such that*

$$\|(T - z)^{-1}\| \leqslant \frac{C}{|z|}, \quad \Re(z) \leqslant 0, \quad z \neq 0.$$

Proof. Let $z \in \mathbb{C}$ with $\Re(z) \leqslant 0$, $z \neq 0$. If $\frac{\pi}{2} \leqslant \arg z \leqslant \frac{\pi}{2} + \theta$, then

$$\|(T - z)^{-1}\| \leqslant \frac{1}{\operatorname{dist}(z, W_T)} \leqslant \frac{1}{|z| \cos\left(\frac{\pi}{2} + \theta_T - \arg z\right)} \leqslant \frac{1}{|z| \cos \theta_T}.$$

The case $-\frac{\pi}{2} - \theta_T \leqslant \arg z \leqslant -\frac{\pi}{2}$ is analogous. If $|\arg z| \geqslant \frac{\pi}{2} + \theta_T$, then

$$\|(T-z)^{-1}\| \leqslant \frac{1}{\operatorname{dist}(z, W_T)} \leqslant \frac{1}{|z|}. \qquad \blacksquare$$

3. BLOCK OPERATOR MATRICES WITH SEPARATED SPECTRUM

In the sequel we consider a block operator matrix (2.1) of the particular form

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \tag{3.1}$$

that is, we assume $C = B^*$. Additionally to the assumptions of Section 2, we suppose:

(α) The operator A is boundedly invertible and its numerical range W_A is contained in the set

$$\{z \in \mathbb{C} : |\arg z| \leqslant \theta_A,\ \Re(z) \geqslant \alpha\}$$

for some θ_A, $0 < \theta_A < \frac{\pi}{2}$, and $\alpha > 0$.

(δ) The operator D is boundedly invertible and its numerical range W_D is contained in the set

$$\{z \in \mathbb{C} : |\arg z| \geqslant \pi - \theta_D,\ \Re(z) \leqslant -\delta\}$$

for some θ_D, $0 < \theta_D < \frac{\pi}{2}$, and $\delta > 0$.

This means that the operators A and $-D$ are m-sectorial (see [4], Chapter V, Section 3) and that their numerical ranges have a positive distance from the imaginary axis. We mention that the role of the imaginary axis can be taken over by any other vertical line in the complex plane.

LEMMA 3.1. *Let a, b, c and d be complex numbers such that $\Re(a) > 0$, $\Re(d) < 0$ and $bc \geqslant 0$. Then the matrix*

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has eigenvalues λ_1, λ_2 such that:

(i) $\Re(\lambda_1) \geqslant \Re(a)$, $\Re(\lambda_2) \leqslant \Re(d)$;

(ii) $\min\{\Im(a), \Im(d)\} \leqslant \Im(\lambda_1), \Im(\lambda_2) \leqslant \max\{\Im(a), \Im(d)\}$;

(iii) $\lambda_1, -\lambda_2 \in \{z \in \mathbb{C} : |\arg z| \leqslant \max\{|\arg a|, \pi - |\arg d|\}$.

Proof. (We thank Mrs. A. Luger for communicating this proof to us.) We can suppose that $\Im(a) \geqslant 0$ (otherwise we consider the matrix $\mathcal{A}^*$) and that

$$\arg a \geqslant \pi - |\arg d| \tag{3.2}$$

(otherwise in the following considerations we start from d instead of a). The assumption (3.2) implies

$$\left|\frac{\Im(a-d)}{\Re(a-d)}\right| \leqslant \tan(\arg a). \tag{3.3}$$

The eigenvalues λ_1, λ_2 satisfy the equation

$$(a-\lambda)(d-\lambda) - t = 0, \quad t = bc \geqslant 0.$$

We consider them as functions of t:

$$\lambda_{1,2}(t) = \frac{a+d}{2} \pm \left(\frac{(a-d)^2}{4} + t\right)^{\frac{1}{2}}.$$

If we decompose $\lambda_{1,2}(t)$ and $\frac{a+d}{2}$ in real and imaginary parts, $\lambda_{1,2}(t) = x(t) + \mathrm{i}y(t)$ and $\frac{a+d}{2} = \beta + \mathrm{i}\gamma$, then we find

$$(x(t)-\beta)^2 - (y(t)-\gamma)^2 = \frac{1}{4}\Re(a-d)^2 + t, \tag{3.4}$$

$$(x(t)-\beta)(y(t)-\gamma) = \frac{1}{8}\Im(a-d)^2. \tag{3.5}$$

The last relation shows that the eigenvalues $\lambda_1(t), \lambda_2(t)$ lie on a hyperbola with centre $\beta + \mathrm{i}\gamma = \frac{a+d}{2}$, and with the asymptotes $\Im(z) = \gamma$ and $\Re(z) = \beta$ parallel to the real and imaginary axis, the right hand branch passing through a and the left hand branch through d. From the identity (3.4) it follows that for $0 \leqslant t \leqslant \infty$ the eigenvalues $\lambda_1(t)$ fill the part of the right hand branch which extends from a to $+\infty + \mathrm{i}\gamma$, and the eigenvalues $\lambda_2(t)$ fill the part of the left hand branch from d to $-\infty + \mathrm{i}\gamma$. This implies (i) and (ii). In order to prove (iii), it is sufficient to show that the derivatives of the hyperbola at d and at a are in modulus less than $\tan(\arg a)$. E.g. for the derivative at d it follows from (3.5)

$$\frac{\dot{y}(0)}{\dot{x}(0)} = -\frac{y(0)-\gamma}{x(0)-\beta} = -\frac{\Im(d) - \frac{1}{2}\Im(a+d)}{\Re(d) - \frac{1}{2}\Re(a+d)} = -\frac{\Im(d-a)}{\Re(d-a)},$$

which is in modulus less than $\tan(\arg a)$ by (3.3). The lemma is proved. ∎

THEOREM 3.2. *Suppose the assumptions* (α) *and* (δ) *are satisfied and define*

$$\Delta := \{z \in \mathbb{C} : |\arg z| \leqslant \max\{\theta_A, \theta_D\}\}.$$

