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C^*-CROSSED PRODUCTS BY TWISTED INVERSE SEMIGROUP ACTIONS

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ABSTRACT. The notions of Busby-Smith and Green type twisted actions are extended to discrete unital inverse semigroups. The connection between the two types, and the connection with twisted partial actions, are investigated. Decomposition theorems for the twisted crossed products are given.

KEYWORDS: *Inverse semigroup, twisted action, partial action, crossed product.*

MSC (2000): 46L55.

1. INTRODUCTION

The work of Renault ([20]) connecting a locally compact groupoid to its ample inverse semigroup makes the study of inverse semigroups interesting. Some of the results on inverse semigroups are Paterson's ([16]) and Duncan and Paterson's work ([3], [4]) on the C^*-algebras of inverse semigroups, Kumjian's localization C^*-algebras ([12]) and Nica's $\tilde{F}$-inverse semigroups ([14]). We have seen in [22] that the theory of crossed products can be generalized to inverse semigroups. The strong connection between the C^*-algebras of locally compact groupoids and inverse semigroups found by Paterson ([17]) promises a similar connection between the groupoid crossed products of [21] and inverse semigroup crossed products.

Green ([9]) and Packer and Raeburn ([15]) showed how to use twisted crossed products to decompose crossed product C^*-algebras. In this paper we partially generalize their results to discrete inverse semigroups. We prove decomposition theorems for both Green and Busby-Smith style twisted actions. We show that unlike in the group case, Green twisted actions seem slightly more general than Busby-Smith twisted actions.

We show that the close connection between partial actions ([5], [13]) and inverse semigroup actions seen in [22] and [7] still holds for the twisted partial actions of [6] and Busby-Smith twisted inverse semigroup actions. It is a natural question to ask, whether there is a similar connection between Green twisted inverse semigroup actions and some sort of Green twisted partial actions.

2. TWISTED INVERSE SEMIGROUP ACTIONS

Twisted actions of locally compact groups were introduced in [1]. The inverse semigroup version closely follows Exel's definition of twisted partial actions in [6].

Recall that a semigroup S is an *inverse semigroup* if for every $s \in S$ there exists a unique element s^* of S so that $ss^*s = s$ and $s^*ss^* = s^*$. The map $s \mapsto s^*$ is an involution. An element $f \in S$ satisfying $f^2 = f$ is called an *idempotent* of S. The set of idempotents of an inverse semigroup is a semilattice. There is a natural partial order on S defined by $s \leqslant t$ if and only if $s = ts^*s$.

Definition 2.1. Let A be a C^*-algebra. A *partial automorphism* of A is an isomorphism between two closed ideals of A.

Definition 2.2. Let A be a C^*-algebra, and let S be a unital inverse semigroup with idempotent semilattice E, and unit e. A *Busby-Smith twisted action* of S on A is a pair (β, w), where for all $s \in S$, $\beta_s : E_{s^*} \to E_s$ is a partial automorphism of A, and for all $s, t \in S$, $w_{s,t}$ is a unitary multiplier of E_{st}, such that for all $r, s, t \in S$ we have

(i) $E_e = A$;
(ii) $\beta_s\beta_t = \operatorname{Ad} w_{s,t} \circ \beta_{st}$;
(iii) $w_{s,t} = 1_{M(E_{st})}$ if s or t is an idempotent;
(iv) $\beta_r(aw_{s,t})w_{r,st} = \beta_r(a)w_{r,s}w_{rs,t}$ for all $a \in E_{r^*}E_{st}$.

We also refer to the quadruple (A, S, β, w) as a *Busby-Smith twisted action.*

Note that $\beta_r(aw_{s,t})$ makes sense since $a = a_1a_2$ for some $a_1, a_2 \in E_{r^*}E_{st}$ and so $aw_{s,t} = a_1a_2w_{s,t} \in a_1E_{st} \subset E_{r^*}$.

Our definition is a generalization of inverse semigroup actions defined in [22]. Every inverse semigroup action is a trivially twisted Busby-Smith inverse semigroup action by taking $w_{r,s} = 1_{M(E_{rs})}$. Conversely, every trivially twisted Busby-Smith inverse semigroup action is an inverse semigroup action. The definition is also a generalization of discrete twisted group actions in case our inverse semigroup S is actually a group.

The basic properties of Busby-Smith twisted inverse semigroup actions are collected in the following.

Lemma 2.3. *If (A, S, β, w) is a Busby-Smith twisted action and $r, s, t \in S$, then*

(i) $E_s = E_{ss^*}$;
(ii) $\beta_{ss^*} = \mathrm{id}_{E_s}$;
(iii) $\beta_e = \mathrm{id}_A$;
(iv) $\beta_{s^*} = \mathrm{Ad}\, w_{s^*,s} \circ \beta_s^{-1}$;
(v) $\beta_r(E_{r^*}E_s) = E_{rs}$;
(vi) $\beta_r(aw_{s,t}^*) = \beta_r(a)w_{r,st}w_{rs,t}^*w_{r,s}^*$ *for all* $a \in E_{r^*}E_{st}$;
(vii) $\beta_r(w_{s,t}a) = w_{r,s}w_{rs,t}w_{r,st}^*\beta_r(a)$ *for all* $a \in E_{r^*}E_{st}$;
(viii) $\beta_r(w_{s,t}^*a) = w_{r,st}w_{rs,t}^*w_{r,s}^*\beta_r(a)$ *for all* $a \in E_{r^*}E_{st}$;
(ix) $w_{s^*r^*r,s} = w_{s^*,r^*rs}$;
(x) $w_{s^*,s}1_{M(E_{s^*r^*})} = w_{s^*,r^*rs}$;
(xi) $\beta_s(w_{s^*,s}) = w_{s,s^*}$.

Proof. (i) and (ii) follow from the calculations

$$E_s = \mathrm{dom}(\beta_s\beta_{s^*}) = \mathrm{dom}(\mathrm{Ad}\, w_{s,s^*} \circ \beta_{ss^*}) = E_{ss^*}$$

$$\beta_f = \beta_f\beta_f\beta_f^{-1} = \mathrm{Ad}\, 1_{M(E_f)} \circ \beta_{ff}\beta_f^{-1} = \beta_f\beta_f^{-1} = \mathrm{id}_{E_f},$$

where $f = ss^*$. (iii) is a special case of (ii). Using (ii) we have (iv) since

$$\beta_{s^*} = \beta_{s^*}\beta_s\beta_s^{-1} = \mathrm{Ad}\, w_{s^*,s} \circ \beta_{s^*s}\beta_s^{-1} = \mathrm{Ad}\, w_{s^*,s} \circ \beta_s^{-1}.$$

We have

$$\beta_r(E_{r^*}E_s) = \mathrm{im}\,(\beta_r\beta_s) = \mathrm{im}(\mathrm{Ad}\, w_{r,s} \circ \beta_{rs}) = E_{rs},$$

which proves (v). Replacing a by $aw_{s,t}^*$ in Definition 2.2 (iv) gives (vi). Applying (vi) we have (vii) because

$$\beta_r(w_{s,t}a) = \beta_r(a^*w_{s,t}^*)^* = (\beta_r(a^*)w_{r,st}w_{rs,t}^*w_{r,s}^*)^* = w_{r,s}w_{rs,t}w_{r,st}^*\beta_r(a).$$

Replacing a by $w_{s,t}^*a$ in (vii) we have (viii). To show (ix) let $b \in E_{s^*r^*}$. Then by (i), (ii) and (v) there is an $a \in E_sE_{r^*} = \beta_{r^*r}(E_{r^*r}E_s) = E_{r^*rs}$ such that $b = \beta_{s^*}(a)$. Hence

$$\begin{aligned} b &= \beta_{s^*}(aw_{r^*r,s}) \qquad \text{(by Definition 2.2 (iii))} \\ &= \beta_{s^*}(a)w_{s^*,r^*r}w_{s^*r^*r,s}w_{s^*,r^*rs}^* \qquad \text{(by Definition 2.2 (iv))} \\ &= bw_{s^*r^*r,s}w_{s^*,r^*rs}^* \qquad \text{(by Lemma 2.3 (i) and Definition 2.2 (iii))}, \end{aligned}$$

which means $w_{s^*r^*r,s}w^*_{s^*,r^*rs}$ is the identity of $E_{s^*r^*}$. This implies our statement since by (i) both $w_{s^*r^*r,s}$ and w_{s^*,r^*rs} are unitary multipliers of $E_{s^*r^*}$. To show (x) let a and b as in the proof of part (ix). Then we have

$$\begin{aligned} b &= \beta_{s^*}(aw_{s,s^*r^*rs}) \qquad \text{(by Definition 2.2 (iii))} \\ &= \beta_{s^*}(a)w_{s^*,s}w_{s^*s,s^*r^*rs}w^*_{s^*,ss^*r^*rs} \qquad \text{(by Definition 2.2 (iv))} \\ &= bw_{s^*,s}1_{M(E_{s^*r^*})}w^*_{s^*,r^*rs} \end{aligned}$$

and the statement follows as in part (ix). Finally, (xi) follows from Definition 2.2 (iii), (iv) if we extend β_s to the multipliers of E_{s^*}. ∎

Recall from [19], [2], [11] that a *congruence relation* on an inverse semigroup S is an equivalence relation $\sim$ on S such that if $s \sim t$ then $rs \sim rt$ and $sr \sim tr$ for all $r \in S$. If E is the idempotent semilattice of S, then the *kernel normal system* of the congruence $\sim$ on S is the set $\{[f] : f \in E\}$ of congruence classes containing idempotents. The kernel normal system is exactly the set of idempotents in the quotient inverse semigroup $S/\sim$. The kernel normal system determines the congruence relation, since $\sim = \{(s,t) : ss^*, tt^*, st^* \in [f] \text{ for some } f \in E\}$.

If $\sim$ is an *idempotent-separating* congruence, that is, no two idempotents are congruent, then every equivalence class $[f]$ in the kernel normal system is a group with identity f. The union $N = \bigcup\{[f] : f \in E\}$ is an inverse subsemigroup of S contained in the centralizer of E. It is also a *normal* subsemigroup, that is, $E \subset N$ and $sNs^* \subset N$ for all $s \in S$. On the other hand, if N is normal subsemigroup of S in the centralizer of E then N determines a kernel normal system $\{[f] : f \in E\}$ of an idempotent-separating congruence $\sim$, where $[f] = \{s \in N : ss^* = f\}$. We also write S/N for $S/\sim$. Note that a normal subsemigroup N of S is contained in the centralizer of E if and only if N is a *Clifford semigroup*, that is, N is an inverse semigroup such that $n^*n = nn^*$ for all $n \in N$. We call such a subsemigroup N a *normal Clifford subsemigroup* of S, that is, N is a normal subsemigroup which is also a Clifford semigroup. We thus get a bijective correspondence between idempotent-separating congruences on S and normal Clifford subsemigroups of S. In the theory of twisted group actions, normal subgroups play an important role. For inverse semigroups the situation is more complicated, but normal Clifford subsemigroups give an appropriate substitute for normal subgroups.

Definition 2.4. If N is a normal Clifford subsemigroup of the inverse semigroup S with idempotent semilattice E, then a cross-section $c : S/N \to S$ is called *order-preserving* if $c([f]) = f$ for all $f \in E$ and $[s] \leqslant [t]$ implies $c([s]) \leqslant c([t])$ for all $s, t \in S$.

Proposition 2.5. *Let T be an inverse semigroup with idempotent semilattice E, and let N be a normal Clifford subsemigroup of T. Let $S = T/N$, and suppose there is an order-preserving cross-section $c : S \to T$. For $s \in S$ define*

$$E_s = \overline{\operatorname{span}} \bigcup \{[f] : f \leqslant c(s)c(s)^*,\ f \in E\},$$

where the closed linear span is taken in $C^*(N)$. Then each E_s is a closed ideal of $C^*(N)$. For $r, s \in S$ define

$$\beta_s : E_{s^*} \to E_s \quad \text{by} \quad \beta_s(a) = c(s)ac(s)^*,$$

and

$$w_{r,s} = c(r)c(s)c(rs)^* \in M(E_{rs}).$$

Then $(C^*(N), S, \beta, w)$ is a Busby-Smith twisted action.

Proof. First notice that since $c(rs)^*c(rs)$ is the identity of the congruence class containing $(c(r)c(s))^*c(r)c(s)$, we have $c(r)c(s) = c(r)c(s)c(rs)^*c(rs) = w_{r,s}c(rs)$ for all $r, s \in S$. Also $c(s)c(s)^* = c(ss^*)$ because $c([f]) = f$ for all $f \in E$, and similarly $c(s)^*c(s) = c(s^*s)$. Each E_s is a right ideal because if $a \in N \cap E_s$, that is, $a \in [f]$ for some $f \leqslant c(s)c(s)^*$ and $b \in [g]$, then $ab \in [fg] \subset E_s$ since $fg \leqslant c(s)c(s)^*$. A similar argument shows that E_s is also a left ideal. It is clear that each β_s is an isomorphism and each $w_{r,s}$ is a unitary multiplier of E_{rs}. It remains to check the conditions of Definition 2.2. Condition (i) holds since $E_e = C^*(N)$.

