J. OPERATOR THEORY
40(1998), 113–132

ON THE POINT SPECTRUM OF SOME JACOBI MATRICES

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Communicated by Nikolai K. Nikolskii

Abstract. The paper deals with spectral properties of selfadjoint tridiagonal Jacobi matrices T acting in l^2. There are given some sufficient conditions for the absence of the point spectrum of T lying over the interior of its essential spectrum.

Keywords: *Jacobi matrix, point spectrum, compact perturbation.*

AMS Subject Classification: Primary 47A75; Secondary 47B35.

1. INTRODUCTION

In this paper we concentrate on a class of Jacobi matrices T of the from

$$T = \begin{pmatrix} q_1 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & q_2 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & q_3 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

In what follows we always assume as one usually does that $\lambda_k \neq 0$, for all k, see [1]. We also suppose that $\lim\limits_k \lambda_k = c \neq 0$ exists and $\lim\limits_k q_k = 0$. Therefore T is a compact perturbation of the operator $c(U + U^*)$, where $Ue_n = e_{n+1}$ is the unilateral shift. We may assume that $c = 1$ and so $\sigma_{\mathrm{ess}}(T) = [-2, 2]$ by the Weyl theorem.

In the paper we consider only eigenvalues of T embeded in $(-2, 2)$. One of the main results of the paper provides a sufficient condition in terms of asymptotic behaviour of $\{\lambda_k\}$ and $\{q_k\}$ which guarantees that $\sigma_{\mathrm{p}}(T) \cap (-2, 2) = \emptyset$. In order to describe this condition more precisely, let us define the sequence $\{\delta_k\}$ by $\delta_k := 1 - \frac{1}{\lambda_k}$, $\delta_0 := 0$.

The aforesaid condition says: if for some $q > 1$

$$\sum_m \exp[-q(c_1(\lambda)F_m + c_2(\lambda)G_m + c_3(\lambda)Q_m)] = +\infty,$$

then $\lambda \notin \sigma_{\rm p}(T) \cap (-2, 2)$; here

$$c_1(\lambda) = 1, \quad c_2(\lambda) = \frac{|\lambda|}{2\sqrt{1-(\lambda/2)^2}}, \quad c_3(\lambda) = \frac{1}{\sqrt{1-(\lambda/2)^2}}$$

$$Q_m = \sum_{k=1}^m |q_k|, \quad F_m = \sum_{k=1}^m |\delta_{k-1} - \delta_k|, \quad G_m = \sum_{k=1}^m |\delta_{k-1} + \delta_k|.$$

Note that the sums F_m (respectively G_m) could be considered as related to the local, (respectively global) behaviour of $\{\lambda_k\}$.

Moreover, we shall prove that the coefficients $c_1(\lambda), c_2(\lambda), c_3(\lambda)$ are essentially determined by the nature of T and are sharp near critical points $\lambda = \pm 2$ and that $c_1(\lambda)$, $c_3(\lambda)$ are sharp at $\lambda = 0$. This will be shown by constructing a sufficiently delicate example, see Section 5, and this is one of the essential goals of the present paper. In particular, sharpness of $c_3(\cdot)$ near $\lambda = \pm 2$ and $\lambda = 0$ has been already proved by Yakovlev and Naboko in [11].

Concerning the sharpness of $c_2(\lambda)$ near $\lambda = 0$ we shall find out in Section 6 that it is not true. Namely, we shall see that $c_2(\lambda)$ can be replaced by $\lambda^2(1+c_2(\lambda))$ near $\lambda = 0$, see Corollary 6.2.

We also point out that the case when $\pm 2 \notin \sigma_{\rm p}(T)$ has been considered in [2]. It should be mentioned here that the problem of absence of eigenvalues of T in $(-2, 2)$ has been studied in [3]. However, the results founded in this work are different from ours. Analysis of related spectral problems has been considered in [6], [10], [4] and [8].

We especially emphasize a recent paper by Khan and Pearson, where a unified approach to the spectral theory of T has been developed ([7]).

Concerning the methods used here, they depend on detailed analysis of a simple dynamical system determined by the eigenvalue equation, $Tu = \lambda u$. Similar methods turn out to be useful in analysis of the absolute continuity of T and we hope to study this case in another paper.

2. PRELIMINARIES

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in a complex Hilbert space H. For a bounded sequence λ_n of positive numbers denote by Λ the diagonal operator given by $\Lambda e_n = \lambda_n e_n$. If U stands for the unilateral shift, $Ue_n = e_{n+1}$, then the operator

$$J = (U\Lambda)^* + U\Lambda$$

is the real part of the weighted shift operator $We_n = 2\lambda_n e_{n+1}$.

We assumed above that $\lim_n \lambda_n = c \neq 0$. Recall that the general case $\lambda_k \neq 0$ can be reduced to positive λ_k by a diagonal unitary rotation. Therefore, without loss of generality, we may assume that $c = 1$. We introduce the notation

$$\frac{\lambda_{n-1}}{\lambda_n} = 1 + \varepsilon_n, \quad 1 - \frac{1}{\lambda_n} = \delta_n \tag{2.1.}$$

Roughly speaking, the parameter ε_n measures local oscilation of weights and δ_n describe their asymptotic behaviour at infinity. The following simple remark will be tacitly used below.

Remark 2.1. Under the above assumptions the series $\sum_k \varepsilon_k$ is not convergent in general. However we have the relation

$$\varepsilon_k = \delta_{k-1} - \delta_k + \mathrm{O}((\delta_{k-1} - \delta_k)\delta_{k-1}) \tag{2.2}$$

and obviously $\sum_k (\delta_{k-1} - \delta_k)$ is always convergent.

This could lead to the convergence of the series $\sum_k \varepsilon_k$ under some weak additional conditions on $\{\delta_k\}$, say $\{\delta_k\} \in l^2$.

For a real number x denote by $(x)_- := \frac{1}{2}(x - |x|)$. The following easy observation will be used later.

Remark 2.2. Denote $\tilde{\varepsilon}_k := \delta_{k-1} - \delta_k$ and $c_n := \sum_{k=1}^{n} \tilde{\varepsilon}_k = -\delta_n = \mathrm{o}(1)$, as $n \to \infty$.

Then

$$2\sum_{k=1}^{n} |(\tilde{\varepsilon}_k)_-| = \sum_{k=1}^{n} |\tilde{\varepsilon}_k| - c_n. \tag{2.3}$$

The equality is immediate by definitions. We also need the next convenient notation

$$\eta_k := 2\delta_k + \varepsilon_k. \tag{2.4}$$

Then we have

$$|\eta_k| = |\delta_{k-1} + \delta_k| + \mathrm{O}((\delta_{k-1} - \delta_k)\delta_{k-1}). \tag{2.5}$$

The rest of our notation is standard or will be defined below.

