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ALMOST MULTIPLICATIVE MORPHISMS AND ALMOST COMMUTING MATRICES

GUIHUA GONG and HUAXIN LIN

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Abstract. We prove that a contractive positive linear map which is approximately multiplicative and approximately injective from $C(X)$ into certain unital simple C^*-algebras of real rank zero and stable rank one is close to a homomorphism (with finite dimensional range) if a necessary K-theoretical obstruction vanishes and dimension of X is no more than two. We also show that the above is false it the dimension of X is greater than 2, in general.

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0. INTRODUCTION

Let X be a compact metric space. A contractive positive linear map $\psi : C(X) \to A$, where A is a C^*-algebra, is said to be δ-$\mathcal{F}$-multiplicative, if

$$\|\psi(fg) - \psi(f)\psi(g)\| < \delta$$

for all $f \in \mathcal{F}$. A homomorphism is certainly δ-$\mathcal{F}$-multiplicative. The purpose of this article is to study when such a δ-$\mathcal{F}$-multiplicative contractive positive linear morphism is actually close to a homomorphism. A classical problem is whether, for any $\varepsilon > 0$ there is $\delta > 0$ such that for any n and any pair of selfadjoint matrices $x, y \in M_n(\mathbb{C})$ such that $\|x\|, \|y\| \leqslant 1$ and $\|xy - yx\| < \delta$, there exists a commuting pair of $x', y' \in M_n(\mathbb{C})$ of selfadjoint matrices with $\|x' - x\| + \|y - y'\| < \varepsilon$. It was an old open problem for decades in linear algebra and operator theory which was solved affirmative recently (see [52]). This result is equivalent to the following: For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \in C(\mathbb{D})$, where $\mathbb{D}$ is the unit disk, there is $\delta > 0$ and a finite subset $\mathcal{G} \in C(\mathbb{D})$ satisfying: for any finite-dimensional C^*-algebra, A and any δ-$\mathcal{G}$-multiplicative contractive positive linear morphism $\psi : C(\mathbb{D}) \to A$, there is a homomorphism $h : C(\mathbb{D}) \to A$ (with finite-dimensional range) such that

$$\|\psi(f) - h(f)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

Perturbation of homomorphisms appear in many area of mathematics. Limited to our knowledge, we encounter almost multiplicative morphisms in operator theory, classification of C^*-algebra extensions and more recently, classification of nuclear C^*-algebras. In fact, the old problem mentioned above attracted many researchers' attention (see [1], [2], [10], [20], [31], [32], [41], [42], [52], [60], [64], [65], [72], [75] and many more).

In the case that A is a unital purely infinite simple C^*-algebra, it is shown ([53]) that a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism $L : C(X) \to A$ is close to a homomorphism on a given finite subset $\mathcal{F} \subset C(X)$, provided that δ is small enough and $\mathcal{G}$ is large enough.

In general, however, a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism, is not close to a homomorphism no matter how small δ is and how large $\mathcal{G}$ is. This was first discovered by D. Voiculescu (see [74]). The K-theoretical obstruction was later explained by T.A. Loring (see [60]). Therefore, what we are hoping for is that a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism is close to a homomorphism, provided that δ is sufficiently small and $\mathcal{G}$ is sufficiently large, as well as the K-theoretical obstruction vanishes. Since every compact metric space X is a subspace of a contractible space Ω, a contractive positive linear morphism $\psi : C(X) \to A$ can always be viewed as a contractive positive linear morphism from $C(\Omega)$ into A. Therefore, in general, some injectivity condition has to be imposed so that we know which obstacle has to vanish.

Among other things, the main results are the following.

The Main Theorem. *Let X be a compact metric space with dimension no more than 2 and let $\mathcal{F}$ be a finite subset of (the unit ball of) $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\mathbf{P}(C(X))$, $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G}$ of (the unit ball of) $C(X)$ such that, whenever A is a unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $\mathrm{K}_0(A)$ and unique normalized quasitrace and whenever $\psi : C(X) \to A$ is a contractive unital positive linear map which is δ-$\mathcal{G}$-multiplicative and is σ-injective with respect to δ and $\mathcal{F}$*

and $\psi_*(\mathcal{P}) \in \mathcal{N}$ *then there exists a unital homomorphism* $\varphi : C(X) \to A$ *with finite dimensional range such that*

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Here $\psi_*(\mathcal{P}) \in \mathcal{N}$ simply means that the KK-obstacle vanishes. $\mathbf{P}(C(X))$, $\mathcal{N}$ and σ-injectivity will be defined below.

In the following corollaries, $\mathbb{A}$ is the set of all unital simple C^*-algebras of real rank zero, stable rank one, with weakly unperforated $\mathrm{K}_0(A)$ and with a unique normalized quasitrace.

Sometimes, however, we do not need to worry about injectivity.

COROLLARY M1. *Let* X *be a compact metric space of dimension no more than* 2. *For any* $\varepsilon > 0$ *and any finite subset* $\mathcal{F}$ *of* $C(X)$, *there exist* $\delta > 0$ *and a finite subset* $\mathcal{P}$ *of* $\mathbf{P}(C(X))$, *and a finite subset of* $C(X)$ *such that whenever* $A \in \mathbb{A}$, $\mathrm{K}_1(A) = 0$ *and* $\mathrm{K}_0(A)$ *is torsion free and whenever* $\psi : C(X) \to A$ *is a contractive unital positive linear map which is* δ-$\mathcal{G}$-*multiplicative and* $\psi_*(\mathcal{P}) \in \mathcal{N}$, *then there exists a unital homomorphism* $h : C(X) \to A$ *with finite dimensional range such that*

$$\|\psi(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

COROLLARY M2. *For any* $\varepsilon > 0$, *there is* $\delta > 0$ *so that, whenever* $A \in \mathbb{A}$, *if* $h_1, h_2 \in A$ *are two selfadjoint elements with* $\|h_1\|, \|h_2\| \leqslant 1$ *and*

$$\|h_1 h_2 - h_2 h_1\| < \delta$$

then there exists a pair of commuting selfadjoint elements $s_1, s_2 \in A$ *such that*

$$\|h_1 - s_1\| < \varepsilon \quad \textit{and} \quad \|h_2 - s_2\| < \varepsilon.$$

COROLLARY M3. *For any* $\varepsilon > 0$, *there is* $\delta > 0$ *so that, whenever* $A \in \mathbb{A}$, *if* u *and* v *are two unitaries in* A,

$$\|uv - vu\| < \delta \quad \textit{and} \quad \kappa(u, v) = 0$$

then there exist commuting unitaries $u_1, v_1 \in A$ *such that*

$$\|u - u_1\| < \varepsilon \quad \textit{and} \quad \|v - v_1\| < \varepsilon.$$

(If $\mathrm{K}_1(A) = 0$*,* u_1 *and* v_1 *can be required to have finite spectrum.) Further, if* $\mathrm{K}_0(A)$ *is a dimension group, then the condition that* $\kappa(u,v) = 0$ *can be replaced by* $\tau(\kappa(u,v)) = 0$*, where* τ *is the normalized quasitrace.*

See [32] for the definition of $\kappa(u,v)$ (also 2.1 in [49]).

It turns out, a little surprise to us, that, even with the injectivity and vanishing KK-obstacle, a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism ψ from a $C(X)$ into C^*-algebra A may not be close to a homomorphism when $\dim(X) \geqslant 3$, no matter how small δ is and how large $\mathcal{G}$ is. Please see Section 4 for higher dimension cases. The main technical lemma is stated in Section 1 which is extracted from the proof in [52]. However, the version in this article is in a much more general form and somewhat complicated because of topological complication. Those readers who care more about matrices than KK-theory could simply ignore anything related to KK-theory or K-theory. One can simply assume that all K-theoretical obstacles vanish at least for the case that X is a compact subset of the plane and algebras has no K_1. We prove this lemma in Section 1 without proving three lemmas which are needed for the proof. These three lemmas together with some other related matters will be proved in Section 2. Section 3 contains the proof of the main theorem and its corollaries. In Section 4, we show that, in general, the main theorem does not hold for a space X with dimension greater than 2.

Here are some conventions which are needed in the rest of this paper.

0.1. DEFINITION. Let A be a C^*-algebra, X be a compact metric space and $\varphi : C(X) \to A$ be a homomorphism. Let B be the weak closure of $\varphi(C(X))$ in A^{**}, the enveloping W^*-algebra of A. Let C be the C^*-algebra of all bounded Borel functions on X. Then φ induces a homomorphism from C into B. Let S be a Borel subset of X and κ_S be the characteristic function of S and p_S be the image of κ_S in B. We call p_S the *spectral projection* (of φ) *corresponding to the subset* S. Let O be an open subset of X and D be the hereditary C^*-subalgebra of A generated by $\varphi(h)$, where $h(\xi) > 0$ for all $\xi \in O$ and $h(\xi) = 0$ for all $\xi \in X \setminus O$ and $h \in C(X)$. The projection p_O is *the open projection corresponding to the hereditary* C^**-subalgebra* D.

0.2. DEFINITION. Let $\psi : C(X) \to C$ be a homomorphism, where C is a C^*-algebra. Let Ω be the compact subset such that

$$\ker(\psi) = \{f \in C(X) \mid f(\xi) = 0 \text{ for all } \xi \in \Omega\}.$$

We will denote Ω by $\mathrm{sp}(\psi)$.

0.3. DEFINITION. (cf. 1.2 of [58]) Let ψ be a contractive positive linear map from $C(X)$ to C^*-algebra A, where X is a compact metric space. Fix a finite subset $\mathcal{F}$ contained in the unit ball of $C(X)$. For $\varepsilon > 0$, we denote by $\Sigma_\varepsilon(\psi, \mathcal{F})$ (or simply $\Sigma_\varepsilon(\psi)$) the closure of the set of those points $\lambda \in X$ for which there is a nonzero hereditary C^*-subalgebra B of A which satisfies

$$\|(f(\lambda) - \psi(f))b\| < \varepsilon \quad \text{and} \quad \|b(f(\lambda) - \psi(f))\| < \varepsilon$$

for $f \in \mathcal{F}$ and $b \in B$ with $\|b\| \leqslant 1$. Note that if $\varepsilon < \sigma$, then $\Sigma_\varepsilon(\psi) \subset \Sigma_\sigma(\psi)$.

We say ψ is *σ-injective with respect to δ and $\mathcal{F}$*, or *σ-$\mathcal{F}$-injective*, if $\Sigma_\delta(\psi, \mathcal{F})$ is σ-dense in X.

It follows from 1.12 in [53] that, for any $\varepsilon > 0$ and $\mathcal{F}$, for any δ-$\mathcal{G}$-contractive positive linear map ψ, if δ is sufficiently small and $\mathcal{G}$ is sufficiently large, $\Sigma_\varepsilon(\psi, \mathcal{F})$ is not empty.

It is important to know, by 1.17 in [53], that, for any $1 > \sigma > 0$, with sufficiently small δ and sufficiently large $\mathcal{G}$, a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism $\psi : C(X) \to A$ can be replaced by a ε-$h(\mathcal{G})$-multiplicative contractive positive linear morphism $\varphi : C(F) \to A$ which is σ-injective with respect to ε and $h(\mathcal{G})$, where F is a compact subset of X and $h : C(X) \to C(F)$ is the surjective map induced by the inclusion $F \to X$ (see Lemma 3.15).

0.4. Definition. Let B be a C^*-algebra and X be a compact metric space. A homomorphism $\psi : C(X) \to B$ has *finite dimensional range* if (and only if) there exist a finite subset $\{\xi_i\}_{i=1}^l \subset X$ and a finite subset of mutually orthogonal projections $\{p_i\}_{i=1}^l \subset B$ such that

$$\psi(f) = \sum_{i=1}^{l} f(\xi_i)p_i \quad \text{for all } f \in C(X).$$

0.5. Definition. Let X be a finite CW-complex and let A be a unital C^*-algebra. Suppose that $\varphi : C(X) \to A \otimes \mathcal{K}$ (or $\varphi : C_0(X) \to A \otimes \mathcal{K}$ if X is not compact) is a homomorphism and $\xi_1, \xi_2, \dots, \xi_m \in X$ are points in each (compact) connected component of X. Let $Y = X \setminus \{x_1, \dots, x_m\}$. Let $\varphi_0 : C_0(Y) \to A \otimes \mathcal{K}$ be the restriction of φ. Let $[\varphi]$ be the element in $\mathrm{KK}(C(X), A)$ and let $[\varphi_0]$ be an element in $\mathrm{KK}(C_0(Y), A)$. We denote by $\mathcal{N}'(X, A)$ (or just $\mathcal{N}'$ if X and A are understood) the set of those elements in $\mathrm{KK}(C(X), A)$ which are represented by those φ such that $[\varphi_0] = 0$. Given m mutually orthogonal projections $p_1, p_2, \dots, p_m \in A \otimes \mathcal{K}$, define $\varphi'(f) = \sum_{i=1}^{m} f(\xi_i)p_i$ for $f \in C(X)$. Then $[\varphi'] \in \mathcal{N}'$. Conversely, if $[\varphi] \in \mathcal{N}'$, let $f_1, f_2, \dots, f_m$ be projections in $C(X)$ corresponding to each component of X, and let $\varphi(f_i) = p_i$, $i = 1, 2, \dots, m$; then $[\varphi] - [\varphi'] = 0$ in $\mathrm{KK}(C(X), A)$. In fact, from the six-term exact sequence in KK-theory, the map from $\mathrm{KK}(C(X), A)$ into $\mathrm{KK}(C_0(Y), A)$ maps both $[\varphi]$ and $[\varphi']$ into zero. So they both are in the image of the map from $\mathrm{KK}(C(X)/C_0(Y), A)$. Note that $C(X)/C_0(Y)$ is m copies of $\mathbb{C}$ corresponding to the m components. From the choice of φ', they are both from the same element in $\mathrm{KK}(C(X), A)$.

Now let X be any compact metric space. Then $C(X) = \lim_{n\to\infty} C(X_n)$, where X_n is a finite CW-complex. There is a surjective map $s : \mathrm{KK}(C(X), A) \to \lim_{n\to\infty} \mathrm{KK}(C(X_n), A)$. We denote by $\mathcal{N}'$ the set of those elements x in $\mathrm{KK}(C(X), A)$ such that $s(x) \in \lim_{n\to\infty} \mathcal{N}'(X_n, A)$ for any sequence of finite CW-complexes $\{X_n\}$.

Recall that $\mathrm{KL}(C(X), A)$ is the quotient of $\mathrm{KK}(C(X), A)$ by the subgroup of pure extensions in $\mathrm{Ext}(\mathrm{K}_*(C(X)), \mathrm{K}_{*-1}(A))$ (see [70]).

We denote by $\mathcal{N}$ the image of $\mathcal{N}'$ in $\mathrm{KL}(C(X), A)$.

We will write $\Gamma(\varphi) = 0$, if φ induces an element in $\mathcal{N}$. If Y is an open subset of X, and $\varphi : C_0(Y) \to A$ is a homomorphism, then we write $\Gamma(\varphi) = 0$, if $\Gamma(\widetilde{\varphi}) = 0$, where $\widetilde{\varphi}$ is the unital homomorphism from $C(\widetilde{Y}) \to A$ and $\widetilde{Y}$ is the one-point compactification of Y.

