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THE AUTOMORPHISM GROUPS OF RATIONAL ROTATION ALGEBRAS

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Abstract. Let A_θ be the universal C^*-algebra generated by two unitaries $U,\ V$ with $VU = \rho UV$, where $\rho = \mathrm{e}^{2\pi\mathrm{i}\theta}$ and θ is rational. Let $\mathrm{Aut}A_\theta$ be the group of $*$-automorphisms of A_θ. It is shown that if $\theta \neq 1/2$ then the image of the natural map from $\mathrm{Aut}A_\theta$ to $\mathrm{Homeo}\mathbb{T}^2$ is the subgroup $\mathrm{Homeo}_+\mathbb{T}^2$ of orientation preserving homeomorphisms of the torus $\mathbb{T}^2$. Hence there exist exact sequences

$$0 \to \mathrm{Inn}\,A_\theta \to \mathrm{Aut}\,A_\theta \to \mathrm{Homeo}\,\mathbb{T}^2 \to 0$$

when $\theta = 1/2$ and

$$0 \to \mathrm{Inn}\,A_\theta \to \mathrm{Aut}\,A_\theta \to \mathrm{Homeo}\,_+\mathbb{T}^2 \to 0$$

when $\theta \neq 1/2$, where $\mathrm{Inn}A_\theta$ is the group of inner automorphisms.

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A rational rotation C^*-algebra is the universal C^*-algebra A_θ generated by a pair U, V of unitaries with $VU = \rho UV$, where $\rho = \mathrm{e}^{2\pi\mathrm{i}\theta}$ and $\theta = p/q$ is rational (with p, q coprime and $0 \leqslant p < q$). A convenient description of A_θ was given in [1], based on a similar description in [4]. This description utilises the $q \times q$ matrices

$$U_0 = \begin{pmatrix} 1 & & & \\ & \rho & & \\ & & \ddots & \\ & & & \rho^{q-1} \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} 0 & I_{q-1} \\ 1 & 0 \end{pmatrix}$$

and the associated matrices $W_1 = U_0^{p'}$ and $W_2 = V_0^{p''}$ where $0 < p',\ p'' < q$, $pp' \equiv -1 \pmod q$ and $pp'' \equiv 1 \pmod q$. Explicitly

$$W_1 = \begin{pmatrix} 1 & & & \\ & \omega^{-1} & & \\ & & \ddots & \\ & & & \omega^{-(q-1)} \end{pmatrix} \quad \text{and} \quad W_2 = \begin{pmatrix} 0 & I_{q-p''} \\ I_{p''} & 0 \end{pmatrix}$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/q}$. The description of A_θ in terms of W_1 and W_2 is then

$$A_\theta = \Big\{ f \in C(\mathbb{R}^2, M_q) : f(\lambda + m, \mu + n) = W_1^n W_2^m f(\lambda, \mu) W_2^{m*} W_1^{n*} \\ \text{for all } (\lambda, \mu) \in \mathbb{R}^2 \text{ and all } (m, n) \in \mathbb{Z}^2 \Big\}.$$

It sometimes proves convenient to specify elements of A_θ by their restriction to $[0,1]^2$, as was done in [1].

In this description, the generating unitaries U and V are given by $U(\lambda, \mu) = \mathrm{e}^{2\pi\mathrm{i}\lambda/q} U_0$ and $V(\lambda, \mu) = \mathrm{e}^{2\pi\mathrm{i}\mu/q} V_0$. The elements U^q and V^q generate the centre $\mathrm{Z}A_\theta$ of A_θ, which is identified with

$$\{ f \in C(\mathbb{R}^2, \mathbb{C}) : f(\lambda + m, \mu + n) = f(\lambda, \mu) \quad \text{for all } (\lambda, \mu) \in \mathbb{R}^2 \text{ and all } (m, n) \in \mathbb{Z}^2 \}$$

and hence to $C(\mathbb{T}^2, \mathbb{C})$, where $\mathbb{T}^2$ denotes the 2-dimensional torus.

Each automorphism α of A_θ restricts to an automorphism of $\mathrm{Z}A_\theta$ and hence gives rise to a homeomorphism $\tilde{\alpha}$ of $\mathbb{T}^2$ such that $(\alpha f)(x) = f(\tilde{\alpha}^{-1} x)$ for each $x \in \mathbb{T}^2$. Let $\sigma : \operatorname{Aut} A_\theta \to \operatorname{Homeo} \mathbb{T}^2$ be the associated group homomorphism, with $\sigma(\alpha) = \tilde{\alpha}$. It is a simple consequence of 2.19 of [6] that the kernel of σ is $\mathrm{Inn} A_\theta$, the group of inner automorphisms of A_θ. The main purpose of the present note is to describe the range of σ and hence to obtain an exact sequence for $\operatorname{Aut} A_\theta$.