3. WHEN $\sigma_{\rm p}(J) \cap (-2,2)$ IS EMPTY ?

In this section we will find a sufficient condition which guarantees a positive answer for the above question. Note that J is always a compact perturbation of $U+U^*$. Therefore $\sigma_{\rm ess}(J) = [-2,2]$ and we are interested in this paper only in the part of $\sigma_{\rm p}(J)$ which lies entirely in $(-2,2)$.

The methods used here to study this question can be divided into complex or real ones. As we will see below, both methods give equivalent sufficient conditions. The real one will be also used in Section 4.

(a) *Complex method*

Suppose that $\lambda \in (-2,2)$ satisfies the equation

$$Ju = \lambda u, \tag{3.1}$$

for a certain sequence $u = l^2$, $u \neq 0$. Let $u = (u_1, u_2, \ldots)^*$. Then (3.1) is equivalent to the system

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\lambda_{n-1}}{\lambda_n} & \frac{\lambda}{\lambda_n} \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \quad n = 1, 2, \ldots$$

where formally $u_0 = 0$. Let $\mu = \frac{\lambda}{2} + {\rm i}\sqrt{1-\lambda^2/4}$. Denote by $B_n(\lambda)$ the matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{\lambda_{n-1}}{\lambda_n} & \frac{\lambda}{\lambda_n} \end{pmatrix} \quad \text{and} \quad \text{let } V = \begin{pmatrix} 1 & 1 \\ \mu & \bar{\mu} \end{pmatrix}.$$

The idea of the complex method consists of diagonalization of the limit matrix

$$B_\infty(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}.$$

It is clear that μ and $\overline{\mu}$ are the eigenvalues of $\mathcal{B}_\infty(\lambda)$. Now

$$V^{-1} = \frac{1}{\bar{\mu} - \mu} \begin{pmatrix} \bar{\mu} & -1 \\ -\mu & 1 \end{pmatrix}.$$

A direct calculation shows that

$$B_n := V^{-1} B_n(\lambda) V = \frac{1}{\bar{\mu} - \mu} \begin{pmatrix} a_n & b_n \\ -\bar{b}_n & -\bar{a}_n \end{pmatrix},$$

where $a_n = 1 + \frac{\lambda_{n-1}}{\lambda_n} - \mu \frac{\lambda}{\lambda_n}$, $b_n = \bar{\mu}^2 + \frac{\lambda_{n-1}}{\lambda_n} - \bar{\mu} \frac{\lambda}{\lambda_n}$.

Hence the minimal s-number s_n of B_n is given by $(|a_n| - |b_n|)|\bar{\mu} - \mu|^{-1}$ (one can easily check that $|a_n| \geqslant |b_n|$ for all n). By a straightforward calculation we have

$$|a_n| = \sqrt{a}\left(1 + \varepsilon_n + \frac{\tau_n}{a}\right)^{\frac{1}{2}}$$
$$|b_n| = \sqrt{\tau_n},$$

where $a := 4 - \lambda^2$ and $\tau_n := \varepsilon_n^2 + \lambda^2(\delta_n^2 + \varepsilon_n\delta_n)$. Denote by θ_k the fraction τ_k/a. Then $s_n = \sqrt{1 + \varepsilon_n + \theta_n} - \sqrt{\theta_n}$. Since $2\sqrt{\theta_n} - \varepsilon_n \geqslant 0$ one can prove that

$$s_n^2 \geqslant 1 + \varepsilon_n - 2\sqrt{\theta_n}. \tag{3.2}$$

Recall that $\|Vf\|^2 \geqslant (2 - |\lambda|)\|f\|^2$. Hence (for $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

$$\|B_n(\lambda)\cdots B_1(\lambda)y\|^2 = \|VB_n \cdot B_{n-1}\cdots B_2B_1V^{-1}y\|^2 \geqslant (2 - |\lambda|)s_n^2\cdots s_1^2\|V^{-1}y\|^2.$$

Obviously, for any $q \geqslant 1$ there exists $\delta \geqslant 0$ such that $\ln(1+x) \geqslant qx$, for $-\delta < x < 0$. Now (using (3.2)) we can estimate from below the l^2-norm of the solution u of (3.1) by

$$\sum_{n=n_0+1}^{\infty} s_n^2\cdots s_{n_0}^2 \geqslant \sum_{n=n_0+1}^{\infty} \exp\left(\sum_{k=n_0+1}^{n} \ln(1 + \varepsilon_k - 2\sqrt{\theta_k})\right),$$

where n_0 is so large that $-\delta < \varepsilon_k - 2\sqrt{\theta_k}$, for $k > n_0$.

Thus

$$\sum_{n_0+1}^{\infty} \exp\left(\sum_{k=n_0+1}^{n} \ln(1 + \varepsilon_k - 2\sqrt{\theta_k})\right) \geqslant \sum_{n_0+1}^{\infty} \exp\left[\sum_{k=n_0+1}^{n} q(\varepsilon_k - 2\sqrt{\theta_k})\right].$$

Put $c(\lambda) = \sqrt{1 - (\frac{\lambda}{2})^2}$. Then $2\sqrt{\theta_k} = \sqrt{\tau_k}c(\lambda)^{-1}$. It is clear that

$$\sqrt{\tau_k} \leqslant c(\lambda)|\varepsilon_k| + |\lambda|\left|\delta_k + \frac{\varepsilon_k}{2}\right| = c(\lambda)|\varepsilon_k| + \frac{|\lambda|}{2}|\eta_k| = c(\lambda)|\varepsilon_k| + \frac{|\lambda|}{2}(|\delta_{k-1} + \delta_k| + \mathrm{o}(\tilde{\varepsilon}_k))$$

(the last equality by (2.5)). Hence

$$\varepsilon_k - \frac{1}{c(\lambda)}\sqrt{\tau_k} \geqslant \varepsilon_k - |\varepsilon_k| - \frac{|\lambda|}{2c(\lambda)}(|\delta_{k-1} + \delta_k| + \mathrm{o}(\tilde{\varepsilon}_k)).$$

But

$$\varepsilon_k - |\varepsilon_k| = (1 + \mathrm{o}(1))(\tilde{\varepsilon}_k - |\tilde{\varepsilon}_k|)$$

(use (2.2)) and so, combining this and Remark 2.2, we have

$$\sum_{k=n_0+1}^{n} [\varepsilon_k - |\varepsilon_k| - \frac{|\lambda|}{2c(\lambda)}|\eta_k|] = -(1 + \mathrm{o}(1))\left(\sum_{k=n_0+1}^{n} (|\tilde{\varepsilon}_k| + \frac{|\lambda|}{2c(\lambda)}|\delta_{k-1} + \delta_k|)\right) + \mathrm{O}(1),$$

where $\mathrm{o}(1)$ refers to n_o tending to infinity. Note that $\mathrm{O}(1)$ term makes no problem in our estimates and $\mathrm{o}(1)$ ones can be absorbed by a change of q. Therefore without loss of generality $1 + \mathrm{o}(1)$ may be replaced by 1.