0.6. Definition. The standard definition of mod-p K-theory for C^*-algebras as given by Schochet in [73], is

$$\mathrm{K}_i(A; \mathbb{Z}/n) = \mathrm{K}_i(A \otimes C_0(C_n)),$$

where C_n is the 2-dimensional CW-complex obtained by attaching a 2-cell to $\mathbb{S}^1$ via the degree n map from $\mathbb{S}^1$ to $\mathbb{S}^1$ (notice that $\mathrm{K}_0(C_0(C_n)) = \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{K}_1(C_0(C_n)) = \{0\}$). Let A be a C^*-algebra; following [18], we denote

$$\underline{\mathrm{K}}(A) = \bigoplus_{\substack{i=0,1\\ n\geqslant 0}} \mathrm{K}_i(A; \mathbb{Z}/n).$$

By [18], there is an isomorphism from $\mathrm{KL}(C(X), A)$ onto $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(C(X)), \underline{\mathrm{K}}(A))$. Note that

$$\mathrm{K}_0(A \otimes C(C_m \times S^1)) \cong \mathrm{K}_0(A) \oplus \mathrm{K}_1(A) \oplus \mathrm{K}_0(A; \mathbb{Z}/m) \oplus \mathrm{K}_1(A; \mathbb{Z}/m).$$

We define $\underline{\mathrm{K}}(A)_+$ to be the semigroup of $\underline{\mathrm{K}}(A)$ generated by $\mathrm{K}_0(A \otimes C(C_m \times \mathbb{S}^1))_+$, $m \geqslant 2$. There is an obvious surjective map from $\bigcup_{m>0} \mathrm{K}_0(A \otimes C(C_m \times \mathbb{S}^1)$ onto $\underline{\mathrm{K}}(A)$.

0.7. Let A be a C^*-algebra. Denote by $\mathbf{P}(A)$ the set of projections in $\bigcup_{m\geqslant 0} M_\infty(A \otimes C(C_m \times \mathbb{S}^1))$. Let $\mathcal{P}$ be a finite subset in $\mathbf{P}(A)$. There exist a finite subset $\mathcal{G}(\mathcal{P}) \subset A$ and $\delta(\mathcal{P}) > 0$ such that if B is any C^*-algebra and $\varphi : A \to B$ is a $*$-preserving linear map which is $\delta(\mathcal{P})$-$\mathcal{G}(\mathcal{P})$-multiplicative, then

$$\|((\varphi \otimes \mathrm{id})(p))^2 - (\varphi \otimes \mathrm{id})(p)\| < \frac{1}{4}$$

for all $p \in \mathcal{P}$. Hence, for each $p \in \mathcal{P}$, there is a projection $q \in \mathbf{P}(B)$ such that

$$\|(\varphi \otimes \mathrm{id})(p) - q\| < \frac{1}{2}.$$

Furthermore, if q' is another projection satisfying the same condition, then $\|q - q'\| < 1$, hence q is unitarily equivalent to q'. Let $\overline{\mathcal{P}}$ be the image of $\mathcal{P}$ in $\underline{\mathrm{K}}(A)$. For each $p \in \mathcal{P}$, we set $\varphi_*([p]) = [q]$. This defines a map $\varphi_* : \overline{\mathcal{P}} \to \underline{\mathrm{K}}(B)$.

Let $\alpha : \overline{\mathcal{P}} \to \underline{\mathrm{K}}(B)$. Suppose that there is a homomorphism $\psi : C(X) \to M_k(B)$ for some integer k with finite dimensional range such that $\psi_* = \alpha : \overline{\mathcal{P}} \to \underline{\mathrm{K}}(B)$. Then we write $\alpha(\overline{\mathcal{P}}) \in \mathcal{N}$.

The results of this paper were reported by the first named author at the 1995 West Coast Operator Algebra Seminar held at Eugene, Oregon. When this paper was being finalized, there have been some related development. First, Friis and Rørdam obtained a short proof of the result in [52] (see [33]) and then, Terry Loring gave further interesting generalizations ([63]). We consider only those simple C^*-algebra of real rank zero with unique normalized quasitrace. The case when the C^*-algebras are purely infinite and simple is considered in [53]. For more general finite simple C^*-algebras, similar results will appear in [57].

1. A TECHNICAL LEMMA

1.1. For any $\varepsilon > 0$, and a fixed finite subset $\mathcal{F} \subset C(X)$, let $\delta_{\mathrm{c}}(\varepsilon, \mathcal{F}) > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $\mathrm{dist}\,(x, y) < \delta_{\mathrm{c}}(\varepsilon, \mathcal{F})$.

Suppose that $\varphi : C(X) \to A$ is a monomorphism, where A is a C^*-algebra of real rank zero and stable rank one and X is a compact metric space.

Suppose that there are ideals

$$A = I_0 \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} = 0,$$

where I_{i+1} is an ideal of I_i. We denote by $\pi_i : A \to A/I_i$ the quotient map.

We will also use π_i for the quotient map from $M_L(A)$ onto $M_L(A/I_i)$ for any integer $L > 0$.

By [77], if $q \in A/I_i$ is a projection, there is a projection $p \in A$ such that $\pi_i(p) = q$. We will use this fact repeatedly without further explanation.

For each i, there is a monomorphism $\varphi_i : C(X_i) \to A/I_i$ induced by $\varphi, i = 1, 2, \ldots, n+1$, where X_i are compact subsets of X. (Note $X_{n+1} = X$.)

Let $Y_1 = X_1, Y_{i+1} = X_{i+1} \setminus X_i$ and $Z_{i+1} = X \setminus X_i$. Let $s_i : C_0(Z_i) \to C_0(Y_i)$ be the natural surjection.

There are also monomorphisms $\psi_i : C_0(Y_i) \to I_{i-1}/I_i$ induced by φ. Note that $C_0(Y_i)$ is an ideal of $C(X_i)$ and $C_0(Z_i)$ is an ideal of $C(X)$. To simplify the notation, we will sometime use ψ_i for $\psi_i \circ s_i$.

We also denote by $C(X)_1$ the unit ball of $C(X)$.

Condition (A). A map $\psi_i : C_0(Y_i) \to I_{i-1}/I_i$ is said to satisfy condition (A), if, for any finite subset $\mathcal{F} \subset C_0(Y_i)$, for any $\varepsilon > 0$, there is a homomorphism $h : C_0(Y_i) \to I_{i-1}/I_i$ such that

$$\|\psi(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, where $h(f) = \sum_{k=1}^{m(i)} f(\xi_k^{(i)}) e_k^{(i)}$, where ξ_k are points in Y_i.

(So far are just notations).

1.2. TECHNICAL LEMMA. *Let X be a compact metric space with covering dimension no more than 2. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Suppose that, for each i, there are projections $e \in I_{i-1}$ such that*

$$p \oplus p \oplus \cdots \oplus p \lesssim e$$

for any number of copies of any projection $p \in I_i$. We also assume the following:

(a) *For any $\lambda \in X_i$, any neighborhood $O(\lambda)$ and any k, there are mutually orthogonal projections $e_1, e_2, \ldots, e_k \in H_{O(\lambda)}$ such that*

$$p \oplus p \oplus \cdots \oplus p \lesssim e_m, \quad m = 1, 2, \ldots, k$$

for any number of copies of any projection $p \in I_i$, where $H_{O(\lambda)}$ is the hereditary C^*-subalgebra generated by $\varphi(h)$, where $h \in C(X)$ with $h > 0$ in $O(\lambda)$ and zero outside $O(\lambda)$.

(b) (*no* KK-*obstacle*) $\Gamma(\varphi_i) = 0$ *for all* i.

(c) *The map* $\varphi_1 : C(Y_1) \to A/I_1$ *satisfies Condition* (A).

(d) *The maps* $\psi_i : C_0(Y_i) \to I_{i-1}/I_i$ *satisfy Condition* (A);

or

(d′) $\Gamma(\psi_i) = 0$ *and*

$$\operatorname{dist}(X_i, \xi) < \frac{\delta_c\left(\frac{\varepsilon}{4}, \mathcal{F}\right)}{2} \quad \text{for all } \xi \in X_{i+1},$$

(*either* (d) *or* (d′)).

(e) $\mathrm{K}_1(J/I) = 0$ *and* $\mathrm{K}_0(J/I)$ *is torsion free for any pair of ideals* $J \supset I \supset I_n$*, and* ψ_{n+1} *satisfies* (d′).

Then there are mutually orthogonal projections $p_1, p_2, \dots, p_m \in A$ *such that*

$$\Big\|\varphi(f) - \sum_{k=1}^{m} f(\lambda_k)p_k\Big\| < \varepsilon,$$

for all $f \in \mathcal{F}$*, where* λ_k *are (fixed) points in* X.

To prove this we need three lemmas.

1.3. LEMMA (Dig). (Lemma 2.5 plus Lemma 2.6 in [52]) *For any* $\varepsilon > 0$*, any* $\eta > 0$*, any positive numbers* $a_2, \dots, a_n$*, and any finite subset* $\mathcal{F}$ *of* $C(X)_1$*, there are* $\delta = \delta_{\mathrm{dig}}(\varepsilon, \eta, \mathcal{F}) > 0$*, a finite subset* $\mathcal{G} = \mathcal{G}_{\mathrm{dig}}(\varepsilon, \eta, \mathcal{F}) \subset C(X)_1$*, a finite subset* $\{\lambda_k^{(i)}\} \subset Y_i$ *which is* η*-dense in* Y_i *and finitely many mutually orthogonal projections* $e_k^{(i)} \in I_{i-1}$ $(I_0 = A)$ *such that*

(i)

$$\Big\|\varphi(f) - \sum_{i=1}^{n} \psi_1^{(i)}(f) - \Big(1 - \sum_{i=1}^{n} e_i\Big)\varphi(f)\Big(1 - \sum_{i=1}^{n} e_i\Big)\Big\| < \varepsilon.$$

(ii)

$$\Big\|\Big(1 - \pi_i\Big(\sum_{j=1}^{n} e_j\Big)\Big)\varphi_i(fg_i) - \varphi_i(fg_i)\Big(1 - \pi_i\Big(\sum_{j=1}^{n} e_j\Big)\Big)\Big\| < \varepsilon$$

for all $f \in \mathcal{G}$*, where* $\sum_k e_k^{(i)} = e_i$ *and* $\psi_1^{(i)}(f) = \sum_k f(\lambda_k^{(i)})e_k^{(i)}$*, and where* $0 \leqslant g_i \leqslant 1$*,* $g_i(t) = 0$ *if* $\operatorname{dist}(t, X_{i-1}) < a_i/4$ *and* $g_i(t) = 1$ *if* $\operatorname{dist}(t, X_{i-1}) \geqslant a_i/2$*,* $i = 2, \dots, n$*;*

(iii) *there is* $b > 0$ *such that, for any* $0 < \beta < b$*,* $\Lambda_i(g_\beta^{(i)} f) = \Lambda_i(g_\beta^{(i)})\Lambda_i(f)$ *for all* $f \in C(X)$*, where* $0 \leqslant g_\beta \leqslant 1$*,* $g_\beta^{(i)}(t) = 0$ *if* $\operatorname{dist}(t, X_{i-1}) < \beta/2$ *and* $g_\beta^{(i)}(t) = 1$ *if* $\operatorname{dist}(t, X_{i-1}) \geqslant \beta$*,* $i = 1, 2, \dots, n$*, and* $\Lambda_i = (1 - \pi_i(e_i))\varphi_i(1 - \pi_i(e_i))$*; and*

(iv) $p \oplus p \oplus \cdots \oplus p \lesssim e_k^{(i)}$ *for any copies of any projection* $p \in I_i$.

If ψ_i *satisfies Condition* (A)*, we can require*

(v)

$$\Big\|(1 - \pi_i(e_i))\varphi_i(fg_i)(1 - \pi(e_i)) - \sum_{k=1}^{m(i)} fg_i(\lambda_k^{(i)})d_k^{(i)})\Big\| < \varepsilon$$

for all $f \in \mathcal{F}$*,* $\{\lambda_k^{(i)}\}$ *is* η*-dense in* Y_i*, and*

$$p \oplus p \oplus \cdots \oplus p \lesssim d_k^{(i)}$$

for any copies of any projection $p \in I_i$.

(vi) *Furthermore, if*

$$\Big\|\Big(1-\pi_i\Big(\sum_{j=1}^{n} e_j\Big)\Big)\varphi_i(f)\Big(1-\pi_i\Big(\sum_{j=1}^{n} e_j\Big)\Big) \oplus H_1(f)-H_2(f)\Big\| < \delta$$

for all $f \in \mathcal{G}$, *where* $H_1 : C(X_i) \to M_{L_i}(A/I_i)$ *and* $H_2 : C(X_i) \to M_{L_i+1}(A/I_i)$ *are homomorphisms with finite dimensional range, then there are finitely many mutually orthogonal projections* $\{p_k\}$ *in* $M_{L_i+1}(A/I_i)$ *with*

$$p \oplus p \oplus \cdots \oplus p \lesssim p_k$$

for any copies of any projection $p \in I_i$, *and a finite subset* $\{\xi_k\}$ *in* X_i *which is* η*-dense in* X_i *such that*

$$\Big\|\Big(1-\pi_i\Big(\sum_{j=1}^{n} e_j\Big)\Big)\varphi_i(f)\Big(1-\pi_i\Big(\sum_{j=1}^{n} e_j\Big)\Big) \oplus H_1(f) - \sum_k f(\xi_k)\pi_i(p_k)\Big\| < \varepsilon$$

for all $f \in \mathcal{F}$.