The fact, established in Theorem 2.22 of [6], that each element $\sigma(\alpha) = \tilde{\alpha}$ fixes the Dixmier-Douady class $\delta(A_\theta) \in H^3(\mathbb{T}^2, \mathbb{Z})$ is no restriction since, by Theorem 3.21 of [5], $H^3(\mathbb{T}^2, \mathbb{Z}) = 0$.

It follows easily from Lemmas 3.1 and 3.2 of [4] and the fact that the image of σ is a subgroup that this image is all of $\mathrm{Homeo}\mathbb{T}^2$ when $\theta = 1/2$ and contains $\mathrm{Homeo}_+\mathbb{T}^2$, the set of orientation preserving homeomorphisms, when $\theta \neq 1/2$. It will now be shown that in the latter case every element in the image of σ is orientation preserving.

LEMMA 1. *If α is an automorphism of A_θ with $(\alpha f)(\lambda, \mu) = f(\mu, \lambda)$ for all $f \in \mathrm{Z}A_\theta$ then there exists a continuous family $(\lambda, \mu) \mapsto X_{\lambda,\mu}$ of unitaries in M_q such that $(\alpha f)(\lambda, \mu) = X_{\lambda,\mu} f(\mu, \lambda) X^*_{\lambda,\mu}$ for all $(\lambda, \mu) \in [0, 1]^2$ and all $f \in A_\theta$.*

Proof. Construct three overlapping, symmetric subsets $R,\ S,\ T$ to cover $[0, 1]^2$ by $R = \{(\lambda, \mu) : \lambda + \mu \geqslant 5/4\}$, $S = \{(\lambda, \mu) : 2/3 \leqslant \lambda + \mu \leqslant 4/3\}$ and $T = \{(\lambda, \mu) : \lambda + \mu \leqslant 3/4\}$. Then the restriction maps from A_θ into $C(R, M_q)$ and $C(T, M_q)$ are surjective, whereas the image of the restriction map from A_θ into $C(S, M_q)$ is $\{f \in C(S, M_q) : f(0, 1) = W_2^* W_1 f(1, 0) W_1^* W_2\}$.

If α is an automorphism of A_θ then an automorphism α_R is well-defined on $C(R, M_q)$ by the formula $\alpha_R f_R = (\alpha f)_R$, where f_R denotes the restriction to R of $f \in A_\theta$. To see this, assume that $f_R = g_R$. If $(\alpha(f - g))_R \neq 0$ then there exists $h \in \mathrm{Z}A_\theta$, supported on R, with $\alpha(f - g)h \neq 0$. However, by the symmetry of R, $\alpha^{-1}(h)$ is supported on R and hence $\alpha^{-1}(\alpha(f - g)h) = (f - g)\alpha^{-1}(h) = 0$, giving a contradiction. Similar arguments then show that automorphisms α_S, α_T are defined by $\alpha_S f_S = (\alpha f)_S$ and $\alpha_T f_T = (\alpha f)_T$.

An automorphism β_R of $C(R, M_q)$ is defined by $(\beta_R f)(\lambda, \mu) = f(\mu, \lambda)$ and the same formula defines an automorphism β_T of $C(T, M_q)$. In the case of the restriction to S, an automorphism β_S is defined by $(\beta_S f)(\lambda, \mu) = U_{\lambda-\mu} f(\mu, \lambda) U^*_{\lambda-\mu}$ where $\{U_t : -1 \leqslant t \leqslant 1\}$ is a continuous path of unitaries joining $U_{-1} = (W_2^* W_1)^2$ to $U_1 = I$.