Summing up we have

THEOREM 3.1. *If λ satisfies the condition*

$$\sum_{n=1}^{\infty} \exp\left[- q \sum_{k=1}^{n} (|\delta_{k-1} - \delta_k| + \frac{|\lambda|}{2c(\lambda)} |\delta_{k-1} + \delta_k|) \right] = +\infty,$$

for some $q > 1$, then $\lambda \notin \sigma_{\mathrm{p}}(J) \cap (-2, 2)$.

REMARK 3.2. By a direct computation one can check that $\lambda = 0 \in \sigma_{\mathrm{p}}(J)$ if and only if

$$\sum_{m=1}^{\infty} \prod_{l=1}^{m} \left(\frac{\lambda_{2l-1}}{\lambda_{2l}} \right)^2 < +\infty.$$

This remark will be used later in proving the sharpness of Theorem 6.1.

We reformulate the condition of Theorem 3.1 by introducing the following sequences

$$F_m := \sum_{k=1}^{m} |\delta_{k-1} - \delta_k|, \quad G_m := \sum_{k=1}^{m} |\delta_{k-1} + \delta_k|.$$

The condition says as follows

$$\sum_{n}^{\infty} \exp\left[-q \left(F_n + \frac{|\lambda|}{2c(\lambda)} G_n \right) \right] = +\infty.$$

From the sequence $\sum\limits_{k=1}^{n} (\delta_{k-1} - \delta_k)$ one can obviously reconstruct δ_k and therefore also G_m, but nevertheless, to some extent, F_n and G_n may have independent behaviour. To show this, take arbitrary sequences $\{\alpha_k\}$ and $\{\beta_k\}$ of positive real numbers such that $\lim\limits_k \alpha_k = \lim\limits_k \beta_k = 0$. We can change the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ into equivalent sequences $\{\tilde{\alpha}_k\}$ and $\{\tilde{\beta}_k\}$ such that for suitable even integers m_s, n_s with $\lim\limits_s m_s = \lim\limits_s n_s = +\infty$, we have

$$\frac{\tilde{\alpha}_s}{m_s} = \frac{\tilde{\beta}_s}{n_s} = \Delta_s \quad \text{and} \quad \lim_s \Delta_s = 0.$$

Now for $m = (m_1 + n_1) + \cdots + (m_l + n_l) = l \cdot r_l$ with $\lim\limits_l r_l = \infty$, define the sequence δ_n by the formulas

$$\delta_k = (-1)^k \Delta_s, \quad \text{if } (m_1 + n_1) + \cdots + (m_{s-1} + n_{s-1}) \leqslant k < (m_1 + n_1) + \cdots m_s$$
$$\delta_k = \Delta_s, \quad \text{if } (m_1 + n_1) + \cdots + m_s \leqslant k < (m_1 + n_1) + \cdots + (m_s + n_s).$$

Hence

$$F_m = \tilde{\beta}_1 + \cdots + \tilde{\beta}_l, \quad G_m = \tilde{\alpha}_1 + \cdots + \tilde{\alpha}_l, \quad m = l \cdot r_l,$$

are quite different.

Remark 3.3. In the case $\sum_k \delta_k^2 < \infty$ we may replace $q > 1$ in Theorem 3.1 by $q = 1$.

(b) *Real method*

There exists an alternative ("real") approach to the localization of $\sigma_{\mathrm{p}}(J) \cap (-2, 2)$. Though it turns out to produce a criterion equivalent again to the one given in Theorem 3.1, we also present this method because it will be used in Section 5. However, the details are omitted as the main idea is the same as that in the complex case.

In what follows we shall use the following notations: for $\lambda \in (-2, 2)$, $\mu = \cos\theta + \mathrm{i}\sin\theta$, where $\mu = \frac{\lambda}{2} + \mathrm{i}\sqrt{1 - \lambda^2/4}$. We start with the following basic relation. If

$$W = \begin{pmatrix} 1 & 0 \\ \operatorname{Re}\mu & \operatorname{Im}\mu \end{pmatrix} \quad \text{and} \quad B(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix},$$

then direct computation shows that

$$B(\lambda)W = W \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.3}$$

Denote for brevity by $P = P_\theta$ the rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. The idea of the real method is based on the similarity of $B(\lambda)$ to the matrix P, the simplest real form for $B(\lambda)$ with the unitary spectrum. Let us compute the product

$$A_n := W^{-1} B_n(\lambda) W.$$

Relation (3.3) and straightforward computations lead to

$$\begin{aligned} A_n &= P - \frac{1}{\operatorname{Im}\mu} \begin{pmatrix} 0 & 0 \\ \varepsilon_n + \frac{\lambda^2\delta_n}{2} & \lambda\delta_n \operatorname{Im}\mu \end{pmatrix} \\ &= \left(I - \frac{1}{\operatorname{Im}\mu} \begin{pmatrix} 0 & 0 \\ \eta_n \operatorname{Re}\mu & -\varepsilon_n \operatorname{Im}\mu \end{pmatrix} \right) P := C_n(\lambda) P. \end{aligned} \tag{3.4}$$

It follows that

$$A_n A_n^* = C_n(\lambda) C_n(\lambda)^* = I - \begin{pmatrix} 0 & \rho_n \\ \rho_n & \varepsilon_n^2 - 2\varepsilon_n + \rho_n^2 \end{pmatrix},$$

where $\rho_n := \eta_n \frac{\operatorname{Re}\mu}{\operatorname{Im}\mu}$. Hence

$$s_{\min}^2(A_n) = s_{\min}^2(C_n(\lambda)) = 1 - \left(\frac{\gamma_n}{2} + \sqrt{\gamma_n^2/4 + \rho_n^2} \right), \tag{3.5}$$

where $\gamma_n := \varepsilon_n^2 - 2\varepsilon_n + \rho_n^2$. Therefore $s_{\min}^2(A_n) \geqslant 1 - (\frac{\gamma_n}{2} + \frac{|\gamma_n|}{2} + |\rho_n|)$.

Repeating the reasoning presented in the complex method case we have

THEOREM 3.4. *If λ satisfies the condition*

$$\sum_{n=1}^{n} \exp\left[-q\sum_{k=1}^{n}\left(\frac{\gamma_k}{2}+\frac{|\gamma_k|}{2}+\frac{|\lambda|}{2c(\lambda)}|\rho_k|\right)\right] = +\infty$$

for a certain $q > 1$, then $\lambda \notin \sigma_{\mathrm{p}}(J) \cap (-2, 2)$.

In the next section we will see the equivalence of Theorems 3.1 and 3.4 or rather their analogs for full tridiagonal Jacobi matrices.