Then

$$\sigma(\widetilde{A}) \subset \{z \in (-\Delta) : \Re(z) \leqslant -\delta\} \cup \{z \in \Delta : \Re(z) \geqslant \alpha\} =: \widetilde{\Delta}.$$

Proof. First we show that $\overline{W^2_{\widetilde{A}}} \subset \widetilde{\Delta}$. To this end consider for $x \in \mathcal{D}(A), \hat{x} \in \mathcal{D}(D)$, $x, \hat{x} \neq 0$, the matrix

$$\begin{pmatrix} \frac{(Ax,x)}{\|x\|^2} & \frac{(B\hat{x},x)}{\|\hat{x}\|\|x\|} \\ \frac{(B^*x,\hat{x})}{\|x\|\|\hat{x}\|} & \frac{(D\hat{x},\hat{x})}{\|\hat{x}\|^2} \end{pmatrix}.$$

According to the assumptions (α) and (δ), it has all the properties of the matrix $\mathcal{A}$ in Lemma 3.1. Hence its eigenvalues are in $\widetilde{\Delta}$ which implies $W^2_{\widetilde{A}} \subset \widetilde{\Delta}$ and hence $\overline{W^2_{\widetilde{A}}} \subset \widetilde{\Delta}$. According to Theorem 2.1 we have $\mathbb{C} \setminus \mathrm{r}(\widetilde{A}) \subset \overline{W^2_{\widetilde{A}}}$ and consequently $\mathbb{C} \setminus \widetilde{\Delta} \subset \mathrm{r}(\widetilde{A})$. On the other hand, $\mathbb{C} \setminus \widetilde{\Delta}$ consists of only one component, hence the theorem will be proved if we show that at least one point λ_0 of this component belongs to $\rho(\widetilde{A})$. To this end we choose λ_0 on the imaginary axis sufficiently large in modulus. According to Lemma 2.2 we can choose λ_0 such that $\|B(D-\lambda_0)^{-1}B^*(A-\lambda_0)^{-1}\| < 1$. Then $\lambda_0 \in \rho(\widetilde{A})$ as

$$(A - \lambda_0 - B(D-\lambda_0)^{-1}B^*)^{-1} = (A-\lambda_0)^{-1}(I - B(D-\lambda_0)^{-1}B^*(A-\lambda_0)^{-1})^{-1}$$

exists and is a bounded everywhere defined operator. The theorem is proved. ∎

REMARK 3.3. *If the numerical ranges of* A *and* D *are even contained in half strips, say* $W_A \subset \{z \in \mathbb{C} : a_1 \leqslant \Im(z) \leqslant a_2,\ \Re(z) \geqslant \alpha\}$ *and* $W_D \subset \{z \in \mathbb{C} : d_1 \leqslant \Im(z) \leqslant d_2,\ \Re(z) \leqslant -\delta\}$*, then the spectrum of* $\widetilde{A}$ *is contained in the set*

$$\{z \in \mathbb{C} : \min\{a_1, d_1\} \leqslant \Im(z) \leqslant \max\{a_2, d_2\},\ \Re(z) \leqslant -\delta \text{ or } \Re(z) \geqslant \alpha\}.$$

4. INVARIANT SUBSPACES

In the sequel we consider a block operator matrix $\widetilde{A}$ as in (3.1) which satisfies the assumptions of Section 3 and for which D is bounded. Note that in this case the assumption (δ) is fulfilled if there exists a $\delta > 0$ such that

$$\Re(D) \leqslant -\delta.$$

From Theorem 3.2 it follows that $\sigma(\widetilde{A})$ splits into the two disjoint subsets

$$\sigma_-(\widetilde{A}) := \sigma(\widetilde{A}) \cap \{z \in (-\Delta) : \Re(z) \leqslant -\delta\},$$
$$\sigma_+(\widetilde{A}) := \sigma(\widetilde{A}) \cap \{z \in \Delta : \Re(z) \geqslant \alpha\}.$$

Here, as A and hence also $\widetilde{A}$ can be unbounded, $\sigma_+(\widetilde{A})$ can be empty. Since D is a bounded operator, $\sigma_-(\widetilde{A})$ is bounded. Let

$$P_-(\widetilde{A}) := -\frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma_-} (\widetilde{A} - z)^{-1} \,\mathrm{d}z$$

be the corresponding Riesz projection. Here Γ_- is a positively oriented Jordan contour in $\{z \in \mathbb{C} : \Re(z) < 0\}$ surrounding $\sigma_-(\widetilde{A})$. If we define the projection

$$P_+(\widetilde{A}) := I - P_-(\widetilde{A}),$$

then we have a decomposition $\widetilde{\mathcal{H}} = \mathcal{L}_- \dot{+} \mathcal{L}_+$ into the spectral subspaces

$$\mathcal{L}_- := P_-(\widetilde{A})\widetilde{\mathcal{H}}, \quad \mathcal{L}_+ := P_+(\widetilde{A})\widetilde{\mathcal{H}},$$

and

$$\sigma(\widetilde{A}|\mathcal{L}_-) = \sigma_-(\widetilde{A}), \quad \sigma(\widetilde{A}|\mathcal{L}_+) = \sigma_+(\widetilde{A}).$$

Here $\widetilde{A}|\mathcal{L}_+$ is, in fact, the restriction of $\widetilde{A}$ to $\mathcal{D}(\widetilde{A}) \cap \mathcal{L}_+$ and has its values in $\mathcal{L}_+$. If $\mathcal{H} \neq \{0\}$, then $\mathcal{L}_+ \neq \{0\}$, even if $\sigma_+(\widetilde{A}) = \emptyset$.