To check (ii), let $a \in \operatorname{dom}(\beta_r\beta_s) \cap \operatorname{dom}(\beta_{rs})$. Then

$$\beta_r\beta_s(a) = c(r)c(s)ac(s)^*c(r)^* = w_{r,s}c(rs)ac(rs)^*w_{r,s}^* = \operatorname{Ad} w_{r,s} \circ \beta_{rs}(a).$$

We need to show that $\beta_r\beta_s$ and $\operatorname{Ad} w_{r,s} \circ \beta_{rs}$ have the same domain. First we show that $\operatorname{dom} \beta_r\beta_s \subset \operatorname{dom} \beta_{rs}$. If $a \in \operatorname{dom} \beta_r\beta_s$, then $a = \lim_i a_i$ for some $a_i \in \operatorname{span} \bigcup \{[f] : f \leqslant c(s)^*c(s)\}$ and $c(s)ac(s)^* \in E_{r^*}$. Hence $c(s)ac(s)^* = \lim_j b_j$ for some $b_j \in \operatorname{span} \bigcup \{[f] : f \leqslant c(r)^*c(r)\}$. Since

$$c(s)^*c(s)ac(s)^*c(s) = \lim_i c(s)^*c(s)a_ic(s)^*c(s) = \lim_i a_i = a,$$

we have $a = \lim_j c(s)^*b_jc(s)$. It suffices to show that $c(s)^*b_jc(s) \in \operatorname{span} \bigcup \{[f] : f \leqslant c(rs)^*c(rs)\}$. This follows from the fact that if $b \in [f]$ such that $f \leqslant c(r)^*c(r)$, then $c(s)^*bc(s) \in [c(s)^*fc(s)]$ and $c(s)^*fc(s) \leqslant c(s)^*c(r)^*c(r)c(s) = c(rs)^*c(rs)$.

Next we show $\operatorname{dom}\beta_r\beta_s \supset \operatorname{dom}\beta_{rs}$. If $a \in \operatorname{dom}\beta_{rs}$, then $a = \lim_i a_i$ for some $a_i \in \operatorname{span}\bigcup\{[f] : f \leqslant c(rs)^*c(rs)\}$. Since

$$c(rs)^*c(rs) = c(s)^*c(r)^*c(r)c(s) \leqslant c(s)^*c(s),$$

$a_i \in \operatorname{span}\bigcup\{[f] : f \leqslant c(s)^*c(s)\}$ and so $a \in E_{s^*}$. We have $c(s)ac(s)^* = c(s)\lim_i a_i c(s)^* = \lim_i c(s)a_i c(s)^*$, so it remains to check that $c(s)a_i c(s)^* \in \bigcup\{[f] : f \leqslant c(r)^*c(r)\}$. This is true since if $b \in [f]$ such that $f \leqslant c(rs)^*c(rs)$, then $c(s)bc(s)^* \in [c(s)fc(s)^*]$ and

$$c(s)fc(s)^* \leqslant c(s)c(rs)^*c(rs)c(s)^* = c(s)c(s)^*c(r)^*c(r)c(s)c(s)^* \leqslant c(r)^*c(r).$$

To check (iii), fix $a \in N \cap E_{[f]s}$ for some $f \in E$. Then $a \in [g]$ for some idempotent $g \in T$ such that

$$g \leqslant c([f]s)c([f]s)^* = c([f]ss^*[f]) = c([f]ss^*) = fc(ss^*).$$

Since c is order-preserving and $[f]s \leqslant s$, we have

$$w_{[f],s} = c([f])c(s)c([f]s)^* \geqslant fc([f]s)c([f]s)^* = fc(ss^*)$$

and so $aw_{[f],s} = w_{[f],s}a = a$. The other part of (iii) follows similarly.

To check (iv), fix $a \in N \cap E_{r^*}E_{st}$. Since the classes in the kernel normal system are disjoint, $a \in [f]$ for some $f \in T$ such that $f \leqslant c(r^*)c(r^*)^* = c(r^*r) = c(r)^*c(r)$ and $f \leqslant c(st)c(st)^*$. Now we have

$$c(r)aw_{s,t}c(r)^*w_{r,st} = c(r)ac(s)c(t)c(st)^*c(r)^*c(r)c(st)c(rst)^*.$$

Since $ac(s)c(t)c(st)^* \in [f]st(st)^*$, the domain projection of $ac(s)c(t)c(st)^*$ is $fc(st)c(st)^* = f$. Hence $ac(s)c(t)c(st)^*c(r)^*c(r) = ac(s)c(t)c(st)^*$, so

$$\begin{aligned} c(r)aw_{s,t}c(r)^*w_{r,st} &= c(r)ac(s)c(t)c(st)^*c(st)c(rst)^* = c(r)ac(s)c(t)c(rst)^* \\ &= c(r)ac(r)^*c(r)c(s)c(t)c(rst)^* \\ &= c(r)ac(r)^*c(r)c(s)c(rs)^*c(rs)c(t)c(rst)^* \\ &= c(r)ac(r)^*w_{r,s}w_{rs,t}. \quad \blacksquare \end{aligned}$$

In Proposition 9.2 we further investigate the action in the previous lemma.

EXAMPLE 2.6. If in Proposition 2.5 N is the idempotent semilattice of T, then the (trivially) twisted Busby-Smith action becomes the *canonical action* of the inverse semigroup on its semilattice ([22]).

Proposition 2.5 shows the significance of an order-preserving cross-section. Unfortunately, these cross-sections do not always exist, as we can see in the following.

Example 2.7. Let $s = (a_i, \dots, a_n)$ denote the partial bijection in the symmetric inverse semigroup $\mathcal{I}(\{1,\dots,n\})$ whose domain is $\{i : a_i \neq 0\}$ and $s(i) = a_i$ for all $i \in \operatorname{dom} s$. Let $n = 6$ and consider the 19-element inverse subsemigroup generated by the elements

$$r = (1,4,5,0,0,0), \quad s = (0,5,4,0,0,6).$$

It is tedious but not hard to check that we can define a kernel normal system consisting of $\{s^*r, s^*rs^*r\}$, $\{rs^*, rs^*rs^*\}$ and the singleton sets containing idempotents other than s^*rs^*r and rs^*rs^*, and the congruence determined by this kernel normal system separates idempotents. Then $[ss^*r] \leqslant [r] = \{r\}$ and $[ss^*r] \leqslant [s] = \{s\}$, but we cannot choose a representative of $[ss^*r]$ which is less than both s and r.

The situation is not as bad as it seems. In many cases we have an order-preserving cross section. Recall [14] that an $\widetilde{F}$*-inverse semigroup* is a unital inverse semigroup in which every non-zero element is majorized by a unique maximal element.

Proposition 2.8. *Let S be a unital inverse semigroup with identity e, and let $\sim$ be an idempotent-separating congruence on S such that $T = S/\sim$ is an $\widetilde{F}$-inverse semigroup. Denote the set of maximal elements in T by $\mathcal{M}$. Choose a cross-section $c : \mathcal{M} \to S$ with $c([e]) = e$. Extend c to T by defining $c(0) = 0$ and $c(t) = c(m_t)f$ for $t \neq 0$, where $m_t \in \mathcal{M}$ is the unique maximal element majorizing t and f is the idempotent in $[t]^*[t]$. Then c is an order-preserving cross-section.*

Proof. It is clear that for all non-zero idempotents $f \in S$ we have $m_{[f]} = [e]$ and hence $c([f]) = f$. If $0 \neq s \leqslant t$ in $S/\sim$ then $m_s = m_t$ and so

$$\begin{aligned} c(t)c(s)^*c(s) &= c(m_t)c(t^*t)c(s^*s)c(m_s)^*c(m_s)c(s^*s) \\ &= c(m_s)c(t^*t)c(s^*s) = c(m_s)c(s^*s) = c(s). \quad \blacksquare \end{aligned}$$

3. BUSBY-SMITH TWISTED CROSSED PRODUCTS

Given a Busby-Smith twisted inverse semigroup action (A, S, β, w), define multiplication and involution on the closed subspace

$$L = \{x \in l^1(S, A) : x(s) \in E_s \text{ for all } s \in S\}$$

of $l^1(S, A)$ by

$$\big(x * y\big)(s) = \sum_{rt=s} \beta_r(\beta_r^{-1}(x(r))y(t))w_{r,t} \quad \text{and} \quad x^*(s) = w_{s,s^*}\beta_s(x(s^*)^*).$$

We will show that the multiplication is well-defined in the proof of Proposition 3.1 below. For $a \in E_s$ we are going to denote by $a\delta_s$ the function in L taking the value a at s and zero at every other element of S, so that L is the closed span of the $a\delta_s$. Then we have

$$\begin{aligned} a_s\delta_s * a_t\delta_t &= \beta_s(\beta_s^{-1}(a_s)a_t)w_{s,t}\delta_{st} \\ (a\delta_s)^* &= \beta_s^{-1}(a^*)w_{s^*,s}^*\delta_{s^*}. \end{aligned}$$

Notice that $a_e\delta_e * a_t\delta_t = a_ea_t\delta_t$ and $a_s\delta_s * a_e\delta_e = \beta_s(\beta_s^{-1}(a_s)a_e)\delta_s$.

Proposition 3.1. *L is a Banach *-algebra.*

Proof. First notice that $\big(x * y\big)(s) \in E_s$ by Lemma 2.3 (v) and so to show that the product is well-defined we only need to check that $\|x * y\|$ is finite:

$$\begin{aligned} \|x * y\| &= \sum_{s\in S}\sum_{rt=s} \|\beta_r(\beta_r^{-1}(x(r))y(t))w_{r,t}\| \\ &\leqslant \sum_{s\in S}\sum_{rt=s} \|\beta_r^{-1}(x(r))y(t)\| \leqslant \sum_{r\in S}\sum_{t\in S} \|\beta_r^{-1}(x(r))\|\,\|y(t)\| \\ &\leqslant \sum_{r\in S} \|\beta_r^{-1}(x(r))\| \sum_{t\in S} \|y(t)\| \leqslant \|x\|\,\|y\|. \end{aligned}$$

One can easily check that $\|x^*\| = \|x\|$ and so $x^* \in L$. For $a \in E_s$ we have

$$\begin{aligned} (a\delta_s)^{**} &= \beta_{s^*}^{-1}(w_{s^*,s}\beta_s^{-1}(a))w_{s,s^*}^*\delta_s \\ &= w_{s,s^*}^*\beta_s((w_{s^*,s}^*\beta_{s^*}(x(s)^*))^*) \\ &= w_{s,s^*}^*\beta_s(w_{s^*,s}\beta_s^{-1}(a))\delta_s \qquad \text{(by Lemma 2.3 (iv))} \\ &= w_{s,s^*}^*w_{s,s^*}w_{ss^*,s}w_{s,s^*s}^*\beta_s(\beta_s^{-1}(a))\delta_s \qquad \text{(by Lemma 2.3 (vii))} \\ &= a\delta_s. \end{aligned}$$

Next we show that the multiplication is associative. Let a, b and c be elements of E_r, E_s and E_t, respectively, and let u_λ be an approximate identity for E_{s^*}. Then we have

$$
\begin{aligned}
(a\delta_r * b\delta_s) * c\delta_t &= \beta_{rs}(\beta_{rs}^{-1}(\beta_r(\beta_r^{-1}(a)b)w_{r,s})c)w_{rs,t}\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_r^{-1}(a)b)w_{r,s}\beta_{rs}(u_\lambda c)w_{rs,t}\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_r^{-1}(a)b)\beta_r(\beta_s(u_\lambda c))w_{r,s}w_{rs,t}\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_r^{-1}(a)b\beta_s(u_\lambda c))w_{r,s}w_{rs,t}\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_r^{-1}(a)\beta_s(\beta_s^{-1}(b)u_\lambda c))w_{r,s}w_{rs,t}\delta_{rst} \\
&= \beta_r(\beta_r^{-1}(a)\beta_s(\beta_s^{-1}(b)c))w_{r,s}w_{rs,t}\delta_{rst} \\
&= \beta_r(\beta_r^{-1}(a)\beta_s(\beta_s^{-1}(b)c)w_{s,t})w_{r,st}\delta_{rst} \qquad \text{(by Definition 2.2 (iii))}, \\
&= a\delta_r * (b\delta_s * c\delta_t).
\end{aligned}
$$