1.4. LEMMA (Ap). (cf. 2.4 and 2.6 in [52]) *For any* $\varepsilon > 0$, $\sigma > 0$, *a finite subset* $\mathcal{F} \subset C(X)_1$, *and* $a_2, \dots, a_n$ *positive numbers there are* $\delta = \delta_{\mathrm{ap}}(\varepsilon, \sigma, \{a_i\}, \mathcal{F}) > 0$, *a finite subset* $\mathcal{G} = \mathcal{G}_{\mathrm{ap}}(\varepsilon, \sigma, \{a_i\}, \mathcal{F}) \subset C(X)_1$ *and positive numbers* $b_2, \dots, b_n$ *satisfying the following:*

(i) $0 \leqslant g_j, g'_j \leqslant 1$ *in* $C(X)$, $g_j(t) = 0$ *if* $\operatorname{dist}(t, X_{j-1}) < a_j/4$, $g_j(t) = 1$ *if* $\operatorname{dist}(t, X_{j-1}) \geqslant a_j/2$;

(ii) $g'_j(t) = 0$ *if* $\operatorname{dist}(t, X_{j-1}) < b_j/4$, $g'_j(t) = 1$ *if* $\operatorname{dist}(t, X_{j-1}) \geqslant b_j/2$, $j = 2, \dots, n$, *and* $g_1 = g'_1 = 1$;

(iii) *if* E *is a projection in* A *and*

$$\|\varphi_j(fg'_j) - (1-\pi_j(E))\varphi_j(fg'_j)(1-\pi_j(E)) - h'(fg'_j)\| < \delta$$

for all $f \in \mathcal{G}$, *where* $h' : C(X) \to \pi_j(E)(I_{j-1}/I_j)\pi_j(E)$ *is a homomorphism with finite dimensional range, then there are homomorphisms* $\psi_2^{(j)} : C_0(Y_j) \to QM_{L_j+1}(I_{j-1})Q$ *and* $\psi_3^{(j)} : C(X) \to M_{L_j}(I_{j-1})$ *with finite dimensional range such that*

$$\|(1-\pi_j(E))\varphi_j(fg_j)(1-\pi_j(E)) \oplus \pi_j \circ \psi_3^{(j)}(fg_j) - \pi_j \circ \psi_2^{(j)}(fg_j)\| < \varepsilon$$

for all $f \in \mathcal{F}$,

$$\psi_2^{(j)}(fg_i) = \sum_{k=1}^{m(j)} f(\zeta_k^{(j)})q_k^{(j)} \quad \textit{and} \quad \psi_3^{(j)}(f) = \sum_k f(\xi_k)d'_k$$

where $\{\zeta_k^{(j)}\}$ *is* σ*-dense in* Y_j*,* $q_k^{(j)}$ *are mutually orthogonal projections in* $QM_{L_j+1}(I_{j-1})Q$ *with*

$$p \oplus p \oplus \cdots \oplus p \lesssim q_k^{(j)}$$

for any copies of projections p *in* I_j*,* $Q = \mathrm{diag}\,((1-E), 1, 1, \ldots, 1)$*, and* $\xi_k \in Y_j$ *and* $g_j(\xi_k) = 1$ *(so* $\psi_3^{(j)}(fg_j) = \psi_3^{(j)}(f)$*, that is that* ξ_k *is at least* $a_j/2$ *distance from* X_{j-1}*).*

1.5. LEMMA (Ab). (Lemma 2.7 and 2.9 in [52]) *Let* A *be a unital* C^**-algebra of real rank zero and* I *be an ideal of* A *with* $\mathrm{K}_1(I) = 0$ *and torsion free* $\mathrm{K}_0(I)$*,* $\psi : C(X) \to A$ *be a unital positive linear map and* $\pi \circ \psi$ *be a unital positive map from* $C(Y) \to A/I$*, where* $\pi : A \to A/I$ *is the quotient map and* Y *is a compact subset of* X*. For any* $\varepsilon > 0$ *and a finite subset* $\mathcal{F} \subset C(X)_1$*, there exists* $\delta = \delta_{\mathrm{ab}}(\varepsilon, \mathcal{F}) > 0$*,* $\mathcal{G} = \mathcal{G}_{\mathrm{ab}}(\varepsilon, \mathcal{F}) \subset C(X)_1$*, a finite subset* $\mathcal{P} = \mathcal{P}_{\mathrm{ab}}(Y, \varepsilon, \mathcal{F}) \subset \mathbf{P}(C(Y))$ *and* $a = a_{\mathrm{ab}}(\varepsilon, \mathcal{F}) > 0$ *satisfying the following: if*

(i) $\|\pi \circ \psi(f) - h_1(f)\| < \delta$ *for all* $f \in \mathcal{G}$*, where* $h_1(f) = \sum_{k=1}^{m} f(\lambda_k')\pi(d_k)$*,* $\{\lambda_k'\}$ *is* $\delta_{\mathrm{c}}(\varepsilon/8, \mathcal{F})$*-dense in* Y *and* $\{d_k\}$ *are mutually orthogonal projections in* A *with* $p \oplus p \oplus \cdots \oplus p \lesssim d_k$ *for any copies of any projections* $p \in I$*;*

(ii) $\psi(g_\beta f) = \psi(g_\beta)\psi(f)$ *for all* $f \in C(X)$ *and* $0 < \beta < a$*; and*

(iii) $\|\psi(fg_{a/4}) - h'(fg_{a/4})\| < \delta$ $(0 \leqslant g_d \leqslant 1,\ g_d(t) = 0,$ *if* $\mathrm{dist}\,(t, Y) < d/4$ *and* $g_d(t) = 1,$ *if* $\mathrm{dist}\,(t, Y) \geqslant d/2)$ *for all* $f \in \mathcal{G}$*, where* $h' : C_0(X \setminus Y)$ *is a homomorphism with finite dimensional range;*

(iv) $\|\psi(fg) - \psi(f)\psi(g)\| < \delta$ *for all* $f \in \mathcal{G}$*; and*

(v) *(no* KK*-obstacle)* $\psi_*(\mathcal{P}) \in \mathcal{N}$*.*

Then there exists a homomorphism $h_2 : C(X) \to A$ *with finite dimensional range such that*

$$\|\psi(f) - h_2(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$*.*

We will prove these lemmas in the next section.

1.6. *Proof of Technical Lemma.* We may assume that $\mathcal{F} \subset C(X)_1$. We will apply Lemma 1.3 and repeatedly apply Lemma 1.5 and Lemma 1.4.

To apply these lemmas repeatedly, we let X be the same as in Lemma 1.5. We note that, in Lemma 1.5, δ and $\mathcal{G}$ do not depend on Y (but $\mathcal{P}$ does). We first let $\delta_1 = \delta_{\mathrm{ab}}(\varepsilon/4, \mathcal{F})/4$, $\mathcal{G}_1' = \mathcal{G}_{\mathrm{ab}}(\varepsilon/4, \mathcal{F})$ in Lemma 1.5, let $\delta_1' = \delta_{\mathrm{c}}(\varepsilon/8, \mathcal{F})$, let $\delta_1'' = \delta_{\mathrm{dig}}(\delta_1', \delta_1, \mathcal{G}_1')/4$ and let $\mathcal{G}_1'' = \mathcal{G}_{\mathrm{dig}}(\delta_1', \delta_1, \mathcal{G}_1')$ in Lemma 1.4. Then let $\delta_{i+1} = \delta_{\mathrm{ab}}(\delta_i'', \mathcal{G}_i'')/4$, $\mathcal{G}_{i+1}' = \mathcal{G}_{\mathrm{ab}}(\delta_{i+1}'', \mathcal{G}_i'')$, $\delta_{i+1}' = \delta_{\mathrm{c}}(\delta_i''/8, \mathcal{G}'')$ and $\delta_{i+1}'' = \delta_{\mathrm{dig}}(\delta_i', \delta_i, \mathcal{G}_i')/4$, $\mathcal{G}_{i+1}'' = \mathcal{G}_{\mathrm{dig}}(\delta_i', \delta_i, \mathcal{G}_i')$ $i = 1, 2, \ldots, n$. We may assume that $\delta_{i+1} \leqslant \delta_i$ and $\delta_1 \leqslant \varepsilon/4$. Set $d_1 = \min\{\delta_i', \delta_i', \delta_i'' \mid i = 1, 2, \ldots, n\}$ and $a_i = a_{\mathrm{ab}}(\delta_i, \mathcal{G}_i')/4$,

$i = 2, \ldots, n$ and $a_1 = 0$. Further, denote $\mathcal{P}_i = \mathcal{P}_{\mathrm{ab}}(X_i, d_1, \mathcal{F})$, $i = 1, 2, \ldots, n$. Let $\mathcal{G}_1 = \bigcup_i^n \mathcal{G}'_i \cup \mathcal{F}$.

To apply Lemma 1.4 later, we let $d_2 = \min(d_1, \delta_{(\mathrm{ap})}(d_1, d_1, a_i, \mathcal{G}_1))$, $\mathcal{G}_2 = \mathcal{G}_{(\mathrm{ap})}(d_1, d_1, a_i, \mathcal{G}_1) \cup \mathcal{G}_1 \cup \mathcal{G}_{\mathrm{dig}}(d_1, d_2, \mathcal{G}_1)$ and $b_1, b_2, \ldots, b_n$ be as in Lemma 1.4.

Let $g_i \in C(X)$ be defined as follows: $g_i(t) = 0$ if $\mathrm{dist}\,(t, X_i) < b_i/4$, $g_i(t) = 1$, if $\mathrm{dist}\,(t, X_i) \geqslant b_i/2$.

(Now we dig a projection E.)

By Lemma 1.3, there are finite subsets $\{\lambda_k^{(i)}\} \subset Y_i$ which are $\delta_{\mathrm{c}}(\varepsilon/4, \mathcal{F})$-dense in Y_i and finitely many mutually orthogonal projections $e_k^{(i)} \in I_{i-1}$ (here $I_0 = A$) such that

$$\Big\|\varphi(f) - \sum_{i=1}^n \psi_1^{(i)}(f) - \Big(1 - \sum_{i=1}^n e_i\Big)\varphi(f)\Big(1 - \sum_{i=1}^n e_i\Big)\Big\| < \frac{\varepsilon}{4},$$

$$\Big\|(1 - \pi_i\Big(\sum_{i=1}^n e_i\Big)\varphi_i(fg_i) - \varphi_i(fg_i)\Big(1 - \pi_i\Big(\sum_{i=1}^n e_i\Big)\Big)\Big\| < \frac{\varepsilon}{4}$$

for all $f \in \mathcal{G}_2$, where $\sum_{k,i} e_k^{(i)} = E$, $\psi_1^{(i)}(g) = \sum_{k=1} g(\lambda_k^{(i)})e_k^{(i)}$, and

$$p \oplus p \oplus \cdots \oplus p \lesssim e_k^{(i)}$$

for any copies of any projection $p \in I_i$, $\varphi_i(g_\beta^{(i)} f) = \varphi_i(g_\beta^{(i)})\varphi_i(f)$ for all $f \in C(X)$ and $0 < \beta < \min_i\{a_i\}$, where $g_\beta^{(i)}$ are as in Lemma 1.3, and if ψ_i satisfies Condition (A), then

$$\Big\|(1 - \pi_i(E))\psi_i(fg_i)(1 - \pi_i(E)) - \sum_{k=1}^{n(i)} fg_i(\zeta_k^{(i)})\pi_i(q_k^{(i)})\Big\| < d_2$$

for all $f \in \mathcal{G}_1$, where $\{\zeta_k^{(i)}\}$ is d_1-dense in Y_i, and $q_k^{(i)}$ are mutually orthogonal projections in I_{i-1} and

$$p \oplus p \oplus \cdots \oplus p \lesssim q_k^{(i)}$$

for any copies of any projection $p \in I_i$. Further, if

$$\Big\|\Big(1 - \pi_i\Big(\sum_{i=1}^n e_i\Big)\Big)\varphi_i(f)\Big(1 - \pi_i\Big(\sum_{i=1}^n e_i\Big)\Big) \oplus H_1(f) - H_2(f)\Big\| < \varepsilon_{\mathrm{dig}}(d_1, d_2, \mathcal{G}_1)$$

for all $f \in \mathcal{G}_2$, and for some homomorphisms $H_1 : C(X_i) \to M_{L_i}(A/I_i)$ and $H_2 : C(X_i) \to M_{L_i+1}(A/I_i)$ with finite dimensional range, then there are some finitely many mutually orthogonal projections $\{p_k\}$ in $M_{L_i+1}(A/I_i)$ with and $\{\xi_k\}$ is d_1-dense in X_i and

$$p \oplus p \oplus \cdots \oplus p \lesssim p_k$$

for any copies of any projection $p \in I_i$, and $\{\xi_k\}$ is d_1-dense in X_i such that

$$\Big\|\Big(1 - \pi_i\Big(\sum_{i=1}^{n} e_i\Big)\Big)\varphi_i(f)\Big(1 - \pi_i\Big(\sum_{i=1}^{n} e_i\Big)\Big) \oplus H_1(f) - \sum_k f(\xi_k)p_k\Big\| < d_2$$

for all $f \in \mathcal{G}_1$. Furthermore, with possibly smaller d_1, since $\Gamma(\varphi_k) = 0$, we may assume that $(1 - \pi_k(E))\varphi_i(1 - \pi_k(E))(\mathcal{P}_k) \in \mathcal{N}$ for $k = 1, 2, \ldots, n$.

To distinguish the cases (d) and (d′), we use i for the case (d) and j for the case (d′).

By Lemma 1.4, for each j, there is a homomorphism $\psi_2^{(j)} : C_0(Y_j) \to QM_{L_j+1}(I_j)Q$ with finite dimensional range and $\psi_3^{(j)} : C(X) \to M_{L_j}(I_j)$ with finite dimensional range ($Q = \mathrm{diag}\,((1 - E, 1, 1, \ldots, 1))$ such that

$$\|(1 - \pi_j(E))\varphi_j(fg_i)(1 - \pi_j(E)) \oplus \pi_i \circ \psi_3^{(j)}(fg_j) - \pi_i \circ \psi_2^{(j)}(fg_j)\| < \delta_2$$

for all $f \in \mathcal{G}_2$ and

$$\psi_2^{(j)}(g) = \sum_{k=1}^{m(j)} g(\zeta_k^{(j)})\pi_j(q_k^{(j)})$$

for all $g \in C_0(Y_j)$ with $\{\zeta_k^{(j)}\}$ d_1-dense in Y_j, and

$$\psi_3^{(j)}(f) = \sum_{k=1}^{n(j)} f(\xi_k^{(j)})d_k^{(j)}$$

with $\xi_k^{(j)} \in Y_j$ and $g_j(\xi_k^{(j)}) = 1$, where $\{d_k^{(j)}\}$ and $\{q_k^{(j)}\}$ are mutually orthogonal projections in $M_{L_j+1}(I_{j-1})$ such that

$$p \oplus p \oplus \cdots \oplus p \lesssim q_k^{(j)}$$

for any number of copies of projection $p \in I_j$.

Let $\Phi(g) = \varphi(g) \oplus \sum_j \psi_3^{(j)}(g)$ for $g \in C(X)$. We would remind to the reader that $\pi_k(e_i) = 0$ if $i > k$ and $\pi_k \circ \psi_3^{(j)} = 0$ if $k < j$.

Now we will apply Lemma 1.5 repeatedly.

Note that since $(1-\pi_m(E))\varphi_m(1-\pi_m(E))(\mathcal{P}_m) \in \mathcal{N}$, we have $(1-\pi_m(E))\pi_m \circ \Phi(1 - \pi_m(E))(\mathcal{P}_m) \in \mathcal{N}$.

We also note that

$$(1-\pi_i(E))\pi_i \circ \Phi((1-\pi_i(E))(fg_i) = (1-\pi_i(E))\psi_i(fg_i)(1-\pi_i(E))$$

and

$$(1-\pi_j(E))\pi_j \circ \Phi((1-\pi_j(E))(fg_j) = (1-\pi_j(E))\psi_j(fg_j)(1-\pi_j(E)) \oplus \pi_j \circ \psi_3^{(j)}(fg_j)$$

for all $f \in \mathcal{G}_2$.

Suppose that ψ_2 satisfies Condition (A), i.e., 2 is one of i.