Lemma 4.1. *We have*

$$-\frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (\widetilde{A} - z)^{-1} \,\mathrm{d}z = \frac{1}{2}\Big(P_-(\widetilde{A}) - P_+(\widetilde{A})\Big).$$

Proof. Let $r_0 > 0$ be such that $\sigma_-(\widetilde{A}) \subset \{z \in \mathbb{C} : \Re(z) < 0, |z| < r_0\}$. Then we have

$$-\frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}r}^{\mathrm{i}r} (\widetilde{A} - z)^{-1} \,\mathrm{d}z = P_-(\widetilde{A}) + \frac{1}{2\pi \mathrm{i}} \int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\widetilde{A} - r\mathrm{e}^{\mathrm{i}t})^{-1} \,\mathrm{i}r\mathrm{e}^{\mathrm{i}t} \,\mathrm{d}t, \quad r \geqslant r_0.$$

The lemma is proved if we show that

$$\frac{1}{2\pi \mathrm{i}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\widetilde{A} - r\mathrm{e}^{\mathrm{i}t})^{-1} \, \mathrm{i}r\mathrm{e}^{\mathrm{i}t} \, \mathrm{d}t \longrightarrow -\frac{1}{2} I, \quad r \to \infty, \tag{4.1}$$

strongly in $\widetilde{\mathcal{H}}$ as $I = P_-(\widetilde{A}) + P_+(\widetilde{A})$. From

$$\begin{aligned} \widetilde{A} - z = &\begin{pmatrix} I & B(D-z)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - z - B(D-z)^{-1}B^* & 0 \\ 0 & D - z \end{pmatrix} \cdot \\ &\cdot \begin{pmatrix} I & 0 \\ (D-z)^{-1}B^* & I \end{pmatrix}, \end{aligned} \tag{4.2}$$

it follows

$$\begin{aligned} &(\widetilde{A} - z)^{-1} \\ &= \begin{pmatrix} M(z) & -M(z)B(D-z)^{-1} \\ -(D-z)^{-1}B^*M(z) & (D-z)^{-1} + (D-z)^{-1}B^*M(z)B(D-z)^{-1} \end{pmatrix} \end{aligned} \tag{4.3}$$

for $z \in \rho(\widetilde{A})$ where

$$M(z) := (A - z - B(D-z)^{-1}B^*)^{-1}.$$

For the proof of (4.1) we first consider the left upper corner of the matrix $(\widetilde{A} - r\mathrm{e}^{\mathrm{i}t})^{-1} r\mathrm{e}^{\mathrm{i}t} + I$ and show that

$$\|\big(M(r\mathrm{e}^{\mathrm{i}t}) r\mathrm{e}^{\mathrm{i}t} + I\big)x\| \to 0, \quad r \to \infty, \tag{4.4}$$

uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for $x \in \mathcal{H}$. Using Lemma 2.2 and

$$\|(D-z)^{-1}\| \leqslant \frac{2}{|z|}, \quad |z| > 2\|D\|, \tag{4.5}$$

we find

$$\begin{aligned} M(r\mathrm{e}^{\mathrm{i}t}) r\mathrm{e}^{\mathrm{i}t} + I &= (A - r\mathrm{e}^{\mathrm{i}t} - B(D - r\mathrm{e}^{\mathrm{i}t})^{-1}B^*)^{-1} r\mathrm{e}^{\mathrm{i}t} + I \\ &= (I - (A - r\mathrm{e}^{\mathrm{i}t})^{-1}B(D - r\mathrm{e}^{\mathrm{i}t})^{-1}B^*)^{-1}(A - r\mathrm{e}^{\mathrm{i}t})^{-1} r\mathrm{e}^{\mathrm{i}t} + I \\ &= (A - r\mathrm{e}^{\mathrm{i}t})^{-1} r\mathrm{e}^{\mathrm{i}t} + I + \mathrm{O}\left(\frac{1}{r}\right) \end{aligned}$$

uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Hence there exists a $\widetilde{C} > 0$ such that

$$\begin{aligned}\|\big(M(re^{it})re^{it} + I\big)x\| &\leqslant \|\big((A - re^{it})^{-1}re^{it} + I\big)x\| + \frac{\widetilde{C}}{r}\|x\| \\ &= \|(A - re^{it})^{-1}Ax\| + \frac{\widetilde{C}}{r}\|x\| \leqslant \frac{C}{r}\|Ax\| + \frac{\widetilde{C}}{r}\|x\|\end{aligned}$$

uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ by Lemma 2.2 which proves (4.4). From (4.4) it follows

$$\|M(re^{it})re^{it}x\| \leqslant K \tag{4.6}$$

with some constant $K > 0$ uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for $x \in \mathcal{H}$. For the off-diagonal elements of $(\widetilde{A} - re^{it})^{-1}re^{it} + I$ we have

$$\begin{aligned}\|M(re^{it})B(D - re^{it})^{-1}re^{it}\hat{x}\| &\leqslant \|\big(M(re^{it})re^{it} + I\big)B(D - re^{it})^{-1}\hat{x}\| \\ &\quad + \|B(D - re^{it})^{-1}\hat{x}\| \to 0, \quad r \to \infty,\end{aligned}$$

uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for $\hat{x} \in \widehat{\mathcal{H}}$ by (4.4) and (4.5). Furthermore,

$$\|(D - re^{it})^{-1}B^*M(re^{it})re^{it}x\| \to 0, \quad r \to \infty,$$

uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for $x \in \mathcal{H}$ by (4.5) and (4.6). For the right lower corner of $(\widetilde{A} - re^{it})^{-1}re^{it} + I$ we have

$$\begin{aligned}&\|\big((D - re^{it})^{-1}re^{it} + (D - re^{it})^{-1}B^*M(re^{it})B(D - re^{it})^{-1}re^{it} + I\big)\hat{x}\| \\ &\quad \leqslant \frac{1}{r}2\|D\|\,\|\hat{x}\| + \frac{1}{r}\|B^*\|\,\|M(re^{it})re^{it}B(D - re^{it})^{-1}\hat{x}\| \to 0, \quad r \to \infty,\end{aligned}$$

uniformly in $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ for $\hat{x} \in \widehat{\mathcal{H}}$. Summarizing, we obtain that all the components in the matrix representation of $(\widetilde{A} - re^{it})^{-1}re^{it} + I$ tend strongly to 0 uniformly for $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and hence

$$\left(\frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\widetilde{A} - re^{it})^{-1}\, ire^{it}\, dt + \frac{1}{2}I\right)\widetilde{x} = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \Big((\widetilde{A} - re^{it})^{-1}re^{it}\widetilde{x} + \widetilde{x}\Big)\, dt \to 0,$$

