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ON CONDITIONAL EXPECTATIONS OF FINITE INDEX

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Abstract. For a conditional expectation E on a (unital) C^*-algebra A there exists a real number $K \geqslant 1$ such that the mapping $K \cdot E - \mathrm{id}_A$ is positive if and only if there exists a real number $L \geqslant 1$ such that the mapping $L \cdot E - \mathrm{id}_A$ is completely positive, among other equivalent conditions. The estimate $(\min K) \leqslant (\min L) \leqslant (\min K)[\min K]$ is valid, where $[\cdot]$ denotes the entire part of a real number. As a consequence the notion of a "conditional expectation of finite index" is identified with that class of conditional expectations, which extends and completes results of M. Pimsner, S. Popa ([27], [28]), M. Baillet, Y. Denizeau and J.-F. Havet ([6]) and Y. Watatani ([35]) and others.

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Treating conditional expectations on C^*-algebras we follow the definition of M. Takesaki's and Ş. Strătilă's monographs [32], [30]: conditional expectations E on a C^*-algebra A are projections of norm one of A into a C^*-subalgebra $B \subseteq A$ leaving B invariant. Immediate consequences are: E is a B-bimodule map on A, and the extension of faithful conditional expectations E to the bidual Banach space and W^*-algebra A^{**} yield normal faithful conditional expectations onto the canonically normally embedded W^*-subalgebra B^{**} preserving the common identity. Consequently, for faithful conditional expectations on C^*-algebras A, the

C^*-algebra can always supposed to be unital, and $E(1_A) = 1_B = 1_A$, cf. [32] and [5], Lemma 4.1.4 for details.

Studying the literature about conditional expectations of finite index on C^*-algebras we obtain two mainstreams of investigations: one direction is concerned with normal conditional expectations of finite index on W^*-factors and several phenomena which occur in more special settings of this case. The other direction treats the algebraically characterizable case (see Y. Watatani [35], [19]) where the original C^*-algebra A is a finitely generated C^*-module over the (unital) image C^*-algebra B.

A first definition for conditional expectations to be of finite index was given by M. Pimsner and S. Popa ([27, [28]) in the context of W^*-algebras M generalizing results of H. Kosaki ([22]) and V.F.R. Jones ([15]): there has to exist a constant $K \geqslant 1$ such that $(K \cdot E - \mathrm{id}_M)$ is a positive mapping on M. However, attempts to describe the more general situation of conditional expectations on C^*-algebras with arbitrary centers to be "of finite index" in some sense(s) get into difficulties, as the gap of knowledge separating Proposition 3.3 and Theorem 3.5 of the paper [6] on W^*-algebras by M. Baillet, Y. Denizeau and J.-F. Havet shows. For non-trivial centers of W^*-algebras M they obtained that even in the case of normal faithful conditional expectations E on M the index value can be calculated only in situations when there exists a number $L \geqslant 1$ such that the mapping $(L \cdot E - \mathrm{id}_M)$ is completely positive, which seems to be more since the difference $(\min L) = (\min K)^2$ at least appears, cf. Example 1.6. S. Popa showed that for normal conditional expectations $E : M \to N \subseteq M$ the conditions of [6], Proposition 3.3, Theorem 3.5 are equivalent, [28], Theorem 1.1.6, Remark 1.1.7. Y. Watatani's attempt to overcome this difficulty in the algebraic way considering the C^*-algebra A as a finitely generated projective module over the (unital) image C^*-algebra B of the conditional expectation E turned out to be unsuitable to give a general solution of the key problems.

Let us give an example which is characteristic for situations for which the algebraic approach does not work but which are, nevertheless, well-behaved in some sense: let A be the C^*-algebra $C([-1,1])$ all continuous functions on the interval $[-1,1]$. Consider the normal conditional expectation E on A defined by

$$E(f)(x) = \frac{f(x) + f(-x)}{2} \quad \text{for } x \in [-1,1].$$

The image C^*-algebra B can be identified with the set $\{f \in A : f(x) = f(-x)$ for $x \in [-1,1]\}$ inheriting the C^*-structure from A. The Hilbert B-module $\{A, E(\langle \cdot , \cdot \rangle_A)\}$ is not self-dual since the bounded B-linear mapping

$$E(f_0^* f)(x) = \frac{f_0^*(x)f(x) + f_0^*(-x)f(-x)}{2} \quad \text{for } x \in [-1,1],\ f \in A,$$

$$f_0(x) = \begin{cases} 1 & x \in (0,1]; \\ 0 & x = 0; \\ -1 & x \in [-1,0); \end{cases}$$

maps A into B, however f_0 does not belong to A. Consequently, the Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ cannot be finitely generated and projective, and E is not of finite index in the sense of Y. Watatani's algebraic definition. Nevertheless, E has a very good property: the mapping $(2 \cdot E - \mathrm{id}_M)$ is obviously completely positive. Beside this a careful analysis yields the B-reflexivity of the Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$.

Investigating such kind of conditional expectations in some more detail we obtain that the existence of a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive should be sufficient to imply very good properties of E, see [6], Proposition 3.3, [3], [14], Theorem 3.4, 3.5 and [28], Theorem 1.1.6, Remark 1.1.7. We use the notations

$$K(E) = \inf\{K : (K \cdot E - \mathrm{id}_A) \text{ is positive on } A\},$$
$$L(E) = \inf\{L : (L \cdot E - \mathrm{id}_A) \text{ is completely positive on } A\}.$$

If there does not exist any finite number K or L with the properties striven for, then $K(E) = \infty$ or $L(E) = \infty$. To close the gap of knowledge separating Proposition 3.3 and Theorem 3.5 of [6] for the general C^*-case and to obtain the right general definition for conditional expectations on C^*-algebras to be of finite index we show the following fact generalizing the partial results by P. Jolissaint ([14], Theorems 3.4, 3.5), by E. Andruchow and D. Stojanoff ([4], Proposition 2.1, Corollary 2.4) and by S. Popa ([28], Theorem 1.1.6, Remark 1.1.7).

Theorem 1. *Let A, B be C^*-algebras, where B is a C^*-subalgebra of A. Let $E : A \to B \subseteq A$ be a conditional expectation with fixed point set B. Then the following three conditions on E are equivalent:*

(i) *E is faithful and the (right) pre-Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ is complete with respect to the norm $\|E(\langle \cdot, \cdot \rangle_A)\|_B^{1/2}$, where $\langle a, b \rangle_A = a^*b$ for every $a, b \in A$;*

(ii) *There exists a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive;*

(iii) *There exists a number $L \geqslant 1$ such that the mapping $(L \cdot E - \mathrm{id}_A)$ is completely positive.*

If the second condition is valid for some K, then every number $L \geqslant K \cdot [K]$ realizes the third condition, where $[K]$ denotes the entire part of K. In some situations the equality $L(E) = K(E)^2 \in \mathbb{N}$ is valid.

DEFINITION 2. (cf. [22], [27], [28]) Let A and B be C^*-algebras, where B is a C^*-subalgebra of A. Let $E : A \to B \subseteq A$ be a conditional expectation with fixed point set B. If there exists a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive, then E *is of finite index.* The index value can be calculated in the center of either the bidual W^*-algebra A^{**} or its discrete part.

Plan of the proof. We start with the simple observation that the conditional expectation E on the C^*-algebra A can be continued to a normal conditional expectation E^{**} on the bidual W^*-algebra A^{**} of A preserving property (ii) above with the same minimal number $K(E^{**}) = K(E) \geqslant 1$. Normality follows from a general extension property of conditional expectations [5], Lemma 4.1.4, whereas the additional property can be derived from monotonicity or from the up-down-up theorem of G.K. Pedersen ([26], II.2.4), for example.

The next step is explained separately in Proposition 1.3 below: the projection of W^*-algebras to their discrete (i.e. atomic type I) part commutes with normal faithful conditional expectations E of finite index if condition (ii) holds for E, i.e. the discrete part can be considered separately without loss of generality. This leads to a detailed study of discrete W^*-algebras and conditional expectations on them. The theorem will be proven for this class, see Propositions 1.2, 1.3.