Working in the A/I_2, applying Lemma 1.5 to $(1-\pi_2(E))\pi_2 \circ \Phi(1-\pi_2(E))$ (with the ideal $I = I_1/I_2$), we obtain a homomorphism $h_1' : C(X_2) \to (1 - \pi_2(E))A/I_2(1-\pi_2(E))$ with finite dimensional range such that

$$\|(1-\pi_2(E))\varphi_2(f)(1-\pi_2(E)) - h_1'(f)\| < \delta_{n-1}''$$

for all $f \in \mathcal{G}_{n-1}''$. By the way we dig, we can find h_1 so that

$$\|(1-\pi_2(E))\varphi_2(f)(1-\pi_2(E)) - h_1(f)\| < \delta_{n-1}$$

for all $f \in \mathcal{G}_{n-1}'$, where $h_1(f)$ has the form $\sum_k f(\xi_k)\pi_2(p_k)$, where the finite subset $\{\xi_k\}$ is d_1-dense in X_i and $\{p_k\}$ are finite many mutually orthogonal projections such that

$$p \oplus p \oplus \cdots \oplus p \lesssim p_k$$

for any copies of any projections in I_2.

Suppose that 2 is one of j (the case (d′)).

Still working in A/I_2 and applying Lemma 1.5 to $(1-\pi_2(E))\circ\Phi(1-\pi_2(E))$, we obtain a homomorphism $h_1' : C(X_2) \to Q_2 M_{L_1}(A/I_2)Q_2$, where $Q_2 \in M_{L_1}(A/I_2)$ with $Q_2 = \operatorname{diag}(1-E, 1, 1, \ldots, 1)$, with finite dimensional range such that

$$\|(1-\pi_2(E))\pi_2 \circ \Phi(f)(1-\pi_2(E)) - h_1'(f)\| < \delta_{n-1}''$$

for all $f \in \mathcal{G}_{n-1}''$.

By the way we dig, we can further assume that

$$\|(1-\pi_2(E))\pi_2 \circ \Phi(f)(1-\pi_2(E)) - h(f)\| < \delta_{n-1}'$$

for all $f \in \mathcal{G}_{n-1}'$, where $h_1(f)$ has the form $\sum_k f(\xi_k)\pi_2(p_k)$, where the finite subset $\{\xi_k\}$ is d_1-dense in X_i and $\{p_k\}$ are finite many mutually orthogonal projections such that

$$p \oplus p \oplus \cdots \oplus p \lesssim p_k$$

for any copies of any projections in I_2.

We then apply Lemma 1.5 to $(1-\pi_3(E)\pi_3\circ\Phi(1-\pi_3(E)$. If 3 is one of i, we obtain a homomorphism $h_2 : C(X_3)\to Q_3M_{L_1+1}(A/I_3)Q$ ($Q=\operatorname{diag}(1-\pi_3(E),1,\dots,1)\in M_{L_1+1}(A/I_3)$) with finite dimensional range such that

$$\|(1-\pi_3(E))\pi_3\circ\Phi(f)(1-\pi_3(E))-h_2(f)\|<\delta'_{n-2}$$

for all $f\in\mathcal{G}'_{n-2}$ (here L_1 could be just zero if 2 is also one of i). If 3 is one of j, we obtain a homomorphism $h_2 : C(X_3)\to Q_3M_{L_1+L_2+2}(A/I_3)Q_3$ ($Q_3=\operatorname{diag}(1-\pi_3(E),1,1,\dots,1)\in M_{L_1+L_2+2}$) with finite dimensional range such that

$$\|(1-\pi_3(E))\pi_3\circ\Phi(f)(1-\pi_3(E))-h_2(f)\|<\frac{\delta_{n-2}}{2}$$

for all $f\in\mathcal{G}_{n-2}$. Furthermore, in both cases (d) and (d′) (by the way we dig), $h_2(f)$ has the form $\sum\limits_k f(\xi_k)p_k$, where finite subset $\{\xi_k\}$ is d_1-dense in X_i and $\{p_k\}$ are finite many mutually orthogonal projections such that

$$p\oplus p\oplus\cdots\oplus p\lesssim p_k$$

for any copies of any projections in I_3.

We will repeat this argument. Note that for $j=n$, we apply Corollary 2.15 (which is simpler than Lemma 1.5).

By repeating this argument, we obtain a homomorphism $h : C(X)\to QM_L(A)Q$ with finite dimensional range such that

$$\|(1-E)\Phi(f)(1-E)-h(f)\|<\frac{\varepsilon}{4}$$

for all $f\in\mathcal{G}_4$, or

$$\Big\|(1-E)\varphi(f)(1-E)\oplus\sum_j\psi_3^{(j)}(f)-h(f)\Big\|<\frac{\varepsilon}{4}$$

for all $f\in\mathcal{F}$.

Since $\{\lambda_k^{(i)}\}$ are $\delta_c(\varepsilon/4,\mathcal{F})$-dense in Y_i, if we replace $\xi_k^{(j)}$ by nearest points in $\{\lambda_k^{(j)}\}$ in the definition of $\psi_3^{(j)}$ and denote it by $\psi_3^{(j)'}$, we have

$$\Big\|(1-E)\varphi(f)(1-E)\oplus\sum_j\psi_3^{(j)'}(f)-h(f)\Big\|<\frac{\varepsilon}{2}$$

for all $f\in\mathcal{F}$. To save the notation, we may write

$$\psi_3^{(j)'}(f)=\sum_k f(\lambda_k^{(j)})d_k^{(j)}$$

with the possibility that some of $d_k^{(j)}$ being zero. By our construction, $d_k^{(j)} \lesssim e_k^{(j)}$ for all k and j. There is a unitary U such that

$$U^* d_k^{(i)} U \leqslant e_k^{(i)}.$$

We have

$$\Big\|(1-E)\varphi(f)(1-E) \oplus \sum_k f(\lambda_k^{(i)}) U^* d_k^{(i)} U - U^* h(f) U\Big\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

for all $f \in \mathcal{F}$.

Since

$$\Big\|\varphi(f) - \sum_{i,k} f(\lambda_k^{(i)}) e_k^{(i)} - (1-E)\varphi(f)(1-E)\Big\| < \frac{\varepsilon}{4},$$

we obtain

$$\begin{aligned}
&\Big\|\varphi(f) - \sum_{i,k} f(\lambda_k^{(i)})(e_k^{(i)} - U^* d_k^{(i)} U) - U^* h(f) U\Big\| \\
&\leqslant \Big\|\varphi(f) - \sum_{i,k} f(\lambda_k^{(i)}) e_k^{(i)} - (1-E)\varphi(f)(1-E)\Big\| \\
&\quad + \Big\|\sum_{i,k} f(\lambda_k^{(i)})(e_k^{(i)} - U^* d_k^{(i)} U) \oplus (1-E)\varphi(f)(1-E) \oplus \sum_k f(\lambda_k^{(i)}) U^* d_k^{(i)} U \\
&\quad - \sum_{i,k} f(\lambda_k^{(i)})(e_k^{(i)} - U^* d_k^{(i)} U) \oplus U^* h(f) U\Big\| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

for all $f \in \mathcal{F}$. ∎

2. LEMMAS

2.1. Let X be a compact metric space. There is a dimension map $d : \mathrm{K}_0(C(X)) \to C(X, \mathbb{Z})$. We denote by $\ker d_X$ the kernel of d.

2.2. LEMMA. *Let X be a compact metric space with $\dim(X) \leqslant 2$, $Y \subset X$ be a compact subset and let $s : C(X) \to C(Y)$ be the canonical surjective map. Then s_* maps $\ker d_X$ onto $\ker d_Y$.*

Proof. There are finite CW-complexes $\{X_n\}$ such that $C(X) = \lim_{n\to\infty} C(X_n)$ and $\ker d_X = \lim_{n\to\infty} \ker d_{X_n}$. Write $\ker d_{X_n} = \widetilde{\mathrm{K}}_0(C(X_n))$. Let $BU = \lim_{n\to\infty} BU(n)$, where each $BU(n)$ is a classifying space of the complex orthogonal unitaries. Then, we have $\ker d_X = \lim_{n\to\infty} \widetilde{\mathrm{K}}_0(C(X_n)) = [X, BU]$, the homotopy equivalent classes of continuous functions to BU. It follows from [4] that for any compact subset Y of X, where X is a compact subspace of $\mathbb{R}^n$, every continuous map $f: Y \to BU$ can be extended to a continuous map $\widetilde{f}: X \to BU$ if and only if $H^q(X, \mathbb{Z}) = 0$ when $q \geqslant 3$. Since $\dim(X) \leqslant 2$, $H^q(X, \mathbb{Z}) = 0$ when $q \geqslant 3$. Therefore a continuous map $f: Y \to BU$ can be extended to a continuous map $\widetilde{f}: X \to BU$. This implies that s_* maps $\ker d_X$ onto $\ker d_Y$. ∎

2.3. Lemma. *Let $\dim(X) \leqslant 2$, Y a compact subset of X, A has real rank zero and stable rank one and I be an ideal of A such that $\mathrm{K}_1(A) = \mathrm{K}_1(I) = 0$, $\mathrm{K}_0(A)$ and $\mathrm{K}_0(I)$ are torsion free. For any finite subset $\mathcal{P} \subset \mathbf{P}(C(Y))$, there are $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P}_1 \subset \mathbf{P}(C(X))$ (none of them depend on A) satisfying: if $\psi = \psi' \circ s \oplus h : C(X) \to A$ is a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism with h being a homomorphism with finite dimensional range, $s: C(X) \to C(Y)$ the surjective map and $\psi' : C(Y) \to eIe$ being a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism such that*

$$\psi_*(\mathcal{P}_1) \in \mathcal{N},$$

then $(\psi')_*(\mathcal{P}) \in \mathcal{N}$.

Proof. Since $\mathrm{K}_1(A) = \mathrm{K}_1(I) = 0$, $\mathrm{K}_0(A)$ and $\mathrm{K}_0(I)$ are torsion free, and $\mathrm{K}_1(C(Y))$ is torsion free, we compute that $\mathrm{K}L(C(Y), eIe) = \mathrm{Hom}(\mathrm{K}_0(C(Y)), \mathrm{K}_0(eIe))$. Therefore it is sufficient to show that, for any finite set $\mathcal{P}$ of projections in $M_\infty(C(Y))$, there is a finite subset $\mathcal{P}_1 \in M_\infty(C(X))$, and there are $\mathcal{G}$ and δ such that if $\psi = \psi' \circ s \oplus h' : C(X) \to A$ is a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism with h' being a homomorphism with finite dimensional range, such that

$$\psi_*(\overline{\mathcal{P}}_1) \in \mathcal{N},$$

(see 0.7 for notation). Then

$$(\psi')_* = h_* : \overline{\mathcal{P}} \to \mathrm{K}_0(eIe),$$

where $h: C(Y) \to eIe$ is a homomorphism with finite dimensional range.

Now write $C(Y) = \lim_{n\to\infty} C(Y_n)$, where Y_n are finite CW-complexes and the maps from $C(Y_n)$ to $C(Y)$ are surjective. So we may assume that Y is a compact subset of Y_n for each n.

Suppose that F is a finite CW-complex and that $f_1, f_2, \ldots, f_l$ are mutually orthogonal projections in $C(F)$ which represent all connected components of F. We claim that if $\alpha : \mathrm{K}_0(F) \to \mathrm{K}_0(eIe)$ maps $\ker d_Y$ into zero and $\alpha([f_i])$ can be represented by l mutually orthogonal projections in eIe, then there is a homomorphism $h_1 : C(F) \to eIe$ such that

$$\alpha = (h_1 \otimes \mathrm{id})_*.$$

Since $\mathrm{K}_0(C(F))$ is finitely generated, we may write $\mathrm{K}_0(C(F)) = C(F, \mathbb{Z}) \oplus \ker d_F$. Let g_i be mutually orthogonal projections in eIe such that $[g_i] = \alpha(f_i)$, $i = 1, 2, \ldots, l$ and ξ_i be a point in the ith component corresponding to f_i. Define

$$h_1(f) = \sum_{i=1}^{l} f(\xi_i) g_i \quad \text{for all } f \in C(F).$$

Then $(h_1)_*|\ker d_F = 0$ and $(h_1)_* = \alpha$. This proves the claim. (Note that the requirement that $\alpha([f_i])$ can be represented by l mutually orthogonal projections in eIe is guaranteed by choosing large enough set $\mathcal{G}$ and small enough δ.)

Given any finite subset $\mathcal{P} \subset M_\infty(C(Y))$, without loss of generality, by replacing projections by equivalent ones, we may assume that $\mathcal{P} \subset M_k(C(Y_n))$ for some n and k. Let $f_1, f_2, \ldots, f_l$ be mutually orthogonal projections in $C(Y_n)$ which represent all connected components of Y_n. We further assume that each of these component intersects with Y. With smaller δ and larger $\mathcal{G}$, we may assume that $(\psi')_*(f_i)$, $i = 1, 2, \ldots$, defines l elements in $\mathrm{K}_0(eIe)$ which can be represented by l mutually orthogonal projections. We may further assume that, since $\mathrm{K}_0(C(Y_n))$ is finitely generated, $(\psi')_*$ gives a homomorphism $\alpha : \mathrm{K}_0(C(Y_n)) \to \mathrm{K}_0(eIe)$.