Despite all the above results one can prove (combining the methods of [11] with the consideration of Sections 3 and 5) the following statement on the dense point spectrum of J.

THEOREM 3.5. *For an arbitrary sequence C_n tending to infinity (arbitrary slowly) there exists a sequence of weights $\{\lambda_n\}$ for which the sequence $\{\delta_n\}$ satisfies the inequality $|\delta_n| \leqslant \frac{C_n}{n}$ and the corresponding Jacobi matrix J has the point spectrum filling densely the interval $(-2, 2)$.*

4. TRIDIAGONAL CASE

It is well known that every bounded cyclic selfadjoint operator A in H can be represented as a tridiagonal matrix with respect to the basis generated by a cyclic vector of A ([9]). In this section we will extend results proved in the previous one to the case of tridiagonal matrices. Namely, we are going to study "weak" diagonal perturbation of the operator J, i.e. the operator $T = J + Q$, where $Qe_n = q_n e_n$ and $\lim_n q_n = 0$. Since $\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(J)$, the problem of eigenvalues of T embedded in $\sigma_{\mathrm{ess}}(T)$ reduces to study the intersection $\sigma_{\mathrm{p}}(T) \cap (-2, 2)$.

Instead of analysing the equation $Tu = \lambda u$, we consider (as in Section 3) the infinite system of equations

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \tilde{B}_n(\lambda)\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \quad n = 1, 2, \ldots \tag{4.1}$$

with $u_0 = 0$, where

$$\tilde{B}_n(\lambda) := \begin{pmatrix} 0 & 1 \\ -1-\varepsilon_n & (\lambda - q_n)(1-\delta_n) \end{pmatrix}.$$

Let us apply the real approach to the system (4.1). For W the same as before, we can compute $W^{-1}\tilde{B}_n(\lambda)W =: \tilde{A}_n$. We have (for $\lambda \neq 0$)

$$\tilde{A}_n = \left(I - \frac{1}{\operatorname{Im}\mu}\begin{pmatrix} 0 & 0 \\ \varepsilon_n \operatorname{Re}\mu + \lambda\tilde{\delta}_n & -\varepsilon_n \operatorname{Im}\mu \end{pmatrix}\right)P, \tag{4.2}$$

where $\tilde{\delta}_n := \delta_n + \frac{q_n(1-\delta_n)}{\lambda}$. Since

$$\tilde{A}_n = \left(I - \frac{1}{\operatorname{Im}\mu}\begin{pmatrix} 0 & 0 \\ \hat{\eta}_n \operatorname{Re}\mu & -\varepsilon_n \operatorname{Im}\mu \end{pmatrix}\right) P$$

where $\hat{\eta}_n := \varepsilon_n + 2\tilde{\delta}_n$, has the same structure as the matrix A_n in Section 3 we may apply formula (3.5). Thus

$$s^2_{\min}(\tilde{A}_n) = 1 - \left(\frac{\tilde{\gamma}_n}{2} + \sqrt{\frac{\tilde{\gamma}_n^2}{4} + \tilde{\eta}_n^2 \left(\frac{\operatorname{Re}\mu}{\operatorname{Im}\mu}\right)^2}\right),$$

where $\tilde{\gamma}_n = \varepsilon_n^2 - 2\varepsilon_n + \tilde{\eta}_n^2 \left(\frac{\operatorname{Re}\mu}{\operatorname{Im}\mu}\right)^2$.

Repeating the reasoning given in the proof of Theorem 3.4 we have (for all $\lambda \in (-2,2)$ including $\lambda = 0$).

Theorem 4.1. *If for some $q > 1$ the condition*

$$\sum_n \exp\left[-q\sum_{k=1}^{n}\left(\frac{\tilde{\gamma}_k}{2} + \frac{|\tilde{\gamma}_k|}{2} + \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k|\right)\right] = +\infty, \tag{4.3}$$

is satisfied, then $\lambda \notin \sigma_{\mathrm{p}}(T) \cap (-2,2)$.

Surely we also have a complex counterpart of Theorem 4.1.

Theorem 4.2. *If for some $q > 1$ the condition*

$$\sum_n \exp\left[-q\sum_{k=1}^{n}\left(|\varepsilon_k| - \varepsilon_k + \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k|\right)\right] = +\infty \tag{4.4}$$

is fulfield, then $\lambda \notin \sigma_{\mathrm{p}}(T) \cap (-2,2)$.

Before we proceed further observe that the above conditions (4.3) and (4.4) are equivalent. In other words, this shows that real and complex approaches are equivalent. To see this, let us define

$$\tilde{R}_k := \frac{\tilde{\gamma}_k}{2} + \frac{|\tilde{\gamma}_k|}{2} + \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k|,$$

$$R_k := |\varepsilon_k| - \varepsilon_k + \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k|.$$

We have three cases:

(a) $\tilde{\gamma}_k \leqslant 0$, then $\varepsilon_k \geqslant 0$ and so $\tilde{R}_k = R_k$.

(b) $\tilde{\gamma}_k > 0$ and $\varepsilon_k < 0$, then

$$\tilde{R}_k = \tilde{\gamma}_k + \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k| = 2|\varepsilon_k|(1+\mathrm{o}(1)) + \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k|(1+\mathrm{o}(1)) = R_k(1+\mathrm{o}(1)).$$

(c) $\gamma_k > 0$ and $\varepsilon_k \geqslant 0$, then

$$0 \leqslant \varepsilon_k \leqslant 2\varepsilon_k\left(1-\frac{\varepsilon_k}{2}\right) \leqslant \left(\frac{\lambda}{2c(\lambda)}\tilde{\eta}_k\right)^2,$$

provided $k \gg 1$. It follows that

$$\tilde{\gamma}_k = \mathrm{O}\left(\left(\frac{\lambda\tilde{\eta}_k}{2c(\lambda)}\right)^2\right)$$

and so

$$\tilde{R}_k = \frac{|\lambda|}{2c(\lambda)}|\tilde{\eta}_k|(1+\mathrm{o}(1)) = R_k(1+\mathrm{o}(1)).$$

Since the o(1) terms make no problem (see the proof of the Theorem 3.1) the above cases (a), (b), (c) complete the proof of the equivalence of (4.3) and (4.4).