To return to the general C^*-case we make use of a theorem of Ch. A. Akemann stating that the $*$-homomorphism of a C^*-algebra A into the discrete part of its bidual W^*-algebra A^{**} which arises as the composition of the canonical embedding of A into A^{**} followed by the projection to the discrete part of A^{**} is an injective $*$-homomorphism, [1], p. 278 and [2], p. 1. A conditional expectation E on A with property (ii) for some real number $K \geqslant 1$ has an extension E^{**} to A^{**} and in particular, to the discrete part of A^{**} where A is faithfully $*$-represented. E^{**} restricted to this $*$-representation of A recovers E and K, and condition (ii) holds for the restriction of E^{**} to the discrete part of A^{**} for the same number $K \geqslant 1$. Since Theorem 1 is fulfilled for E^{**}, it holds for its restriction E to the faithfully represented original C^*-algebra A with the same minimal numbers $K(E), L(E)$, and condition (iii), and the estimates turn out to be valid.

The equivalence of the conditions (i) and (ii) follows from [6], Proposition 3.3 for the W^*-case. To give an independent argument we show how to derive condition (ii) from condition (i). If A is complete with respect to the norm $\|E(\langle \cdot , \cdot \rangle_A)\|^{1/2}$, then there exists a number K such that the inequality $K\|E(x^*x)\| \geqslant \|x^*x\|$ holds for every $x \in A$ by the general theory of Banach spaces. Set $x = a(\varepsilon + E(a^*a))^{-1/2}$ with $a \in A$ and observe that

$$(\varepsilon + E(a^*a))^{-\frac{1}{2}} \cdot E(a^*a) \cdot (\varepsilon + E(a^*a))^{-\frac{1}{2}} \leqslant 1_A.$$

This implies the inequality $K \cdot 1_A \geqslant (\varepsilon + E(a^*a))^{-1/2} a^*a (\varepsilon + E(a^*a))^{-1/2}$, and multiplying by $(\varepsilon + E(a^*a))^{1/2}$ from both sides we obtain $K(\varepsilon + E(a^*a)) \geqslant a^*a$ for every $\varepsilon > 0$, every $a \in A$. This yields condition (ii). The converse is obvious by spectral theory. ∎

In the next four sections we explain some details and consequences of our investigations. The first section is concerned with the property of normal conditional expectations E of finite index to commute with the abstract projection of W^*-algebras to their discrete part, as well as with the proof of Theorem 1 for the discrete W^*-case. In Section 2 we show that those mappings E commute with the abstract projections of W^*-algebras to their finite, infinite, continuous type I, type II_1, type II_∞ and type III parts, too. This extends results of S. Sakai ([29]) and J. Tomiyama ([33]) for general normal conditional expectations. Section three is devoted to the investigation of the index value of general conditional expectations of finite index and of Jones' tower constructions. There exists an index value for $E : A \to B \subseteq A$ inside the center of A iff the discrete part of the index value of the extended normal conditional expectation E^{**} inside the center of the bidual W^*-algebra A^{**} of A belongs to the canonical embedding of A into the discrete part of A^{**}. For normal conditional expectations $E : M \to N \subseteq M$ of finite index on W^*-algebras M the Jones' tower always exists. The Jones' tower exists in the general C^*-case if $\mathrm{Ind}(E)$ is contained in the center of B. The last section collects some interpretations of the results obtained in terms of non-commutative topology and some dimension estimates in the case of finite centers.

1. THE DISCRETE W^*-CASE

We want to consider *discrete* W^*-algebras, i.e. W^*-algebras for which the supremum of all minimal projections contained equals their identity. First of all, we have to recall the structure of normal conditional expectations on type I factors over separable and finite dimensional Hilbert spaces the image of which is a (type I) subfactor. As a partial case we describe normal states. Subsequently we will make use of the inner structure of these mappings.

Example 1.1. Let E be a normal conditional expectation on the set $M = B(l_2)$ of all bounded linear operators on the separable Hilbert space l_2 or on $M = M_n(\mathbb{C})$, respectively. Suppose, its image is a W^*-subfactor $N \subseteq M$. Then the Hilbert space l_2 (or $\mathbb{C}^n$) can be decomposed as the tensor product of two Hilbert spaces H_o and K_o, $l_2 = H_\mathrm{o} \otimes K_\mathrm{o}$, ($\mathbb{C}^n = H_\mathrm{o} \otimes K_\mathrm{o}$, resp.) such that the representation of M as the W^*-tensor product $B(H_o) \otimes B(K_o)$ allows the description of E by the formula

$$E(T \otimes S) = T \otimes \mathrm{Tr}(C \cdot S)\mathrm{id}_{K_o}$$

for a unique trace class operator $C \in \mathrm{B}(K_o)_h^+$ with $\mathrm{Tr}(C) = 1$ and for arbitrary $T \in B(H_o)$, $S \in B(K_o)$ on elementary tensors, (cf. [8], Lemma 3.3.1, 3.3.2, [34], Proposition 2.4). Note that E is faithful if and only if zero is not an eigenvalue of the operator C. To satisfy the condition (ii) of Theorem 1 the inequality $\|T\| \leqslant K\|E(T)\|$ has to be valid for a positive number $K \in \mathbb{N}$ (to be fixed) and for every operator $T \in M$. Denote by $P_n \in B(K_o)$ that projection which maps K_o onto the eigenspace of the n-th eigenvalue λ_n of the trace class operator C. Then this inequality can be rewritten as

$$1 = \|\mathrm{id}_{H_o} \otimes P_n\| \leqslant K\|E(\mathrm{id}_{H_o} \otimes P_n)\| = \lambda_n K. \tag{1.1}$$

If K_o is infinite dimensional, then the eigenvalues of C form a sequence converging to zero. Such an assumption contradicts to the finiteness and universality of the constant K in the inequality (1.1). Hence, E satisfies condition (ii) of Theorem 1 if and only if the dimension of K_o is finite.

The image $N = E(M)$ can be identified with the W^*-subalgebra $B(H_o) \otimes \mathrm{id}_{K_o} \simeq B(H_o)$. Therefore, the set $\{\mathrm{id}_{H_o} \otimes P_{n,i} : i = 1, \ldots, \dim(P_n(K_o)), n \in \mathbb{N}\} \cup$ $\{$minimal partial isometries between them$\}$ is an orthogonal basis of the Hilbert N-module $\{M, E(\langle \cdot, \cdot \rangle_M)\}$, where the projections $\{P_{n,i} : i \in \mathbb{N}\}$ vary over a set of pairwise orthogonal minimal subprojections of the corresponding eigenspace projection P_n of C. In case we assume Theorem 1 (ii) to be valid, the index of E exists in the sense of Y. Watatani ([35]) and equals

$$\mathrm{Ind}(E) = \left(\sum_{n=1}^{\dim(K_o)} (\lambda_n)^{-1} \right) \cdot \mathrm{id}_M.$$

Note that this special structure of some conditional expectations of finite index on σ-finite type I factors is not preserved for similar ones over non-separable Hilbert spaces, as we shall obtain during our investigations. The structure of E described above implies that $\mathrm{Ind}(E) \in \{1\} \cup [4, \infty)$, whereas in the non-separable case the discrete series of possible index values between one and four arises additionally.

Now we are going to investigate normal conditional expectations on W^*-algebras satisfying condition (ii) of Theorem 1 and their restrictions to the discrete part. We start with a proposition which was obtained by J. Tomiyama ([33]) investigating normal conditional expectations. For completeness, we include a short proof of it which is different from J. Tomiyama's:

Proposition 1.2. (J. Tomiyama) *Let M be a discrete W^*-algebra and $E : M \to N \subseteq M$ be a normal faithful conditional expectation with respect to N. Then N is a discrete W^*-algebra.*

Proof. Let $P_N \in N$ be the projection which maps N onto its discrete part. The projection P_N is known to be contained in the center of N. Since $E(1_M - P_N)$ is positive and belongs to the center of N too, it has a carrier projection $r \in N$ which is contained in the center of N. Note that $(1_M - P_N)$ and N commute inside M. Consider the W^*-subalgebra $(1_M - P_N)M(1_M - P_N)$ of M. Let r_1 be the smallest central projection of N satisfying the equality $nr_1(1_M - P_N) = n(1_M - P_N)$ for every $n \in N$. Then $(1_M - P_N) = (1_M - P_N)r_1$ and $r_1 \leqslant r$. Applying E we obtain $E(1_M - P_N) = E(1_M - P_N)r_1$ and hence, $r_1 = r$. That means, the mapping $nr \to nr(1_M - P_N)$ is a faithful normal $*$-homomorphism from Nr into $(1_M - P_N)M(1_M - P_N)$.