$r \to \infty$, for $\widetilde{x} \in \widetilde{\mathcal{H}}$. The lemma is proved. ∎

As $\widetilde{A}$ is not supposed to be selfadjoint, the spectral subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ need not be orthogonal. However, the block operator matrix $\widetilde{A}^*$ adjoint to $\widetilde{A}$ given by

$$\widetilde{A}^* = \begin{pmatrix} A^* & B \\ B^* & D^* \end{pmatrix}$$

fulfills the assumptions of the present section as well. Hence $P_-(\widetilde{A}^*)$ and $P_+(\widetilde{A}^*)$ are defined, and $\widetilde{A}^*$ has the spectral subspaces

$$\mathcal{L}_-^* := P_-(\widetilde{A}^*)\widetilde{\mathcal{H}}, \quad \mathcal{L}_+^* := P_+(\widetilde{A}^*)\widetilde{\mathcal{H}}.$$

LEMMA 4.2. $\mathcal{L}_-^{\perp} = \mathcal{L}_+^*$, $\mathcal{L}_+^{\perp} = \mathcal{L}_-^*$.

Proof. We have

$$\begin{aligned}\mathcal{L}_-^{\perp} &= R(P_-(\widetilde{A}))^{\perp} = \ker(P_-(\widetilde{A})^*) = \ker(P_-(\widetilde{A}^*))\\ &= \ker(I - P_+(\widetilde{A}^*)) = R(P_+(\widetilde{A}^*)) = \mathcal{L}_+^*\end{aligned}$$

as $\sigma_-(\widetilde{A}^*) = \overline{\sigma_-(\widetilde{A})}$ and hence $P_-(\widetilde{A})^* = P_-(\widetilde{A}^*)$ (see [4], Chapter III, Theorem 6.22 and (6.25)). The proof of the second statement is similar. ∎

Now we are ready to show that the spectral subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ of $\widetilde{A}$ can be represented by means of angular operators. For $1 \leqslant p \leqslant \infty$, we denote by $\mathcal{S}_p$ the von Neumann–Schatten classes of linear operators in $\mathcal{H}$ (see [3]); in particular, $\mathcal{S}_\infty$ is the class of all compact operators and $\mathcal{S}_1$ is the class of all nuclear or trace class operators.

THEOREM 4.3. *There exist bounded linear operators $K_- \in L(\widehat{\mathcal{H}}, \mathcal{H})$ and $K_+ \in L(\mathcal{H}, \widehat{\mathcal{H}})$ such that:*

(i) *The spectral subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ have the representations*

$$\mathcal{L}_- = \left\{ \begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} : \hat{x} \in \widehat{\mathcal{H}} \right\}, \quad \mathcal{L}_+ = \left\{ \begin{pmatrix} x \\ K_+x \end{pmatrix} : x \in \mathcal{H} \right\}.$$

(ii) *The operator K_- has the property $R(K_-) \subset \mathcal{D}(A)$ and K_-, K_+ satisfy the Riccati equations*

$$\begin{aligned} K_-B^*K_- - B - AK_- + K_-D = 0 \quad &on\ \ \widehat{\mathcal{H}},\\ K_+BK_+ - B^* - DK_+ + K_+A = 0 \quad &on\ \ \mathcal{D}(A).\end{aligned}$$

(iii) *The restriction $\widetilde{A}|\mathcal{L}_-$ is unitarily equivalent to the operator $D + B^*K_-$ in the Hilbert space $\big(\widehat{\mathcal{H}}, [\,\cdot\,,\cdot\,]_\wedge\big)$ where*

$$[\hat{x}, \hat{y}]_\wedge := \big((I + K_-^*K_-)\hat{x}, \hat{y}\big), \quad \hat{x}, \hat{y} \in \widehat{\mathcal{H}}.$$

There is a $\hat{\gamma} > 0$ such that

$$\Re[(D + B^*K_-)\hat{x}, \hat{x}]_\wedge \leqslant -\hat{\gamma}[\hat{x}, \hat{x}]_\wedge, \quad \hat{x} \in \widehat{\mathcal{H}}.$$

(iv) *The restriction* $\widetilde{A}|\mathcal{L}_+$ *is unitarily equivalent to the operator* $A+BK_+$ *in the Hilbert space* $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]\big)$ *where*

$$[x,y] := \big((I+K_+^*K_+)x,y\big), \quad x,y\in\mathcal{H}.$$

There is a $\gamma>0$ *such that*

$$\Re[(A+BK_+)x,x] \geqslant \gamma[x,x], \quad x\in\mathcal{D}(A).$$

If for one (and hence for all) $z\in\rho(A)$ *the resolvent* $(A-z)^{-1}$ *of* A *belongs to some class* $\mathcal{S}_p$, $1\leqslant p\leqslant\infty$, *then the operators* K_- *and* K_+ *belong to the same class* $\mathcal{S}_p$.