Finally, we may reformulate Theorem 4.1 in a way similar to the statement of Theorem 3.1. Indeed, by the definition of $\tilde{\eta}_k$

$$|\tilde{\eta}_k| \leqslant |\eta_k| + \frac{2|q_k|}{|\lambda|}(1-\delta_k)$$

and so (4.4) implies that

$$\sum_n \exp\left[-q\left(\sum_{k=1}^m\left(|\varepsilon_k| - \varepsilon_k + \frac{|\lambda|}{2c(\lambda)}|\eta_k| + \frac{|q_k|}{c(\lambda)}\right)\right)\right] = +\infty. \tag{4.5}$$

Denote by $Q_m := \sum_{k=n_0}^{m} |q_k|$. Then, by repeating the reasoning from the proof of Theorem 3.1, we have

COROLLARY 4.3. *If for a certain $q > 1$ a number λ satisfies the condition*

$$\sum_m \exp\left[-q\left(F_m + \frac{|\lambda|}{2c(\lambda)}G_m + \frac{1}{c(\lambda)}Q_m\right)\right] = +\infty,$$

then $\lambda \notin \sigma_{\mathrm{p}}(T) \cap (-2,2)$.

This result turns out to be almost sharp (in the sense that will be definded in the next section).

5. SHARPNESS OF RESULTS

This section is devoted to the analysis of exactness of criterions presented in Section 4. In particular, we concentrate on sharpness of Corollary 4.3. Let us explain this more precisely.

If we consider for some functions $c_1(\lambda), c_2(\lambda)$ and $c_3(\lambda)$ and $q > 1$ a class of conditions

$$\text{“}\sum_m \exp[-q(c_1(\lambda)F_m + c_2(\lambda)G_m + c_3(\lambda)Q_m)] = +\infty \Rightarrow \lambda \notin \sigma_{\mathrm{p}}(T) \cap (-2,2)\text{”},$$

then Corollary 4.3 shows this is true for $c_1(\lambda) \equiv 1, c_2(\lambda) \equiv \frac{|\lambda|}{2c(\lambda)}, c_3(\lambda) \equiv \frac{1}{c(\lambda)}$.

Therefore if $\lambda \in \sigma_{\mathrm{p}}(T) \cap (-2,2)$, then

$$\sum_m \exp\left[-q\left(F_m + \frac{|\lambda|}{2c(\lambda)}G_m + \frac{1}{c(\lambda)}Q_m\right)\right] < +\infty, \tag{5.1}$$

for any $q > 1$.

Fix $\lambda = 2\cos\theta \in (-2,2) \setminus \{0\}$. Assume that for any $\gamma > 0$ one can construct the weights $\tilde{\lambda}_k = \tilde{\lambda}_k(\gamma)$ such that $F_m \sim f\gamma \ln m,\ G_m \sim g\gamma \ln m$ for certain constants f and g independent of γ.

Next, suppose that for these weights $\{\tilde{\lambda}_k\}$ and some functions $\tilde{c}_1(\lambda), \tilde{c}_2(\lambda)$ (which do not depend on γ) we have proved that the convergence of the series

$$\sum_m \exp[-q'(\tilde{c}_1(\lambda)f + \tilde{c}_2(\lambda)g)\gamma \ln m] \tag{5.2}$$

(for a certain $q' < 1$) implies that $\lambda \in \sigma_{\mathrm{p}}(\tilde{J})$, where $\tilde{J}$ corresponds to the above constructed weights $\{\tilde{\lambda}_k\}$. Note that convergence in (5.2) imposes the only restriction on γ. The series in (5.2) is convergent if and only if

$$(\tilde{c}_1(\lambda)f + \tilde{c}_2(\lambda)g)\gamma q' > 1. \tag{5.3}$$

On the other hand, applying Corollary 4.3 (remember that $\lambda \in \sigma_{\mathrm{p}}(\tilde{J})$!) we must have

$$\gamma\left(f + \frac{|\lambda| g}{2c(\lambda)}\right) > \frac{1}{q}, \tag{5.4}$$

for every $q > 1$. Therefore the inequality (5.3) implies (5.4). Letting now $q' \to 1$ in (5.3) we deduce that the inequality $\tilde{c}_1(\lambda)f + \tilde{c}_2(\lambda)g > 1/\gamma$ clearly forces that $f + \frac{|\lambda|}{2c(\lambda)}g > \frac{1}{q\gamma}$, for any $q > 1$.

Finally, due to the arbitrary choice of $\gamma > 0$, we obtain the inequality

$$f + \frac{|\lambda|}{2c(\lambda)} g \geqslant \tilde{c}_1(\lambda) f + \tilde{c}_2(\lambda) g. \tag{5.5}$$

Below we shall construct a specific example of weights which satisfy the above discussed assumptions and so (5.5) will be used. This example will confirm sharpness of $c_1(\lambda)$ near $\lambda = 0$ and the same for $c_2(\lambda)$ near $\lambda = \pm 2$. We shall not consider sharpness of $c_3(\lambda)$ near $\lambda = \pm 2$ as it has been shown already by Yakovlev and Naboko ([11], Theorem 3).

Before we start constructions of the above mentioned example, let us prove two elementary results of general character. Both these results will be useful below.

LEMMA 5.1. *If θ/π is an irrational number, then for any $\varepsilon > 0$ there exists $N = N(\varepsilon, \theta)$ such that for every unit vectors e, e_0 in $\mathbb{R}^2$ there is a number $0 \leqslant n < N$ for which*

$$\|P^n e - e_0\| < \varepsilon, \qquad \text{where } P = P_\theta.$$

Proof. Standard reasoning based on ergodicity of irrational rotations. ∎

LEMMA 5.2. *Let $A = \begin{pmatrix} s_2 & 0 \\ 0 & s_1 \end{pmatrix}$, where $0 < s_1 = 1 - \Delta < 1 + \tilde{\Delta} = s_2, \Delta \geqslant 0$, and $|\tilde{\Delta}| \leqslant const \cdot \Delta$. Then for any $\varepsilon > 0$ and the unit vector $e = \begin{pmatrix} \sin\varepsilon \\ \cos\varepsilon \end{pmatrix}$,*

$$(Ae, e) \leqslant 1 - \Delta(1 - \mathrm{O}(\varepsilon^2)).$$

Proof. If $P_\varepsilon := \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix}$ is the rotation matrix for ε-angle, then

$$e = P_\varepsilon \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\varepsilon \\ \cos\varepsilon \end{pmatrix}.$$

Hence,

$$\begin{aligned}(Ae, e) &= (1 + \tilde{\Delta}) \sin^2\varepsilon + (1 - \Delta)\cos^2\varepsilon = 1 - \Delta\cos^2\varepsilon + \tilde{\Delta}\sin^2\varepsilon \\ &\leqslant 1 - (1 - \varepsilon^2)\Delta + C\varepsilon^2\Delta = 1 - \Delta(1 - \mathrm{O}(\varepsilon^2))\end{aligned}$$

and the proof is complete. ∎

Note that the estimation of (Ae, e) in Lemma 5.2 is better than the easy one: $(Ae, e) \leqslant (1 - \Delta)(1 - \mathrm{O}(\varepsilon))$.

Now we shall find an example which will confirm the sharpness of $c_2(\lambda)$ near $\lambda = \pm 2$ as well as the sharpness of $c_1(\lambda)$ near $\lambda = 0$.