Furthermore, E maps the discrete W^*-algebra $(1_M - P_N)M(1_M - P_N)$ faithfully onto Nr, and Nr does not possess any discrete part. Fix a normal state f on Nr with support projection p_f. The centralizer $c(f)$ of the state f inside $p_f(Nr)p_f$ coincides with the fixed point algebra of the associated KMS-group σ_f on $p_f(Nr)p_f$, and there exists a normal faithful conditional expectation $E' : p_f(Nr)p_f \to c(f)$ such that $f \equiv f \circ E'$ on $p_f(Nr)p_f$, cf. [31]. Composing the normal conditional expectations E, E' and the normal state f we obtain a normal state $f \circ E' \circ E = f \circ E$ on the discrete W^*-algebra $p_f(1_M - P_N)M(1_M - P_N)p_f$ which possesses a non-discrete centralizer since $c(f)$ is non-discrete and contained in it. This is a contradiction, since the intersection of the centralizer of the normal state $f \circ E$ with $p_f(1_M - P_N)M(1_M - P_N)p_f$ has to be isomorphic to a direct integral of orthogonal direct sums of finite matrix algebras over the discrete center of $p_f(1_M - P_N)M(1_M - P_N)p_f$, which can be supposed to be finite by a special choice of f. (This structure of the centralizers can be derived from the structure of normal states on discrete W^*-factors, cf. Example 1.1 and [32].) Consequently, $r = 0$, $1_M = P_N = 1_N$ inside M, and N has to be discrete. ∎

We want to remark that we have always to be careful of the structure of the centers of M and of N and of their interrelation. For example, set $M = \sum_{k \in \mathbb{Z}} M_{2,(k)}(\mathbb{C})$ and consider the normal faithful conditional expectation

$$M = \sum_{k \in \mathbb{Z}} M_{2,(k)}(\mathbb{C}) \to N = \sum_{k \in \mathbb{Z}} \mathbb{C}_{(k)}$$

$$\left\{ \begin{pmatrix} a_{(k)} & b_{(k)} \\ c_{(k)} & d_{(k)} \end{pmatrix}_{(k)} : k \in \mathbb{Z} \right\} \to \left\{ \lambda_{(k)} = \frac{d_{(k)} + a_{(k+1)}}{2} : k \in \mathbb{Z} \right\},$$

where the normal embedding of N into M is defined by the formula

$$\left\{ \begin{pmatrix} \lambda_{(k-1)} & 0 \\ 0 & \lambda_{(k)} \end{pmatrix}_{(k)} : k \in \mathbb{Z} \right\}.$$

The centers of M and of N are both isomorphic to $l_\infty(\mathbb{Z})$, however their intersection is the smallest possible one — the complex multiples of the identity. By the way, the index of this mapping equals $4 \cdot 1_M$. The most unpleasant circumstance is the non-existence of a common central direct integral decomposition of M and N commuting with the conditional expectation $E : M \to N$ described above, causing difficulties in proving. As the referee pointed out to us the following proposition was independently observed by S. Popa ([28], 1.1.2):

PROPOSITION 1.3. *Let M be a W^*-algebra and $E : M \to N \subseteq M$ be a normal conditional expectation leaving N invariant, for which there exists a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_M)$ is a positive mapping. Then the essential part of the preimage of the discrete part of N is contained in the discrete part of M, and the image of the discrete part of M is exactly the discrete part of N. That is, the projection to the discrete part of W^*-algebras commutes with normal conditional expectations E on them possessing this additional property.*

Proof. Recall that E is faithful by the additional condition. Let $p \in N$ be a minimal projection of N and let $q \leqslant p$ be a projection of M. Since E is faithful, $E(q) \neq 0$ and the inequality $0 < E(q) \leqslant E(p) = p$ holds. Since p is minimal inside N, there exists a number $\mu \in (0, 1]$ such that $E(q) = \mu p$. That is,

$$E(\mu^{-1} \cdot q) = p,$$

and because of the additional condition on E we obtain

$$K \cdot p \geqslant K \cdot E(\mu^{-1} q) \geqslant \mu^{-1} q > 0$$

and the estimate $\mu \geqslant K^{-1}$.

Suppose, $p \in N$ can be decomposed into a sum of pairwise orthogonal (arbitrary) projections $\{q_\alpha : \alpha \in I\}$ inside M. Obviously, $q_\alpha \leqslant p$ for every $\alpha \in I$, and

$$p = E(p) = \sum_{\alpha \in I} E(q_\alpha) = \left(\sum_{\alpha \in I} \mu_\alpha \right) p \geqslant \left(\sum_{\alpha \in I} K^{-1}_{(\alpha)} \right) p.$$

Consequently, the sum has to be finite and the maximal number of non-trivial summands is $[K]$, the entire part of K. We see that the projections $\{q_\alpha : \alpha \in I\} \in M$ possess only finitely many subprojections and hence, belong to the discrete part of M. This shows that the discrete part of $N = E(M)$ is completely contained in the discrete part of M.

By the way, we obtain that every minimal projection $p \in N$ has to be represented only in a small part of the central direct integral decomposition of M with finite-dimensional center. Extending $p \in N \subseteq M$ by the partial isometries of $N \subseteq M$ every W^*-factor block of N turns out to be represented in a part of the central direct integral decomposition of M with finite-dimensional center. Conversely, E maps each W^*-factor block of M into a part of the central direct integral decomposition of N with finite-dimensional center, cf. [6], Corollary 3.18.

Let P_N denote the projection which maps N onto its discrete part. Subject to the first part of the present proof, P_N belongs to the discrete part of M since P_N is the supremum of all minimal projections of N. Applying Proposition 1.2 to the discrete part $P_M \cdot M$ of M (where P_M denotes the (central) projection of M which carries its discrete part) we obtain the equality $P_M = P_N$ in the center of M by the faithfulness of E. ∎

Corollary 1.4. *Let M be a discrete W^*-algebra and $E : M \to N \subseteq M$ be a normal conditional expectation for which there exists a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_M)$ is positive. Then:*

(i) *The center of M is finite-dimensional if and only if the center of N is finite-dimensional.*

(ii) *The positive part of the preimage of a minimal projection $p \in N$ is contained in the finite dimensional W^*-subalgebra pMp of M generated by the minimal projections of M which are subprojections of p. The dimension of pMp is at most $[K]^2$, where $[K]$ denotes the entire part of K.*

(iii) *For a minimal projection $p \in N$ the minimal projections generating pMp are mapped by E to the set $\{\mu \cdot p : K^{-1} \leqslant \mu \leqslant 1\}$. The image of pMp is $\mathbb{C}p$, i. e. E acts on pMp as a normal state.*

(iv) *If q_1, q_2 are two orthogonal minimal projections of M, then either $E(q_1Mq_1) \equiv E(q_2Mq_2)$ or $E(q_1Mq_1) \cap E(q_2Mq_2) = \{0\}$.*

We restrict our attention to the situation when N is a type I W^*-factor. Then the center of M has to be finite dimensional. If we consider M as a self-dual (right) Hilbert N-module with the N-valued inner product $E(\langle \cdot, \cdot \rangle_M)$, (where $\langle a, b \rangle_M = a^*b$ for $a, b \in M$), we can try to count the index of E applying [6], Theorem 3.5. We have to find a suitable Hilbert N-module basis $\{m_\alpha : \alpha \in I\}$ of M and to count the sum $\sum_\alpha m_\alpha m_\alpha^*$. If this sum is finite in M in the sense of w^*-convergence for some basis, then it is the same for every other Hilbert N-module basis of M.