Proof. From the representation (4.3) and Lemma 4.1 it follows that

$$\begin{aligned}\Re(P_+(\widetilde{A})-P_-(\widetilde{A}))_{11} &= \Re\left(\frac{1}{\pi \mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}(A-z-B(D-z)^{-1}B^*)^{-1}\,\mathrm{d}z\right)\\ &= \Re\left(\frac{1}{\pi}\int\limits_{-\infty}^{\infty}(A-\mathrm{i}\eta-B(D-\mathrm{i}\eta)^{-1}B^*)^{-1}\,\mathrm{d}\eta\right)\\ &\gg 0\end{aligned}$$

since for $-\delta<\Re(z)<\alpha$,

$$\Re((A-z-B(D-z)^{-1}B^*)x,x)\geqslant\widetilde{\alpha}(x,x), \quad x\in\mathcal{D}(A), \tag{4.7}$$

with some $\widetilde{\alpha}>0$ by the assumptions (α) and (δ). Here and in the following we use the notation $(X_{ij})_{i,j=1}^2 := X$ for the components of a block operator matrix X in $\widetilde{\mathcal{H}}=\mathcal{H}\times\widehat{\mathcal{H}}$. On the other hand, $\Re(P_+(\widetilde{A})+P_-(\widetilde{A}))_{11}=I$ and hence

$$\Re(P_+(\widetilde{A}))_{11}\gg\frac{1}{2}. \tag{4.8}$$

(i) Let $x\in\mathcal{H}$ be such that $\begin{pmatrix}x\\0\end{pmatrix}\in\mathcal{L}_-$. Then $P_+(\widetilde{A})\begin{pmatrix}x\\0\end{pmatrix}=(I-P_-(\widetilde{A}))\begin{pmatrix}x\\0\end{pmatrix}=0$ and thus $(\Re(P_+(\widetilde{A}))_{11}x,x)=0$. By (4.8) this implies $x=0$.

Now consider a sequence $\left(\begin{pmatrix}x_n\\\hat{x}_n\end{pmatrix}\right)_1^\infty\subset\mathcal{L}_-$ with $x_n\in\mathcal{H}$, $\hat{x}_n\in\widehat{\mathcal{H}}$, $\|x_n\|=1, n=1,2,\dots$, and $\|\hat{x}_n\|\to0$ for $n\to\infty$. By Lemma 4.2,

$$0=\left(P_+(\widetilde{A}^*)\begin{pmatrix}x_n\\0\end{pmatrix},\begin{pmatrix}x_n\\\hat{x}_n\end{pmatrix}\right)=\big((P_+(\widetilde{A}^*))_{11}x_n,x_n\big)+\big((P_+(\widetilde{A}^*))_{21}x_n,\hat{x}_n\big).$$

The last term tends to 0 for $n \to \infty$ and hence

$$\big((P_+(\widetilde{A}^*))_{11}x_n, x_n\big) \to 0, \quad n \to \infty.$$

By means of the inequality (4.8) for $\widetilde{A}^*$, we find $\|x_n\| \to 0$ for $n \to \infty$. This proves that $\mathcal{L}_-$ can be represented as

$$\mathcal{L}_- = \left\{ \begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} : \hat{x} \in \widehat{\mathcal{H}}' \right\}$$

where K_- is a bounded linear operator from some closed subspace $\widehat{\mathcal{H}}'$ of $\widehat{\mathcal{H}}$ into $\mathcal{H}$. It remains to be proved that $\widehat{\mathcal{H}}' = \widehat{\mathcal{H}}$. To this end let $\hat{z} \in \widehat{\mathcal{H}}'^{\perp}$. Then $\hat{z} = 0$ as

$$\begin{pmatrix} 0 \\ \hat{z} \end{pmatrix} \in \mathcal{L}_-^{\perp} = \left\{ \begin{pmatrix} x \\ -K_-^* x \end{pmatrix} : x \in \mathcal{D}(K_-^*) \right\}.$$

The proof of the assertion for $\mathcal{L}_+$ is analogous. Here we have to use that

$$\Re(D - z - B^*(A - z)^{-1}B) \ll 0$$

for $-\delta < \Re(z) < \alpha$ instead of (4.7) and the representation

$$\widetilde{A} - z = \begin{pmatrix} I & 0 \\ B^*(A-z)^{-1} & I \end{pmatrix} \begin{pmatrix} A - z & 0 \\ 0 & D - z - B^*(A-z)^{-1}B \end{pmatrix} \begin{pmatrix} I & (A-z)^{-1}B \\ 0 & I \end{pmatrix}$$

for $z \in \rho(A)$ instead of (4.2) in order to obtain

$$\Re(P_-(\widetilde{A}))_{22} \gg \frac{1}{2}. \tag{4.9}$$

(ii) Since $P_-(\widetilde{A})$ is a Riesz projection, $\mathcal{L}_- = R(P_-(\widetilde{A})) \subset \mathcal{D}(\widetilde{A})$. On the other hand,

$$\mathcal{D}(\widetilde{A}) = \left\{ \begin{pmatrix} x \\ \hat{x} \end{pmatrix} : x \in \mathcal{D}(A),\ \hat{x} \in \hat{\mathcal{H}} \right\},$$

and hence the representation of $\mathcal{L}_-$ according to (i) yields $R(K_-) \subset \mathcal{D}(A)$.

The Riccati equations for K_- and K_+ follow from the invariance of the subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ under $\widetilde{A}$ and the representations of $\mathcal{L}_-$ and $\mathcal{L}_+$ in (i).