Example 5.3. Fix $\lambda \in (-2,2)\setminus\{0\}$. Suppose that $\lambda = 2\cos\theta$ with θ/π irrational. Let ε and N be as in Lemma 5.1. For any $\gamma > 0$ choose a sequence of weights $\{\lambda_k\} = \{\lambda_k(\gamma)\}$ such that $\delta_k = \frac{\gamma}{k}$. Then

$$\sum_{j=1}^{m} \delta_j \sim \gamma \ln m. \tag{5.6}$$

Define a new sequence of weights $\{\tilde\lambda_k\}$ by

$$\tilde\lambda_n = \begin{cases} 1, & (l-1)N \leqslant n < lN, \quad n \neq n_l, \\ \frac{1}{1-\Delta_l}, & \Delta_l := \sum\limits_{j=(l-1)N+1}^{lN} \delta_j, \quad n = n_l; \end{cases} \tag{5.7}$$

where $n_l \in [(l-1)N, lN)$ will be chosen later (according to Lemma 5.1).

If $\tilde\varepsilon_k$ and $\tilde\delta_k$ are sequences defined as before by $\{\tilde\lambda_k\}$, then $\tilde\delta_n = 0$ for $n \neq n_l$, $(l-1)N \leqslant n < lN$ and $\tilde\delta_{n_l} = \Delta_l$. Similarly, all $\tilde\varepsilon_n$ vanish except $\tilde\varepsilon_{n_l} = -\Delta_l$ and $\tilde\varepsilon_{n_l+1} = \frac{\Delta_l}{1-\Delta_l} = \Delta_l(1+\mathrm{O}(\Delta_l))$. It follows that for $n \in [(l-1)N, lN)$

$$\tilde A_n = P, \quad n \neq n_l, n_l+1$$

and

$$\begin{aligned} \tilde A_{n_l} &= P - \frac{1}{\operatorname{Im}\mu}\begin{pmatrix} 0 & 0 \\ -\Delta_l + \frac{\lambda^2}{2}\Delta_l & \lambda\Delta_l \operatorname{Im}\mu \end{pmatrix} \\ \tilde A_{n_l+1} &= P - \frac{\Delta_l}{\operatorname{Im}\mu}\begin{pmatrix} 0 & 0 \\ 1+\mathrm{O}(\Delta_l) & 0 \end{pmatrix}. \end{aligned}$$

Let us explain how to choose the sequence n_l. Note that for the above N and l, the matrices $\tilde A_{n_l}, \tilde A_{n_l+1}$ do not depend on n_l. Define the unit vector $e_0 = e_0(l)$ by the condition

$$\|\tilde A_{n_l+1}\tilde A_{n_l} e_0\| = \inf\big\{\|\tilde A_{n_l+1}\tilde A_{n_l} f\|,\ \|f\| = 1\big\}.$$

Then the sequence $\{n_l\}$ is defined according to Lemma 5.1 as follows. Choose $n_0 = 2$ and define $n_1 \in (0, N)$ such that

$$\left\| \left(\left\| W^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| \right)^{-1} P^{n_1} W^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - e_0(1) \right\| < \varepsilon.$$

This is possible by Lemma 5.1

Suppose that $n_1, \dots, n_{l-1}$ are already defined. Let $n \in [(l-1)N, lN)$. We have (assuming that $n > n_l$)

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = WP^{n-n_l-1}\tilde{A}_{n_l+1}\tilde{A}_{n_l}P^{n_l-(l-1)N-1}\cdot$$
$$\cdot P^{(l-1)N-n_{l-1}-1}\tilde{A}_{n_{l-1}+1}\tilde{A}_{n_{l-1}}\cdots\tilde{A}_{n_1+1}\tilde{A}_{n_1}P^{n_1}W^{-1}y$$
$$= WP^{n-n_l-1}\tilde{A}_{n_l+1}\tilde{A}_{n_l}P^{n_l-n_{l-1}-2}x_l,$$

where $x_l = \tilde{A}_{n_{l-1}+1}\tilde{A}_{n_{l-1}}\cdots P^{n_1}W^{-1}y$. Thus

$$\left\| \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \right\|^2 \leqslant \|W\|^2 \|\tilde{A}_{n_l+1}\tilde{A}_{n_l}P^{n_l-n_{l-1}-2}x_l\|^2, \tag{5.8}$$

and we define n_l by the condition

$$\|P^{n_l-n_{l-1}-2}x_l\|x_l\|^{-1} - e_0(l)\| < \varepsilon,$$

(using Lemma 5.1).

Now compute the product

$$\tilde{A}_{n_l+1}\tilde{A}_{n_l} = P\left[I - \frac{\Delta_l}{\operatorname{Im}\mu}P^*\begin{pmatrix} 0 & 0 \\ 1+\mathrm{O}(\Delta_l) & 0 \end{pmatrix}\right]$$
$$\times\left[I - \frac{\Delta_l}{\operatorname{Im}\mu}\left(\begin{pmatrix} 0 & 0 \\ \frac{\lambda^2}{2}-1 & \lambda\operatorname{Im}\mu \end{pmatrix}\right)P^*\right]P$$
$$= P\left[I - \Delta_l\begin{pmatrix} -1 & 0 \\ d(\lambda) & 1 \end{pmatrix}\right]P + \mathrm{O}_1(\Delta_l^2),$$

where $d(\lambda) = 2\frac{\operatorname{Re}\mu}{\operatorname{Im}\mu}$ and $\mathrm{O}_1(\Delta_l^2)$ is a 2×2 matrix with the norm of order $\mathrm{O}(\Delta_l^2)$.

Denote $T_l := \tilde{A}_{n_l+1}\tilde{A}_{n_l}$. We shall compute the square of the minimal s-number of T_l. First note that

$$T_l^*T_l = P^*\left[I - \Delta_l\begin{pmatrix} -2 & d(\lambda) \\ d(\lambda) & 2 \end{pmatrix} + \mathrm{O}_2(\Delta_l^2)\right]P,$$

where $\mathrm{O}_2(\Delta_l^2)$ is a 2×2 matrix with the norm of order $\mathrm{O}(\Delta_l^2)$. It follows that

$$s^2_{\min}(T_l^*T_l) = 1 - 2\frac{|\Delta_l|}{c(\lambda)} + O(\Delta_l^2).$$

Applying now Lemma 5.2 to the matrix $A = T_l^*T_l$ and the minimal eigenvector $e_0(l)$ for A, we have

$$\|T_lx_l\|^2 = (Ax_l, x_l) \leqslant [1 - \overline{\Delta}_l(1-\mathrm{O}(\varepsilon^2))]\,\|x_l\|^2, \tag{5.9}$$

provided x_l is a vector such that $\left\| \frac{x_l}{\|x_l\|} - e_0(l) \right\| < \frac{\varepsilon}{2}$ and $\overline{\Delta}_l = \frac{2\Delta_l}{c(\lambda)} - \mathrm{O}(\Delta_l^2)$. Due to the above construction of $\{u_n\}$ the series $\sum\limits_n \|u_n\|^2$ is majorized by $M \sum\limits_i \|u_{n_i}\|^2$, where $M = M(\varepsilon, \theta)$.