Let $j : C(Y_n) \to C(Y)$ be the map in the direct limit. Then $j_*(\ker d_{Y_n}) \subset \ker d_Y$. Choose a finite subset $\mathcal{P}_1 \in M_\infty(C(X))$ such that $s_*(\mathcal{P}_1)$ generates a subgroup which contains $j_*(\ker d_{Y_n})$. This is possible because of Lemma 2.2 and because that $\ker d_{Y_n}$ is finitely generated. With a sufficiently large $\mathcal{P}_1$, sufficiently large $\mathcal{G}$ and sufficiently small δ, $\psi_*(\overline{\mathcal{P}}_1) \in \mathcal{N}$ (notation as in 0.7) implies that

$$(\psi)_*(s_*^{-1}(j_*(\ker d_{Y_n})) = 0.$$

Thus $(\psi' \circ j)_*(\ker d_{Y_n}) = 0$ (we use the injectivity of the map $\mathrm{K}_0(eIe) \to \mathrm{K}_0(A)$). So, by the claim, there is a homomorphism $h_1 : C(Y_n) \to eIe$ with finite dimensional range such that

$$(\psi' \circ j)_* = (h_1)_* : \mathcal{P} \to \mathrm{K}_0(eIe).$$

Write $h_1(f) = \sum_{i=1}^{l} f(\xi_i) g_i$ for $f \in C(Y_n)$. Note that we may assume that there are $\xi_i \in Y$ since each component intersects with Y. So $h(f) = \sum_{i=1}^{l} f(\xi_i) g_i$ for $f \in C(Y)$ defines a homomorphism from $C(Y)$ into eIe and

$$(\psi')_*|\mathcal{P} = h_*|\mathcal{P}.$$

From the reduction of the beginning of the proof, this ends the proof. ∎

2.4. Lemma. *Let X be a compact metric space with dimension no more than two, A be a (unital) C^*-algebra and I be an ideal of A with $\mathrm{K}_1(A/I)=0$ and torsion free $\mathrm{K}_0(A/I)$, and let $\varphi : C(X) \to A$ be a homomorphism. Suppose that $Y = \mathrm{sp}\,(\pi \circ \varphi)$, where $\pi : A \to A/I$ is the quotient map. Denote $\psi : C(Y) \to A/I$ be the monomorphism induced by φ. If $\Gamma(\varphi) \in \mathrm{K}L(C(X), A) \cap \mathcal{N}$ then $\Gamma(\psi) \in \mathrm{K}L(C(Y), A/I) \cap \mathcal{N}$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_0(C(X)) & \longrightarrow & \mathrm{K}_0(A/I) \\ \downarrow & \nearrow & \\ \mathrm{K}_0(C(Y)) & & \end{array} .$$

Since $\Gamma(\varphi) \in \mathrm{K}L(C(X)) \cap \mathcal{N}$, $(\pi \circ \varphi)_*(\ker d_X) = 0$. Thus, by Lemma 2.2, $(\pi \circ \varphi)_*(\ker d_Y) = 0$. We note that, since $\mathrm{K}_1(A/I) = 0$ and $\mathrm{K}_0(A/I)$ is torsion free, $\mathrm{K}L(C(Y), A/I) = \mathrm{Hom}(\mathrm{K}_0(C(Y), \mathrm{K}_0(A/I))$. Now the argument used in Lemma 2.3 shows that

$$\Gamma(\psi) \in \mathrm{K}L(C(Y), A/I) \cap \mathcal{N}. \quad \blacksquare$$

2.5. Lemma. *Let A be a C^*-algebra of real rank zero and stable rank one and H be a hereditary C^*-subalgebra of A. Suppose that $\mathrm{K}_i(A/I(H))$ is torsion free, where $I(H)$ is the ideal generated by H and $i = 0, 1$. Let $B = C(C_n \times \mathbb{S}^1)$. Then the map from $\mathrm{K}_0(H \otimes B) \to \mathrm{K}_0(A \otimes B)$ is injective.*

Proof. We first assume that $H = I$ is an ideal of A. Since A and I have real rank zero and stable rank one, the map from $\mathrm{K}_0(I) \to \mathrm{K}_0(A)$ is injective. It follows from 2.1 in [51] that the map from $\mathrm{K}_1(I) \to \mathrm{K}_1(A)$ is also injective. Therefore we have a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Tor}(\mathrm{K}_i(I), \mathrm{K}_j(B)) \to \mathrm{Tor}(\mathrm{K}_i(A), \mathrm{K}_i(B)) \to \mathrm{Tor}(\mathrm{K}_i(A/I), \mathrm{K}_j(B)) \\ &\to \mathrm{K}_i(I) \otimes \mathrm{K}_j(B) \to \mathrm{K}_i(A) \otimes \mathrm{K}_j(B) \to \mathrm{K}_i(A/I) \otimes \mathrm{K}_j(B) \to 0 \end{aligned}$$

which gives two short exact sequences

$$\begin{aligned} &0 \to \mathrm{Tor}(\mathrm{K}_i(I), \mathrm{K}_j(B)) \to \mathrm{Tor}(\mathrm{K}_i(A), \mathrm{K}_j(B)) \to 0 \\ &0 \to \mathrm{K}_i(I) \otimes \mathrm{K}_j(B) \to \mathrm{K}_i(A) \otimes \mathrm{K}_j(B) \to \mathrm{K}_i(A/I) \otimes \mathrm{K}_j(B) \to 0. \end{aligned}$$

Using Kunneth formula, we obtain

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{i=0}^{1} \mathrm{K}_i(I) \otimes \mathrm{K}_i(B) & \to & \mathrm{K}_0(I \otimes B) & \to & \bigoplus_{i=0}^{1} \mathrm{Tor}(\mathrm{K}_i(I), \mathrm{K}_{i+1}(B)) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \bigoplus_{i=0}^{1} \mathrm{K}_i(A) \otimes \mathrm{K}_i(B) & \to & \mathrm{K}_0(A \otimes B) & \to & \bigoplus_{i=0}^{1} \mathrm{Tor}(\mathrm{K}_i(A), \mathrm{K}_{i+1}(B)) & \to & 0. \end{array}$$

Since the left and the right arrows are injective, as we have proved, by Lemma 2.5, the middle one is injective.

Now let e be a projection in A. We will show that the map $\mathrm{K}_0(eAe \otimes B) \to \mathrm{K}_0(A \otimes B)$ is injective. Let I_e be the ideal of A generated by e. From what has been proved, it suffices to show that the map $\mathrm{K}_0(eAe \otimes B) \to \mathrm{K}_0(I_e \otimes B)$ is injective. Note that the ideal of $I_e \otimes B$ generated by $eAe \otimes B$ is $I_e \otimes B$. So the injectivity follows from 5.3 in [30]. This shows that Lemma 2.5 holds for unital hereditary C^*-subalgebras. To get the general case, we note, since any hereditary C^*-subalgebra H of A has an approximate identity consisting of projections, that $H \otimes B$ has an approximate identity consisting of projections. ∎

2.6. LEMMA. *Let A be a unital C^*-algebra of real rank zero and stable rank one, and I be an ideal. Suppose that $d \in A$ is a projection such that*

$$p \oplus p \oplus \cdots \oplus p \lesssim d$$

for any number of copies of any projection $p \in I$ and $e \in A$ is another projection with $e - d \in I$. Then

$$p \oplus p \oplus \cdots \oplus p \lesssim e$$

for any number of copies of any projection $p \in I$.

Proof. Note that I has the real rank zero. Let $\{q_n^{(1)}\}$ and $\{q_n^{(2)}\}$ be approximate identities of dId and $(1-d)I(1-d)$ consisting of projections, respectively. Set $q_n = q_n^{(1)} + q_n^{(2)}$. Then $\{q_n\}$ is an approximate identity for A consisting of projections and $q_n d = dq_n$ for all n. Since $e - d \in I$, there is n such that

$$\|(e-d)(1-q_n)\| < \frac{1}{4}.$$

Note that $d_1 = (1-q_n)d(1-q_n)$ is a projection, $d_1 \leqslant d$ and $d - d_1 \in I$. Thus

$$p \oplus p \oplus \cdots \oplus p \oplus (d - d_1) \lesssim d$$

for any number of copies of any projection $p \in I$. Since A has real rank zero and stable rank one,

$$p \oplus p \oplus \cdots \oplus p \lesssim d_1$$

for any number of copies of any projection $p \in I$. Since

$$\|e(1-q_n)e - d_1\| < \frac{1}{2},$$

there is a projection $e_1 \leqslant eAe$ such that

$$\|e_1 - d_1\| < 1.$$

This implies that $d_1 \lesssim e$. ∎

2.7. LEMMA. *Let A be a C^*-algebra of real rank zero and stable rank one, I be an ideal of A, $\varphi : C(X) \to A$ be a homomorphism, and $\pi : A \to A/I$ be the quotient map. Let $X_1 = \mathrm{sp}\,(\pi\circ\varphi)$. Suppose that, for any $\lambda \in X_1$, any neighborhood $O(\lambda)$ and k, there are mutually orthogonal projection $e_1, e_2, \dots, e_k \leqslant H_{O(\lambda)}$ such that*

$$p \oplus p \oplus \cdots \oplus p \lesssim e_m, \quad m = 1, 2, \dots, k$$

for any number of copies of any projection $p \in I_i$, where $H_{O(\lambda)}$ is the hereditary C^*-subalgebra $\varphi(h)$, where $h \in C(X)$ with $h > 0$ in $O(\lambda)$ and zero outside $O(\lambda)$. Then, for any $\varepsilon > 0$, $\sigma > 0$ and a finite subset $\mathcal{F} \in C(X)$, there is $\eta = \eta(\varepsilon, \sigma, \mathcal{F}) > 0$ and a finite subset $\mathcal{G} = \mathcal{G}(\varepsilon, \sigma, \mathcal{F})$ of $C(X)$ such that, if

$$\|\pi\circ\varphi(f) - h'(f)\| < \eta$$

for all $f \in \mathcal{G}$ for some homomorphism $h : C(X_1) \to A/I$ with finite dimensional range, then there is a homomorphism $h(f) = \sum_{k=1}^{m} f(\xi_k)\pi(p_k)$, where $\{\xi_k\}$ is σ-dense in X_1 and $\{p_k\}$ are m mutually orthogonal projections in A with $\pi(p_k) \neq 0$ and

$$p \oplus p \oplus \cdots \oplus p \lesssim p_k$$

for any copies of any projections in I such that

$$\|\pi\circ\varphi(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. Let $a, b > 0$ be positive numbers with $a < b/4$ and a finite subset $\{\zeta_i\}_{i=1}^n \subset X_1$ be a b-dense set such that $\mathrm{dist}\,(\zeta_i, \zeta_j) \geqslant a$ if $i \neq j$. Let $f_i \in C(X)$ such that $0 \leqslant f_i \leqslant 1$, $f_i(t) = 1$ if $\mathrm{dist}\,(t, \zeta_i) < a/2$ and $f_i(t) = 0$ if $\mathrm{dist}\,(t, \zeta_i) \geqslant a$. Let $\mathcal{G} = \mathcal{F} \cup \{f_1, f_2, \dots, f_n\}$. Suppose that

$$\|\pi\circ\varphi(f) - h'(f)\| < \eta$$

for all $f \in \mathcal{G}$, where $h' : C(X_1) \to A/I$ is a homomorphism with finite dimensional range and $\eta > 0$ (to be determined later). Suppose that $q_l^{(1)}, q_l^{(2)}$ are two mutually orthogonal projections in $H_{O(\zeta_l)}$, where

$$O_{\zeta_l} = \Big\{x \in X \mid \mathrm{dist}\,(x, \zeta_l) < \frac{a}{4}\Big\}.$$

(The only reason that we take two projections is to be used in the proof of Lemma 1.3.) We have

$$\|\pi(q_l^{(j)})(\pi\circ\varphi)(f_l) - \pi(q_l^{(j)})h'(f_l)\| < \eta, \quad j = 1, 2.$$

Note that $\pi(q_l^{(j)})(\pi\circ\varphi)(f_l)=\pi(q_l^{(j)})$. Write $h'(f)=\sum_{k=1}^{m} f(\lambda_k)d_k$, where $\{\lambda_k\}$ is a subset of X_1 and d_k are mutually orthogonal projections in A/I. Let d_l' be a sum of some $\{d_k\}$ with the property that $\operatorname{dist}(\lambda_k,\zeta_l)\leqslant b$ and $\sum_{l=1}^{n} d_l'=\sum_{k=1}^{m} d_k$. (Note that $\{\zeta_l\}$ is b-dense in X_1.) The above inequality implies that

$$\|\pi(q_l^{(j)})-\pi(q_l^{(j)})d_l'\|<\eta,$$

whence

$$\|\pi(q_l^{(j)})-d_l'\pi(q_l^{(j)})d_l'\|<2\eta.$$

A standard argument, with $\eta<1/16$, shows that there are mutually orthogonal projections $c_l^{(1)},c_l^{(2)}\leqslant d_l'$ such that

$$\|\pi(q_l^{(j)})-c_l^{(j)}\|<4\eta,\quad j=1,2.$$

This also implies that $\pi(q_l^{(j)})$ is equivalent to $c_l^{(j)}$, $j=1,2$.

Let

$$h(f)=\sum_{l=1}^{n} f(\zeta_l)d_l'.$$

Now if a,b and η are small enough,

$$\|h'(f)-h(f)\|<\frac{\varepsilon}{2}$$

for all $f\in\mathcal{F}$. Then, (if η is also less than $\varepsilon/2$),

$$\|\pi\circ\varphi(f)-h(f)\|<\varepsilon$$

for all $f\in\mathcal{F}$. From assumptions, we may assume that

$$p\oplus p\oplus\cdots\oplus p\lesssim q_l^{(j)}.$$

Suppose that $p_1,p_2,\ldots,p_n$ are mutually orthogonal projections in A such that $\pi(p_l)=d_l'$. Then, by applying Lemma 2.6 with $b\leqslant\sigma$, h meets the requirements in the lemma. ∎

Note that both Lemma 1.3 and Lemma 1.4 assume the conditions in Lemma 1.2.

2.8. *Proof of Lemma* 1.3. If we just want to obtain (i) and (ii) in Lemma 1.3, then it follows from [27] (in fact, (iii) follows too — we will explain later).

We now use Lemma 2.7 to obtain (v) and (vi).

Let $a, b > 0$ be as in the proof of Lemma 2.7. Let $\{\zeta_k^{(i)}\}$ be finite subsets of Y_i which are b-dense in Y_i and $\operatorname{dist}(\zeta_k^{(i)}, \zeta_{k'}^{(j)}) \geqslant a$, if $i \neq j$ or $k \neq k'$. We may also assume that

$$\operatorname{dist}(\zeta_k^{(i+1)}, X_i) \geqslant a, \quad i = 1, 2, \dots, n.$$

Let $H_{O(\zeta_k^{(i)})}$ be as in the proof of Lemma 2.7. So, $H_{O(\zeta_k^{(i)})} \subset I_i$. As in [27], if $e_k^{(i)} \in H_{O(\zeta_k^{(i)})}$, with small b, (i) and (ii) are satisfied. We can choose those $e_k^{(i)}$ so that (iv) are satisfied.

Suppose that ψ_i satisfies Condition (A), then the argument in the proof of Lemma 2.7 applies. Suppose that

$$\|\psi_i(f) - h_i(f)\| < \varepsilon_1$$

for all $f \in \mathcal{G}_i$ for some finite subset $\mathcal{G}_i \in C_0(Y_i)$ and for some small ε_1. We may assume that $h_i(f) = \sum_{k=1}^{n(i)} f(\xi_k^{(i)}) d_k^{(i)}$ with $\xi_k^{(i)} \in Y_i$. It is clear that, with larger $\mathcal{G}_i$ and small η, we may assume that $\{\xi_k^{(i)}\}$ is σ-dense in Y_i.

Now fix i (and forget it so that we save some notation) and denote φ_i by F. In the proof of Lemma 2.7, we have

$$\|\pi(q_l^{(j)}) - c_l^{(j)}\| < 4\eta, \quad j = 1, 2.$$

So, as in the proof of Lemma 2.7, we have that (with $\pi(e_l) = \pi(q_l^{(1)})$)

$$\Big\|\Big(1 - \sum_l \pi(e_l)\Big)\pi \circ F(f)\Big(1 - \sum_i \pi(e_l)\Big) - \sum_i f(\zeta_l)\big(d_l' - c_l^{(1)}\big)\Big\|$$

is small, provided that η is small enough and $\mathcal{G}_i$ is large enough. Furthermore, the presence of the second projection $q_l^{(2)}$ and $c_l^{(2)} \leqslant d_l'$ in the proof of Lemma 2.7 implies that

$$p \oplus p \oplus \cdots \oplus p \lesssim d_l' - c_l^{(1)}.$$

Remember that we can do this for each i. This proves (vi). To prove (v), we can apply a similar argument to the homomorphism $\psi_i \oplus H$.