Finally we show that the involution is anti-multiplicative. Fix $a \in E_r$ and $b \in E_s$. Then we have

$$
\begin{aligned}
&(a\delta_r * b\delta_s)^* \\
&= (\beta_r(\beta_r^{-1}(a)b)w_{r,s}\delta_{rs})^* \\
&= \beta_{rs}^{-1}(w_{r,s}^*\beta_r(b^*\beta_r^{-1}(a^*)))w_{s^*r^*,rs}^*\delta_{s^*r^*} \\
&= w_{s^*r^*,rs}^*\beta_{s^*r^*}(w_{r,s}^*\beta_r(b^*\beta_r^{-1}(a^*)))\delta_{s^*r^*} \qquad \text{(by Lemma 2.3 (iv))} \\
&= w_{s^*r^*,rs}^*\beta_{s^*r^*}(w_{r,s}^*\beta_r(\beta_s(\beta_s^{-1}(b^*\beta_r^{-1}(a^*)))))\delta_{s^*r^*} \\
&= w_{s^*r^*,rs}^*\beta_{s^*r^*}(w_{r,s}^*w_{r,s}\beta_{rs}(\beta_s^{-1}(b^*\beta_r^{-1}(a^*)))w_{r,s}^*)\delta_{s^*r^*} \qquad \text{(by Definition 2.2 (ii))} \\
&= w_{s^*r^*,rs}^*\beta_{s^*r^*}(\beta_{rs}(\beta_s^{-1}(b^*\beta_r^{-1}(a^*))))w_{s^*r^*,rs}w_{s^*r^*r,s}^*w_{s^*r^*,r}^*\delta_{s^*r^*} \qquad \text{(by 2.3 (vi))} \\
&= \beta_{rs}^{-1}(\beta_{rs}(\beta_s^{-1}(b^*\beta_r^{-1}(a^*))))w_{s^*r^*r,s}^*w_{s^*r^*,r}^*\delta_{s^*r^*}\,.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&(b\delta_s)^* * (a\delta_r)^* \\
&= (\beta_s^{-1}(b^*)w_{s^*,s}^*\delta_{s^*}) * (\beta_r^{-1}(a^*)w_{r^*,r}^*\delta_{r^*}) \\
&= \beta_{s^*}(\beta_{s^*}^{-1}(\beta_s^{-1}(b^*)w_{s^*,s}^*)\beta_r^{-1}(a^*)w_{r^*,r}^*)w_{s^*,r^*}\delta_{s^*r^*} \\
&= \beta_{s^*}(w_{s,s^*}^*\beta_s(\beta_s^{-1}(b^*)w_{s^*,s}^*)w_{s,s^*}\beta_r^{-1}(a^*)w_{r^*,r}^*)w_{s^*,r^*}\delta_{s^*r^*} \\
&= \beta_{s^*}(w_{s,s^*}^*\beta_s(\beta_s^{-1}(b^*))w_{s,s^*s}w_{ss^*,s}^*w_{s,s^*}^*w_{s,s^*}\beta_r^{-1}(a^*)w_{r^*,r}^*)w_{s^*,r^*}\delta_{s^*r^*} \quad \text{(by 2.3 (vi))} \\
&= \beta_{s^*}(w_{s,s^*}^*b^*\beta_r^{-1}(a^*)w_{r^*,r}^*)w_{s^*,r^*}\delta_{s^*r^*} \qquad \text{(by Definition 2.2 (iii))} \\
&= w_{s^*,ss^*}w_{s^*s,s^*}^*w_{s^*,s}^*\beta_{s^*}(b^*\beta_r^{-1}(a^*)w_{r^*,r}^*)w_{s^*,r^*}\delta_{s^*r^*} \qquad \text{(by Lemma 2.3 (viii))} \\
&= w_{s^*,s}^*\beta_{s^*}(b^*\beta_r^{-1}(a^*))w_{s^*,r^*r}w_{s^*r^*,r}^*w_{s^*,r^*}^*w_{s^*,r^*}\delta_{s^*r^*} \qquad \text{(by Lemma 2.3 (vi))} \\
&= \beta_s^{-1}(b^*\beta_r^{-1}(a^*))w_{s^*,s}^*1_{M(E_{s^*r^*r})}w_{s^*r^*,r}^*\delta_{s^*r^*} \qquad \text{(by Lemma 2.3 (iv))},
\end{aligned}
$$

and so the equality $(a\delta_r * b\delta_s)^* = (b\delta_s)^* * (a\delta_r)^*$ follows from Lemma 2.3 (ix) and (x). ∎

DEFINITION 3.2. A *covariant representation* of a Busby-Smith twisted action (A, S, β, w) is a triple (π, v, H), where π is a nondegenerate representation of A on the Hilbert space H and v_s is a partial isometry for all $s \in S$, such that for all $r, s \in S$ we have

(i) $\pi(\beta_s(a)) = v_s \pi(a) v_s^*$ for $a \in E_{s^*}$;

(ii) $v_r v_s = \pi(w_{r,s}) v_{rs}$;

(iii) v_s has initial space $\pi(E_{s^*})H$ and final space $\pi(E_s)H$.

To evaluate $\pi(w_{r,s})$ we extend π to the enveloping von Neumann algebra A^{**} of A.

We sometimes use the shortened notation (π, v) if we do not need a symbol for the Hilbert space. Note that the above definition is a generalization of a covariant representation of an inverse semigroup action ([22]), which is a trivially twisted Busby-Smith twisted action.

THEOREM 3.3. (Cohen-Hewitt) *If A is a Banach algebra with bounded approximate identity, V is a Banach space and $\pi : A \to B(V)$ is a bounded homomorphism then $\pi(A)V = \{\pi(a)v : a \in A,\, v \in V\}$ is a closed subspace of V.*

LEMMA 3.4. *If I is a closed ideal of the C^*-algebra A, m is a unitary multiplier of I and $\pi : A \to B(H)$ is a representation of A then $\pi(m)$ is a partial isometry with initial and final space $\pi(I)H$.*

Proof. $\pi(m)$ is a partial isometry since $\pi(m)\pi(m)^*\pi(m) = \pi(m)$. If $a \in I$ and $h \in H$ then $\pi(a)h = \pi(mm^{-1}a)h = \pi(m)\pi(m^{-1}a)h \in \pi(m)H$. On the other hand, $\pi(m)$ is in the strong operator closure of $\pi(I)$ and so there is a net $\{a_\lambda\} \subset I$ so that $\pi(a_\lambda)h \to \pi(m)h$ for all $h \in H$. Thus $\pi(m)H \subset \pi(I)H$ and so $\pi(m)$ has the required final space. The statement about the initial space follows from the fact that $\pi(m)^* = \pi(m^{-1})$. ∎

PROPOSITION 3.5. *Let (π, v, H) be a covariant representation. If $s \in S$ and f is an idempotent in S then*

(i) *v_f is the orthogonal projection onto $\pi(E_f)H$;*

(ii) *$v_e = 1_{B(H)}$;*

(iii) *$v_{s^*} = \pi(w_{s^*,s}) v_s^*$;*

(iv) *$v_s^* = v_{s^*} \pi(w_{s,s^*}^*)$;*

(v) *$v_s^* = \pi(w_{s^*,s}^*) v_{s^*}$.*

Proof. We have $v_f = v_f v_f v_f^* = \pi(w_{f,f}) v_f v_f^* = \pi(1_{M(E_f)})$. Since $E_e = A$, (ii) is a special case of (i). (iii) follows from the calculation $v_{s^*} = v_{s^*} v_s v_s^* = \pi(w_{s^*,s}) v_{s^*s} v_s^* = \pi(w_{s^*,s}) v_s^*$. We can get (iv) from (iii) upon taking adjoints. Finally using (iii) we have $\pi(w_{s^*,s}^*) v_{s^*} = \pi(w_{s^*,s}^* w_{s^*,s}) v_s^* = \pi(1_{E_{s^*s}}) v_s^* = \pi(1_{E_{s^*}}) v_s^* = v_s^*$. ∎

DEFINITION 3.6. Let (π, v, H) be a covariant representation. Define $\pi \times v : L \to B(H)$ by

$$\big(\pi \times v\big)(x) = \sum_{s \in S} \pi(x(s)) v_s,$$

where the series converges in norm.

PROPOSITION 3.7. *$\pi \times v$ is a nondegenerate representation of L.*

Proof. It is clear that $\pi \times v$ is linear. It suffices to show multiplicativity for $a\delta_s$ and $b\delta_t$ where $a \in E_s$ and $b \in E_t$:

$$\begin{aligned}\big(\pi \times v\big)(a\delta_s * b\delta_t) &= \big(\pi \times v\big)(\beta_s(\beta_s^{-1}(a)b)w_{s,t}\delta_{st}) = \pi(\beta_s(\beta_s^{-1}(a)b)w_{s,t})v_{st} \\ &= v_s\pi(\beta_s^{-1}(a)b)v_s^*\pi(w_{s,t})v_{st} = v_s\pi(w_{s^*,s}^*\beta_{s^*}(a)w_{s^*,s})\pi(b)v_s^*v_sv_t \\ &= v_s\pi(w_{s^*,s}^*)v_{s^*}\pi(a)v_{s^*}^*\pi(w_{s^*,s})\pi(b)v_s^*v_sv_t \\ &= v_sv_s^*\pi(a)v_s\pi(w_{s^*,s}^*)\pi(w_{s^*,s})\pi(b)v_s^*v_sv_t = \pi(a)v_s\pi(b)v_t.\end{aligned}$$

To show that $\pi \times v$ preserves adjoints let $a \in E_s$. Then we have

$$\begin{aligned}\big(\pi \times v\big)(a\delta_s)^* &= (\pi(a)v_s)^* = v_s^*\pi(a^*) = v_s^*v_s\pi(\beta_s^{-1}(a^*))v_s^* \\ &= \pi(\beta_s^{-1}(a^*))v_s^* = \pi(\beta_s^{-1}(a^*))\pi(w_{s^*,s}^*)\pi(w_{s^*,s})v_s^* \\ &= \pi(\beta_s^{-1}(a^*)w_{s^*,s}^*)v_{s^*} = \big(\pi \times v\big)(\beta_s^{-1}(a^*)w_{s^*,s}^*\delta_{s^*}) \\ &= \big(\pi \times v\big)((a\delta_s)^*).\end{aligned}$$

If $\{u_\lambda\}$ is a bounded approximate identity for A then $\{u_\lambda\delta_e\}$ is a bounded approximate identity for L, since for $a \in E_s$ we have $\lim_\lambda u_\lambda\delta_e * a\delta_s = \lim_\lambda u_\lambda a\delta_s = a\delta_s$, and $\lim_\lambda a\delta_s * u_\lambda\delta_e = \lim_\lambda \beta_s(\beta_s^{-1}(a)u_\lambda)\delta_s = \beta_s(\beta_s^{-1}(a))\delta_s = a\delta_s$. Since π is a nondegenerate representation, $\big(\pi \times v\big)(u_\lambda\delta_e) = \pi(u_\lambda)$ converges strongly to $1_{B(H)}$ and so $(\pi \times v)$ is nondegenerate. ∎

DEFINITION 3.8. Let (A, S, β, w) be a Busby-Smith twisted action. Define a C^*-seminorm $\|\cdot\|_{\mathrm{c}}$ on L by

$$\|x\|_{\mathrm{c}} = \sup\{\|(\pi \times v)(x)\| : (\pi, v) \text{ is a covariant representation of } (A, S, \beta, w)\}.$$

Let $I = \{x \in L : \|x\|_{\mathrm{c}} = 0\}$. The *Busby-Smith twisted crossed product* $A \times_{\beta,w} S$ is the completion of the quotient L/I with respect to $\|\cdot\|_{\mathrm{c}}$. We denote the quotient map of L onto L/I by Φ.

Note that for trivially twisted Busby-Smith actions this definition gives the inverse semigroup crossed product of [22].

Let (π, v) be a covariant representation of (A, S, β, w), and let $\pi \times v$ be the associated representation of L. Since $\ker \Phi \subset \ker \pi \times v$, we can factor $\pi \times v$

through the quotient L/I and extend to $A \times_{\beta,w} S$ by continuity. We denote this extension also by $\pi \times v$. Thus every covariant representation gives a nondegenerate representation of the crossed product. Proposition 3.10 will show how to reverse this process. The following lemma shows that the ideal I may be nontrivial.

LEMMA 3.9. *If $s \leqslant t$ in S, that is, $s = ft$ for some idempotent $f \in S$, then $\Phi(a\delta_s) = \Phi(a\delta_t)$ for all $a \in E_s$. In particular $\Phi(a\delta_s) = \Phi(a\delta_e)$ if s is an idempotent.*

Proof. It is clear that $a \in E_t$. If (π, v) is a covariant representation of (A, S, β, w) then

$$\begin{aligned}\big(\pi \times v\big)(a\delta_{ft}) &= \pi(a)v_{ft} = \pi(a)\pi(w^*_{f,t})v_f v_t \\ &= \pi(a)v_t \qquad \text{(by Proposition 3.5 (i))} \\ &= \big(\pi \times v\big)(a\delta_t),\end{aligned}$$

which shows that $\Phi(a\delta_s - a\delta_t) = 0$. The second statement follows from the fact that $s = se$. ∎

In spite of the above lemma, we identify $a\delta_s$ with its image in $A \times_{\beta,w} S$.

PROPOSITION 3.10. *Let (Π, H) be a nondegenerate representation of $A \times_{\beta,w} S$. Define a representation π of A on H and a map $v : S \to B(H)$ by*

$$\pi(a) = \Pi(a\delta_e) \quad \text{and} \quad v_s = \operatorname*{s\text{-}lim}_{\lambda} \Pi(u_\lambda \delta_s),$$

where $\{u_\lambda\}$ is an approximate identity for E_s and s-lim *denotes strong operator limit. Then (π, v, H) is a covariant representation of (A, S, β, w).*

Proof. π is a nondegenerate representation, since $\{u_\lambda\delta_e\}$ is an approximate identity for $A \times_{\beta,w} S$ whenever $\{u_\lambda\}$ is an approximate identity for A. We show that v_s is well-defined. If $h \in \pi(E_{s^*})H$ then $h = \Pi(a\delta_e)k$ for some $a \in E_{s^*}$ and $k \in H$. Hence

$$\begin{aligned}\lim_\lambda \Pi(u_\lambda\delta_s)h &= \lim_\lambda \Pi(u_\lambda\delta_s)\Pi(a\delta_e)k = \lim_\lambda \Pi(u_\lambda\delta_s * a\delta_e)k \\ &= \lim_\lambda \Pi(\beta_s(\beta_s^{-1}(u_\lambda)a)\delta_s)k = \Pi(\beta_s(a)\delta_s)k,\end{aligned}$$

since $\beta_{s^*}(u_\lambda)$ is an approximate identity for E_{s^*}. Note that the limit is independent of the choice of $\{u_\lambda\}$ since the expression $h = \Pi(a\delta_e)k$ was. On the other hand if $h \perp \pi(E_{s^*})H$ then

$$\begin{aligned}\lim_\lambda \Pi(u_\lambda\delta_s)h &= \lim_\lambda \Pi(\beta_s(\beta_s^{-1}(\sqrt{u_\lambda})\beta_s^{-1}(\sqrt{u_\lambda}))\delta_s)h \\ &= \lim_\lambda \Pi(\sqrt{u_\lambda}\delta_s * \beta_s^{-1}(\sqrt{u_\lambda})\delta_e)h \\ &= \lim_\lambda \Pi(\sqrt{u_\lambda}\delta_s)\pi(\beta_s^{-1}(\sqrt{u_\lambda}))(h).\end{aligned}$$