We start with a decomposition of the identity $1_N = 1_M$ into a w*-sum of pairwise orthogonal minimal projections $\{p_\nu\} \subset N$, and with a subdecomposition of this sum into a w*-sum of pairwise orthogonal minimal projections $\{q_\alpha : \alpha \in I\} \subset M$. In our special setting, a suitable basis contains this maximal set of pairwise orthogonal minimal projections $\{q_\alpha\}$ of M weighted down by the inverse of the number μ_α arising by the equality $E(q_\alpha) = \mu_\alpha p_\nu$ for some minimal projection p_ν of N with $\mu_\alpha \in [K^{-1}, 1]$ (compare with Example 1.1):

$$\left\{\sqrt{\mu_\alpha^{-1}} \cdot q_\alpha : \alpha \in I\right\} \subseteq \text{basis}.$$

If M is commutative, then the Hilbert N-module basis of M is complete, and

$$\operatorname{Ind}(E) = \sum_{\alpha \in I} \sqrt{\mu_\alpha^{-1}} \cdot q_\alpha \cdot \left(\sqrt{\mu_\alpha^{-1}} \cdot q_\alpha\right)^* \leqslant K \sum_{\alpha \in I} q_\alpha \leqslant K \cdot 1_M$$

by [6], Theorem 3.5. But, if M is non-commutative, then we have to add all the minimal partial isometries $\{u_\beta : \beta \in J\}$ of M each connecting two minimal projections, but also weighted down by the inverse of the number μ_β arising by the equality $E(u_\beta^* u_\beta) = \mu_\beta p_\nu$ for a minimal projection p_ν of N, $\mu_\beta \in [K^{-1}, 1]$, (compare with Example 1.1 again):

$$\left\{\sqrt{\mu_\alpha^{-1}} \cdot q_\alpha : \alpha \in I\right\} \cup \left\{\sqrt{\mu_\beta^{-1}} \cdot u_\beta : \beta \in J\right\} \equiv \text{basis}.$$

The Hilbert N-module basis of M is complete, but rather big. However, based on special Hilbert W^*-module isomorphisms ([10]) we can reduce the generating set of partial isometries $\{u_\beta : \beta \in J\} \subset M$. For this aim, we define equivalence classes of them by the rule: $u_\beta \sim u_\gamma$ if and only if $q_\alpha u_\beta = q_\alpha u_\gamma \neq 0$ for some minimal projection $q_\alpha \in M$ of our choice and $u_\beta = u_\gamma v$ for some partial isometry $v \in N$ linking the projections $\{p_\nu\} \subset N$. By application of Hilbert W^*-module isomorphisms we obtain a sufficiently large set of generators of M as right Hilbert N-module if we select only one representative of each equivalence class supplementary to our choice of minimal projections. As the result, the number of selected partial isometries $\{u_\beta : \beta \in J\} \subset M$ satisfying $q_\alpha u_\beta = u_\beta$ for a fixed minimal projection $q_\alpha \in M$ is limited by $([K] - 1)$.

Now we can easily estimate the index value in norm to check the w*-convergence of the appropriate series inside M:

$$\begin{aligned} \operatorname{Ind}(E) &= \sum_{\alpha \in I} \sqrt{\mu_\alpha^{-1}} \cdot q_\alpha \cdot \left(\sqrt{\mu_\alpha^{-1}} \cdot q_\alpha\right)^* + \sum_{\beta \in J} \sqrt{\mu_\beta^{-1}} \cdot u_\beta \cdot \left(\sqrt{\mu_\beta^{-1}} \cdot u_\beta\right)^* \\ &\leqslant K \left(\sum_{\alpha \in I} q_\alpha + \sum_{\beta \in J} u_\beta u_\beta^* \right) \leqslant (K \cdot [K]) \sum_{\alpha \in I} q_\alpha \leqslant (K \cdot [K]) \cdot 1_M. \end{aligned} \tag{1.2}$$

Consequently, $\|\mathrm{Ind}(E)\|_M \leqslant K \cdot [K]$, and E is of finite index in the sense of M. Baillet, Y. Denizeau and J.-F. Havet ([6], Theorem 3.5). Furthermore, the mapping $((K \cdot [K]) \cdot E - \mathrm{id}_M)$ is completely positive by the same theorem. ∎

Proposition 1.5. *If M is supposed to be a discrete W^*-algebra, then Theorem 1 is valid.*

For a proof we have only to realize that the equality $L(E) = \|\mathrm{Ind}(E)\|$ is valid by [6], Theorem 3.5, (a)–(b), and the value $\|\mathrm{Ind}(E)\|$ was estimated by the number $(K(E) \cdot [K(E)])$ from above by (1.2). (Of course, in special situations this estimate may be far from being sharp.)

Example 1.6. ([6], Examples 3.7) Let N be a W^*-algebra. Let M be the C^*-algebra of all 2×2-matrices with entries from N. The embedding of N into M can be described as that subset of M consisting of the N-multiples of the identity matrix. Consider the conditional expectation

$$E \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denote by e_{jk} those elements of M which possess only one non-zero element at that place where the j-th row and the k-th column intersect, the identity of N. Then the set $\{\sqrt{2} \cdot e_{jk} : j,k = 1,2\}$ is a Hilbert N-module basis of M, and the index of E has the value $\mathrm{Ind}(E) = 4 \cdot \mathrm{id}_{M_2}$.

If N is commutative, then the mapping $(K(i \circ E) - \mathrm{id}_M)$ is positive for $K \geqslant 2$ already, whereas it is completely positive for $L \geqslant 4$ only. That is, the minimal constants of item (ii) and of item (iii) of Theorem 1 may be different, in general, and in our special setting $4 = L(E) = K(E)^2 = 2^2 \in \mathbb{N}$.

On the contrary, if N is a type I_∞, type II_1 or separably representable infinite W^*-factor, then $L(E) = K(E)$, see [6], [35]. That is, our estimate of the number $L(E)$ by the number $K(E)$ does not give a formula to calculate the value of $L(E)$ precisely, in general.

Furthermore, a general estimate $L(E) \leqslant [K(E)]^2$ is not true: modify the preceding example with $\lambda \in (0,1)$ setting

$$E_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\lambda a + (1-\lambda)d) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $K(E_\lambda) = \max\{\lambda^{-1}, (1-\lambda)^{-1}\}$ and $\mathrm{Ind}(E_\lambda) = L(E_\lambda) = \lambda^{-1} + (1-\lambda)^{-1}$. For every small $\varepsilon > 0$ and the choice $\lambda(\varepsilon) = 1/(2+\varepsilon)$ the assumption $L(E_\lambda) \leqslant [K(E_\lambda)]^2$ leads to the contradiction $\varepsilon^2 < 0$ since

$$L(E_{\lambda(\varepsilon)}) = 4 + \frac{\varepsilon^2}{1+\varepsilon}, \quad K(E_{\lambda(\varepsilon)}) = 2 + \varepsilon.$$

COROLLARY 1.7. *In contrast to the separable situation (see Example* 1.1*) there exist type* I *factors M on non-separable Hilbert spaces and normal conditional expectations E of finite index on them with factor image realizing the discrete index series between one and four.*

Proof. Start with the hyperfinite type II_1 factor A and the appropriate (normal) conditional expectation E of finite index on it the image W^*-algebra of which is a factor again, (see [15], [27]). Turn to the discrete part M of the bidual W^*-algebra A^{**} of A. The extended normal conditional expectation E^{**} has the same invariant $K(E^{**}) = K(E)$, hence, also its restriction to the discrete part M of A^{**} does. But M is a type I factor, i.e. M has not any non-trivial central projection since A has not any non-trivial two-sided norm-closed ideal. Indeed, every central projection $p \in M$ can be decomposed into an orthogonal sum of minimal projections $\{p_\alpha\}$. For every projection p_α the orthogonal complement $(1_M - p_\alpha)$ is the carrier projection of the w*-closure of an appropriate maximal norm-closed left (or right) ideal I_α of A. Consequently, $(1_M - p) \in Z(M)$ should be the carrier projection of the intersection $\bigcap_\alpha I_\alpha \equiv \bigcap_\alpha I_\alpha^* \subseteq A$ which should be a norm-closed two-sided ideal, and the claim is obvious. ∎

As a corollary we obtain the structure of the relative commutant in the general C^*-setting:

COROLLARY 1.8. *Let $E : A \to B \subseteq A$ be a conditional expectation of finite index on a C^*-algebra A. The relative commutant of B inside A is a subhomogeneous C^*-algebra of finite type, i.e. it is a C^*-subalgebra of some matrix algebra $M_n(C)$ with $n < \infty$ and with entries from a commutative W^*-algebra C, for example $C = Z((B^{**})_{\mathrm{discr}})$.*