Now, to get (iii), let $\beta_1 > 0$ such that

$$\operatorname{dist}(X_i, \zeta_k^{(i+1)}) > \beta_1$$

for all k and i. Then let b in the lemma be $\beta_1/2$. We note that, in the above, $e_k^{(i)} \in H_{O(\zeta_k^{(i)})}$. Thus, if $\beta > 0$ and $\beta < \beta_1/2$,

$$\varphi_i(g_\beta^{(i)}) e_k^{(i)} = e_k^{(i)} \varphi_i(g_\beta^{(i)}) = e_k^{(i)}$$

for all k and i. So

$$\begin{aligned}\Lambda_i(g_\beta^{(i)}f) &= (1-\pi_i(e_i))\varphi_i(g_\beta^{(i)}f)((1-\pi_i(e_i))\\ &= (1-\pi_i(e_i))\varphi_i(g_\beta^{(i)})\varphi_i(f)(1-\pi_i(e_i))\\ &= (1-\pi_i(e_i))\varphi_i(g_\beta^{(i)})(1-\pi_i(e_i))\varphi_i(f)(1-\pi_i(e_i))\\ &= \Lambda_i(g_\beta^{(i)})\Lambda_i(f)\end{aligned}$$

for all $f \in C(X)$ $(\Lambda_i(f) = (1-\pi_i(e_i))\varphi_i(f)(1-\pi_i(e_i)))$. ∎

In what follows we will use 1.6 in [53]. However, when $K_i(C(X))$ has torsion, the proof needs to be modified. We include a brief modification here.

Let $C_n = B_n \otimes \mathcal{K}$ and $D = \prod B_n \otimes \mathcal{K}$, where each B_n is unital.

We have the following lemma:

2.9. LEMMA. *For the above D,*

$$K_i(D, \mathbb{Z}/k\mathbb{Z}) = \prod K_i(B_n, \mathbb{Z}/k\mathbb{Z})$$

and

$$K_i(D/\oplus C_n, \mathbb{Z}/k\mathbb{Z}) = \prod K_i(B_n, \mathbb{Z}/k\mathbb{Z})/\oplus K_i(B_n, \mathbb{Z}/k\mathbb{Z})$$

$i = 0, 1$.

Proof. It is easy to check (since each $C_n = B_n \otimes \mathcal{K}$) that

$$K_0(D) = \prod K_0(B_n).$$

To show that $K_1(D) = \prod K_1(B_n)$, we will show that $\{u_n\} \in \widetilde{D}$ connects to the identity if each $u_n \in U_0((B_n \otimes \mathcal{K})\tilde{\ }$. This requires that there are equi-continuous paths of unitaries which connecting u_n to $\mathrm{id}_{(B_n\otimes\mathcal{K})\tilde{\ }}$. This follows form the proof (not the statement) of 3.7 in [67]. The proof implies that any unitary in $U_0((B_n \otimes \mathcal{K})\tilde{\ })$ is close to a product of two unitaries connecting to the identity each of which has an exponential length no more than 2π. This implies that the equi-continuous path exists. This proves that $K_1(D) = \prod K_1(B_n)$. The other two identities follow easily, using the facts that projections and normal partial isometries in $\prod C_n/\oplus C_n$ lift to projections and normal partial isometries in $\prod C_n$.

Now, for $k > 0$, we follow an argument in Section 5 of [38]. We have the following long exact sequences (see 23.15.7 (c) in [3] also [73]), for any C^*-algebras A,

$$K_1(A) \xrightarrow{k} K_1(A) \longrightarrow K_1(A, \mathbb{Z}/k\mathbb{Z}) \longrightarrow K_0(A) \xrightarrow{k} K_0(A)$$

and

$$K_0(A) \xrightarrow{k} K_0(A) \longrightarrow K_0(A, \mathbb{Z}/k\mathbb{Z}) \longrightarrow K_1(A) \xrightarrow{k} K_1(A),$$

where k is the multiplication by k. We also have the same long exact sequences for each B_n. Now since we have shown that $K_i(D) = \prod K_i(B_n)$, $i = 0, 1$, it follows that

$$K_i(D, \mathbb{Z}/k\mathbb{Z}) = \prod K_i(B_n, \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1.$$

The other two identities for $k > 0$ follow. ∎

2.10. Now in the proof of 1.6 in [53], we replace B by $\oplus C_n$ and $M(B)$ by D and consider maps $\bar{\Psi}$, $\bar{\Phi} : C(X) \to (C/\oplus C_n)\tilde{}$. One should verify

$$\bar{\Psi}_*([p]) = \bar{\Phi}_*([p])$$

as follows. Assume that $[p]$ is an element in $K_i(C(X), \mathbb{Z}/k\mathbb{Z})$ ($i = 0$ or $i = 1$) with $k \geqslant 0$, or $[p] \in K_1(C(X))$. We identify $\bar{\Psi}_*([p])$ and $\bar{\Phi}_*([p])$ with elements $\bar{z}'$ and $\bar{z}''$ in $\prod K_i(B_n, \mathbb{Z}/k\mathbb{Z})/\oplus K_i(B_n, \mathbb{Z}/k\mathbb{Z})$, respectively. Let $\pi_{*i} : \prod K_i(B_n, \mathbb{Z}/k\mathbb{Z}) \to \prod K_i(B_n, \mathbb{Z}/k\mathbb{Z})/\oplus K_i(B_n, \mathbb{Z}/k\mathbb{Z})$. Let $z' = \{z'_n\}$, $z'' = \{z''_n\} \in \prod K_i(B_n, \mathbb{Z}/k\mathbb{Z})$ such that $\pi_{*i}(z') = \bar{z}'$ and $\pi_{*i}(z') = \bar{z}''$. On the other hand, we know that

$$y_n = (\psi_n)_*(p) = (\varphi_n)_*(p) \text{ in } K_i(B_n, \mathbb{Z}/k\mathbb{Z}) \quad (\text{for } n \geqslant s).$$

However, $\pi_{*i}(\{(\psi_n)_*(p)\} = \bar{z}'$ and $\pi_{*i}(\{(\varphi_n)_*(p)\} = \bar{z}''$. Therefore $\bar{z}' = \pi_{*i}(\{y_n\}) = \bar{z}''$. This implies that

$$(\bar{\Psi}_*)|_{K_i(C(X), \mathbb{Z}/k\mathbb{Z})} = (\bar{\Phi}_*)|_{K_i(C(X), \mathbb{Z}/k\mathbb{Z})}.$$

Therefore $\bar{\Psi}_* = \bar{\Phi}_* : \underline{K}(C(X)) \to \underline{K}(D/\oplus C_n)$. The rest of the proof of 1.6 in [55] remains the same (but replace $M(B)$ by $\widetilde{D}$, B by $\oplus C_n$ and $M(B)/B$ by $(D/\oplus C_n)\tilde{}$, respectively. It is important to note that a unitary u_n in $\widetilde{C}_n$ is (arbitrarily) close to a unitary with the form $\lambda(1 - e_m) + u'_n$, where u'_n is a unitary in $M_m(B_n)$, $e_m = 1_{M_m(B_n)}$. Furthermore, for a finite set of projections $d_1, d_2, \ldots, d_k$ and a unitary $u_n \in \widetilde{C}_n$, one can find large $k(n)$ such that $1_{M_{k(n)}(B_n)}$ approximately commutes with each d_i and u. So a standard perturbation shows that we can assume that u_n is in $M_{k(n)}(B_n)$ and $\varphi_n^{(1)} : C(X) \to M_{k(n)}(B_n)$.

One should note that, in the statement of 1.6 in [53], from the modification above, we do not need to assume that $A \in \mathbb{A}_r$. However, if we want to have the integer L independent of A, φ and ψ, one needs to assume that $A \in \mathbb{A}_r$, A has real rank zero and $K_0(A)$ is weakly unperforated.

2.11. *Proof of Lemma* 1.4. We would like to remind to the reader that the conditions in the Lemma 1.2 are assumed.

Suppose that

$$\|\varphi_j(fg'_j) - (1-\pi_j(E))\varphi_j(fg'_j)(1-\pi_j(E)) - h'(fg'_j)\| < \delta$$

for all $f \in \mathcal{G}$. For any finite subset $\mathcal{G}_j \subset C_0(Y_j)$, and $\delta_1 > 0$, with small enough $b_1, b_2, \dots, b_n$, small enough δ and large enough $\mathcal{G}$,

$$\|\psi_j(f) - (1-\pi_j(E))\psi_j(f)(1-\pi_j(E)) - h'(f)\| < \delta_1$$

for all $f \in \mathcal{G}_i$. Let $\Lambda_j = (1-\pi_j(E))\psi_i(1-\pi_j(E))$. To save the notation, let us fix j. For any $\delta_1 > 0$, since I_{j-1}/I_j has real rank zero, there is a projection $e \in (1-\pi_j(E))I_{j-1}/I_j(1-\pi_j(E))$ such that

$$\|e\Lambda_j(f)e - \Lambda_j(f)\| < \delta_1$$

for all $f \in \mathcal{G}_i$. Thus, we have

$$\|\psi_j(f) - e\Lambda_j(f)e - h'(f)\| < \delta_1$$

for all $f \in \mathcal{G}_j$. Note that $e\Lambda_j e$ is a contractive positive linear morphism which is $2\delta_1$-$\mathcal{G}_j$-multiplicative.

Since $\Gamma(\psi_i) = 0$ and $\mathrm{K}_0(e(I_{j-1}/I_j)e \otimes B) \to \mathrm{K}_0(I_{j-1}/I_j \otimes B)$ is injective for each $B = C(C_n \times \mathbb{S}^1)$ by (e) and by 2.5, we conclude that, for any given finite subset $\mathcal{P}_j \in \mathbf{P}(C_0(Y_j)^{\sim})$, if δ_1 is small enough and $\mathcal{G}'_j$ is large enough,

$$(\Lambda'_j)_* : \mathcal{P}_j \to \underline{\mathrm{K}}(e(I_{j-1}/I_j)e)$$

is in $\mathcal{N}$, where $\Lambda'_j : C_0(Y_j)^{\sim} \to e(I_{j-1}/I_j)e$ is the unital contractive positive linear morphism induced by $e\Lambda_j e$.

Now we apply 1.6 in [53] (see 2.12 and also [15]). Note that δ and $\mathcal{G}$ in 1.6 in [53] do not depend on the C^*-algebra A. So, for any $\varepsilon_1 > 0$, with small enough δ_1 and large enough $\mathcal{G}_j$ (this require to have small enough δ, small enough $b_1, b_2, \dots, b_n$, and large enough $\mathcal{G}$), there are homomorphisms $H_j : C_0(Y_j)^{\sim} \to M_L(e(I_{j-1}/I_j)e)$ and $h_j : C_0(Y_j)^{\sim} \to M_{L+1}(e(I_{j-1}/I_j)e)$ both with finite dimensional range such that

$$\|\Lambda'_j(f) \oplus H_j(f) - h_j(f)\| < \frac{\varepsilon_1}{2}$$

for all $f \in \mathcal{G}_j$. Thus

$$\|\Lambda_j(f) \oplus H_j(f) - h_j(f)\| < \varepsilon_1$$

for all $f \in \mathcal{G}_j$, if δ_1 is small enough, where L is some positive integer. For any $a > 0$, let the finite subset $\{x_i^{(j)}\}_{i=1}^{k(j)}$ be a-dense in Y_j. There are nonzero projections $q_1, q_2, \dots, q_{k(j)}$ mutually orthogonal projections in $M_N(I_{j-1})$ (for some large N) such that

$$p \oplus p \oplus \cdots \oplus p \lesssim q_i$$

for any number of copies of any projection $p \in I_j$ for each i. Set $H_j'(f) = \sum_i f(x_i)\pi_j(q_i)$ for all $f \in C_0(Y_j)$. Then

$$\|\Lambda_j(f) \oplus H_j(f) \oplus \pi_j \circ H_j'(f) - h_j(f) \oplus \pi \circ H_j'(f)\| < \varepsilon_1$$

for all $f \in \mathcal{G}_j$. Suppose that $H_j(f) \oplus H_j'(f) = \sum_{k=1}^{m(j)} f(\zeta_k^{(j)})\pi_j(q_k^{(j)})$ and $h_j(f) \oplus H_j'(f) = \sum_k f(\xi_k)\pi_j(d_k')$ for all $f \in C_0(Y_j)$, where $\{\zeta_k^{(j)}\}$ and the finite subset $\{\xi_k\}$ are in Y_j, $\{q_k^{(j)}\}$ and $\{d_k'\}$ are both mutually orthogonal projections in $M_{L+N}(I_{j-1})$ and $M_{L+1+N}(I_{j-1})$, respectively. We now define $\psi_2^{(j)}(f) = \sum_{k=1}^{m(j)} f(\zeta_k^{(j)})q_k^{(j)}$ and $\psi_3^{(j)}(f) = \sum_k f(\xi_k)d_k'$ for all $f \in C(X)$. Thus, with small enough ε_1 and $a > 0$, we see that $\psi_2^{(j)}$ and $\psi_3^{(j)}$ satisfy the requirements. ∎

2.12. Lemma. (L.G. Brown, [6]) *Let A be a C^*-algebra of real rank zero, q and p be two projections in A^{**}. Suppose that p is an open projection and there is a positive element $a \in A$ such that $q \leqslant e \leqslant p$. Then there exists a projection $e \in A$ such that $q \leqslant e \leqslant p$.*

Proof. This follows from [6] ($1 \Rightarrow 2$).