But $\pi(E_{s^*})h = 0$, so $v_s(h) = 0$. Hence v_s is well-defined. Clearly v_s is a bounded linear transformation, and if f is an idempotent then v_f is the orthogonal projection onto $\pi(E_f)H$. Notice that

$$v_s^* = \operatorname*{s-lim}_{\lambda} \Pi(u_\lambda \delta_s)^* = \operatorname*{s-lim}_{\lambda} \Pi(\beta_s^{-1}(u_\lambda) w_{s^*,s}^* \delta_{s^*}) = \pi(w_{s^*,s}^*) v_{s^*}.$$

For $s,\, t \in S$ let $\{u_\lambda^s\}$ and $\{u_\mu^t\}$ be bounded approximate identities for E_s and E_t respectively. Then

$$\begin{aligned} v_s v_t &= \operatorname*{s-lim}_{\lambda,\mu} \Pi(u_\lambda^s \delta_s * u_\mu^t \delta_t) = \operatorname{s-lim} \Pi(\beta_s(\beta_{s^*}(u_\lambda^s) u_\mu^t) w_{s,t} \delta_{st}) \\ &= \Pi(w_{s,t}\delta_{st}) = \operatorname{s-lim} \Pi(w_{s,t}\beta_s(\beta_{s^*}(u_\lambda^s) u_\mu^t)\delta_{st}) \\ &= \operatorname{s-lim} \Pi(w_{s,t}\delta_e) \Pi(\beta_s(\beta_{s^*}(u_\lambda^s) u_\mu^t)\delta_{st}) = \pi(w_{s,t}) v_{st}, \end{aligned}$$

since the net $\{\beta_s(\beta_{s^*}(u_\lambda^s)u_\mu^t)\}$ with the product direction is an approximate identity for $\beta_s(E_{s^*}E_t) = E_{st}$ (using boundedness of $\{u_\lambda^s\}$ and $\{u_\mu^t\}$). Thus v is multiplicative. We have $v_s^* v_s = \pi(w_{s^*,s}^*) v_{s^*} v_s = v_{s^*s}$, which is the projection onto $\pi(E_{s^*s})H = \pi(E_{s^*})H$. Hence v_s is a partial isometry with initial space $\pi(E_{s^*})H$, hence final space $\pi(E_s)H$ (since $v_s^* = \pi(w_{s^*,s}^*) v_{s^*}$).

The covariance condition is satisfied since if $a \in E_{s^*}$ then

$$\begin{aligned} v_s \pi(a) v_s^* &= v_s \pi(a) \pi(w_{s^*,s}^*) v_{s^*} = \operatorname*{s-lim}_{\mu,\lambda} \Pi(u_\lambda \delta_s * a w_{s^*,s}^* \delta_e * \beta_s^{-1}(u_\mu)\delta_{s^*}) \\ &= \operatorname*{s-lim}_{\mu,\lambda} \Pi(\beta_s(\beta_s^{-1}(u_\lambda) a w_{s^*,s}^*)\delta_s * \beta_s^{-1}(u_\mu)\delta_{s^*}) \\ &= \operatorname*{s-lim}_{\mu,\lambda} \Pi(u_\lambda \beta_s(a w_{s^*,s}^*) u_\mu w_{s,s^*} \delta_{ss^*}) = \Pi(\beta_s(a)\delta_e), \end{aligned}$$

since $\{\beta_s^{-1}(u_\mu)\}$ is an approximate identity for E_{s^*}. ∎

Proposition 3.11. *The correspondence $(\pi, v, H) \leftrightarrow (\pi \times v, H)$ is a bijection between covariant representations of (A, S, β, w) and nondegenerate representations of $A \times_{\beta,w} S$.*

Proof. Let $(\tilde\pi, \tilde v, H)$ be a covariant representation of (A, S, β, w), and let (π, v) be the covariant representation associated to $\tilde\pi \times \tilde v$ by Proposition 3.10. Then

$$\begin{aligned} \pi(a) &= \big(\tilde\pi \times \tilde v\big)(a\delta_e) = \tilde\pi(a)\tilde v_e = \tilde\pi(a) \\ v_s &= \operatorname*{s-lim}_{\lambda} \big(\tilde\pi \times \tilde v\big)(u_\lambda \delta_s) = \operatorname*{s-lim}_{\lambda} \tilde\pi(u_\lambda)\tilde v_s = \tilde v_s, \end{aligned}$$

since $\tilde\pi(u_\lambda)$ converges strongly to the projection of H onto $\tilde\pi(E_s)H$. On the other hand, if Π is a representation of $A \times_\beta S$ and (π, v) is the covariant representation associated to Π, then for $a \in E_s$ we have

$$\begin{aligned} \big(\pi \times v\big)(a\delta_s) &= \pi(a) v_s = \Pi(a\delta_e) \operatorname*{s-lim}_{\lambda} \Pi(u_\lambda \delta_s) = \operatorname*{s-lim}_{\lambda} \Pi(a\delta_e * u_\lambda \delta_s) \\ &= \operatorname*{s-lim}_{\lambda} \Pi(a u_\lambda \delta_s) = \Pi(a\delta_s). \end{aligned}$$

Thus the correspondence is a bijection. ∎

4. CONNECTION WITH TWISTED PARTIAL ACTIONS

We have seen in [22] and [7] that there is a close connection between partial actions of [13] and untwisted inverse semigroup actions. There is a similar connection between *twisted partial actions* of [6] and Busby-Smith twisted inverse semigroup actions, which is the topic of this section. Recall the definition of a twisted partial action from [6].

DEFINITION 4.1. A *twisted partial action* of a group G on a C^*-algebra A is a pair (α, u), where for all $s \in G$, $\alpha_s : D_{s^{-1}} \to D_s$ is a partial automorphism of A, and for all r, $s \in G$, $u_{r,s}$ is a unitary multiplier of $D_r D_{rs}$, such that for all r, s, $t \in G$ we have

(i) $D_e = A$, and α_e is the identity automorphism of A;
(ii) $\alpha_r(D_{r^{-1}} D_s) = D_r D_{rs}$;
(iii) $\alpha_r(\alpha_s(a)) = u_{r,s}\alpha_{rs}(a)u^*_{r,s}$ for all $a \in D_{s^{-1}} D_{s^{-1}r^{-1}}$;
(iv) $u_{e,t} = u_{t,e} = 1_{M(A)}$;
(v) $\alpha_r(a u_{s,t})u_{r,st} = \alpha_r(a)u_{r,s}u_{rs,t}$ for all $a \in D_{r^{-1}} D_s D_{st}$.

Recall from [22], that if α is a partial action of G on A then the partial automorphism $\alpha_{s_1} \cdots \alpha_{s_n}$ has domain $D_{s_n^{-1}} D_{s_n^{-1}s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}}$ and range $D_{s_1} D_{s_1 s_2} \cdots D_{s_1 \cdots s_n}$ for all $s_1, \ldots, s_n \in G$. A similar proof shows that this is also true for twisted partial actions, even if α_{s_i} is replaced by $\alpha^{-1}_{s_i^{-1}}$ for some i.

Recall from [7] that for a group G, the *associated inverse semigroup* $S(G)$ has elements written in canonical form $[g_1][g_1^{-1}] \cdots [g_m][g_m^{-1}][s]$, where $g_1, \ldots, g_n, s \in G$, and the order of the $[g_i][g_i^{-1}]$ terms is irrelevant. Multiplication and inverses are defined by

$$
\begin{aligned}
&[g_1][g_1^{-1}] \cdots [g_m][g_m^{-1}][s] \cdot [h_1][h_1^{-1}] \cdots [h_m][h_m^{-1}][t] \\
&\qquad = [g_1][g_1^{-1}] \cdots [g_m][g_m^{-1}][s][s^{-1}][sh_1][(sh_1)^{-1}] \cdots [sh_m][(sh_m)^{-1}][st]
\end{aligned}
$$

and

$$
([g_1][g_1^{-1}] \cdots [g_m][g_m^{-1}][s])^* = [s^{-1}g_m][(s^{-1}g_m)^{-1}] \cdots [s^{-1}g_1][(s^{-1}g_1)^{-1}][s^{-1}].
$$

The idempotents of $S(G)$ are the elements in the form $[g_1][g_1^{-1}] \cdots [g_m][g_m^{-1}][e]$, where e is the identity of G. The next theorem shows that every twisted partial action of G determines a Busby-Smith twisted action of $S(G)$.

THEOREM 4.2. *Let (A, G, α, u) be a twisted partial action. For all $p = [g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s]$ and $q = [h_1][h_1^{-1}]\cdots[h_n][h_n^{-1}][t]$ in $S(G)$, define $E_p = D_{g_1}\cdots D_{g_m}D_s$ and let*

$$\beta_p = \alpha_{g_1}\alpha_{g_1}^{-1}\cdots\alpha_{g_m}\alpha_{g_m}^{-1}\alpha_s.$$

Also let

$$w_{p,q} = 1_{M(E_{pq})}u_{s,t}.$$

Then $(A, S(G), \beta, w)$ is a Busby-Smith twisted action.

Proof. Throughout the proof, let p and q be in the form

$$p = [g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][s]$$

and

$$q = [h_1][h_1^{-1}]\cdots[h_n][h_n^{-1}][t].$$

First note that for all $p \in S(G)$, β_p is an isomorphism between the closed ideals E_{p^*} and E_p of A. Also note that $w_{p,q}$ is a unitary multiplier of $E_{p,q}$, since $u_{s,t}$ is a unitary multiplier of D_sD_{st} and $E_{pq} = D_{g_1}\cdots D_{g_m}D_sD_{sh_1}\cdots D_{sh_n}D_{st}$ is contained in D_sD_{st}. If e is the identity of G then $E_{[e]} = D_e = A$, verifying Definition 2.2 (i). For all $p, q \in S(G)$ we have

$$\beta_p\beta_q = \alpha_{g_1}\alpha_{g_1}^{-1}\cdots\alpha_{g_m}\alpha_{g_m}^{-1}\alpha_s\alpha_{h_1}\alpha_{h_1}^{-1}\cdots\alpha_{h_n}\alpha_{h_n}^{-1}\alpha_t\,,$$

which is the restriction of $\alpha_s\alpha_t$ to

$$D_{t^{-1}}D_{t^{-1}h_n}\cdots D_{t^{-1}h_1}D_{t^{-1}s^{-1}}D_{t^{-1}s^{-1}g_m}\cdots D_{t^{-1}s^{-1}g_1} = E_{(pq)^*}.$$

On the other hand

$$\beta_{pq} = \alpha_{g_1}\alpha_{g_1}^{-1}\cdots\alpha_{g_m}\alpha_{g_m}^{-1}\alpha_s\alpha_s^{-1}\alpha_{sh_1}\alpha_{sh_1}^{-1}\cdots\alpha_{sh_n}\alpha_{sh_n}^{-1}\alpha_{st},$$

which is the restriction of α_{st} to $E_{(pq)^*}$. Thus, Definition 2.2 (ii) follows from Definition 4.1 (iii), since $E_{(pq)^*} \subset D_{t^{-1}}D_{t^{-1}s^{-1}}$. To check Definition 2.2 (iii), note that if $f = [g_1][g_1^{-1}]\cdots[g_m][g_m^{-1}][e]$ is an idempotent in $S(G)$, then $w_{f,q} = 1_{M(E_{fq})}u_{e,t} = 1_{M(E_{fq})}$ for all $q \in S(G)$. Similarly, $w_{q,f} = 1_{M(E_{qf})}$ for all $q \in S(G)$. Finally to check Definition 2.2 (iv), let $k = [f_1][f_1^{-1}]\cdots[f_l][f_l^{-1}][r]$, p and q be arbitrary elements of $S(G)$, and fix

$$a \in E_{k^*}E_{pq} = D_{r^{-1}}D_{r^{-1}f_l}\cdots D_{r^{-1}f_1}D_{g_1}\cdots D_{g_n}D_sD_{sh_1}\cdots D_{sh_n}D_{st}.$$

Then $a \in D_{r^{-1}}D_sD_{st}$, and so we have

$$\begin{aligned}\beta_k(aw_{p,q})w_{k,pq} &= \beta_k(a1_{M(E_{pq})}u_{s,t})1_{M(E_{kpq})}u_{r,st} = \alpha_r(au_{s,t})u_{r,st}\\ &= \alpha_r(a)u_{r,s}u_{rs,t} = \alpha_r(a)1_{M(E_{kp})}u_{r,s}1_{M(E_{kpq})}u_{rs,t}\\ &= \beta_k(a)w_{k,p}w_{kp,q}. \quad \blacksquare\end{aligned}$$

This process works the other way too. Starting with a Busby-Smith twisted action β of $S(G)$, the restriction of β to the canonical image of G in $S(G)$ gives a twisted partial action:

THEOREM 4.3. *Let* $(A, S(G), \beta, w)$ *be a Busby-Smith twisted action. If* $\alpha_s = \beta_{[s]}$ *and* $u_{s,t} = w_{[s],[t]}$ *for all* $s,t \in G$*, then* (A, G, α, u) *is a twisted partial action.*

Proof. Let D_s be the range of α_s for all $s \in G$. If e is the identity of G, then $D_e = E_{[e]} = A$. We have

$$\begin{aligned}\alpha_r(D_{r^{-1}}D_s) &= \operatorname{im}\beta_{[r]}\beta_{[s]} = \operatorname{im}\beta_{[r][s]} = \operatorname{im}\beta_{[r][r^{-1}][rs]}\\ &= \operatorname{im}\beta_{[r]}\beta_{[r^{-1}]}\beta_{[rs]} = E_{[r]}E_{[rs]} = D_rD_{rs},\end{aligned}$$

which verifies Definition 4.1 (ii). Similar calculations can be used to verify the other conditions in Definition 4.1. ∎

It is clear that the processes above are the inverses of each other. Thus we have a bijective correspondence between twisted partial actions of G and Busby-Smith twisted actions of $S(G)$. Although the theory of crossed products by twisted partial actions has not been developed, we believe that the crossed products of corresponding twisted partial actions mentioned in [6] and Busby-Smith twisted actions are isomorphic. This is, in fact, the case for untwisted actions, since the semigroup action constructed in Theorem 4.2 factors through the semigroup action giving the isomorphic crossed product constructed in [22]. We plan to pursue this in an upcoming paper.