(iii) The operator $U_- \in L(\mathcal{L}_-, (\widehat{\mathcal{H}}, [\,\cdot\,,\cdot\,]_\wedge))$ defined by

$$U_- \begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} := \hat{x}, \quad \hat{x} \in \widehat{\mathcal{H}},$$

is isometric and bijective. Using the Riccati equation for K_-, we find

$$\widetilde{A}|\mathcal{L}_- = U_-^{-1}(D + B^*K_-)U_-$$

and

$$\begin{aligned}\Re[(D + B^*K_-)\hat{x}, \hat{x}]_\wedge &= \Re\big((I + K_-^*K_-)(D + B^*K_-)\hat{x}, \hat{x}\big)\\ &= \Re\big((D + B^*K_- + K_-^*(K_-D + K_-B^*K_-))\hat{x}, \hat{x}\big)\\ &= \Re\big((D + K_-^*AK_- + B^*K_- - K_-^*B)\hat{x}, \hat{x}\big)\\ &= \Re(D\hat{x}, \hat{x}) - \Re(AK_-\hat{x}, K_-\hat{x})\\ &\leqslant -\hat{\gamma}(\hat{x}, \hat{x})\end{aligned}$$

for $\hat{x} \in \widehat{\mathcal{H}}$ with some $\hat{\gamma} > 0$ by the assumptions (α) and (δ).

(iv) The operator $U_+ \in L(\mathcal{L}_+, (\mathcal{H}, [\,\cdot\,,\cdot\,]))$ defined by

$$U_+ \begin{pmatrix} x \\ K_+x \end{pmatrix} := x, \quad x \in \mathcal{H},$$

is isometric and bijective. The Riccati equation for K_+ yields

$$\widetilde{A}|\mathcal{L}_+ = U_+^{-1}(A + BK_+)U_+$$

and, similar as in the proof of (iii),

$$\Re[(A + BK_+)x, x] = \Re(Ax, x) - \Re(DK_+x, K_+x) \geqslant \gamma(x, x)$$

for $x \in \mathcal{D}(A)$ with some $\gamma > 0$ by the assumptions (α) and (δ).

Finally, the Riccati equations for K_- and K_+ can be written in the form

$$\begin{aligned}K_-B^*K_- - B - (A - \mu)K_- + K_-(D - \mu) = 0,\\ K_+BK_+ - B^* - (D - \mu)K_+ + K_+(A - \mu) = 0,\end{aligned}$$

where $\mu \in \mathbb{C}$ is arbitrary. If we choose $\mu \in \rho(A)$ and multiply the first relation from the left hand side and the second equation from the right hand side by $(A - \mu)^{-1}$, it follows that K_- and K_+ are in $\mathcal{S}_p$ if $(A - \mu)^{-1}$ is in $\mathcal{S}_p$ for some p, $1 \leqslant p \leqslant \infty$. The theorem is proved. ∎

If in Theorem 4.3 the operators A and D are selfadjoint, then $K_+ = -K_-^*$, which is immediate from the relation $\mathcal{L}_+ = \mathcal{L}_-^\perp$. Thus, in the selfadjoint case the assertions of Theorem 4.3 coincide with the respective statements established in [1], Theorem 2.3, which were proved by a different method. Additionally, it was shown therein that in this case K_- and K_+ are strict contractions. To prove a generalization of this property, we use the following two results.

LEMMA 4.4. *The operators $I - K_+K_-$ and $I - K_-K_+$ are bijective.*

Proof. Let $\hat{x} \in \ker(I - K_+K_-)$. Then $\hat{x} = K_+K_-\hat{x}$,

$$\begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} K_-\hat{x} \\ K_+K_-\hat{x} \end{pmatrix} \in \mathcal{L}_- \cap \mathcal{L}_+ = \{0\}$$

and consequently $\hat{x} = 0$. Now let $\hat{z} \in \widehat{\mathcal{H}}$. As $\mathcal{H} = \mathcal{L}_- \dot{+} \mathcal{L}_+$, there exist $x \in \mathcal{H}$ and $\hat{x} \in \widehat{\mathcal{H}}$ such that

$$\begin{pmatrix} 0 \\ \hat{z} \end{pmatrix} = \begin{pmatrix} K_-\hat{x} + x \\ \hat{x} + K_+x \end{pmatrix}.$$

This implies $\hat{z} = \hat{x} - K_+K_-\hat{x} \in R(I - K_+K_-)$. The proof for $I - K_-K_+$ is analogous. ∎

PROPOSITION 4.5. *The projections $P_-(\widetilde{A})$ and $P_+(\widetilde{A})$ have the matrix representations*

$$P_-(\widetilde{A}) = \begin{pmatrix} -K_-(I - K_+K_-)^{-1}K_+ & K_-(I - K_+K_-)^{-1} \\ -(I - K_+K_-)^{-1}K_+ & (I - K_+K_-)^{-1} \end{pmatrix},$$

$$P_+(\widetilde{A}) = \begin{pmatrix} (I - K_-K_+)^{-1} & -(I - K_-K_+)^{-1}K_- \\ K_+(I - K_-K_+)^{-1} & -K_+(I - K_-K_+)^{-1}K_- \end{pmatrix}.$$

Proof. As $P_-(\widetilde{A})$ is the projection onto $\mathcal{L}_-$ along $\mathcal{L}_+$, we have

$$P_-(\widetilde{A}) = \begin{pmatrix} K_-X & K_-Y \\ X & Y \end{pmatrix}$$

with operators $X \in L(\mathcal{H}, \widehat{\mathcal{H}})$ and $Y \in L(\widehat{\mathcal{H}})$. From $P_-(\widetilde{A})|\mathcal{L}_+ = 0$, it follows that

$$0 = \begin{pmatrix} K_-X & K_-Y \\ X & Y \end{pmatrix} \begin{pmatrix} x \\ K_+x \end{pmatrix} = \begin{pmatrix} K_-(X + YK_+)x \\ (X + YK_+)x \end{pmatrix}, \quad x \in \mathcal{H},$$

and hence $X = -YK_+$. Then $P_-(\widetilde{A})|\mathcal{L}_- = I$ reads

$$\begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} -K_-YK_+ & K_-Y \\ -YK_+ & Y \end{pmatrix} \begin{pmatrix} K_-\hat{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} K_-Y(-K_+K_- + I)\hat{x} \\ Y(-K_+K_- + I)\hat{x} \end{pmatrix}, \ \hat{x} \in \widehat{\mathcal{H}}.$$