Proof. Consider the restriction of the (normal) extended conditional expectation E^{**} of E to the discrete part of A^{**}. Since E maps the relative commutant $B' \cap A$ of B with respect to A to the center of B , the same is true for the restricted mapping E^{**} and the appropriate injectively $*$-represented C^*-subalgebras of the discrete part of A^{**}. Note that $(B^{**})' \cap A^{**} \supseteq B' \cap A$. The center of the discrete part of B^{**} is discrete, and the preimage of every minimal projection of it is a matrix algebra of dimension lower-equal $[K(E)]^2$ by Corollary 1.4(i). Consequently, the injectively $*$-represented C^*-algebra $B' \cap A$ is contained in the C^*-algebra of all $n \times n$-matrices with $n = [K(E)]^2$ and with entries from the center of the discrete part of B^{**}. ∎

2. DECOMPOSITION PRESERVING PROPERTIES

By the work of S. Sakai ([29]) and of J. Tomiyama ([33]) it has been known that normal conditional expectations preserve semi-finiteness, type I and discreteness of W^*-algebras as properties of their image W^*-subalgebras. We want to show that normal conditional expectations E of finite index not only commute with the abstract projection of W^*-algebras to their discrete part, but also with many canonical abstract projections of W^*-algebras to other parts of them. We obtain that such mappings E possess a decomposition into their restrictions to the appropriate type components of the W^*-algebras involved. The same observations were made by S. Popa ([28], 1.1.2) independently, a fact which was brought to our attention by the referee. Another result on projections which belongs to both $Z(N)$ and $Z(M)$ can be found in [7], Theorem 1.

We would like to point out that the word "preimage" subsequently refers to that part of the preimage of N via E which is spanned by the positive part of it inside M. Of course, the kernel of E is spread out over all parts of the central decomposition of M into W^*-types, in general.

Proposition 2.1. *Let M be a W^*-algebra and $E : M \to N \subseteq M$ be a normal faithful conditional expectation leaving N invariant. Then the image of the finite part of M is exactly the finite part of N.*

If E is additionally of finite index, then the preimage of the finite (resp., infinite) part of N is contained in the finite (resp., infinite) part of M. That is, the projection to the finite (resp., infinite) part of W^*-algebras commutes with normal conditional expectations E of finite index. Moreover, E commutes with the projection to the type II_1 part, to the continuous type $\mathrm{I}_{\mathrm{fin}}$ part and with the projection to the non-discrete infinite part of W^*-algebras.

Proof. Suppose E to be faithful. Denote a normal faithful center-valued normalized trace on the finite part $M \cdot P_{\mathrm{fin}}$ of M by $\mathrm{tr}(\cdot)$. Suppose $r \in N$ is the central carrier projection of $E(P_{\mathrm{fin}})$. By the same arguments as in the proof of Proposition 1.2, the C^*-subalgebras Nr and $N \cdot E(P_{\mathrm{fin}})$ are identical W^*-subalgebras. The mapping $E \circ \mathrm{tr} \circ E$ is a normal faithful positive normalized mapping on Nr taking values in the center of Nr. Beside this, $E \circ \mathrm{tr} \circ E$ is tracial on Nr. Commutative W^*-algebras are generally known to possess a decomposition into σ-finite W^*-subalgebras like

$$\sum_{\alpha \in I}^{\oplus} \mathbb{C}_{(\alpha)} \oplus \sum_{\beta \in J}^{\oplus} L^{\infty}([0,1], \lambda)_{(\beta)}$$

up to $*$-isomorphism by [32], III.1.22 and [14], Proposition 1.14.10, where λ denotes the Lebesgue measure on $[0,1]$. Considering the direct integral decomposition of

Nr over its center and taking faithful normal states on the σ-finite components of $Z(Nr)$ to be composed with the mapping $E \circ \mathrm{tr} \circ E$, we obtain a separating set of tracial normal states on Nr. Consequently, Nr is finite, cf. [5], Proposition 6.5.15.

The second statement is an easy consequence of the proof of [14], Theorem 1.6 (1) if we consider arbitrary finite W^*-algebras N and apply the criterion on finiteness of W^*-algebras of Sakai ([5], Theorem 6.3.12) in this generalized setting.

The continuous type $\mathrm{I}_{\mathrm{fin}}$ part of M (i.e. the part with continuous center and finite-dimensional fibres in the appropriate direct integral decomposition over it) is mapped to the continuous type $\mathrm{I}_{\mathrm{fin}}$ part of N since E maps the type I part of M to the type I part of N by the results of J. Tomiyama ([33]) and since E preserves the finite part and the discrete part by our previous results. Conversely, the projection of N to its continuous type $\mathrm{I}_{\mathrm{fin}}$ part can be decomposed into the sum of finitely many pairwise orthogonal abelian projections $\{p_n : n = 1, \ldots, m\} \subseteq N$. Since

$$E(x) = \sum_{n_1,n_2=1}^{m} p_{n_1} E(x) p_{n_2} = \sum_{n_1,n_2=1}^{m} E(p_{n_1} x p_{n_2})$$

for all $E(x)$ of the continuous type $\mathrm{I}_{\mathrm{fin}}$ part of N, and since the W^*-subalgebras $\{p_{n_1} N p_{n_2} : n_1, n_2 = 1, \ldots, m\}$ are commutative, the preimage of the continuous type $\mathrm{I}_{\mathrm{fin}}$ part of N has to be of finite subhomogeneous type inside M by Corollary 1.8. That is, it is contained in the continuous type $\mathrm{I}_{\mathrm{fin}}$ part of M since E preserves the discrete and the finite W^*-parts.

Since the projections of W^*-algebras to their discrete, their finite and their infinite parts, respectively, commute with conditional expectations E of finite index on them and because they are realized by central projections P_{discr}, $P_{I_{\mathrm{contfin}}}$, P_{fin} and P_{infin} of $Z(M) \cap Z(N)$ the mapping E commutes with the product projections $P_{\mathrm{II}_1} = P_{\mathrm{fin}}(1_M - P_{\mathrm{discr}} - P_{I_{\mathrm{contfin}}})$ and $P_{\mathrm{infin}}(1_M - P_{\mathrm{discr}})$, too. ∎

PROPOSITION 2.2. *Let M be a W^*-algebra and $E : M \to N \subseteq M$ be a normal faithful conditional expectation leaving N invariant. If E is of finite index, then:*

(i) *The image of the continuous type* I_∞ *part of M is contained in the continuous type* I_∞ *part of N, and the preimage of the continuous type* I_∞ *part of N is contained in the continuous type* I_∞ *part of M.*

(ii) *The image of the type* II_∞ *part of M is contained in the type* II_∞ *part of N, and the preimage of the type* II_∞ *part of N is contained in the type* II_∞ *part of M.*

(iii) *The image of the type* III *part of* M *is contained in the type* III *part of* N*, and the preimage of the type* III *part of* N *is contained in the type* III *part of* M.

Proof. By [32], Lemma V.2.29, the preimage of the type III part of N is completely contained in the type III part of M even if the faithful conditional expectation E is not assumed to be of finite index. This implies that the semi-finite part of M is mapped by E into the semi-finite part of N.

Let E be of finite index. The type II_∞ part of M is denoted by $M \cdot P_{\mathrm{II}_\infty}$. There exists a faithful semi-finite normal center-valued trace tr on $M \cdot P_{\mathrm{II}_\infty}$, cf. [5], Proposition 6.5.8. Since E commutes with the projection of W^*-algebras to their finite part and to their discrete part by Proposition 1.3 and 2.1, the W^*-subalgebra $E(M \cdot P_{\mathrm{II}_\infty})$ is contained in the direct sum of the continuous type I_∞ and of the type II_∞ part of N. Consider the mappings

$$\{f \circ E \circ \mathrm{tr} \circ E \,:\, f \in Z(N)_*\}$$

on the W^*-subalgebra $E(M \cdot P_{\mathrm{II}_\infty}) \subseteq N$. These mappings form a faithful family of semi-finite normal traces on the positive cone of $E(M \cdot P_{\mathrm{II}_\infty})$. By [5], Proposition 6.5.7 the W^*-algebra $E(M \cdot P_{\mathrm{II}_\infty})$ has to be semi-finite, and hence, of type II_∞.

Conversely, let N be of type II_∞ and M arbitrary. Let $p \in N$ be a finite projection. Then the preimage of the type II_1 W^*-subalgebra pNp is exactly the W^*-subalgebra pMp. By Proposition 2.1, pMp has to be of type II_1, too, since E is supposed to be of finite index. By [5], Theorem 6.5.10 there exists an increasing net $\{p_\alpha : \alpha \in I\}$ of finite projections in N with strong limit 1_N inside N. Since E is faithful and normal, the least upper bound of the net $\{p_\alpha\} \subset N \subseteq M$ with respect to M is $1_M = 1_N$, again. Consequently, M equals the w*-closure of the union of all type II_1 W^*-subalgebras $\{p_\alpha M p_\alpha : \alpha \in I\}$, and it has to be of type II_∞.