It would be interesting to know if there is a similar correspondence for the Green twisted actions defined in Section 6 below. It would first be necessary to find a satisfactory definition of Green twisted partial actions of a group.

5. EXTERIOR EQUIVALENCE

Exterior equivalence is defined by Packer and Raeburn for twisted group actions in [15]. We extend this definition to Busby-Smith twisted inverse semigroup actions.

DEFINITION 5.1. Two Busby-Smith twisted actions (α, u) and (β, w) of S on A are *exterior equivalent* if for all $s \in S$ there is a unitary multiplier V_s of E_s such that for all $s, t \in S$

(i) $\beta_s = \operatorname{Ad} V_s \circ \alpha_s$;
(ii) $w_{s,t} = V_s\alpha_s(1_{M(E_{s^*})}V_t)u_{s,t}V_{st}^*$.

We say that the exterior equivalence is *implemented* by V. Notice that for all $s \in S$, α_s and β_s have to have the same domain and range. Also note that $1_{M(E_{s^*})}V_t \in E_{s^*}^{**}$ since $E_{s^*}^{**} = 1_{M(E_{s^*})}A^{**}$. Hence we can evaluate $\alpha_s(1_{M(E_{s^*})}V_t)$ if we extend α_s to the double dual of E_{s^*}.

PROPOSITION 5.2. *Exterior equivalence is an equivalence relation.*

Proof. Let V implement an exterior equivalence between the Busby-Smith twisted actions (α, u) and (β, w). Then taking adjoints we have $\alpha_s = \operatorname{Ad} V_s^* \circ \beta_s$ and $u_{s,t} = \alpha_s(1_{M(E_{s^*})}V_t^*)V_s^* w_{s,t} V_{st} = V_s^* \beta_s(1_{M(E_{s^*})}V_t^*)w_{s,t}V_{st}$, so V^* implements an exterior equivalence between (β, w) and (α, u), showing symmetry. Reflexivity is clear by letting $V_s = 1_{M(E_s)}$ for all $s \in S$. To show transitivity let V implement an exterior equivalence between (α, u) and (β, w), and X implement an exterior equivalence between (β, w) and (γ, z). Then

$$\begin{aligned}
\gamma_s &= \operatorname{Ad} X_s V_s \circ \alpha_s,\\
z_{s,t} &= X_s \beta_s(1_{M(E_{s^*})}X_t)w_{s,t}X_{st}^*\\
&= X_s V_s \alpha_s(1_{M(E_{s^*})}X_t)V_s^* V_s \alpha_s(1_{M(E_{s^*})}V_t)u_{s,t}V_{st}^* X_{st}^*\\
&= X_s V_s \alpha_s(1_{M(E_{s^*})}X_t V_t)u_{s,t}(X_{st}V_{st})^*\,,
\end{aligned}$$

which shows that XV implements an exterior equivalence between (α, u) and (γ, z). ∎

PROPOSITION 5.3. *If (α, u) and (β, w) are exterior equivalent Busby-Smith twisted inverse semigroup actions of S on A, then $A \times_{\alpha,u} S$ and $A \times_{\beta,w} S$ are isomorphic.*

Proof. Suppose V implements an exterior equivalence between (α, u) and (β, w). Let (π, z) be a covariant representation of (β, w), and define $v_s = \pi(V_s^*)z_s$. We show that (π, v) is a covariant representation of (α, u). If $s \in S$ and $a \in E_{s^*}$ then

$$\pi(\alpha_s(a)) = \pi(V_s^* \beta_s(a) V_s) = \pi(V_s^*)z_s \pi(a) z_s^* \pi(V_s) = v_s \pi(a) v_s^*.$$

We also have

$$\begin{aligned}
v_s v_t &= \pi(V_s^*)z_s \pi(1_{M(E_{s^*})}V_t^*)z_t\\
&= \pi(V_s^*)\pi(\beta_s(1_{M(E_{s^*})}V_t^*))z_s z_t\\
&= \pi(V_s^*)\pi(V_s)\pi(\alpha_s(1_{M(E_{s^*})}V_t^*))\pi(V_s^*)\pi(w_{s,t})z_{st}\\
&= \pi(u_{s,t})\pi(V_{st}^*)z_{st} = \pi(u_{s,t})v_{st}.
\end{aligned}$$

The other conditions of Definition 3.2 are clearly satisfied. Note that the images of $\pi \times v$ and $\pi \times z$ are the same, that is, $\big(\pi \times v\big)(A \times_{\alpha,u} S) = \big(\pi \times z\big)(A \times_{\beta,w} S)$.

Now suppose that $\pi \times z$ is a faithful representation of $A \times_{\beta,w} S$. Then $\Phi = (\pi \times z)^{-1} \circ (\pi \times v) : A \times_{\alpha,u} S \to A \times_{\beta,w} S$ is a surjective homomorphism. Similarly, starting with a faithful representation $\rho \times l$ of $A \times_{\alpha,u} S$, we can find a covariant representation (ρ, m) of (β, w) such that $\Psi = (\rho \times l)^{-1} \circ (\rho \times m)$:

$A \times_{\beta,w} S \to A \times_{\alpha,u} S$ is a homomorphism. We are going to show that $\Psi \circ \Phi$ is the identity map, which implies that Φ is an isomorphism. For $a \in E_s$ we have

$$\begin{aligned}\Psi \circ \Phi(a\delta_s) &= \Psi \circ (\pi \times z)^{-1}(\pi(a)v_s) = \Psi \circ (\pi \times z)^{-1}(\pi(a)\pi(V_s^*)z_s) \\ &= \Psi \circ (\pi \times z)^{-1}(\pi(aV_s^*)z_s) = \Psi(aV_s^*\delta_s) \\ &= (\rho \times l)^{-1}(\rho(aV_s^*)m_s) = (\rho \times l)^{-1}(\rho(aV_s^*)\rho(V_s)l_s) \\ &= (\rho \times l)^{-1}(\rho(aV_s^*V_s)l_s) = a\delta_s. \quad ∎\end{aligned}$$

EXAMPLE 5.4. If we have two order-preserving cross-sections $c, d : S \to T$ in Proposition 2.5, then the corresponding Busby-Smith twisted actions (β, w) and (γ, z) defined by c and d respectively, are exterior equivalent, and so $C^*(N) \times_{\beta,w} S$ and $C^*(N) \times_{\gamma,z} S$ are isomorphic.

To see this define $V_s = d(s)c(s)^*$ for all $s \in S$. Then we have

$$\begin{aligned}\gamma_s &= \operatorname{Ad} d(s) = \operatorname{Ad} d(s) \circ \operatorname{Ad} c(s)^* \circ \operatorname{Ad} c(s) \\ &= \operatorname{Ad} d(s)c(s)^* \circ \operatorname{Ad} c(s) = \operatorname{Ad} V_s \circ \beta_s,\end{aligned}$$

and

$$\begin{aligned}&V_s\beta_s(1_{M(E_{s^*})}V_t)w_{s,t}V_{st}^* \\ &\quad = d(s)c(s)^*c(s)1_{M(E_{s^*})}d(t)c(t)^*c(s)^*c(s)c(t)c(st)^*(d(st)c(st)^*)^* \\ &\quad = d(s)d(t)d(st)^* = z_{s,t}\end{aligned}$$

showing that V implements an exterior equivalence between (β, w) and (γ, z). ∎

6. GREEN TWISTED ACTIONS

Green studies another type of twisted group action in [9], and here we adapt this to inverse semigroups:

DEFINITION 6.1. Let A be a C^*-algebra, let S be a unital inverse semigroup with idempotent semilattice E, and let N be a normal Clifford subsemigroup of S. A *Green twisted action* of (S, N) on A is a pair (γ, τ), where γ is an inverse semigroup action of S on A, that is, a semigroup homomorphism $s \mapsto (\gamma_s, E_{s^*}, E_s) : S \to \operatorname{PAut}(A)$ with $E_e = A$, and for all $n \in N$, τ_n is a unitary multiplier of E_n, such that for all $n, l \in N$ we have

(i) $\gamma_n = \operatorname{Ad} \tau_n$;
(ii) $\gamma_s(\tau_n) = \tau_{sns^*}$ for all $s \in S$ with $n^*n \leqslant s^*s$;
(iii) $\tau_n\tau_l = \tau_{nl}$.

We call τ the twisting map, and we also refer to (A, S, N, γ, τ) as a Green twisted action.

EXAMPLE 6.2. Let S, E, N be as in Definition 6.1. Define $E_s = \overline{\operatorname{span}} \bigcup \{[f] : f \leqslant ss^* \text{ for some idempotent } f \in E\} \subset C^*(N)$. For all $s \in S$ define $\gamma_s : E_{s^*} \to E_s$ by $\gamma_s = \operatorname{Ad} s$, and for all $n \in N$ define $\tau_n = n$. Then $(C^*(N), S, N, \gamma, \tau)$ is a Green twisted action.

DEFINITION 6.3. Let (A, S, N, γ, τ) be a Green twisted action. A *covariant representation* of the Green twisted action (γ, τ) is a covariant representation (π, u) of the action γ that *preserves the twist* τ, that is, π is a nondegenerate representation of A on the Hilbert space H, and $u : S \to B(H)$ is a multiplicative map such that

(i) $u_s \pi(a) u_s^* = \pi(\gamma_s(a))$ for all $a \in E_{s^*}$;

(ii) u_s is a partial isometry with initial space $\pi(E_{s^*})H$ and final space $\pi(E_s)H$;

(iii) $u_n = \pi(\tau_n)$ for all $n \in N$.

To evaluate $\pi(\tau_n)$ we again extend π to the enveloping von Neumann algebra A^{**} of A.

DEFINITION 6.4. Let (A, S, N, γ, τ) be a Green twisted action. The *Green twisted crossed product* $A \times_{\gamma,\tau} S$ is the quotient of $A \times_\gamma S$ by the ideal

$$I_\tau = \bigcap \{\ker(\pi \times u) : (\pi, u) \text{ is a covariant representation of } (\gamma, \tau)\}.$$

Our definitions of Green twisted inverse semigroup action and Green twisted crossed product are generalizations of inverse semigroup action and crossed product defined in [22]. Every action β of S may be regarded (trivially) as a Green twisted inverse semigroup action by taking $\tau_n = 1_{M(E_n)}$ for all n in the idempotent semilattice E of S, and we have $A \times_\beta S \cong A \times_{\beta,\tau} S$.

THEOREM 6.5. *There is a bijective correspondence between nondegenerate representations of the Green twisted crossed product and covariant representations of the Green twisted action.*

Proof. Let $\Psi : A \times_\gamma S \to A \times_{\gamma,\tau} S$ be the quotient map. If Π is a representation of $A \times_{\gamma,\tau} S$ then $\Pi \circ \Psi$ is a representation of $A \times_\gamma S$ and so $\Pi \circ \Psi = \pi \times u$ for some covariant representation (π, u) of γ by Proposition 3.11. We show that (π, u) preserves the twist. If $a \in E_n$ then for every covariant representation (ρ, z) of (γ, τ) we have $(\rho \times z)(a\tau_n\delta_e - a\delta_n) = \rho(a)(\rho(\tau_n) - z_n) = 0$ and so

$$\pi(a)(\pi(\tau_n) - u_n) = (\pi \times u)(a\tau_n\delta_e - a\delta_n) = \Pi \circ \Psi(a\tau_n\delta_e - a\delta_n) = \Pi(0) = 0.$$

Thus $\big(\pi(\tau_n)^* - u_n^*\big)\pi(E_n)H = \{0\}$. Since $\pi(\tau_n)^*$ and u_n^* are partial isometries with initial space $\pi(E_n)H$, we must have $\pi(\tau_n) - u_n = 0$ and so (π, u) is a covariant representation of (γ, τ).

On the other hand, if (π, u) is a covariant representation of the twisted action (γ, τ), then $I_\tau \subset \ker\,(\pi \times u)$, and so there is a unique representation $\pi \times_\tau u$ of $A \times_{\gamma,\tau} S$ defined by $\pi \times_\tau u(\Psi(x)) = \big(\pi \times u\big)(x)$.

For the uniqueness, note that Proposition 3.11, in the absence of the Busby-Smith twist, gives a bijection between nondegenerate representations Π of $A \times_\gamma S$ and covariant representations (π, u) of (A, S, γ). By the above argument, Π kills the ideal I_τ if and only if (π, u) preserves the twist τ, so we are done. ∎

7. CONNECTION BETWEEN BUSBY-SMITH AND GREEN TWISTED ACTIONS

There is a close connection between Busby-Smith twisted and Green twisted actions just like in the group case [15]. In this section we show that starting with a Busby-Smith twisted action we can construct a Green twisted action with the same crossed product. Conversely, starting with a Green twisted action we construct a Busby-Smith twisted action with the same crossed product. This latter construction depends on the existence of an order-preserving cross-section, suggesting that Green twisted actions are “more general” than Busby-Smith twisted actions. This may seem surprising since in the discrete group case there is an essentially bijective correspondence between Busby-Smith twisted and Green twisted actions.