Note that by the choice of δ_n the series $\sum\limits_j \Delta_j^2$ is convergent. Let

$$w = \left\| W^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|^2 .$$

Combining (5.8) and (5.9) we have

(5.10)

$$\begin{aligned}
\sum_l \left\| \begin{pmatrix} u_{n_l} \\ u_{n_l+1} \end{pmatrix} \right\|^2 &\leqslant \|W\|^2 \sum_l [1 - \overline{\Delta}_l(1 - \mathrm{O}(\varepsilon^2))] \cdots [1 - \overline{\Delta}_1(1 - \mathrm{O}(\varepsilon^{\,2}))] w \\
&= \|W\|^2 \sum_l \exp\left(\sum_{j=1}^{l} \ln[1 - \overline{\Delta}_j(1 - \mathrm{O}(\varepsilon^{\,2}))] \right) w \\
&\leqslant w\|W\|^2 \sum_l \exp\left(- \sum_{j=1}^{l} \overline{\Delta}_j (1 - \mathrm{O}(\varepsilon^{\,2})) \right) \\
&\leqslant w\|W\|^2 \sum_l \exp\left(- q' \sum_{j=1}^{l} \overline{\Delta}_j \right),
\end{aligned}$$

for a certain $0 < q' < 1$, provided $\varepsilon \ll 1$. It follows that $u = \{u_n\} \in l^2$ if the last series is convergent, i.e. when

$$\sum_l \exp\,[-2q' c(\lambda)^{-1} \sum_{j=1}^{l} \Delta_j] < +\infty.$$

Evoking the definition of Δ_j we can rewrite the last condition as follows

$$\sum_m \exp\left(- 2q' c(\lambda)^{-1} \sum_{j=1}^{m} \delta_j \right) \sim \sum_m \exp(-2q' \gamma c(\lambda)^{-1} \ln m) \; < +\infty.$$

Therefore for $\gamma > \frac{c(\lambda)}{2q'}$ the equation (3.1) has a nontrivial solution. Note that for any $m \in [(l-1)N, lN)$ we have

$$F_m = 2 \sum_{i=1}^{l} |\tilde{\delta}_{n_i}| = 2 \sum_{i=1}^{l} \Delta_i \sim 2\gamma \sum_{i=1}^{l} i^{-1} \sim 2\gamma \ln l \sim 2\gamma \ln m$$

and similary

$$G_m = 2 \sum_{i=1}^{l} |\tilde{\delta}_{n_i}| \sim 2\gamma \ln l \sim 2\gamma \ln m.$$

Thus $f = g = 2$ and $2(\tilde{c_1}(\lambda) + \tilde{c_2}(\lambda)) = \frac{2}{c(\lambda)}$. Applying (5.5) we have

$$2(c_1(\lambda) + c_2(\lambda)) \geqslant \frac{2}{c(\lambda)}.$$

The last inequality proves sharpness of $c_2(\lambda) = \frac{|\lambda|}{2c(\lambda)}$ near $\lambda = \pm 2$ and simultanously confirms that $c_1(\lambda) = 1$ is sharp near $\lambda = 0$.

6. GROUPING PAIRWISE

In this section we propose yet another approach to the same problem of finding when $\sigma_{\mathrm{p}}(J) \cap (-2,2) = \emptyset$. The idea is based on the simple observation that $P_\theta \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while $P_\theta^2 \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, as $\theta \to \frac{\Pi}{2}$. Therefore P_θ^2 alters vectors "less" than P_θ. Write

$$\begin{aligned} A_n &= \left[I - \frac{1}{\operatorname{Im}\mu} \begin{pmatrix} 0 & 0 \\ \eta_n \operatorname{Re}\mu & -\varepsilon_n \operatorname{Im}\mu \end{pmatrix} \right] P \\ A_{n+1} &= P \left[I - \frac{1}{\sin\theta} \begin{pmatrix} -(\varepsilon_{n+1} + \frac{\lambda^2}{2}\delta_{n+1})\sin\theta & -\lambda\delta_{n+1}\sin^2\theta \\ (\varepsilon_{n+1} + \frac{\lambda^2}{2}\delta_{n+1})\cos\theta & \lambda\delta_{n+1}\sin\theta\cos\theta \end{pmatrix} \right]. \end{aligned}$$

We have

$$A_{n+1}A_n = P[I + T_n + R_n]P,$$

where $R_n = \lambda\tilde{R}_n$, $\|\tilde{R}_n\| = \mathrm{O}(\delta_{n-1}^2 + \delta_n^2 + \delta_{n+1}^2)$ and T_n can be explicitely computed,

$$T_n = \frac{-1}{\sin\theta} \begin{pmatrix} -(\varepsilon_{n+1} + \frac{\lambda^2}{2}\delta_{n+1})\sin\theta & -\lambda\delta_{n+1}\sin^2\theta \\ (\varepsilon_{n+1} + \varepsilon_n + 2\delta_n + \frac{\lambda^2\delta_{n+1}}{2})\cos\theta & (2\delta_{n+1}\cos^2\theta - \varepsilon_n)\sin\theta \end{pmatrix}.$$

It follows that $(A_{n+1}A_n)^*(A_{n+1}A_n) = P^*[I + T_n^* + T_n + S_n]P$, where $\|S_n\| = \mathrm{O}(\delta_{n-1}^2 + \delta_n^2 + \delta_{n+1}^2)$. Hence the minimal s-number s_n of $A_{n+1}A_n$ is given by

$$\begin{aligned} s_n^2 = 1 + \varepsilon_{n+1} + \varepsilon_n - \Big[&(\varepsilon_{n+1} - \varepsilon_n + 4\delta_{n+1}\cos^2\theta)^2 \\ &+ \frac{\cos^2\theta}{\sin^2\theta} \big(\varepsilon_{n+1} + \varepsilon_n + 2(\delta_n - \delta_{n+1}) + 4\delta_{n+1}\cos^2\theta\big)^2 \Big]^{\frac{1}{2}} \\ &+ \mathrm{O}(\delta_{n-1}^2 + \delta_n^2 + \delta_{n+1}^2). \end{aligned}$$