The automorphisms $\alpha_R \beta_R^{-1},\ \alpha_S \beta_S^{-1},\ \alpha_T \beta_T^{-1}$ are inner, by the results of 2.19 of [6]; this uses the facts, from Corollary 3.8 of [5], that $H^2(R, \mathbb{Z}) = 0,\ H^2(\tilde{S}, \mathbb{Z}) = 0$ and $H^2(T, \mathbb{Z}) = 0$, where $\tilde{S}$ denotes S with the points $(0, 1)$ and $(1, 0)$ identified. Hence there are continuous families $(\lambda, \mu) \mapsto X_{\lambda,\mu},\ (\lambda, \mu) \mapsto Y_{\lambda,\mu}$ and $(\lambda, \mu) \mapsto Z_{\lambda,\mu}$ of unitaries defined on $R,\ S,\ T$ respectively such that

$$\begin{aligned} (\alpha f)(\lambda, \mu) &= X_{\lambda,\mu} f(\mu, \lambda) X^*_{\lambda,\mu} && \text{for } (\lambda, \mu) \in R, \\ (\alpha f)(\lambda, \mu) &= Y_{\lambda,\mu} f(\mu, \lambda) Y^*_{\lambda,\mu} && \text{for } (\lambda, \mu) \in S \text{ and} \\ (\alpha f)(\lambda, \mu) &= Z_{\lambda,\mu} f(\mu, \lambda) Z^*_{\lambda,\mu} && \text{for } (\lambda, \mu) \in T. \end{aligned}$$

There then exists a continuous scalar valued family $(\lambda,\mu)\mapsto g_{\lambda,\mu}$ on $R\cap S$ such that $g_{\lambda,\mu}Y_{\lambda,\mu}=X_{\lambda,\mu}$ on $R\cap S$. Let this family be extended to S and define $X_{\lambda,\mu}=g_{\lambda,\mu}Y_{\lambda,\mu}$ for $(\lambda,\mu)\in S$. After a similar extension to T, a family of unitaries with the required properties is obtained. ∎

Lemma 2. *If there exists a continuous family $(\lambda,\mu)\mapsto X_{\lambda,\mu}$ of unitaries in M_q, for $(\lambda,\mu)\in[0,1]^2$, such that $(\lambda,\mu)\mapsto X_{\lambda,\mu}f(\mu,\lambda)X^*_{\lambda,\mu}$ is an element of A_θ for each $f\in A_\theta$, then $\theta=1/2$.*

Proof. If a continuous family of unitaries exists as in the statement of the lemma, then the restriction of this family to the boundary of the square is a closed path homotopic to a point and therefore the restriction of $(\lambda,\mu)\mapsto\det X_{\lambda,\mu}$ to the boundary of the square has winding number zero. The consequences of this will now be explored.

From the conditions $X_{\lambda,1}f(1,\lambda)X^*_{\lambda,1}=W_1X_{\lambda,0}f(0,\lambda)X^*_{\lambda,0}W_1^*$ and $f(1,\lambda)=W_2f(0,\lambda)W_2^*$ it follows that there exists a continuous scalar valued map $\lambda\mapsto\psi_\lambda$ with $X_{\lambda,1}W_2=\psi_\lambda W_1X_{\lambda,0}$ for each $\lambda\in[0,1]$. Similarly, from $X_{1,\mu}f(\mu,1)X^*_{1,\mu}=W_2X_{0,\mu}f(\mu,0)X^*_{0,\mu}W_2^*$ and $f(\mu,1)=W_1f(\mu,0)W_1^*$, there exists a continuous scalar valued map $\mu\mapsto\varphi_\mu$ with $X_{1,\mu}W_1=\varphi_\mu W_2X_{0,\mu}$ for all $\mu\in[0,1]$. Then $X_{1,1}=\psi_1W_1X_{1,0}W_2^*=\psi_1\varphi_0W_1W_2X_{0,0}W_1^*W_2^*$ and $X_{1,1}=\varphi_1W_2X_{0,1}W_1^*=\varphi_1\psi_0W_2W_1X_{0,0}W_2^*W_1^*=\rho^{2p''p'}\varphi_1\psi_0W_1W_2X_{0,0}W_1^*W_2^*$ (where $W_1=U_0^{p'}$ and $W_2=V_0^{p''}$). Thus $\psi_1\varphi_0=\rho^{2p''p'}\varphi_1\psi_0$.