THEOREM 7.1. *Let (A, S, N, γ, τ) be a Green twisted action, and suppose there is an order-preserving cross-section $c : S/N \to S$. For $q, r \in S/N$ define*

$$\beta_q = \gamma_{c(q)} \quad \text{and} \quad w_{q,r} = \tau_{c(q)c(r)c(qr)^*}.$$

Then $(A, S/N, \beta, w)$ is a Busby-Smith twisted action, and the crossed products $A \times_{\gamma,\tau} S$ and $A \times_{\beta,w} S/N$ are isomorphic.

Proof. First we show that (β, w) is a Busby-Smith twisted action. If q, $r \in S/N$ then $w_{q,r}$ is a unitary multiplier of $E_{c(q)c(r)} = E_{c(q)c(r)c(r)^*c(q)^*}$ since $\tau_{c(q)c(r)c(qr)^*}$ is a unitary multiplier of $E_{c(q)c(r)c(qr)^*} = E_{c(q)c(r)c(qr)^*c(qr)c(r)^*c(q)^*} = E_{c(q)c(r)c(r)^*c(q)^*}$. It is clear that $E_{[e]} = E_e = A$. For q, $r \in S/N$ we have

$$\begin{aligned}\beta_q\beta_r = \gamma_{c(q)}\gamma_{c(r)} = \gamma_{c(q)c(r)} &= \gamma_{c(q)c(r)c(qr)^*c(qr)} \\ &= \operatorname{Ad}\tau_{c(q)c(r)c(qr)^*} \circ \gamma_{c(qr)} = \operatorname{Ad} w_{q,r} \circ \beta_{qr},\end{aligned}$$

which verifies Definition 2.2 (ii). To check Definition 2.2 (iii) let $f \in E$ and $q \in S/N$. Then we have

$$w_{[f],q} = \tau_{fc(q)c([f]q)^*} = 1_{M(E_{[f]qq^*})} = 1_{M(E_{[f]q})}\,.$$

To see this notice that since c is order-preserving and $q \geq [f]q$, we have $c(q) \geq c([f]q)$. Hence $fc(q)c([f]q)^* = fc([f]q)c([f]q)^*$ is an idempotent in $[f]qq^*$. Similarly $w_{q,[f]} = 1_{M(E_{q[f]})}$. It remains to check Definition 2.2 (iv). If p, q, $r \in S/N$ and $a \in E_{c(p^*)}E_{c(qr)}$, then

$$\begin{aligned}
\beta_p(aw_{q,r})w_{p,qr} &= \gamma_{c(p)}(a\tau_{c(q)c(r)c(qr)^*})\tau_{c(p)c(qr)c(pqr)^*}\\
&= \gamma_{c(p)}(a)\tau_{c(p)c(q)c(r)c(qr)^*c(p)^*c(p)c(qr)c(pqr)^*}\\
&= \gamma_{c(p)}(a)\tau_{c(p)c(q)c(r)c(pqr)^*}\\
&= \gamma_{c(p)}(a)\tau_{c(p)c(q)c(pq)^*c(pq)c(r)c(pqr)^*}\\
&= \beta_p(a)w_{p,q}w_{pq,r}.
\end{aligned}$$

Next, we investigate the connection between (γ, τ) and (β, w). Let (π, v) be a covariant representation of (β, w). Define $u_s = \pi(\tau_{sc([s])^*})v_{[s]}$. We show that (π, u) is a covariant representation of (γ, τ). First notice that u is a homomorphism since for s, $t \in S$ we have

$$\begin{aligned}
u_su_t &= \pi(\tau_{sc([s])^*})v_{[s]}\pi(\tau_{tc([t])^*})v_{[t]}\\
&= \pi(\tau_{sc([s])^*})\pi(\beta_{[s]}(\tau_{tc([t])^*}))v_{[t]}v_{[t]}\\
&= \pi(\tau_{sc([s])^*}\tau_{c([s])tc([t])^*c([s])^*})\pi(w_{[s],[t]})v_{[st]}\\
&= \pi(\tau_{stc([t])^*c([s])^*})\pi(\tau_{c([s])c([t])c([st])^*})v_{[st]}\\
&= \pi(\tau_{stc([st])^*})v_{[st]} = u_{st}.
\end{aligned}$$

To check the covariance condition let $s \in S$ and $a \in E_{s^*}$, then

$$\begin{aligned}
u_s\pi(a)u_s^* &= \pi(\tau_{sc([s])^*})v_{[s]}\pi(a)v_{[s]}^*\pi(\tau_{sc([s])^*}^*)\\
&= \pi(\gamma_{sc([s])^*}(\beta_{[s]}(a))) \quad \text{(by Definition 6.1 (i))}\\
&= \pi(\gamma_{sc([s])^*c([s])}(a)) = \pi(\gamma_s(a))
\end{aligned}$$

where we used the fact that $sc([s])^* \in N$. It is clear that u_s has the right initial and final spaces. The following calculation shows that (π, u) preserves the twist

$$\begin{aligned}
u_n &= \pi(\tau_{nc([n])^*})v_{[n]} = \pi(\tau_n)v_{[n]}\\
&= \pi(\tau_n)P_{\pi(E_{[n]})H} \quad \text{(by Proposition 3.3 (i))}\\
&= \pi(\tau_n)P_{\pi(E_n)H} = \pi(\tau_n).
\end{aligned}$$

We show that $\operatorname{im}(\pi \times_\tau u) = \operatorname{im}(\pi \times v)$. For $a \in E_s$ we have

$$\pi(a)u_s = \pi(a)\pi(\tau_{sc([s])^*})v_{[s]} = \pi(a\tau_{sc([s])^*})v_{[s]} \in \operatorname{im}(\pi \times v)$$

and so $\operatorname{im}(\pi \times_\tau u) \subset \operatorname{im}(\pi \times v)$. On the other hand,

$$\pi(a)v_{[s]} = \pi(a\tau^*_{sc([s])^*})\pi(\tau_{sc([s])^*})v_{[s]} = \pi(a\tau^*_{sc([s])^*})u_s \in \operatorname{im}(\pi \times_\tau u)$$

and so $\operatorname{im}(\pi \times_\tau u) \supset \operatorname{im}(\pi \times v)$.

Next let (π, u) be a covariant representation of (γ, τ). Define $v_q = u_{c(q)}$. We show that (π, v) is a covariant representation of (β, w). If $a \in E_{q^*}$ then

$$\pi(\beta_q(a)) = \pi(\gamma_{c(q)}(a)) = u_{c(q)}\pi(a)u^*_{c(q)} = v_q\pi(a)v_{q^*}\,,$$

which shows that the covariance condition holds. We also have

$$\begin{aligned} v_q v_r &= u_{c(q)}u_{c(r)} = u_{c(q)c(r)c(qr)^*}u_{c(qr)} \\ &= \pi(\tau_{c(q)c(r)c(qr)^*})u_{c(qr)} = \pi(w_{q,r})v_{qr}. \end{aligned}$$

It is clear that v_q has the right initial and final spaces. It is also clear that $\operatorname{im}(\pi \times_\tau u) \supset \operatorname{im}(\pi \times v)$ since for $a \in E_{[s]}$ we have $\pi(a)v_{[s]} = \pi(a)u_{c([s])} \in \operatorname{im}(\pi \times_\tau u)$.

Now let $\pi \times v$ be a faithful representation of $A \times_{\beta,w} S/N$. Then by the above correspondence, $\pi \times_\tau u$ is a representation of $A \times_{\gamma,\tau} S$ such that $\operatorname{im}(\pi \times v) = \operatorname{im}(\pi \times_\tau u)$, and so we have a surjective homomorphism $\Phi = (\pi \times v)^{-1} \circ (\pi \times_\tau u) : A \times_{\gamma,\tau} S \to A \times_{\beta,w} S/N$. Similarly, starting with a faithful representation $\rho \times_\tau z$ of $A \times_{\gamma,\tau} S$, the representation $\rho \times (z \circ c)$ can be used to find a homomorphism $\Psi = (\rho \times_\tau z)^{-1} \circ (\rho \times (z \circ c)) : A \times_{\beta,w} S/N \to A \times_{\gamma,\tau} S$. We show that $\Psi \circ \Phi$ is the identity map, hence Φ is an isomorphism. For $a \in E_s$ we have

$$\begin{aligned} \Psi \circ \Phi(a\delta_s) &= \Psi \circ (\pi \times v)^{-1}(\pi(a)u_s) = \Psi \circ (\pi \times v)^{-1}(\pi(a)\pi(\tau_{sc([s])^*})v_{[s]}) \\ &= \Psi(a\tau_{sc([s])^*}\delta_{[s]}) = (\rho \times_\tau z)^{-1} \circ \big(\rho \times (z \circ c)\big)(a\tau_{sc([s])^*}\delta_{[s]}) \\ &= (\rho \times_\tau z)^{-1}(\rho(a)\rho(\tau_{sc([s])^*})z_{c[s]}) = (\rho \times_\tau z)^{-1}(\rho(a)z_{sc([s])^*}z_{c[s]}) \\ &= (\rho \times_\tau z)^{-1}(\rho(a)z_s) = a\delta_s. \quad \blacksquare \end{aligned}$$

Proposition 7.2. *If we use a different order-preserving cross-section $b : S/N \to S$ in Theorem 7.1 then we get an exterior equivalent Busby-Smith twisted action (α, u).*

Proof. The exterior equivalence is implemented by $V_q = \tau_{c(q)b(q)^*}$ because for $q \in S/N$ we have

$$\beta_q = \gamma_{c(q)} = \gamma_{c(q)b(q)^*b(q)} = \gamma_{c(q)b(q)^*}\gamma_{b(q)} = \operatorname{Ad}\tau_{c(q)b(q)^*} \circ \gamma_{b(q)},$$

and for $q, r \in S/N$ we have

$$\begin{aligned} V_q\alpha_q(1_{M(E_{q^*})}V_r)u_{q,r}V_{qr}^* &= \tau_{c(q)b(q)^*}\gamma_{b(q)}(1_{M(E_{q^*})}\tau_{c(r)b(r)^*})\tau_{b(q)b(r)b(qr)^*}\tau_{c(qr)b(qr)^*}^* \\ &= \tau_{c(q)}1_{M(E_{q^*})}\tau_{c(r)b(r)^*}\tau_{b(r)b(qr)^*}\tau_{c(qr)b(qr)^*}^* \\ &= \tau_{c(q)}1_{M(E_{q^*})}\tau_{c(r)}\tau_{b(qr)^*}\tau_{b(qr)^*}^*\tau_{c(qr)}^* = w_{q,r}. \quad \blacksquare \end{aligned}$$

Our next goal is to show that starting with a Busby-Smith twisted action we can build a Green twisted action with the same crossed product. The construction mimics that of Fell's in [8], Theorem I.9.1.

LEMMA 7.3. *Let (A, T, β, w) be a Busby-Smith twisted action. The set*

$$S = \{u\partial_t : u \in UM(E_t),\, t \in T\}$$

with multiplication and adjoint defined by

$$\begin{aligned} u_r\partial_r * u_t\partial_t &= \beta_r(\beta_r^{-1}(u_r)u_t)w_{r,t}\partial_{rt} \\ (u_t\partial_t)^* &= \beta_t^{-1}(u_t^*)w_{t^*,t}^*\partial_{t^*} \end{aligned}$$

is an inverse semigroup with idempotent semilattice

$$E_S = \{1_{M(E_f)}\partial_f : f \in E_T\},$$

where E_T is the idempotent semilattice of T. If for $u_t\partial_t \in S$, $a \in E_{t^*}$ and $u_f\partial_f$ in

$$N := \{u\partial_f : u \in UM(E_f),\, f \in E_T\}$$

we define

$$\gamma_{u_t\partial_t}(a) = u_t\beta_t(a)u_t^* \quad \textit{and} \quad \tau_{u_f\partial_f} = u_f,$$

then (A, S, N, γ, τ) is a Green twisted action.

Proof. To verify that S is an inverse semigroup note that the operations are well-defined and as in the proof of Proposition 3.1, associativity holds for the multiplication. Also if $u_t\partial_t \in S$ then by the calculation in the proof of Proposition 3.1 and Lemma 2.3 (xi) we have

$$\begin{aligned} u_t\partial_t(u_t\partial_t)^*u_t\partial_t &= \beta_t(\beta_t^{-1}(u_t)\beta_{t^*}(\beta_{t^*}^{-1}(\beta_t^{-1}(u_t^*)w_{t^*,t}^*)u_t))w_{t,t^*}w_{tt^*,t}\partial_{tt^*t} \\ &= \beta_t(\beta_t^{-1}(u_t)\beta_t^{-1}(u_t^*)w_{t^*,t}^*)\beta_{t^*}(u_t))w_{t,t^*}\partial_t \\ &= u_tu_t^*\beta_t(w_{t^*,t}^*)\beta_t(\beta_{t^*}(u_t))w_{t,t^*}\partial_t \\ &= w_{t,t^*}^*\beta_t(\beta_{t^*}(u_t))w_{t,t^*}\partial_t = u_t\partial_t \end{aligned}$$

and similarly $(u_t\partial_t)^*u_t\partial_t(u_t\partial_t)^* = (u_t\partial_t)^*$. It is easy to see that E_S is in fact the idempotent semilattice of S. The definition

$$u\partial_r \sim v\partial_t \text{ if and only if } r = t$$

gives an idempotent-separating congruence on S. The corresponding normal Clifford subsemigroup is N.