Now let N be of continuous type I_∞ and M arbitrary. Let $p \in N$ be an abelian projection. The preimage of the commutative W^*-subalgebra pNp of N equals the W^*-subalgebra $pMp \subseteq M$. It has to be of subhomogeneous type by Corollary 1.8, and hence, it has to be contained in a matrix algebra of finite size with entries from some commutative W^*-algebra. That is, pMp is of continuous type $\mathrm{I}_{\mathrm{fin}}$ and possesses sufficiently many abelian subprojections of p with least upper bound p. Since the set of all abelian projections $\{p_\alpha : \alpha \in I\}$ of N has least upper bound 1_N with respect to N, the least upper bound of them with respect to M has to be $1_M = 1_N$, too, since E is normal and faithful. Then M is the w*-closure of the union of continuous type $\mathrm{I}_{\mathrm{fin}}$ W^*-subalgebras $\{p_\alpha M p_\alpha : \alpha \in I\}$. The set of abelian projections of M has least upper bound 1_M and hence, M is of continuous type I_∞.

Summing up the considerations above, E maps the type III part of M into the type III part of N, and the continuous type I_∞ part of M into the continuous type I_∞ part of N. ∎

Since $\mathrm{End}_N(M)$ and N have a common center and since the W^*-type of $p \cdot \mathrm{End}_N(M)$ is the same as for pN for any suitable $p \in Z(N)$, we conclude that for finite W^*-algebras M with finite-dimensional center and normal conditional expectations of finite index $E : M \to N \subseteq M$ the W^*-algebra M is always a finitely generated projective N-module in the standard decomposition of M as a Hilbert N-module via E, cf. [14], Theorem 2.2 (2).

By the examples of P. H. Loi ([23]) we conclude that conditional expectations of finite index on type III_λ factors, $0 < \lambda < 1$, can map them to type III_{λ^m} subfactors for any natural number $m \in \mathbb{N}$. Consequently, we can expect type preserving properties for conditional expectations of finite index on type III W^*-algebras only in the type III_0 and III_1 cases, if at all. This question remains unsolved for the time being.

3. THE GENERAL JONES' TOWER CONSTRUCTION

The next step should be the construction of the Jones' tower in the general C^*-case and the estimate of a hopefully existing index value for conditional expectations $E : A \to B \subseteq A$. The general construction of the Jones' tower in the W^*-case was shown by S. Popa ([28], 1.2.2). We are going to obtain a general index notion and a general Jones' tower construction in some specific C^*-cases. Unfortunately, the index value belongs to the discrete part of A^{**} only, in general, and not to the original C^*-algebra A itself.

We consider the conditional expectation $E : A \to B \subseteq A$ of finite index. The C^*-algebra A has the structure of a (right) Hilbert B-module setting $\{A, E(\langle \cdot, \cdot \rangle_A)\}$, where $\langle a, b\rangle_A = a^*b$ for $a, b \in A$ by Theorem 1. The conditional expectation E acts as a bounded (right-)B-linear mapping on A. Furthermore, it can be identified with an elementary "compact" operator $e = \theta_{1_A,1_A}$ on the Hilbert B-module A by the formula $e(x) = \theta_{1_A,1_A}(x) = 1_A \cdot E(1_A^* x)(= E(x))$. The projection e is the first projection to build the Jones' tower. It always exists since the Hilbert B-module B is a direct summand of the Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$.

The set of "compact" B-linear operators $K_B(A)$ on the Hilbert B-module A is defined as the norm closure of the linear hull of the elementary "compact" operators

$$\{\theta_{a_1,a_2} : \theta_{a_1,a_2}(x) = a_2 \cdot E(a_1^* x), \quad (x \in A)\}.$$

There is also a (faithful) $*$-representation π_E of A in the set of all bounded adjointable module operators $\mathrm{End}_B^*(A)$ on A by multiplication operators from the left

$$\{\pi_E(a) : \pi_E(a)(x) = ax, \quad (x \in A)\}.$$

Note that not every bounded B-linear operator on the Hilbert B-module $\{A, E(\langle \cdot , \cdot \rangle_A)\}$ should possess an adjoint bounded B-linear operator on it, cf. [10], Theorems 5.6, 6.8, for criteria. The unital C^*-algebra $\mathrm{End}_B^*(A)$ of all bounded adjointable B-linear operators on the Hilbert B-module A is the multiplier C^*-algebra of the C^*-algebra $\mathrm{K}_B(A)$ by [13], Lemma 16 and [17], Theorem 1. Its center can be identified with the center of B by the formula

$$b \in Z(B) \to \pi'_E(b) \in Z(\mathrm{End}_B^*(A)) \quad \pi'_E(b)(x) = xb, \quad (x \in A).$$

Obviously, $\pi_E(a_2)\theta_{1_A,1_A}\pi_E(a_1^*) = \theta_{a_1,a_2}$ for $a_1, a_2 \in A$, i. e. the emphasized operator e generates all elementary "compact" operators on A by two-sided ideal operations with respect to the $*$-represented C^*-algebra A inside $\mathrm{End}_B^*(A)$. The linear hull of the elementary "compact" module operators on A coincide with the C^*-algebra $K_B(A)$, and the latter coincides with its multiplier C^*-algebra $\mathrm{End}_B^*(A)$ if and only if A is a projective finitely generated B-module, which recovers the algebraic case described by Y. Watatani, cf. [10], Proposition 1.1, [35], [9].

What about a conditional expectation $E_1 : \mathrm{End}_B^*(A) \to \pi_E(A)$? Unfortunately, we can only obtain a finite faithful operator-valued weight

$$F_1 : K_B(A) \to \pi_E(A) \subseteq \mathrm{End}_B^*(A), \quad F_1(\theta_{a_1,a_2}) = \pi_E(a_2 a_1^*).$$

In the algebraically characterizable case, F_1 maps $\mathrm{id}_A \in K_B(A)$ to the index value $\mathrm{Ind}(E)$ of E existing and belonging to the center of A. The index value is greater-equal the identity and hence, invertible. The sought for conditional expectation E_1 arises as $E_1 = \pi_E(\mathrm{Ind}(E)^{-1}) \cdot F_1$ in that case.

But, this easy construction does not work in the general case if the identity operator on the Hilbert B-module A is non-"compact". P. Jolissaint gave a criterion in the W^*-case showing that the right construction can be obtained if and only if the number $L(E)$ is finite and, hence, the index value exists for the normal conditional expectation E, [14], Proposition 1.5 (3). By the results of S. Popa ([28]) or by Theorem 1 this gives the general solution of the problem in the W^*-case. To overcome this difficulty in the general C^*-case we make the following definition for the index value of conditional expectations of finite index and show its correctness afterwards:

DEFINITION 3.1. Let A be a C^*-algebra and $E : A \to B \subseteq A$ be a conditional expectation of finite index leaving B invariant. The index value $\mathrm{Ind}(E)$ of E is the projection of the index value $\mathrm{Ind}(E^{**})$ of the extended conditional expectation $E^{**} : A^{**} \to B^{**} \subseteq A^{**}$ to the discrete part of A^{**}.

THEOREM 3.2. *Let A be a C^*-algebra and $E : A \to B \subseteq A$ be a conditional expectation of finite index leaving B invariant. Then the index value $\mathrm{Ind}(E)$ is contained in the center of A, and there exists a conditional expectation $E_1 : \mathrm{End}^*_B(A) \to \pi_E(A)$ mapping e to $\pi_E(\mathrm{Ind}(E)^{-1})$ if and only if $\mathrm{Ind}(E)$ belongs to the standardly embedded image of A inside the discrete part of A^{**}.*

For normal conditional expectations $E : M \to N \subseteq M$ of finite index on W^*-algebras M the Jones' tower always exists.

The Jones' tower exists in the general C^*-case if $\mathrm{Ind}(E)$ is contained in the center of B.