2.13. Lemma (Cut). (cf Lemma 2.1 in [52]) *Let X be a locally compact metric space, $G \subset X$ be an open subset,*

$$I = \{f \in C_0(X) \mid f(x) = 0 \text{ if } x \notin G\}.$$

For any $\varepsilon > 0$, $\sigma > 0$ and a finite subset $\mathcal{F} \in C_0(X)$, there exist $\delta > 0$, $a > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: if A is a C^*-algebra of real rank zero and $\varphi : C_0(X) \to A$ is a contractive positive linear map, if $\varphi(g_\beta f) = \varphi(g_\beta)\varphi(f)$ for all $f \in C(X)$ and all $0 < \beta < a < \sigma$ for some $\sigma > \beta > 0$, and if

$$\left\|\varphi(g_{a/16}f) - \sum_{k=1}^{m} g_{a/16}f(\xi_k)p_k\right\| < \delta$$

for all $f \in \mathcal{G}$*, where* $\xi_k \in G$*,* $\{p_k\}$ *are mutually orthogonal projections in* A *and where* $g_d \in C(X)$*,* $0 \leqslant g_d \leqslant 1$*,* $g_d(t) = 0$ *if* $\operatorname{dist}(t, X \setminus G) < d/2$ *and* $g_d(t) = 1$ *if* $\operatorname{dist}(t, X \setminus G) \geqslant d$ *(*$d > 0$*), then there exists a projection* $p \in A$ *such that*

$$\varphi(g_\sigma) \leqslant p, \quad \|p\varphi(f) - \varphi(f)p\| < \varepsilon$$

and

$$\left\| p\varphi(f)p - \sum_{k=1}^{m} f(\xi_k)p_k \right\| < \varepsilon$$

for all $f \in \mathcal{F}$*, where* $\xi_k \in \Omega_r$*,*

$$\Omega_r = \{\xi \in G \mid \operatorname{dist}(\xi, X \setminus G) > r\}$$

for some $r < a/2$ *and* $\{p_k\}$ *are mutually orthogonal projections in* pAp*.*

Proof. Let $F = X \setminus G$. Fix $\varepsilon > 0$ and $\sigma > 0$. For any positive number $d > 0$, denote by Ω_d the set

$$\{\xi \in G \mid \operatorname{dist}(\xi, F) \geqslant d\}.$$

Let $\mathcal{G} = \mathcal{F} \cup \{g_d \mid d = a/2^i, 0 \leqslant i \leqslant 4\} \cup \{g_d f \mid f \in \mathcal{F}, d = a/2^i, 0 \leqslant i \leqslant 4\}$. Suppose that there are

$$\xi_1, \xi_2, \ldots, \xi_m \in G$$

and mutually orthogonal projections $p_1, p_2, \ldots, p_m \in A$ such that

$$\left\| \varphi(g_d) - \sum_{j=1}^{m} g_d(\xi_j)p_j \right\| < \eta,$$

and

$$\left\| \varphi(g_d f) - \sum_{j=1}^{m} g_d f(\xi_j)p_j \right\| < \eta, \quad d = a/2^i,\ 0 \leqslant i \leqslant 4,$$

for all $f \in \mathcal{F}$ and for some small $\eta < 1/16$. Let

$$p_{a/8} = \sum_{\xi_j \in \Omega_{a/8}} p_j.$$

From the above inequalities,

$$\|\varphi(g_{a/4}) - \varphi(g_{a/4})p_{a/8}\| < 2\eta \quad \text{and} \quad \|\varphi(g_{a/8})p_{a/8} - p_{a/8}\| < 2\eta.$$

Since $\varphi(g_a^k) = \varphi(g_a)^k \leqslant \varphi(g_{a/2})$ for all k,

$$\varphi(g_\sigma) \leqslant \varphi(g_a) \leqslant q_a \leqslant \varphi(g_{a/2}) \leqslant q_{a/2} \leqslant \varphi(g_{a/4}),$$

where q_d is the open projection corresponding to the hereditary C^*-subalgebra generated by $\varphi(g_d)$.

By Lemma 2.12, there is a projection $p' \in A$ such that

$$\varphi(g_\sigma) \leqslant \varphi(g_a) \leqslant q_a \leqslant p' \leqslant q_{a/2}.$$

We have $\|p'p_{a/8} - p'\| < 2\eta$. This implies that

$$\|p_{a/8}p'p_{a/8} - p'\| < 4\eta.$$

By 2.1 in [21], there is a unitary $v \in A$ such that

$$\|v - 1\| < 8\eta \quad \text{and} \quad v^*p_{a/8}v \geqslant p' \geqslant q_a.$$

We also have

$$\|p_{a/8}\varphi(g_{a/16}f) - \varphi(g_{a/16}f)p_{a/8}\| < 2\eta$$

for all $f \in \mathcal{F}$ and

$$\Big\|p_{a/8}\varphi(f)p_{a/8} - \sum_{\xi_j \in \Omega_{a/8}} g_{a/16}f(\xi_j)p_j\Big\| < \eta$$

for all $f \in \mathcal{F}$. From $\|\varphi(g_{a/8})p_{a/8} - p_{a/8}\| < 2\eta$, we obtain

$$\|p_{a/8} - p_{a/8}q_{a/8}\| < 2\eta.$$

Thus

$$\|p_{a/8}\varphi(f) - p_{a/8}q_{a/8}\varphi(f)\| < 2\eta$$

for all $f \in \mathcal{F}$. Since $\varphi(g_{a/8}g_{a/16}) = \varphi(g_{a/8})\varphi(g_{a/16})$ and $\varphi(g_{a/16}f) = \varphi((g_{a/16})\varphi(f)$ for all $f \in C(X)$, we have $q_{a/8}\varphi(f) = q_{a/8}\varphi(g_{a/16}f)$. Then

$$\|p_{a/8}\varphi(f) - p_{a/8}\varphi(g_{a/16}f)\| < 4\eta$$

for all $f \in \mathcal{F}$. Similarly,

$$\|\varphi(f)p_{a/8} - \varphi(g_{a/16}f)p_{a/8}\| < 4\eta$$

for all $f \in \mathcal{F}$. Therefore

$$\|p_{a/8}\varphi(f) - \varphi(f)p_{a/8}\| < 10\eta$$

for all $f \in \mathcal{F}$ and

$$\Big\|p_{a/8}\varphi(f)p_{a/8} - \sum_{\xi_j \in \Omega_{a/8}} f(\xi_j)p_j\Big\| < 10\eta.$$

Notice that

$$\|v^*p_{a/8}v - p_{a/8}\| < 16\eta.$$

We take $p = v^*p_{a/8}v$ and $\delta = \eta < \varepsilon/64$. ∎

2.14. *Proof of Lemma* 1.5. Let $F = Y \subset X$ and $G = X \setminus F$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)_1$. Let $\sigma = (1/2)\delta_{\rm c}(\varepsilon/8, \mathcal{F})$ and let $\mathcal{P}_1 = P(X, \varepsilon, \mathcal{F})$ and $\delta_0 = \delta(X, \varepsilon, \mathcal{F})$ be as in Theorem 1.6 in [53]. Suppose that $\{x_1, x_2, \dots, x_N\}$ is $\sigma/16$-dense in X. Denote $Y_i = \{\xi \in X \mid \mathrm{dist}\,(\xi, x_i) \leqslant \sigma/8\}$ and denote $X_1, X_2, \dots, X_{2^N}$ all possible finite union of Y_i's. Let $\mathcal{P}_i' = \mathcal{P}(X_i, \varepsilon/8, s_i(\mathcal{F}))$ be as in Theorem 1.6 in [53], where $s_i : C(X) \to C(X_i)$ is the natural surjective map. Among $\{X_i\}$, there is one, which we denote by X_σ, satisfying the following

$$\{\xi \in X \mid \mathrm{dist}\,(\xi, F) \leqslant \sigma\} \subset X_\sigma \subset \{\xi \in X \mid \mathrm{dist}\,(\xi, F) \leqslant 2\sigma\}.$$

Let $\mathcal{G}_1 = \mathcal{G}(\varepsilon/8, \mathcal{F}) \in C(X)_1$ and $\mathcal{G}_2 = \bigcup_{i=1}^{2^n} \mathcal{G}(\varepsilon/8, s_i(\mathcal{F})) \in C(X_i)_1$ be as in Theorem 1.6 in [53]. Note that $\mathcal{G}_2$ does not depend on F but on X, ε and $\mathcal{F}$. Let $\delta_1 = \delta(\varepsilon/8, s(\mathcal{F}))$ be as in Theorem 1.6 in [53]. Every function in $C(X_\sigma)_1$ can be extended to a function in $C(X)_1$. Let $\mathcal{G}_2'$ be the set of such extensions of functions in $\mathcal{G}_2$. Let $P_3 = \mathcal{P}_{\rm KK}(\varepsilon/8, \mathcal{F}, \mathcal{P}_2)$, $\mathcal{G}_3 = \mathcal{G}_{\rm KK}(\varepsilon/8, \mathcal{F}, \mathcal{P}_2)$ and $\delta_2 = \delta_{\rm KK}(\varepsilon/8, \mathcal{F}, \mathcal{P}_2)$ be as in Lemma 2.3. Let $\mathcal{G}_4 = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2' \cup \mathcal{G}_3$ and let $G_5 = \mathcal{G}_{\rm cut}(\varepsilon/8, \sigma, \mathcal{G}_4)$ be as in Lemma 2.13 (with $G = X \setminus F$). We then set $\mathcal{G}_6 = \mathcal{G}_4 \cup \mathcal{G}_5$. Let $\mathcal{P}_3 = \mathcal{P}_1 \cup \mathcal{P}_2$ and let $\delta_3 = (1/8)\min\{\delta_0, \delta_1, \delta_2, \varepsilon/2\}$.

Now suppose that the conditions of Lemma 1.5 hold with $\delta < \min\{\delta_{\rm cut}(\delta_3/4, \sigma, \mathcal{G}), \delta_3/4\}$, $\mathcal{G} = \mathcal{G}_6$, $\mathcal{P} = \mathcal{P}_3$ and $a < \sigma$. By Lemma 2.13, there are $\xi_i \in \Omega_{\sigma/16} = \{\xi \in G \mid \mathrm{dist}\,(\xi, F) \geqslant \sigma/16\}$, and nonzero mutually orthogonal projections $p_i \in I$ such that

$$\psi(g_\sigma) \leqslant p_\sigma, \quad \|p_\sigma \psi(f) - \psi(f) p_\sigma\| < \delta_3,$$

where g_σ is as in Lemma 2.13, and

$$\left\| p_\sigma \psi p_\sigma - \sum_{i=1}^{n} f(\xi_i) p_i \right\| < \delta_3,$$

where $p_\sigma = \sum_{i=1}^{n} p_i$ (when $G = \emptyset$, we let $p_i = 0$) for all $f \in \mathcal{G}_4$. Set $p = 1 - \sum_{i=1}^{n} p_i$. Then $\pi(p\psi p) = \pi \circ \psi$. Since pAp has real rank zero, by 2.4 in [79], every projection in pAp/pIp lifts to a projection in pAp. In fact, there are mutually orthogonal projections $\{q_j\} \in pAp$ such that $\pi(q_j) = \pi(d_j)$. Therefore there are $a_f \in pIp$ such that

$$\left\| p\psi(f)p - \sum_{j=1}^{m} f(\lambda_j) q_j - a_f \right\| < \delta$$

for all $f \in \mathcal{G}$. We may assume that $\sum_{j=1}^{m} q_j = p$. Since eIe has real rank zero, there are projections $e_j \in q_j I q_j$ such that

$$\|ea_f e - a_f\| < \frac{\delta_3}{4},$$

for all $f \in \mathcal{G}$, where $e = \sum_{j=1}^{m} e_j$. Then it is easy to see that

$$\|e\psi(f) - \psi(f)e\| < \frac{\delta_3}{2} \quad \text{and} \quad \Big\|e\psi(f)e - \sum_{j=1}^{m} f(\lambda_j)e_j - ea_f e\Big\| < \frac{\delta_3}{2},$$

for all $f \in \mathcal{G}_4$.

For any $g \in C(X_\sigma)$, there is $g' \in C(X)$ such that $g'(\xi) = g(\xi)$ for all $\xi \in X_\sigma$. Define $\psi'(g) = e\psi(g')e$. We have to check that this is well defined. Since $g'(\xi) = g''(\xi)$ for all $\xi \in X_\sigma$, we have $(g' - g'')g_\sigma = g' - g''$. Since $p_\sigma \geqslant \psi(g_\sigma)$, $\psi'(g_\sigma) = 0$. Since ψ' is positive, this implies that $\psi'(g' - g'') = 0$. This checks that ψ' is a well defined contractive positive linear map which is δ_2-$\mathcal{G}_4$-multiplicative.

Since $\mathrm{K}_1(A) = \{0\}$, $\dim(X) \leqslant 2$ and

$$\Big\|\psi'(f) \oplus \sum_{j=1}^{m} f(\xi_j)(q_j - e_j) - p\varphi(f)p\Big\| < 3\delta_3$$

and

$$\Big\|\varphi(f) - \sum_{i=1}^{n} f(\xi_i)p_i + p\varphi(f)p\Big\| < 6\delta_3$$

for all $f \in \mathcal{G}_4$, by Lemma 2.3,

$$(\psi')_* : \overline{\mathcal{P}}(X_\sigma) \to \underline{\mathrm{K}}(eIe) \text{ is in } \mathcal{N}.$$

It follows from 1.6 in [53] (cf. Theorem 6.2 in [15] and 1.6 in [58]) that there are homomorphisms $h_1 : C(X_\sigma) \to M_L(eIe)$ and $h_2 : C(X_\sigma) \to M_{L+1}(eIe)$ with finite dimensional range such that

$$\|\psi'(f) \oplus h_1(f) - h_2(f)\| < \frac{\varepsilon}{8}$$

for some integer L and all $f \in \mathcal{F}$. Without loss of generality, since $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is $\delta_{\mathrm{c}}(\varepsilon/8, \mathcal{F})$-dense in F, we may assume that $h(g) = \sum_{i=1}^{m} g(\lambda_i)d_i$ for all $g \in C(X)$, where d_i are mutually orthogonal projections in $M_L(eIe)$ such that $\sum_{i=1}^{k} d_i$ is the identity of $M_L(eIe)$, and

$$\left\|\psi'(f) \oplus \sum_{i=1}^{m} f(\lambda_i)d_i - h_2(f)\right\| < \frac{\varepsilon}{4}.$$

Now we will use the part $\sum_{i=1}^{k} f(\xi_i)(q_i - e_i)$ in $p\varphi(f)p$ to "absorb" h_1. By Lemma 2.6, for any integer $N > L$ and any projection $e \in I$,

$$e \oplus e \oplus \cdots \oplus e \lesssim (q_j - e_j)$$

(there are N copies of e). There is a partial isometry

$$v \in (p \oplus e \oplus \cdots \oplus e)M_{L+1}(M(A)/A)(p \oplus e \oplus \cdots \oplus e)$$

(there are L copies of e) such that $v^*d_iv \leqslant q_i - e_i$, $i = 1, 2, \ldots, m$,

$$v^*v = \sum_{i=1}^{m} v^*d_iv \quad \text{and} \quad vv^* = e \oplus e \oplus \cdots \oplus e$$

(there are L copies of e). There is then a partial isometry u such that

$$u^*[\psi'(f) \oplus h_1(f)]u = \psi'(f) \oplus \sum_{i=1}^{m} f(\lambda_i)d_i'$$

and u^*h_2u has finite dimensional range, where $d_i' = v^*d_iv$. So

$$\left\|\left[\psi'(f) \oplus \sum_{i=1}^{m} f(\lambda_i)(q_i - e_i)\right] - \left[u^*h_2(f)u \oplus \sum_{i=1}^{m} f(\lambda_i)((q_i - e_i) - d_i')\right]\right\| < \frac{\varepsilon}{2}.$$

Therefore,

$$\left\|p\psi(f)p - \left[u^*h_2(f)u \oplus \sum_{i=1}^{m} f(\lambda_i)((q_i - e_i) - d_i')\right]\right\| < \frac{\varepsilon}{2}.$$

Thus

$$\left\|\varphi(f) - \left[u^*h_2(f)u \oplus \sum_{i=1}^{m} f(\lambda_i)((q_i - e_i) - d_i')\right] + \sum_{i=1}^{n} f(\xi_i)p_i\right\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$. ∎

2.15. Corollary. *Let A be a unital C^*-algebra of real rank zero and I be an ideal of A, $\psi : C(X) \to A$ be a unital positive linear map and $\pi \circ \psi$ be a unital positive map from $C(Y) \to A/I$, where $\pi : A \to A/I$ is the quotient map and Y is a compact subset of X. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)_1$, there exists $\delta = \delta_{\mathrm{ab}}(\varepsilon, \mathcal{F}) > 0$, $\mathcal{G} = \mathcal{G}_{\mathrm{ab}}(\varepsilon, \mathcal{F}) \subset C(X)_1$ and a finite subset $\mathcal{P} = \mathcal{P}_{\mathrm{ab}}(\varepsilon, \mathcal{F}) \subset \mathbf{P}(C(X))$ satisfying the following: if*