If $a \in E_{(rt)^*}$ then we have

$$\begin{aligned}\gamma_{u_r\partial_r}(\gamma_{u_t\partial_t}(a)) &= u_r\gamma_r(u_t\gamma_t(a)u_t^*)u_r^* = \beta_r(\beta_r^{-1}(u_r)u_t\beta_t(a)u_t^*\beta_r^{-1}(u_r^*))\\ &= \beta_r(\beta_r^{-1}(u_r)u_t)w_{r,t}\beta_{rt}(a)w_{r,t}^*\beta_r(u_t^*\beta_r^{-1}(u_r^*))\\ &= \gamma_{\beta_r(\beta_r^{-1}(u_r)u_t)w_{r,t}\partial_{rt}}(a) = \gamma_{u_r\partial_r * u_t\partial_t}(a),\end{aligned}$$

showing that γ is a homomorphism. To verify Definition 6.1 (ii) note that by Lemma 2.3 (xi), for $s = u_t\partial_t$ we have

$$ss^* = \beta_t(\beta_t^{-1}(u_t)\beta_t^{-1}(u_t^*)w_{t^*,t}^*)w_{t,t^*}\partial_{tt^*} = \beta_t(w_{t^*,t}^*)w_{t,t^*}\partial_{tt^*} = 1_{M(E_{tt^*})}\partial_{tt^*},$$

for all $s \in S$. Thus if $n = u_f\partial_f \in N$ and $s = u_t\partial_t \in S$ with $n^*n \leqslant s^*s$ then

$$1_{M(E_{ft^*t})}\partial_{ft^*t} = 1_{M(E_f)}\partial_f 1_{M(E_{tt^*})}\partial_{tt^*} = n^*ns^*s = n^*n = 1_{M(E_f)}\partial_f,$$

and so $f \leqslant t^*t$. Hence $\tau_n = u_f \in E_{t^*t} = E_t$ and we have

$$\begin{aligned}\tau_{sns^*} &= \beta_t(\beta_t^{-1}(u_t)u_f\beta_t^{-1}(u_t^*)w_{t^*,t}^*)w_{t,f}w_{tf,t}\\ &= u_t\beta_t(u_f)u_t^*w_{t,t^*t}w_{tt^*,t}^*w_{t,t^*}^*\\ &= u_t\beta_t(u_f)u_t^* = \gamma_s(\tau_n).\end{aligned}$$

The conditions in Definition 6.1 (i) and (iii) are easy to verify. ∎

DEFINITION 7.4. The Busby-Smith twisted actions (A, S, α, u) and (B, T, β, w) are called *conjugate*, if there is a pair (ρ, φ) such that $\rho : A \to B$ and $\varphi : S \to T$ are isomorphisms such that for all $s, t \in S$ we have

$$\rho \circ \alpha_s = \beta_{\varphi(s)} \circ \rho \quad \text{and} \quad \rho(u_{s,t}) = w_{\varphi(s),\varphi(t)},$$

where ρ is extended to the double dual of A.

The proof of the following is straightforward.

LEMMA 7.5. *Conjugate Busby-Smith twisted actions have isomorphic crossed products.*

THEOREM 7.6. *If* (A, T, β, w) *is a Busby-Smith twisted action and* (A, S, N, γ, τ) *is the corresponding Green twisted action constructed in Lemma* 7.3, *then the crossed products* $A \times_{\beta,w} T$ *and* $A \times_{\gamma,\tau} S$ *are isomorphic.*

Proof. It is easy to see that the cross-section $c : S/N \to S$ defined by $c([u\partial_t]) = 1_{M(E_t)}\partial_t$ is order-preserving so if for all $q, r \in S/N$

$$\tilde{\beta}_q = \gamma_{c(q)} \quad \text{and} \quad \tilde{w}_{q,r} = \tau_{c(q)*c(r)*c(qr)^*},$$

then by Theorem 7.1, $(A, S/N, \tilde{\beta}, \tilde{w})$ is a Busby-Smith twisted action and the crossed products $A \times_{\gamma,\tau} S$ and $A \times_{\tilde{\beta},\tilde{w}} S/N$ are isomorphic. We finish the proof by showing that (id, φ) is a conjugacy between the Busby-Smith twisted actions (A, T, β, w) and $(A, S/N, \tilde{\beta}, \tilde{w})$, where

$$\varphi(t) = [1_{M(E_t)}\partial_t].$$

It is easy to see that φ is an isomorphism. For $s, t \in T$ we have

$$\tilde{\beta}_{\varphi(s)} = \gamma_{c([1_{M(E_s)}\partial_s])} = \beta_s$$

and

$$\begin{aligned}
\tilde{w}_{\varphi(s),\varphi(t)} &= \tau_{c([1_{M(E_s)}\partial_s])c([1_{M(E_t)}\partial_t])c([1_{M(E_{st})}\partial_{st}])^*} \\
&= \tau_{1_{M(E_s)}\partial_s * 1_{M(E_t)}\partial_t * (1_{M(E_{st})}\partial_{st})^*} \\
&= \beta_{qr}(\beta_{qr}^{-1}(w_{q,r})w^*_{(qr)^*,qr})w_{qr,(qr)^*} = w_{q,r},
\end{aligned}$$

and so we are done by Lemma 7.5. ∎

8. DECOMPOSITION OF GREEN TWISTED ACTIONS

Now we prove that in the presence of a normal Clifford subsemigroup, a Green twisted crossed product can be decomposed as an iterated Green twisted crossed product. The close analogy with Green's decomposition theorem [9] gives further evidence that in the inverse semigroup case, normal Clifford subsemigroups play the same role as normal subgroups do in the group case.

Theorem 8.1. *Let (A,S,N,γ,τ) be a Green twisted action and K be a normal Clifford subsemigroup containing N. The restriction of γ to K gives a Green twisted action (A,K,N,γ,τ). Let $\mu\times_\tau m$ be the universal representation of $A\times_{\gamma,\tau}K$, and identify $A\times_{\gamma,\tau}K$ with its image under $\mu\times_\tau m$. Let*

$$\widetilde{E}_s=\overline{\operatorname{span}}\,\{\mu(a)m_k : a\in E_k, k\in K, kk^*\leqslant ss^*\}$$

and define

$$\tilde{\gamma}_s:\widetilde{E}_{s^*}\to\widetilde{E}_s \quad by \quad \tilde{\gamma}_s=\operatorname{Ad}m_s.$$

Let $\tilde{\tau}_k=m_k$ for all $k\in K$. Then $(A\times_{\gamma,\tau}K,S,K,\tilde{\gamma},\tilde{\tau})$ is a Green twisted action and

$$A\times_{\gamma,\tau}S\cong(A\times_{\gamma,\tau}K)\times_{\tilde{\gamma},\tilde{\tau}}S.$$

Proof. First we show that $(\tilde{\gamma},\tilde{\tau})$ is a Green twisted action. $\widetilde{E}_s$ is an ideal by the calculation in the proof of Proposition 3.7 and the fact that $k,l\in K$ and $kk^*\leqslant ss^*$ imply $kl(kl)^*\leqslant ss^*$ and $lk(lk)^*\leqslant ss^*$. It is clear that $\tilde{\gamma}_s$ is multiplicative and preserves adjoints. Since

$$\tilde{\gamma}_s(\mu(a)m_k)=m_s\mu(a)m_km_s^*=m_s\mu(a)m_s^*m_{sks^*}=\mu(\gamma_s(a))m_{sks^*}$$

for all $\mu(a)m_k\in\widetilde{E}_k$, $\tilde{\gamma}_s$ is a bijection and therefore an isomorphism of $\widetilde{E}_{s^*}$ to $\widetilde{E}_s$, hence a partial automorphism of $A\times_{\gamma,\tau}K$. For $\mu(a)m_k\in\widetilde{E}_{(st)^*}$ we have

$$\tilde{\gamma}_s\tilde{\gamma}_t(\mu(a)m_k)=\tilde{\gamma}_s(\mu(\gamma_t(a))m_{tkt^*})=\mu(\gamma_s(\gamma_t(a))m_{stkt^*s^*}=\tilde{\gamma}_{st}(\mu(a)m_k),$$

so to show that $\tilde{\gamma}$ is a homomorphism from S to $\operatorname{PAut}A\times_{\gamma,\tau}K$ we only need to check that $\operatorname{dom}\tilde{\gamma}_s\tilde{\gamma}_t=\operatorname{dom}\tilde{\gamma}_{st}$. First note that if $T\in\widetilde{A}_t$ then $T=\lim_i T_i$ for some $T_i\in\operatorname{span}\{\mu(a)m_k : a\in A_k, k\in K, kk^*\leqslant tt^*\}$. Hence, $\mu(1_{M(A_t)})T=\lim_i\mu(1_{M(A_t)})T_i=T$ since for $a\in A_k$, $k\in K$ and $kk^*\leqslant tt^*$ we have $A_k\subset A_t$ and so

$$\mu(1_{M(A_t)})\mu(a)m_k=\mu(1_{M(A_t)}a)m_k=\mu(a)m_k.$$

Now, if $T\in\operatorname{dom}\tilde{\gamma}_s\tilde{\gamma}_t$ then $\tilde{\gamma}_t(T)=\lim_i T_i$ for some $T_i\in\operatorname{span}\{\mu(a)m_k : a\in A_k, k\in K, kk^*\leqslant s^*s\}$. Then

$$T=m_t^*\mu(1_{M(A_t)})\tilde{\gamma}_t(T)m_t=\lim_i m_t^*\mu(1_{M(A_t)})T_im_t,$$

so to see that $T\in\operatorname{dom}\tilde{\gamma}_{st}$, it suffices to show that if $a\in A_k$, $k\in K$ and $kk^*\leqslant s^*s$, then

$$m_t^*\mu(1_{M(A_t))}\mu(a)m_km_t=\mu(\gamma_{t^*}(1_{M(A_t)}a))m_{t^*kt}\in\widetilde{A}_{(st)^*}.$$

This is true since $\gamma_{t^*}(1_{M(A_t)}a) \in A_{t^*k} = A_{t^*kk^*t}$ and $t^*kk^*t \leqslant t^*s^*st = (st)^*st$.

To see that $\operatorname{dom}\tilde{\gamma}_s\tilde{\gamma}_t \supset \operatorname{dom}\tilde{\gamma}_{st}$, first note that $\widetilde{A}_{t^*} \supset \widetilde{A}_{(st)^*}$ since $kk^* \leqslant (st)^*st$ implies $kk^* \leqslant t^*s^*st \leqslant t^*t$. So it suffices to show that if $a \in A_k$, $k \in K$ and $kk^* \leqslant (st)^*st$ then

$$\tilde{\gamma}_t(\mu(a)m_k = m_t\mu(a)m_km_t^* = \mu(\gamma_t(a))m_{tkt^*} \in \widetilde{A}_{s^*}.$$

But this follows since

$$\gamma_t(a) \in A_{tk} = A_{tkk^*t^*} = A_{tkt^*tk^*t^*} = A_{tkt^*(tkt^*)^*} = A_{tkt^*}$$

and $tkt^*(tkt^*)^* = tkk^*t^* \leqslant tt^*s^*stt^* \leqslant s^*s$. We used the fact that $tkk^*t^* = tkt^*tk^*t^*$. This is true because tkk^*t^* is the unique idempotent in $[t][kk^*][t^*]$ and $tkt^*tk^*t^*$ is the unique idempotent in $[t][k][t^*t][k^*][t^*] = [t][kk^*][t^*t][kk^*][t^*] = [t][kk^*][t^*]$.

Next we show that $\tilde{\tau}_k$ is a unitary multiplier of $\widetilde{E}_k$. If $a \in E_l$ for some $l \in K$ with $ll^* \leqslant kk^*$ then we have $\mu(a)m_lm_k = \mu(a)m_{lk} \in \widetilde{E}_k$ since $lk(lk)^* = ll^*$ and so $lk(lk)^* \leqslant kk^*$ and $a \in E_l = E_{ll^*} = E_{lk(lk)^*} = E_{lk}$. We also have $m_k\mu(a)m_l = (\mu(\tau_l^*a^*)m_{k^*})^* \in \widetilde{E}_k$ since $\tau_l^*a^* \in E_l$ and so $\tau_l^*a^* \in E_l = E_{kk^*l} = \gamma_{kk^*}(E_{kk^*}E_l) \subset E_{kk^*} = E_k$. Hence $\tilde{\tau}_k$ is a multiplier of $\widetilde{E}_k$. The multiplier m_k is clearly unitary with inverse m_k^*.

To check Definition 6.1 (i) let $k \in K$ and $\mu(a)m_l \in \widetilde{E}_{k^*}$. Then $a \in E_l$ for some $l \in K$ such that $l^*l \leqslant k^*k$, so

$$\tilde{\gamma}_k(\mu(a)m_l) = m_k\mu(a)m_lm_k^* = \tilde{\tau}_k\mu(a)m_l\tilde{\tau}_k^*.$$

Since for $s \in S$ and $k \in K$ with $k^*k \leqslant s^*s$ we have

$$\tilde{\gamma}_s(\tilde{\tau}_k) = \tilde{\gamma}_s(m_k) = m_sm_km_s^* = m_{sks^*} = \tilde{\tau}_{sks^*},$$

Definition 6.1 (ii) is verified and so we are finished showing that $(\tilde{\gamma}, \tilde{\tau})$ is a Green twisted action.