By Lemma 4.4, $I - K_+K_-$ is invertible and hence

$$Y = (I - K_+K_-)^{-1}, \quad X = -(I - K_+K_-)^{-1}K_+.$$

This shows the representation of $P_-(\widetilde{A})$. The proof for $P_+(\widetilde{A})$ is analogous. ∎

THEOREM 4.6. *The operators K_-K_+ and K_+K_- are strict contractions in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, that is,*

$$\|K_-K_+\| < 1, \quad \|K_+K_-\| < 1.$$

Proof. From (4.8) and the preceding proposition it follows that

$$\Re((I - K_-K_+)^{-1}) \gg \frac{1}{2}.$$

Consequently,

$$\begin{aligned} I &\ll ((I - K_-K_+)^{-1} + (I - (K_-K_+)^*)^{-1} \\ &= (I - K_-K_+)^{-1}(2I - K_-K_+ - (K_-K_+)^*)(I - (K_-K_+)^*)^{-1}. \end{aligned}$$

This implies

$$(I - K_-K_+)(I - (K_-K_+)^*) \ll 2I - K_-K_+ - (K_-K_+)^*$$

and hence

$$(K_-K_+)(K_-K_+)^* \ll I,$$

which shows $\|K_-K_+\| < 1$. The proof of $\|K_+K_-\| < 1$ is analogous. ∎

If A and D are selfadjoint, then it follows from Theorem 4.6 and from the relation $K_+ = -K_-^*$ that

$$\|K_-\| = \|K_+\| < 1.$$

5. HALF RANGE COMPLETENESS

Additionally to the assumptions (α) and (δ) we suppose now that A has a compact resolvent. Then $\sigma_+(\widetilde{A})$ is discrete and ∞ is the only accumulation point as

$$\sigma_+(\widetilde{A}) = \sigma(\widetilde{A}|\mathcal{L}_+) = \sigma(A + BK_+)$$

by Theorem 4.3 where K_+ is compact.

In the following we denote by P_1 the projection from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$. If $\lambda \in \sigma(\widetilde{A})$, then $\mathcal{L}_\lambda(\widetilde{A})$ denotes the root subspace of $\widetilde{A}$ at λ.

THEOREM 5.1. *Suppose additionally to the assumptions* (α) *and* (δ) *that the resolvent* $(A-z)^{-1}$ *of* A *belongs to the class* $\mathcal{S}_1$ *for some* $z \in \rho(\widetilde{A})$. *Then*

$$\overline{\bigcup_{\lambda \in \sigma_+(\widetilde{A})} P_1 \mathcal{L}_\lambda(\widetilde{A})} = \mathcal{H},$$

that is, the first components of the root vectors of $\widetilde{A}$ *corresponding to the eigenvalues in the right half plane form a complete system in* $\mathcal{H}$.

Proof. Let $\lambda_0 \in \sigma_+(\widetilde{A})$ and assume that $\{\widetilde{x}_0, \widetilde{x}_1, \dots, \widetilde{x}_h\} \subset \widetilde{\mathcal{H}}$ is a Jordan chain of $\widetilde{A}$ at λ_0. Then $\widetilde{x}_j \in \mathcal{L}_+$ and hence $\widetilde{x}_j = \begin{pmatrix} x_j \\ K_+ x_j \end{pmatrix}$ with $x_j \in \mathcal{H}$ for $j = 0, 1, \dots, h$ by Theorem 4.3. Then the relation

$$\begin{pmatrix} A-\lambda_0 & B \\ B^* & D-\lambda_0 \end{pmatrix} \begin{pmatrix} x_j \\ K_+ x_j \end{pmatrix} = \begin{pmatrix} x_{j-1} \\ K_+ x_{j-1} \end{pmatrix}, \quad j = 0, 1, \dots, h,$$

implies that $\{x_0, x_1, \dots, x_h\} = \{P_1 \widetilde{x}_0, P_1 \widetilde{x}_1, \dots, P_1 \widetilde{x}_h\}$ is a Jordan chain of the operator $A + BK_+$ at λ_0. Hence

$$\bigcup_{\lambda \in \sigma_+(\widetilde{A})} P_1 \mathcal{L}_\lambda(\widetilde{A}) = \bigcup_{\lambda \in \sigma(A+BK_+)} \mathcal{L}_\lambda(A + BK_+).$$

For sufficiently large $\zeta > 0$, the operator $A + BK_+ + \zeta$ is accretive, that is, $\Re((A + BK_+ + \zeta)x, x) \geqslant 0$ for $x \in \mathcal{D}(A)$. If $z \in \rho(A + BK_+ + \zeta)$, $\Re(z) < 0$, the resolvent

$$(A + BK_+ + \zeta - z)^{-1} = (A - z)^{-1}(I + (BK_+ + \zeta)(A - z)^{-1})^{-1}$$

belongs to $\mathcal{S}_1$ as $(A - z)^{-1} \in \mathcal{S}_1$ according to the assumption. Then the statement follows from [3], Chapter V, Theorem 2.3, applied to the dissipative operator $-\mathrm{i}(A+BK_++\zeta)^{-1}$ as $\lambda \in \sigma(A+BK_+)$ if and only if $-\frac{\mathrm{i}}{\lambda+\zeta} \in \sigma(-\mathrm{i}(A+BK_++\zeta)^{-1})$ and

$$\mathcal{L}_\lambda(A + BK_+) = \mathcal{L}_{-\frac{\mathrm{i}}{\lambda+\zeta}}(-\mathrm{i}(A + BK_+ + \zeta)^{-1}), \quad \lambda \in \sigma(A + BK_+). \quad ∎$$

EXAMPLE 5.2. We consider the λ-rational boundary eigenvalue problem

$$\begin{aligned} &y'' + \lambda y + \frac{q}{u-\lambda} y = 0, \\ &y(0) - \beta y'(0) = 0, \quad y(1) = 0, \end{aligned} \tag{5.1}$$

in $L_2(0,1)$ where $q, u \in C[0,1]$, $q > 0$, $\Re(u) < 0$, and $\beta \in \mathbb{C}$, $\Re(\beta) > 0$.