From $\det X_{\lambda,1}=\psi_\lambda^q\det W_1W_2^*\det X_{\lambda,0}$ and $\det X_{1,\mu}=\varphi_\mu^q\det W_2W_1^*\det X_{0,\mu}$ for each $\lambda,\mu\in[0,1]$ it follows that the winding number of $(\lambda,\mu)\mapsto\det X_{\lambda,\mu}$ around the boundary of the square is equal to that of $\{\psi_\lambda^q/\varphi_\lambda^q:0\leqslant\lambda\leqslant1\}$, which is a closed path because $\psi_1\varphi_0=\rho^{2p''p'}\varphi_1\psi_0$ and hence $\psi_1^q\varphi_0^q=\varphi_1^q\psi_0^q$. However $\{\psi_\lambda/\varphi_\lambda:0\leqslant\lambda\leqslant1\}$ is a path joining ψ_0/φ_0 to $\psi_1/\varphi_1=\rho^{2p''p'}\psi_0/\varphi_0=\mathrm{e}^{4\pi\mathrm{i}p'/q}\psi_0/\varphi_0$. Since the winding number of $\{(\psi_\lambda/\varphi_\lambda)^q:0\leqslant\lambda\leqslant1\}$ is zero it therefore follows that $\mathrm{e}^{4\pi\mathrm{i}p'/q}=1$. From $0<p'<q$ it follows that $p'/q=1/2$ and so $q=2p'$ is even with $2p'p\equiv0\pmod q$. However, by the definition of p', $p'p\equiv-1\pmod q$ so $q=2$ and therefore $p=1$. ∎

The following result can now be proved.

Proposition 3. *Let A_θ be the rational rotation algebra corresponding to $\theta=p/q$, with p,q coprime and $0\leqslant p<q$. If $\theta\neq1/2$, then* $\mathrm{Homeo}_+\mathbb{T}^2$ *is the range of the natural map $\sigma:\mathrm{Aut}\,A_\theta\to\mathrm{Homeo}\,\mathbb{T}^2$.*

Proof. If there exists $\alpha\in\mathrm{Aut}\,A_\theta$ with $\sigma(\alpha)$ orientation reversing then there exists $\beta\in\mathrm{Aut}\,A_\theta$ with $\sigma(\beta)=\sigma(\alpha)\tau$ where $\tau(\lambda,\mu)=(\mu,\lambda)$. Hence $\sigma(\alpha^{-1}\beta)=\tau$, contradicting Lemmas 1 and 2. ∎

When Proposition 3 is combined with the known results of [4] and [6] summarised earlier it gives the existence of exact sequences

$$0 \to \operatorname{Inn} A_\theta \to \operatorname{Aut} A_\theta \xrightarrow{\sigma} \operatorname{Homeo} \mathbb{T}^2 \to 0 \tag{1}$$

when $\theta = 1/2$ and

$$0 \to \operatorname{Inn} A_\theta \to \operatorname{Aut} A_\theta \xrightarrow{\sigma} \operatorname{Homeo}_+ \mathbb{T}^2 \to 0 \tag{2}$$

when $\theta \neq 1/2$.

For each homeomorphism ψ of $\mathbb{T}^2$ there is an associated group automorphism ψ_* of $H_1(\mathbb{T}^2)$. This gives rise, via the identification of $H_1(\mathbb{T}^2)$ with $\mathbb{Z}^2$ described in Example 7.14 of [7], to an element A_ψ of $\mathrm{GL}(2, \mathbb{Z})$, which belongs to $\mathrm{SL}(2, \mathbb{Z})$ when $\psi \in \operatorname{Homeo}_+ \mathbb{T}^2$. The map $\pi : \psi \mapsto A_\psi$ is a group homomorphsim with kernel the set $\operatorname{Homeo}_0 \mathbb{T}^2$ of homeomorphisms homotopic to the identity and so $\pi\sigma$ is a group homomorphism with kernel $K = \{\alpha : \sigma(\alpha) \in \operatorname{Homeo}_0 \mathbb{T}^2\}$. Hence the exact sequences above yield the sequences

$$0 \to K \to \operatorname{Aut} A_\theta \xrightarrow{\pi\sigma} \mathrm{GL}(2, \mathbb{Z}) \to 0 \tag{3}$$

when $\theta = 1/2$ and

$$0 \to K \to \operatorname{Aut} A_\theta \xrightarrow{\pi\sigma} \mathrm{SL}(2, \mathbb{Z}) \to 0 \tag{4}$$

when $\theta \neq 1/2$. In both cases

$$0 \to \operatorname{Inn} A_\theta \to K \to \operatorname{Homeo}_0 \mathbb{T}^2. \tag{5}$$

The exact sequence (4) provides a contrast with the behaviour for irrational θ where, by the results of [3], the corresponding map is onto $\mathrm{GL}(2, \mathbb{Z})$.

Recall from [2] and [8] that the map $A \mapsto \beta_A$, where $\beta_A(U) = \rho^{ac/2} U^a V^c$ and $\beta_A(V) = \rho^{bd/2} U^b V^d$ gives an action of $\mathrm{SL}(2, \mathbb{Z})$ on A_θ, which can easily be adapted to give a splitting of the exact sequence (4). The following result shows that, at least in general, there is no splitting of the exact sequences (1) and (2). It seems likely that the exact sequence (3) also does not split.