Next we find a correspondence between covariant representations of our actions. First let (Π, v) be a covariant representation of $(\tilde{\gamma}, \tilde{\tau})$. Then $\Pi = \pi \times_\tau u$ for some covariant representation (π, u) of $(\gamma|K, \tau)$. We show that (π, v) is a covariant representation of (γ, τ) by checking the conditions of Definition 6.3. Condition (i) follows from the calculation

$$\begin{aligned} v_s\pi(a)v_s^* &= v_s\Pi(\mu(a)m_{s^*s})v_s^* = \Pi(\tilde{\gamma}_s(\mu(a)m_{s^*s})) \\ &= \Pi(\gamma_s(a)m_{ss^*ss^*}) = \pi(\gamma_s(a)). \end{aligned}$$

Since v_s has final space

$$\begin{aligned}\Pi(\widetilde{E}_s)H &= \overline{\operatorname{span}}\,\{\Pi(\mu(E_k)m_k) : k \in K, kk^* \leqslant ss^*\}\\ &= \Pi(\mu(E_{ss^*})m_{ss^*}) = \pi(E_s)\,,\end{aligned}$$

v_s has the required initial and final spaces. The equality $v_n = \Pi(\tilde{\tau}_n) = \Pi(m_n) = \Pi(\mu(\tau_n)) = \pi(\tau_n)$ for all $n \in N$ shows that (π, v) preserves the twist.

Next we show that if (π, v) is a covariant representation of (γ, τ), then $(\pi \times_\tau v|K, v)$ is a covariant representation of $(\tilde{\gamma}, \tilde{\tau})$. Condition (a) follows from the calculation

$$\begin{aligned}v_s\big(\pi \times_\tau v|K\big)(\mu(a)m_k)v_s^* &= v_s\pi(a)v_kv_s^* = v_s\pi(a)v_s^*v_sv_kv_s^*\\ &= \pi(\gamma_s(a))v_{sks^*} = \big(\pi \times_\tau v|K\big)(\mu(\gamma_s(a))m_{sks^*})\\ &= \big(\pi \times_\tau v|K\big)(\tilde{\gamma}_s(\mu(a)m_k))\ ,\end{aligned}$$

where $a \in E_k$ for some $k \in K$ satisfying $kk^* \leqslant ss^*$. Conditions (ii) and (iii) are clearly satisfied.

Now we investigate the ranges of the corresponding representations. If $(\pi \times_\tau u, v)$ is a covariant representation of $(\tilde{\gamma}, \tilde{\tau})$ and (π, v) is a covariant representation of (γ, τ) then for $a \in E_s$ and $s \in S$ we have

$$\big(\pi \times_\tau v\big)(\mu(a)m_s) = \pi(a)v_s = \pi(a)u_{ss^*}v_s = \big((\pi \times_\tau u) \times_{\tilde{\tau}} v\big)(\mu(a)m_{ss^*}\delta_s).$$

Hence the range of $\pi \times_\tau v$ is contained in the range of $(\pi \times_\tau u) \times v$ since $\mu(a)m_{ss^*} \in \widetilde{E}_s$. On the other hand, for $\mu(a)m_k \in \widetilde{E}_{s^*}$, that is, $a \in E_k$ for some $k \in K$ with $kk^* \leqslant ss^*$, we have

$$\big((\pi \times_\tau u) \times_{\tilde{\tau}} v\big)(\mu(a)m_k\delta_s) = \pi(a)\big(\pi \times_\tau u\big)(\tilde{\tau}_k)v_s = \pi(a)v_kv_s = \big(\pi \times_\tau v\big)(\mu(a)m_{ks}).$$

Hence the range of $(\pi \times_\tau u) \times v$ is contained in the range of $\pi \times_\tau v$.

Now starting with a faithful representation $\rho \times_\tau z$ of $A \times_{\gamma,\tau} S$ we know that $(\rho \times_\tau z|K) \times_{\tilde{\tau}} z$ is a representation of $(A \times_{\gamma,\tau} K) \times_{\tilde{\gamma},\tilde{\tau}} S$ and $\Psi = (\rho \times_\tau z)^{-1} \circ (\rho \times_\tau z|K) \times_{\tilde{\tau}} z$ is a surjective homomorphism. Similarly, starting with a faithful representation $(\pi \times_\tau u) \times_{\tilde{\tau}} v$ of $(A \times_{\gamma,\tau} K) \times_{\tilde{\gamma},\tilde{\tau}} S$ we know that $\pi \times_\tau v$ is a representation of $A \times_{\gamma,\tau} S$ and $\Phi = ((\pi \times_\tau u) \times_{\tilde{\tau}} v)^{-1} \circ (\pi \times_\tau v)$ is a surjective homomorphism. The relationship among these maps is expressed by the commutative diagram:

$$\begin{array}{ccc} A \times_{\gamma,\tau} S & \overset{\rho \times_\tau z}{\hookrightarrow} & \operatorname{im}(\rho \times_\tau z)\\ {\scriptstyle \pi \times_\tau v}\big\downarrow & & \big\uparrow{\scriptstyle (\rho \times_\tau z|K) \times_{\tilde{\tau}} z}\\ \operatorname{im}((\pi \times_\tau u) \times_{\tilde{\tau}} v) & \underset{(\pi \times_\tau u) \times_{\tilde{\tau}} v}{\hookleftarrow} & (A \times_{\gamma,\tau} K) \times_{\tilde{\gamma},\tilde{\tau}} S. \end{array}$$

We show that $\Psi \circ \Phi$ is the identity map, which implies that Φ is an isomorphism. For $a \in E_s$ we have

$$\begin{aligned}\Psi \circ \Phi(a\delta_s) &= \Psi \circ ((\pi \times_\tau u) \times_{\tilde{\tau}} v)^{-1}(\pi(a)v_s) = \Psi(\mu(a)m_{ss^*}\delta_s)\\ &= (\rho \times_\tau z)^{-1}(\rho(a)z_{ss^*}z_s) = \mu(a)m_s. \quad ∎\end{aligned}$$

Recall [22] that if β is the canonical action of the inverse semigroup S on its semilattice E (Example 2.6), then $C^*(S) \cong C^*(E) \times_\beta S$. Also recall that if β is the trivial action of S on $\mathbb{C}$, that is, β_s is the identity for all $s \in S$, then $C^*(G_S) \cong \mathbb{C} \times_\beta S$, where G_S is the maximal group homomorphic image of S. These results can be extended to Green twisted inverse semigroup actions. Similar results will be given for Busby-Smith twisted actions in Proposition 9.2.

Proposition 8.2. *Let S be an inverse semigroup with idempotent semilattice E, and let K be a normal Clifford subsemigroup of S. For all $s \in S$ let E_s be the closed span of $\{k \in K : kk^* \leqslant ss^*\}$ in $C^*(K)$, and define $\alpha_s : E_{s^*} \to E_s$ by $\alpha_s(a) = sas^*$. For $k \in K$ let $\sigma_k = k$. Then $(C^*(K), S, K, \alpha, \sigma)$ is a Green twisted action and $C^*(S) \cong C^*(K) \times_{\alpha,\sigma} S$.*

Proof. Let β be the canonical action of S on its semilattice E. The result can be obtained by following through the isomorphism chain

$$C^*(S) \cong C^*(E) \times_\beta S \cong (C^*(E) \times_\beta K) \times_{\tilde{\beta},\tilde{\tau}} S \cong C^*(K) \times_{\alpha,\sigma} S,$$

using Theorem 8.1. ∎

Proposition 8.3. *Let S be an inverse semigroup with idempotent semilattice E, and let K be a normal Clifford subsemigroup of S. For all $s \in S$ let E_s be the closed span in $C^*(G_K)$ of $\{[k] : k \in K, kk^* \leqslant ss^*\}$, where G_K is the maximal group homomorphic image of K and $[k]$ is the canonical image of k in $C^*(G_K)$, and define $\alpha_s : E_{s^*} \to E_s$ by $\alpha_s(a) = sas^*$. For $k \in K$ let $\sigma_k = [k]$. Then $(C^*(G_K), S, K, \alpha, \sigma)$ is a Green twisted action and $C^*(G_S) \cong C^*(G_K) \times_{\alpha,\sigma} S$, where G_S is the maximal group homomorphic image of S.*

Proof. Let β be the trivial action of S on $\mathbb{C}$. The result can be obtained by following through the isomorphism chain

$$C^*(G_S) \cong \mathbb{C} \times_\beta S \cong (\mathbb{C} \times_\beta K) \times_{\tilde{\beta},\tilde{\tau}} S \cong C^*(K) \times_{\alpha,\sigma} S,$$

using Theorem 8.1. ∎

Recall from [20] that the Cuntz algebra is the C^*-algebra $C^*(O_n)$ of the Cuntz groupoid O_n. Paterson recently found a connection between groupoid C^*-algebras and inverse semigroup crossed products [18]. The details of this connection will be included in his upcoming book. Using his results it is possible to identify $C^*(O_n)$ with a crossed product of $C_0(O_n^0)$ by the Cuntz inverse semigroup $\mathcal{O}_n$.

Since the only normal Clifford subsemigroup of the Cuntz inverse semigroup is its idempotent semilattice, there is no nontrivial decomposition of this crossed product. This is further evidence of the rigidity of the Cuntz relations. The question remains whether the Cuntz-Krieger inverse semigroups ([10]) have normal Clifford subsemigroups supplying a possible decomposition of the Cuntz-Krieger algebras.

9. DECOMPOSITION OF BUSBY-SMITH TWISTED ACTIONS

Since every Busby-Smith twisted action corresponds to a Green twisted action, we can apply our Decomposition Theorem 8.1 to Busby-Smith twisted actions.

THEOREM 9.1. *If (A, T, β, w) is a Busby-Smith twisted action and L is a normal Clifford subsemigroup of T with an order-preserving cross-section $c : T/L \to T$, then the crossed product can be decomposed as an iterated Busby-Smith twisted crossed product*

$$A \times_{\beta,w} T \cong (A \times_{\beta,w} L) \times_{\tilde{\beta},\tilde{w}} T/L.$$

Proof. Let (A, S, N, γ, τ) be the Green twisted action corresponding to (A, T, β, w), constructed in Lemma 7.3. Let $\sim_L$ denote the idempotent-separating congruence on T determined by L. Define a normal Clifford subsemigroup

$$K = \{u_t \partial_t : u_t \in UM(E_t),\, t \in L\}$$

of S. The corresponding congruence relation on S is given by

$$u\partial_r \sim_K v\partial_t \quad \text{if and only if} \quad r \sim_L t.$$

By Theorem 8.1 there is a Green twisted action $(A \times_{\gamma,\tau} K, S, K, \tilde{\gamma}, \tilde{\tau})$ such that $A \times_{\gamma,\tau} S$ is isomorphic to $(A \times_{\gamma,\tau} K) \times_{\tilde{\gamma},\tilde{\tau}} S$. Define a cross-section

$$d : S/K \to S, \qquad d([u_t\partial_t]) = 1_{M(E_{c([t])})}\partial_{c([t])}.$$

To see that d is order-preserving, let $[u_s\partial_s] \leqslant [u_t\partial_t]$ in S/K. Then $[u_s\partial_s] = [u_t\partial_t * (u_s\partial_s)^* * u_s\partial_s]$ and so $s \sim_L ts^*s$. This means $[s] \leqslant [t]$ in T/L and so $d([u_s\partial_s]) \leqslant d([u_t\partial_t])$ because

$$\begin{aligned} d([u_t\partial_t])d([u_t\partial_t])^*d([u_t\partial_t]) &= 1_{M(E_{c([t])})}\partial_{c([t])} * 1_{M(E_{c([t])^*})}\partial_{c([t])^*c([t])} \\ &= w_{c([t]),c([s])^*c([s])}\partial_{c([t])c([s])^*c([s])} \\ &= 1_{M(E_{c([s])})}\partial_{c([s])}. \end{aligned}$$

Hence by Theorem 7.1 there is a Busby-Smith twisted action $(A \times_{\gamma,\tau} K, S/K, \tilde{\alpha}, \tilde{u})$ such that $(A \times_{\gamma,\tau} K) \times_{\tilde{\gamma},\tilde{\tau}} S$ and $(A \times_{\gamma,\tau} K) \times_{\tilde{\alpha},\tilde{u}} S/K$ are isomorphic. By Theorem 7.6 there is an isomorphism $\rho : A \times_{\beta,w} L \to A \times_{\gamma,\tau} K$, and it is easy to see that

$$\varphi : T/L \to S/K, \qquad \varphi([t]) = [1_{M(E_t)}\partial_t]$$

is an isomorphism. So if for all $s, t \in T/L$ we define

$$\tilde{\beta}_s = \rho^{-1} \circ \tilde{\alpha}_{\varphi(s)} \circ \rho \quad \text{and} \quad \tilde{w}_{s,t} = \rho^{-1}(\tilde{u}_{\varphi(s),\varphi(t)}),$$

then $(A \times_{\beta,w} L, T/L, \tilde{\beta}, w)$ is clearly a Busby-Smith twisted action, which is conjugate to $(A \times_{\gamma,\tau} K, S/K, \tilde{\alpha}, \tilde{u})$. Thus we have the following chain of isomorphisms

$$\begin{aligned} A \times_{\beta,w} T &\cong A \times_{\gamma,\tau} S \cong (A \times_{\gamma,\tau} K) \times_{\tilde{\gamma},\tilde{\tau}} S \cong (A \times_{\gamma,\tau} K) \times_{\tilde{\alpha},\tilde{u}} S/K \\ &\cong (A \times_{\beta,w} L) \times_{\tilde{\beta},\tilde{w}} T/L. \quad \blacksquare \end{aligned}$$

Using Theorem 7.1 we get analogs of Propositions 8.2 and 8.3 for Busby-Smith twisted actions:

PROPOSITION 9.2. *If $(C^*(N), T/N, \beta, w)$ is the Busby-Smith twisted action of Proposition* 2.5*, then*

$$C^*(T) \cong C^*(N) \times_{\beta,w} T/N.$$

PROPOSITION 9.3. *Let S be an inverse semigroup and let N be a normal Clifford subsemigroup with an order-preserving cross-section from S/N to S. Then*

$$C^*(G_S) \cong C^*(G_N) \times_{\beta,w} S/N.$$

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