Denote $c_4(\lambda) = \lambda^2(1 + c_2(\lambda))$. Then

$$s_n^2 \geqslant 1 + \varepsilon_{n+1} + \varepsilon_n - |\varepsilon_{n+1} - \varepsilon_n| \\ - \frac{|\lambda|}{2c(\lambda)}|\varepsilon_{n+1} + \varepsilon_n + 2(\delta_n - \delta_{n+1})| - |\delta_{n+1}|c_4(\lambda) + \mathrm{O}(\delta_{n-1}^2 + \delta_n^2 + \delta_{n+1}^2).$$

It turns out that the appearance of λ^2 instead of λ^1 in $c_4(\lambda)$ is connected with the following algebraic fact: the rotation matrix P_θ becomes an antihermitian one as $\lambda \to 0$. Additionally, we had to refer here to simple properties of anti-commutator. Actually, the anti-commutator $\{A, B\}$ of the anti-hermitian operator A and the hermitian operator B is an anti-hermitian operator. This phenomenon is a consequence of pairwise grouping procedure. The method used in Section 3 gave the comutator $[A, B]$, a hermitian matrix for the above mentioned A and B, and this caused that $c_2(\lambda) = \mathrm{O}(\lambda), \lambda \to 0$. In order to check the above comments we have to omit all terms with ε_n and the ones of order $\mathrm{O}(\lambda^2)$ in the formula for A_n. We do not enter into details.

Considering the product $(A_{2n}A_{2n-1}) \cdots (A_2A_1)$ and repeating the reasoning given in the proof of Theorem 3.1 we obtain :

Theorem 6.1. *If λ satisfies the condition*

$$\sum_n \exp\Big\{ -q\sum_{k=1}^{n}[-2\min(\varepsilon_{2k-1}, \varepsilon_{2k}) \\ + c_2(\lambda)|\varepsilon_{2k-1} + \varepsilon_{2k} + 2(\delta_{2k-1} - \delta_{2k})| + |\delta_{2k}|c_4(\lambda)]\Big\} = +\infty$$

for some $q > 1$, then $\lambda \notin \sigma_{\mathrm{p}}(J) \cap (-2, 2)$.

Observe that the last condition is not comparable to the previous one (given in Theorem 3.1). This can be easily seen by considering the following examples.

(a) Let $\lambda_k = 1 + \frac{c}{k}$, $c > 0$.

Direct computation and Theorem 6.1 show that $\lambda_0 \notin \sigma_{\mathrm{p}}(J) \cap (-2, 2)$ provided λ_0 satisfies the inequality:

$$qcc_4(\lambda_0) \leqslant 2 \quad \text{for some } q > 1.$$

On the other hand it may happen (for the same λ_0) that

$$q|\lambda_0|c > c(\lambda_0),$$

and then, in turn, the condition of Theorem 3.1 does not hold for this λ_0.

(b) Let $\lambda_k = [1 - (-1)^k \frac{c}{k}]^{-1}$, $c > 0$.

The condition of Theorem 3.1 can be expressed as $\sum_n n^{-2qc} = +\infty$, and so it holds, if $2qc \leqslant 1$, for some $q > 1$. Now the condition of Theorem 6.1 is equivalent to

$$\sum_n n^{-2qc[1+c_2(\lambda)+\frac{c_4(\lambda)}{4}]} = +\infty.$$

By choosing c and λ such that $2qc \leqslant 1$ and $2qc(1+c_2(\lambda)+\frac{c_4(\lambda)}{4}) > 1$ we conclude that Theorem 6.1 does not apply but Theorem 3.1 does. Assume additionaly that $\{\delta_k\} \in l^2$.

By definition of F_n and G_n one can easily verify that

$$\sum_{k=1}^{n}[-2\min(\varepsilon_{2k-1},\varepsilon_{2k}) + c_2(\lambda)|\varepsilon_{2k-1}+\varepsilon_{2k}+2(\delta_{2k-1}-\delta_{2k})| + c_4(\lambda)|\delta_{2k}|]$$
$$\leqslant [1+3c_2(\lambda)+\frac{1}{2}c_4(\lambda)]F_{2n} + \frac{1}{2}c_4(\lambda)G_{2n} + r_n,$$

where $r_n = \mathrm{O}(1)$.

COROLLARY 6.2. *If* $\{\delta_k\} \in l^2$ *and*

$$\sum_n \exp\left\{-\left[\left(1+3c_2(\lambda)+\frac{1}{2}c_4(\lambda)\right)F_n + \frac{1}{2}c_4(\lambda)G_n\right]\right\} = +\infty,$$

then $\lambda \notin \sigma_{\mathrm{p}}(J)$.

Therefore Theorem 6.1 is better near $\lambda = 0$ than Theorem 3.1 and vice versa, Theorem 3.1 is more precise near the boundary points ± 2.

REMARK 6.3. Note that all results found in this paper give criteria when an individual number λ does not belong to $\sigma_{\mathrm{p}}(J)$. Therefore one can find examples of weights which satisfy conditions of some of our theorems but they do not ones of Dombrowski. Her result [3], Theorem 1 may be expressed as follows: if $\sum_k(\delta_k - \delta_{k-1})^- < \infty$ then $\sigma_{\mathrm{p}}(J)$ is empty, where

$$a^- = \begin{cases} |a|, & a < 0; \\ 0, & a > 0. \end{cases}$$

Put

$$\delta_k = \begin{cases} \frac{a_1}{k}, & \text{when } k \text{ is even}, \\ \frac{a_2}{k}, & \text{when } k \text{ is odd}; \end{cases}$$

where $-1 < a_1 - a_2 < 0$. Then $\sum_k(\delta_k - \delta_{k-1})^- = +\infty$ but one can easily check that Theorem 6.1 works for $|\lambda|$ sufficiently small if $a_1 - a_2 \in (-1, 0)$.

We end up the paper with a simple comment on sharpness of Theorem 6.1 at $\lambda = 0$. Namely as we noticed in Remark 3.2, $0 \notin \sigma_{\mathrm{p}}(J)$ iff

$$\sum_n \left(\prod_{l=1}^{n} \frac{\lambda_{2l-1}}{\lambda_{2l}} \right)^2 = +\infty. \tag{6.1}$$

In other words, $0 \notin \sigma_{\mathrm{p}}(J)$ iff

$$\sum_n \exp\left(2 \sum_{k=1}^{n} \varepsilon_{2k} \right) = +\infty$$

for some $q > 1$. By Theorem 6.1 we know that $0 \notin \sigma_{\mathrm{p}}(J)$ provided

$$\sum_n \exp\left(2q \sum_{k=1}^{n} \min(\varepsilon_{2k}, \varepsilon_{2k-1}) \right) = +\infty.$$

Therefore in the case $\min(\varepsilon_{2k}, \varepsilon_{2k-1}) = \varepsilon_{2k}$ for k sufficiently large, the condition of Theorem 6.1 reduces to (6.1) and so Theorem 6.1 is sharp for $\lambda = 0$.

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Received February 18, 1997.