Proof. First, consider normal conditional expectations $E : M \to N \subseteq M$ of finite index on W^*-algebras M. Then the normal conditional expectation E_1 mapping the W^*-algebra $\mathrm{End}_N(M)$ to $\pi_E(M)$ is faithful, and there exists a number $K(E_1) = \|\mathrm{Ind}(E)^{-1}\|$ such that the mapping $(K(E_1) \cdot E_1 - \mathrm{id}_{\mathrm{End}})$ is positive on $\mathrm{End}_N(M)$, see [6], Theorem 3.5. But, by Theorem 1, E_1 turns out to be of finite index, and the Jones' tower can be built up repeating the basic construction countably many times. This shows that the Jones' tower construction always exists in the W^*-case.

Now consider the general case. Suppose, $\mathrm{Ind}(E)$ belongs to A. The finite faithful operator-valued weight F_1 on the C^*-algebra $K_B(A)$ extends uniquely to a normal finite faithful operator-valued weight $F_{1,**}$ on the C^*-algebra $K_{B^{**}}(A^{**})$ by the way in which E^{**} is derived from E. Obviously, $K_B(A)$ can be considered as a C^*-subalgebra of $K_{B^{**}}(A^{**})$, and $F_{1,**}$ restricted to $K_B(A)$ recovers F_1. If the projection of $\mathrm{Ind}(E^{**})$ to the discrete part of A^{**} is contained in the standard injective $*$-representation of A in the discrete part of A^{**}, then the restriction of $E_{1,**}$ (which exists as a normal conditional expectation from the W^*-algebra $\mathrm{End}_{B^{**}}(A^{**})$ to $\pi_{E^{**}}(A^{**})$) to the C^*-subalgebra $\mathrm{End}^*_B(A)$ of $\mathrm{End}_{B^{**}}(A^{**})$ gives a conditional expectation E_1 from $\mathrm{End}^*_B(A)$ to $\pi_E(A)$ which equals F_1 multiplied by the inverse of the projection of $\mathrm{Ind}(E^{**})$ to the discrete part of A^{**}, which belongs to $Z(A)$ by assumption.

Conversely, the extension F_1^{**} of F_1 to the bidual linear space and W^*-algebra $K_B(A)^{**}$ yields an image $F_1^{**}(\mathrm{id})$ of the identity inside the center of $\pi_E(A^{**})$. If there exists an extension of F_1 to $\mathrm{End}^*_B(A)$ at all, then the projection of $F_1^{**}(\mathrm{id}) \in Z(A^{**})$ to the discrete part of A^{**} should equal $F_1(\mathrm{id}_A)$ because of the canonical embedding of multiplier C^*-algebras into bidual W^*-algebras. Obviously, $F_1(\mathrm{id}_A)$

equals the projection of the index value $\mathrm{Ind}(E^{**}) \in Z(A^{**})$ to the discrete part of A^{**}, i. e. $\mathrm{Ind}(E)$. Consequently, $F_1(\mathrm{id}_A)$ belongs to A if and only if $\mathrm{Ind}(E)$ is contained in the standard injective $*$-representation of A in the discrete part of A^{**}, and $F_1(\mathrm{id}_A)$ equals $\mathrm{Ind}(E)$.

If $\mathrm{Ind}(E)$ is contained in the center of B, then the basic construction of the Jones' tower can be repeated countably many times by [6], Theorem 3.5, 3.10, since $\mathrm{Ind}(E_1) = \mathrm{Ind}(E)$ and the index value is stabilized throughout the tower. ∎

Let us give an example showing that the index value can be outside A, and that $\mathrm{Ind}(E)$ is very different from $\mathrm{Ind}(E^{**})$, in general.

Example 3.3. Let $A = C(\mathbb{S}^1)$ be the C^*-algebra of all continuous functions on the unit circle $\mathbb{S}^1 = \{e^{i\varphi} : \varphi \in [0, 2\pi)\}$, where $\mathbb{S}^1$ is equipped with the usual topology. Consider the conditional expectation

$$E(f)(x) = \frac{f(x) + f(\overline{x})}{2}, \quad (x \in \mathbb{S}^1),$$

for $f \in A$, where $\overline{x}$ denotes the complex conjugate of $x \in S^1$. Obviously, the mapping $(2 \cdot E - \mathrm{id}_A)$ is positive on A and $L(E) = K(E) = 2$. There does not exist any finite quasi-basis in the sense of Y. Watatani. The discrete part of the bidual linear space and W^*-algebra A^{**} of A is isomorphic to $l_\infty(\mathbb{S}^1)$ by the Gel'fand theorem. The formula defining E^{**} is the same as for E. The index can be counted by [6], Theorem 3.5 (but not algebraically, anyway). The value is

$$\mathrm{Ind}(E) = \begin{cases} 2 & : \quad x^2 \neq 1 \\ 1 & : \quad x^2 = 1 \end{cases} \quad \in l_\infty(\mathbb{S}^1),$$

and $\mathrm{Ind}(E) \notin A$. Since A^{**} is commutative (hence, type I) let us have a look on the non-discrete part of it. By the decomposition theory of commutative W^*-algebras it is a direct sum of W^*-algebras $L^\infty([0,1], \lambda)$, where λ denotes the Lebesgue measure, cf. [32]. Consider the canonical embedding $A = \mathrm{C}(\mathbb{S}^1) \subset L^\infty(\mathbb{S}^1, \lambda) \cong L^\infty([0,1], \lambda)$. Again, the index of E^{**} reduced to $L^\infty(\mathbb{S}^1, \lambda)$ can be found, it equals $f(x) \equiv 2$, which is different from the value obtained for the discrete part of $\mathrm{Ind}(E^{**})$.

The previous example is closely related to a Stinespring theorem for conditional expectations. We found a more general result in a recent paper of G.J. Murphy ([25], Theorem 2.4), but the construction is rather different and more complicated because of the greater generality grasped there.

THEOREM 3.4. *Let $E : A \to B \subseteq A$ be a conditional expectation of finite index. Then there exists a Hilbert B-module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$, a $*$-homomorphism $\pi : A \to \mathrm{End}_B(\mathcal{M})$ and a partial isometry V mapping the Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ to $\mathcal{M}$ so that $E(a) = V^* \circ \pi(a) \circ V$ for every $a \in A$ and the linear hull of the set $\{\pi(a_1)(V(a_2)) : a_1, a_2 \in A\}$ is norm-dense in $\mathcal{M}$.*

Proof. First of all we have to construct a suitable right Hilbert B-module $\mathcal{M}$ to represent A on it. We consider A as a right B-module multiplying with elements of B from the right. Start with the algebraic tensor product $A \odot A$ and factorize the resulting linear space by the kernel of the B-valued inner pre-product

$$\langle a \otimes x, b \otimes y \rangle = E(x^* E(a^* b) y), \quad a, b, x, y \in A.$$

The completion of the resulting linear factor space with respect to the norm derived from this B-valued inner product is denoted by $\mathcal{M}$. Define the $*$-homomorphism $\pi : A \to \mathrm{End}_B(\mathcal{M})$ by the formula $\pi(a)(x \otimes y) = ax \otimes y$ for $a, x, y \in A$. It is a faithful representation of A. The partial isometry V mapping the Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ to $\mathcal{M}$ can be obtained by the rule $V(x) = 1_A \otimes x$ for $x \in A$. Respectively, $V^*(x \otimes y) = E(x)y$ for $x, y \in A$. Consequently, for each $a \in A$ the element $V^* \circ \pi(a) \circ V$ acts as the B-linear operator $E(a)$ on A multiplying elements of A by $E(a)$ from the left. ∎

4. FURTHER RESULTS

We would like to describe the local action of conditional expectations of finite index in a similar way as we have been able to do it in the discrete W^*-case in Section one. But, we cannot give such a description on the elements of the C^*-algebra A where the conditional expectation E acts, in general. Using a fundamental principle of non-commutative topology we should switch from the minimal projections of the bidual Banach space and W^*-algebra A^{**} either to the maximal (modular) one-sided ideals of A or to the maximal (modular) hereditary C^*-subalgebras of A or to the pure states of A.