Problems of this type with real-valued u and Dirichlet boundary conditions, that is, $\beta = 0$, were studied in [5] and [1]. If we define $\hat{y} := -\frac{q^{1/2}}{u-\lambda} y$, then the problem (5.1) is equivalent on $\rho(u)$, which is the complement of the set of all values of the continuous function u, to the λ-linear problem

$$(\widetilde{A} - \lambda)\widetilde{y} = 0, \quad \widetilde{y} \in \mathcal{D}(\widetilde{A}),$$

in $L_2(0,1) \times L_2(0,1)$ where $\widetilde{A}$ is a block operator matrix of the form (3.1) given by

$$\widetilde{A} := \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} & q^{\frac{1}{2}} \\ q^{\frac{1}{2}} & u \end{pmatrix}$$

and

$$\mathcal{D}(\widetilde{A}) := \{\widetilde{y} = (y, \hat{y})^{\mathrm{t}} \in W_2^2(0,1) \times L_2(0,1) : y(0) - \beta y'(0) = 0,\ y(1) = 0\}.$$

In the following we prove that the operator $\widetilde{A}$ fulfills the assumptions of Theorem 5.1. As $\Re(u) < 0$, the assumption (δ) is fulfilled with $\theta_D := \pi - \min\limits_{x\in[0,1]} |\arg u(x)| < \frac{\pi}{2}$ and $\delta := -\max\limits_{x\in[0,1]} \Re(u(x))$. The operator A in $L_2(0,1)$ given by

$$Ay := -y'', \quad \mathcal{D}(A) := \{y \in W_2^2(0,1) : y(0) - \beta y'(0) = 0,\ y(1) = 0\}$$

is densely defined, closed, and it satisfies the assumption (α). Indeed, A is boundedly invertible, and

$$(Ay, y) = \beta |y'(0)|^2 + \int_0^1 |y'(x)|^2 \,\mathrm{d}x, \quad y \in \mathcal{D}(A). \tag{5.2}$$

First we show that $\Re(Ay, y) \geqslant \alpha$ for $y \in \mathcal{D}(A), \|y\| = 1$, for some $\alpha > 0$. From (5.2) and the assumption $\Re(\beta) > 0$ it follows that

$$\Re(Ay, y) = \Re(\beta)|y'(0)|^2 + \int_0^1 |y'(x)|^2 \,\mathrm{d}x \geqslant 0, \quad y \in \mathcal{D}(A). \tag{5.3}$$

Now suppose that there exists a sequence $(y_n)_1^\infty \subset \mathcal{D}(A), \|y_n\| = 1$, with $\Re(Ay_n, y_n) \to 0$ for $n \to \infty$. Then $y_n'(0) \to 0$ and $\|y_n'\| \to 0$ for $n \to \infty$ by (5.3) and hence

$$|y_n(x) - y_n(0)| \leqslant \int_0^x |y_n'(t)| \,\mathrm{d}t \leqslant \sqrt{x} \left(\int_0^x |y_n'(t)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \leqslant \|y_n'\| \to 0, \quad n \to \infty,$$

uniformly for $x \in [0,1]$, and hence $\|y_n\| \to 0$ as $y_n(0) = \beta y_n'(0) \to 0$ for $n \to \infty$, a contradiction. Furthermore, (5.3) implies

$$\frac{|\Im(Ay,y)|}{\Re(Ay,y)} = \frac{|\Im(\beta)||y'(0)|^2}{\Re(\beta)|y'(0)|^2 + \|y'\|^2} \leqslant \frac{|\Im(\beta)|}{\Re(\beta)}$$

which proves that the numerical range W_A of A is contained in a set $\{z \in \mathbb{C} : \arg z \leqslant \theta_A,\ \Re(z) \geqslant \alpha\}$ with $\theta_A = |\arg\beta| < \frac{\pi}{2}$.

It remains to be shown that $A^{-1} \in \mathcal{S}_1$. In order to see this, we introduce the selfadjoint operator A_0 in $L_2(0,1)$,

$$A_0 y := -y'', \quad \mathcal{D}(A_0) := \{y \in W_2^2(0,1) : y(0) = 0,\ y(1) = 0\}.$$

Its eigenvalues are the numbers $\mu_n := \pi^2 n^2$, $n = 1,2,\dots$, hence $A_0^{-1} \in \mathcal{S}_1$. As the difference $A^{-1} - A_0^{-1}$ is a one-dimensional operator, it follows that $A^{-1} \in \mathcal{S}_1$.

Thus Theorem 5.1 can be applied to the operator $\widetilde{A}$ and we get:

THEOREM 5.3. *The spectrum σ_+ of the eigenvalue problem* (5.1) *in the right half plane is discrete, ∞ is its only accumulation point, and σ_+ is contained in the sector*

$$\Big\{z \in \mathbb{C} : |\arg z| \leqslant \max\big\{\pi - \min_{x\in[0,1]} |\arg u(x)|, |\arg\beta|\big\}\Big\}.$$

The root vectors corresponding to the eigenvalues in σ_+ form a complete system in $L_2(0,1)$.

Proof. The theorem follows from Theorem 5.1 and from the fact that $\lambda \in \rho(u)$ is an eigenvalue of the problem (5.1) if and only if it is an eigenvalue of $\widetilde{A}$ and the first components of the corresponding Jordan chains of $\widetilde{A}$ coincide with the Jordan chains of (5.1) at λ (compare [5], Proposition 1.2). ∎

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