Proposition 4. *If q is even there is no element α of order 2 in* $\mathrm{Aut}\,A_\theta$ *with* $\sigma(\alpha)(\lambda,\mu)=(\lambda+1/2,\mu)$ *on* $\mathbb{R}^2/\mathbb{Z}^2$.

Proof. The automorphism β of A_θ defined by $\beta(U)=\mathrm{e}^{\pi\mathrm{i}/q}U$, $\beta(V)=V$ satisfies $(\beta f)(\lambda,\mu)=f(\lambda+1/2,\mu)$ for each $f\in A_\theta$ and each $(\lambda,\mu)\in\mathbb{R}^2$. By 2.19 of [4], any other automorphism α with $\sigma(\alpha)=\sigma(\beta)$ is of the form $(\mathrm{Ad}\,g)\beta$ for some unitary $g\in A_\theta$. If $\alpha^2=\mathrm{id}$ then, for each $f\in A_\theta$ and $(\lambda,\mu)\in\mathbb{R}^2$,

$$\begin{aligned} f(\lambda,\mu)&=g(\lambda,\mu)\beta(\mathrm{Ad}\,g\beta f)(\lambda,\mu)g(\lambda,\mu)^*\\ &=g(\lambda,\mu)g\left(\lambda+\frac{1}{2},\mu\right)f(\lambda+1,\mu)g\left(\lambda+\frac{1}{2},\mu\right)^*g(\lambda,\mu)^*\\ &=g(\lambda,\mu)g\left(\lambda+\frac{1}{2},\mu\right)W_2f(\lambda,\mu)W_2^*g\left(\lambda+\frac{1}{2},\mu\right)^*g(\lambda,\mu)^*\end{aligned}$$

and hence $g(\lambda,\mu)g(\lambda+1/2,\mu)W_2=h_{\lambda,\mu}1$ for some $h_{\lambda,\mu}\in\mathbb{C}$. Thus $(\lambda,\mu)\mapsto h_{\lambda,\mu}W_2^*=g\beta(g)(\lambda,\mu)$ and so $(\lambda,\mu)\mapsto h_{\lambda,\mu}W_2^*$ belongs to A_θ. Noting that each element of A_θ can be written uniquely in the form $\sum f_{ij}U^iV^j$, where $f_{ij}\in \mathrm{Z}A_\theta$ for $0\leqslant i,j\leqslant q-1$, it follows that $h_{\lambda,\mu}=f(\lambda,\mu)\mathrm{e}^{-2\pi\mathrm{i}\mu p''/q}$ for some $f\in\mathrm{Z}A_\theta$. Thus

$$g\left(\lambda+\frac{1}{2},\mu\right)=g(\lambda,\mu)^*f(\lambda,\mu)\mathrm{e}^{-2\pi\mathrm{i}\mu p''/q}W_2^*$$

for each $\lambda,\mu\in[0,1]^2$. Furthermore, since g is a unitary, $|f(\lambda,\mu)|=1$ for each λ,μ.

As in the proof of Lemma 2, the winding number of $\det(g(\lambda,\mu))$ around the path $\{(0,\mu):0\leqslant\mu\leqslant1\}\cup\{(\lambda,1):0\leqslant\lambda\leqslant1/2\}\cup\{(1/2,1-\mu):0\leqslant\mu\leqslant1\}\cup\{(1-\lambda,0):1/2\leqslant\lambda\leqslant1\}$ must be zero. However $g(\lambda,1)=W_1g(\lambda,0)W_1^*$, so the appropriate condition is that the winding number k of $\mu\mapsto\det g(0,\mu)$ for $0\leqslant\mu\leqslant1$ is equal to that of $\mu\mapsto\det(g(0,\mu)^*f(0,\mu)\mathrm{e}^{-2\pi\mathrm{i}\mu p''/q})$, i.e. to $\mu\mapsto\mathrm{e}^{-2\pi\mathrm{i}\mu p''}f(0,\mu)^q\det(g(0,\mu)^*)$. (Note that $\det g(0,0)=\det g(0,1)$ and $\det g(1/2,0)=\det g(1/2,1)$.) Thus $k=-k-p''+\ell q$ where ℓ is the winding number of $\mu\mapsto f(0,\mu)$. Hence, if q is even, then so is p'', which contradicts the definition $pp''\equiv1\ (\mathrm{mod}\,q)$. ∎

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