COROLLARY 4.1. *Let A be a C^*-algebra and E be a conditional expectation on it for which there exists a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive. Then the preimage of every maximal (modular, norm-closed) left ideal of the image C^*-algebra $B = E(A) \subseteq A$ contains the set union of at most $[K(E)]$ maximal (modular, norm-closed) left ideals of A with pairwise orthogonal complements of their* w**-carrier projections, where $[K(E)]$ denotes the entire part* *of $K(E)$. The intersection of these maximal left ideals of A with B gives that maximal left ideal of B back we started with.*

Proof. Let $p \in B^{**}$ be minimal. Recall, that the carrier projections of maximal (modular, norm-closed) one-sided ideals of the C^*-algebra A (which are contained in the bidual W^*-algebra A^{**}) correspond to the minimal projections of the bidual W^*-algebra A^{**} one-to-one by taking orthogonal complements. By Corollary 1.4 (i), p is a finite sum of minimal projections $\{q_\alpha\} \in A$, which corresponds to the relation of w*-closed left ideals

$$\bigcap_\alpha q_\alpha A^{**} \equiv pA^{**}$$

for every such decomposition. Moreover, $E(pA^{**}) = pB^{**}$ and $E(q_\alpha A^{**}) \subseteq pB^{**}$ for every index α. Suppose $E(rA^{**}) \subseteq pB^{**}$ for a minimal projection $r \in A^{**}$. Then $E(r) = \mu p$ with $\mu \in (0,1]$ since $p \in B^{**}$ is minimal and E is faithful. This implies $r \leqslant p$, and $r \in pA^{**}p$, which is a finite dimensional C^*-algebra of dimension not greater than $[K(E)]^2$ by Corollary 1.4 (i). Hence, the number of minimal w*-closed left ideals of A^{**} mapped into the minimal w*-closed ideal pB^{**} of B^{**} which possess pairwise orthogonal carrier projections is limited by the number $[K(E)]$, as well as the number of pairwise orthogonal minimal projections $\{q_\alpha\} \in A^{**}$ mapped to a multiple of $p \in B^{**}$. Applying again Akemann's correspondence and observing that E preserves the maximality and the ideal property of maximal one-sided ideals of A, we are done. ∎

Because of the interrelation of pure states, maximal (modular, norm-closed) left ideals and maximal (modular) hereditary C^*-subalgebras for arbitrary C^*-algebras we derive the following fact, cf. [26], [24] and [12]:

Corollary 4.2. *Let A be a C^*-algebra and E be a conditional expectation on it for which there exists a number $K \geqslant 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive. Then the preimage of every maximal (modular) hereditary C^*-subalgebra of $B = E(A) \subseteq A$ contains the set union of at most $[K(E)]$ maximal (modular) hereditary C^*-subalgebras $p_\alpha A^{**} p_\alpha \cap A$ of A with pairwise orthogonal complements of their* w*-*carrier projections p_α.*

Every pure state on B has at most $[K(E)]$ different extensions to pure states on A such that their restrictions to $B \subseteq A$ equal the original pure state and that they possess pairwise orthogonal w*-*carrier projections.*

As a by-product of our considerations we reobtain a fact which was observed in the case of type II_1 factors by V.F.R. Jones (16], p. 6) and which was proven for the general W^*-case by E. Andruchow and D. Stojanoff ([4], Corollary 2.4) and

S. Popa ([28]). Moreover, we can estimate the dimension of the relative commutant $N' \cap M$ in terms of the constants $K(E)$ and $\dim(Z(N))$.

COROLLARY 4.3. *Let $E : M \to N$ be a normal conditional expectation on a W^*-algebra M such that there exists a number $K \geqslant 1$ for which the mapping $(K \cdot E - \mathrm{id}_A)$ is positive. Then the center of M is finite-dimensional if and only if the center of N is finite-dimensional if and only if the relative commutant $N' \cap M$ is finite-dimensional, and*

$$\dim(N' \cap M) \leqslant [K(E)]^2 \cdot \dim(Z(N)).$$

(Note that $\dim(Z(N)) \leqslant \dim(N' \cap M)$, $\dim(Z(M)) \leqslant \dim(N' \cap M)$ because $Z(N) \subseteq (N' \cap M)$, $Z(M) \subseteq (N' \cap M)$.)

Proof. The equivalence of the three claimed conditions follows from [6], Corollary 3.19 and from Theorem 1 immediately. Since the center of the relative commutant $N' \cap M$ contains $Z(N)$ and since the minimal projections of $Z(N)$ commute with E, the conditional expectation E is the direct sum of $\dim(Z(N))$ states on $N' \cap M$ with non-intersecting areas of definition. But, for states on matrix algebras and conditional expectations of finite index on them the dimension of the matrix algebra is bounded by $[K(E)]^2$ for $K(E)$ being the structural constant of E, cf. Examples 1.6 and 1.1. This gives the argument. ∎

Our next result generalizes [4], Corollary 2.3 and gives some estimates of the dimensions of M and N:

COROLLARY 4.4. *Let M be a C^*-algebra, N be a finite-dimensional C^*-subalgebra of M and $E : M \to N \subseteq M$ be a faithful conditional expectation. Then M is finite-dimensional if and only if E is of finite index, and the estimate*

$$\dim(M) \leqslant [K(E)]^2 \cdot \dim(N)^2$$

is valid. If M and N are commutative, then we have the estimate

$$\dim(M) \leqslant [K(E)] \cdot \dim(N).$$

Proof. Obviously, if M is finite-dimensional, then it is finitely generated as a N-module and hence, E is of finite Watatani index. To show the converse, let B be a maximal commutative C^*-subalgebra of N. Let A be a maximal commutative C^*-subalgebra of M containing B. Then $E(A) = B$ since $E(a)b = E(ab) = E(ba) = bE(a)$ for every $b \in B$, every $a \in A$ and B is maximal commutative in N by assumption. Similar to the proof of Proposition 1.3, every minimal projection of B has no more than $[K(E)]$ minimal projection-summands in its decomposition inside A. This shows the statement for commutative C^*-algebras.

By the general theory of C^*-algebras the estimate $\dim(M) \leqslant \dim(A)^2$ is valid for every C^*-algebra M and for every maximal commutative C^*-subalgebra A of M. ∎

Applying a result of R.V. Kadison ([16]) we can show the following:

Proposition 4.5. *Let $E : A \to B \subseteq A$ be a conditional expectation of finite index. Then the inequality*

$$0 \leqslant (E(a) - a)^2 \leqslant (K(E) - 1) \cdot (E(a^2) - E(a)^2)$$

holds for self-adjoint elements $a \in A$. Moreover, if $p \in A$ is a projection, then $E(p)$ is a projection if and only if $E(p) = p \in B$. Otherwise, $E(p)$ has a spectral value $0 < \lambda < 1$. Beside this, $E(a^2) = E(a)^2$ if and only if $E(a) = a \in B$, and $K(E) = 1$ if and only if $E = \mathrm{id}_M$. Furthermore, $K(E) \in \{1\} \cup [2, \infty)$.

Proof. We apply the theorem on page 29 of [16] to our situation. R.V. Kadison states that for positive mappings ψ between unital C^*-algebras with $\psi(1) \leqslant 1$, the inequality $\psi(a)^2 \leqslant \psi(a^2)$ holds for every self-adjoint element a. In our situation the equality $(K(E) \cdot E - \mathrm{id}_A)(1_A) = (K(E) - 1)1_A$ holds. Dividing by $(K(E) - 1)$ we obtain the inequality

$$((K(E) \cdot E - \mathrm{id}_A)(a))^2 \leqslant (K(E) - 1) \cdot \big((K(E) \cdot E(a^2) - a^2\big)$$

which is valid for every self-adjoint $a \in A$. After some obvious transformations we arrive at

$$K(E)^2 \cdot E(a)^2 - K(E) \cdot aE(a) - K(E) \cdot E(a)a \leqslant K(E) \cdot \big((K(E) - 1) \cdot E(a^2) - a^2\big),$$

an inequality which can be transformed to the inequality claimed above. Setting $E(p) = E(p)^2$ for a projection $p \in A$ we derive $0 = (E(p) - p)^2$ and hence, $E(p) = p$. Since in general $1_A \geqslant E(p) = E(p^2) \geqslant E(p)^2 \geqslant 0$ for every projection $p \in A$ the condition $p \notin B$ implies the existence of spectral values of $E(p)$ strongly between zero and one.

The case $K(E) = 1$ is obvious. Applying E to the derived above inequality and investigating the resulting inequality for $K(E) \in [1, 2)$ we obtain $K(E) \in \{1\}$ as the only possibility. Example 1.6 realizes any possible value of $K(E)$ inside $[2, \infty)$. ∎

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