(i) $\|\pi \circ \psi(f) - h_1(f)\| < \delta$ *for all* $f \in \mathcal{G}$, *where* $h_1(f) = \sum_{k=1}^{m} f(\lambda'_k)\pi(d_k)$, $\{\lambda'_k\}$ *is* $\delta_{\mathrm{c}}(\varepsilon/8, \mathcal{F})$*-dense in* Y *and* $\{d_k\}$ *are mutually orthogonal projections in* A *with*

$$p \oplus p \oplus \cdots \oplus p \lesssim d_k$$

for any copies of any projections $p \in I$,

(ii) $\sup_{\xi \in X}\{\mathrm{dist}\,(\xi, Y)\} < \delta_{\mathrm{c}}(\varepsilon/2, \mathcal{F})$,

(iii) $\|\psi(fg) - \psi(f)\psi(g)\| < \delta$ *for all* $f \in \mathcal{G}$ *and*

(iv) (*no* KK*-obstacle*) $\psi_*(\mathcal{P}) \in \mathcal{N}$;

then there exists a homomorphism $h_2 : C(X) \to A$ with finite dimensional range such that

$$\|\psi(f) - h_2(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. The proof is much easier than that of Lemma 1.5 (see the proof of 1.12 in [39]). Here we do not need Lemma 2.13. In the proof of Lemma 1.5, we let $p = 1$ (i.e., $p_\sigma = 0$). So $e\psi e$ is a positive linear map from $C(X)$ into eIe. From (iv), we check that $e\psi e$ has no KK-obstacle. ∎

3. THE MAIN THEOREM AND ITS COROLLARIES

3.1. Theorem. *Let X be a compact metric space with dimension no more than two and let $\mathcal{F}$ be a finite subset of* (*the unit ball of*) $C(X)$. *For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\mathbf{P}(C(X))$, $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G}$ of* (*the unit ball of*) $C(X)$ *such that whenever $A \in \mathbb{A}$ and whenever $\psi : C(X) \to A$ is a contractive unital positive linear map which is δ-$\mathcal{G}$-multiplicative and is σ-injective with respect to δ and $\mathcal{G}$ and $\psi_*(\mathcal{P}) \in \mathcal{N}$, then there exists a unital homomorphisms $\varphi : C(X) \to A$ with finite dimensional range such that*

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

We will prove Theorem 3.1 in several steps.

Step 1 *of the proof.* There is an increasing sequence of finite subsets $\mathcal{P}(n)$ of projections in $\bigcup_{m=1}^{\infty} M_\infty(C(X)\otimes C(C_m\times \mathbb{S}^1))$ such that $\bigcup_{m=1}^{\infty}\overline{\mathcal{P}}(n)$ forms a generating set of the semigroup $\underline{\mathrm{K}}(C(X))_+$. Suppose that the theorem is false. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots$ be a sequence of finite subsets of the unit ball of $C(X)$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and the union $\bigcup_{n=1}^{\infty}\mathcal{F}_n$ is dense in the unit ball of $C(X)$, and $\mathcal{G}(\mathcal{P}(n)) \subset \mathcal{F}_n$, where $\mathcal{G}(\mathcal{P})$ is defined in 0.7. Then there are a positive number $\varepsilon > 0$, a finite subset $\mathcal{F}$, a sequence of positive numbers $\delta_n \to 0$ with $\delta_n \leqslant \delta(\mathcal{P}(n))$, a sequence of positive numbers $\sigma_n \to 0$, the unital simple C^*-algebras B_n of real rank zero, stable rank one, weakly unperforated $\mathrm{K}_0(B_n)$ and unique quasitrace, and unital contractive positive linear maps $\psi_n : C(X) \to B_n$ which are δ_n-$\mathcal{F}_n$-multiplicative, σ_n-δ-$\mathcal{F}_n$-injective and $(\psi_n)_*(\mathcal{P}(n)) \in \mathcal{N}$, and for all n,

$$\inf_{k,\varphi,u}\left\{\sup_{f\in\mathcal{F}}\{\|\psi_n(f)-\varphi(f)\|\}\right\} \geqslant \varepsilon.$$

Here the infimum is taken for all $k \in \mathbb{N}$, all $\varphi : C(X) \to M_k(B_n)$ homomorphisms with finite dimensional range.

Now let

$$A = \bigoplus_{n=1}^{\infty} B_n,$$

the set of all sequences b with $b_n \in B_n$ and $\|b_n\| \to 0$. Then A is a σ-unital C^*-algebra of real rank zero. The multiplier algebra $M(A)$ of B is

$$M(A) = \prod_{n=1}^{\infty} B_n,$$

the set of all sequences b with $b_n \in B_n$ and $\sup_n \|b_n\| < \infty$. Let $\pi : M(A) \to M(A)/A$ be the quotient map. We note that both $M(A)$ and $M(A)/A$ has real rank zero and stable rank one. Let $\varphi' = \{\psi_n\} : C(X) \to M(A)$ be the contractive positive linear map defined by the sequences $\{\psi_n\}$. Since $\bigcup_{n=1}^{\infty}\mathcal{F}_n$ is dense in the unital ball of $C(X)$ and $\delta_n \to 0$, it follows that $\varphi = \pi\circ\varphi'$ is a homomorphism from $C(X) \to M(A)/A$. Since ψ_n are σ_n-δ_n-$\mathcal{F}_n$-injective, $F_n = \sum_{\delta_n}(\psi_n, \mathcal{F}_n)$ converges to X. As in the proof of 1.12 in [53], this implies that φ is a monomorphism.

Let $h_n : C(X) \to B_n$ be homomorphisms with finite dimensional range such that

$$(h_n)_*(\mathcal{P}(n)) = (\psi_n)_*(\mathcal{P}(n)).$$

Denote $H = \{h_n\}$. As in the proof of 1.6 in [53], since $M(A)/A$ has stable rank one,

$$\Gamma(\varphi) = \Gamma(\pi \circ H) \in \mathcal{N}.$$

The proof will be completed if we show that φ is approximated by homomorphisms with finite dimensional range. In fact, if there is a homomorphism $h : C(X) \to M(A)/A$ with finite dimensional range such that

$$\|\varphi(f) - h(f)\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$. Then, by expressing $h(f) = \sum_{k=1}^{m} f(\xi_k)p_k$, where $\{\xi_k\} \subset X$ is a fixed finite subset and $\{p_k\}$ is a set of mutually orthogonal projections in $M(A)/A$, we have the following:

$$\left\|\psi_n(f) - \sum_{k=1}^{m} f(\xi_k)q_k^{(n)}\right\| < \varepsilon$$

for all $f \in \mathcal{F}$, provided that n is large enough, where $\{q_k^{(n)}\}$ are mutually orthogonal projections (for each n) in B_n. Note that both A and $M(A)$ have real rank zero, whence orthogonal projections can be lifted (by a result of Zhang ([76])). This leads to a contradiction.

To show that such homomorphism h exists, we will apply the Lemma 1.2. But before we do that, we will introduce some notation, state some easy facts and one key lemma.

3.2. Let $\mathbb{N}' \subset \mathbb{N}$ be an infinite subset, let $Q_{\mathbb{N}'} = \{b_n\}$, where $b_n = 1$ if $n \in \mathbb{N}'$ and $b_n = 0$ if $n \notin \mathbb{N}'$. We see that $Q_{\mathbb{N}'}$ is a projection in $M(A)$. It is clear that $\mathrm{sp}\,(\varphi) = \mathrm{sp}\,(\pi(Q_{\mathbb{N}'}) \cdot \varphi) = X$. We also note that it suffices to show that $\pi(Q_{\mathbb{N}'}) \cdot \varphi$ can be approximated by a homomorphism $h : C(X) \to \pi(\mathbb{N}')(M(A)/A)\pi(Q_{\mathbb{N}'})$ with finite dimensional range. In other words, we are free to pass to subsequences.

3.3. Let τ_n be the normalized quasitrace on B_n. Fix a nonzero projection $p \in M(A)/A$. Suppose that $\pi(\{p_m\}) = p$ and $p_m \neq 0$. Define

$$\tau_n^p = \tau_n/\tau_n(p_m).$$

Let J be the ideal generated by those projections $e \in M(A)/A$ such that $\sup_n\{\tau_n^p(e_n)\} < \infty$, where $\pi(\{e_n\}) = e$ and I be the ideal generated by those projections $e \in M(A)/A$ such that $\lim \tau_n^p(e_m) = 0$. By 1.8 in [52], this definition does not depend on the choice of $\{p_n\}$ or the choice of $\{e_n\}$. Moreover, J is the ideal generated by the projection p. Clearly, $p \in J$. Suppose that $e = \pi(\{e_n\})$ and that for some integer $K > 0$, $\sup_n\{\tau_n^p(e_n)\} < K$. Then

$$K \cdot \tau_n(p_n) > \tau_n(e_n)$$

for n large. This implies that

$$e \lesssim p \oplus p \oplus \cdots \oplus p \quad \text{in } M_K(M(A)/A)$$

(there are K copies of p). Therefore e is in the ideal generated by p. On the other hand, if $e \in J$, by 1.13 in [58], there is an integer $K > 0$ such that

$$e \lesssim p \oplus p \oplus \cdots \oplus p \quad \text{in } M_K(M(A)/A)$$

(there are K copies of p). Thus, if $e \in J$, $\sup_n\{\tau_n^p(e_n)\} < \infty$, where $\pi(\{e_m\}) = e$.

Now we show that, for any projection $e \in I$, $\lim \tau_m^p(e_m) = 0$, if $\pi(\{e_m\}) = e$. In fact, if $e \in I$, then by 1.13 in [52], there are $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in I$ such that $\lim \tau_m^p(\varepsilon_i^m) = 0$, where $\pi(\{\varepsilon_i^m\}) = \varepsilon_i$, $i = 1, 2, \ldots, k$, and $e \lesssim d_1 \oplus d_2 \oplus \cdots \oplus d_k$, where $d_i = \varepsilon_i \oplus \varepsilon_i \oplus \cdots \oplus \varepsilon_i$ (there are K copies of ε_i), $i = 1, 2, \ldots, k$ and K is some positive integer. This implies that $\lim \tau_m^p(e_m) = 0$, where $\pi(\{e_m\}) = e$.

3.4. Consider a homomorphism $\psi : C_0(Y) \to M(A)/A$, where Y is a locallly compact metric space (if Y is compact, then φ is a homomorphism from $C(Y)$ into $M(A)/A$). Let $\{\xi_m\}$ be a dense sequence in $\mathrm{sp}(\varphi)$. Let $\{S_k\}$ be the sequence of all finite subset of $\{\xi_m\}$. Fix this $\{S_k\}$. Set

$$O_{k,n} = \{\xi \in \mathrm{sp}(\varphi) \mid \mathrm{dist}(\xi, S_k) < r_n\},$$

where $\{r_n\}$ is dense in $(0,1)$. By [6] (see also 1.4 in [52]), there exists a projection $p_{k,n} \in M(A)/A$ such that

$$p_{O_{k,n}} \leqslant p_{k,n} \leqslant p_{\overline{O}_{k,n}},$$

where $p_{O_{k,n}}, p_{\overline{O}_{k,n}} \in (M(A)/A)^{**}$ are the spectral projections of ψ corresponding to the sets $O_{k,n}$ and $\overline{O}_{k,n}$. We will use this notation later.

The following is an easy fact which is proved in 1.18 in [52].

3.5. LEMMA. *Let $\overline{G} \subset O \subset \mathrm{sp}(\varphi)$, where G and O are (relative) open subsets of* $\mathrm{sp}(\varphi)$*. Suppose that $p \in M(A)/A$ is a projection such that*

$$p_G \leqslant p \leqslant p_{\overline{G}}.$$

Suppose also that the sequence $(\tau_m^p(p_{k,n}^m))_m$ *converges (to a finite number or to infinity) for every* $p_{k,n}$. *Let* J *be the ideal generated by* p, $\pi_J : M(A)/A \to (M(A)/A)/J$ *be the quotient map,* I *be the ideal generated by those projections* $e \in M(A)/A$ *such that*

$$\lim \tau_m^p(e_m) = 0,$$

where $\pi(\{e_m\}) = e$ *and* $\pi_I : M(A)/A \to (M(A)/A)/I$ *be the quotient map. If*

$$\mathrm{sp}\,(\pi_J \circ \varphi) \cap O = \mathrm{sp}\,(\pi_I \circ \varphi)) \cap O,$$

then the following hold:

(1) *if* $\overline{G} \subset O_{k,n} \subset \overline{O}_{k,n} \subset O$, *then* $\lim \tau_m^p(p_{k,n}^m) = \infty$;
(2) *if* $\overline{O}_{k,n} \subset G$, *then* $\lim \tau_m^p(p_{k,n}^m) = 0$;
(3) *if* $\overline{O}_{k,n} \subset O \setminus \overline{G}$, *then* $\lim \tau_m^p(p_{k,n}^m) = 0$, *or* ∞;
(4) *there exists* $\xi \in \overline{G} \setminus G$ *such that* $\xi \in \mathrm{sp}\,(\pi_J \circ \varphi)$;
(5) *if* $\xi \in G$, $\xi \notin \mathrm{sp}\,(\pi_I \circ \varphi)$.

(Notice that, for the case (1), the condition $\mathrm{sp}\,(\pi_J \circ \varphi) \cap O = \mathrm{sp}\,(\pi_I \circ \varphi)) \cap O$ implies that $\mathrm{sp}\,(\pi_J \circ \varphi) \cap O \neq 0$. For other cases, the proof is similar, see 1.18 in [52] for details.)

Let I_1 be the (closed) ideal generated by those projections $e \in M(A)/A$ such that

$$\lim_{m\to\infty} \tau_m(e_m) = 0$$

and $\pi_1 : M(A) \to M(A)/I_1$ is the quotient map.

3.6. Lemma. *Let* I_1 *be as above. Suppose that* $J_1 \supset J_2$ *are two ideals of* $M(A)/A$ *such that* J_2 *is generated by a projection* $p \notin I_1$ *and* I_1. *Then* $\mathrm{K}_1(J_1/J_2) = 0$ *and* $\mathrm{K}_0(J_1/J_2)$ *is torsion free.*

Proof. Since J_1 is an ideal of $M(A)/A$, it has real rank zero and stable rank one. It suffices to show that, for any projection $q \in J_1$, $\mathrm{K}_1(qJ_1q/J_2) = 0$ and $\mathrm{K}_0(qJ_1q/J_2)$ is torsion free.

Suppose that $Q = \{q^{(m)}\}$ is a projection in $M(A)$ such that $\pi(Q) = q$. Since $J_2 \supset I_1$, we may assume that $q^{(m)} \neq 0$ for all but finitely many of m. Replacing B_m by $q^{(m)}B_mq^{(m)}$, we may assume that $qJ_1q = M(A)/A$, without loss of generality. It follows from 1.10 in [39] that $\mathrm{K}_1(M(A)/I_1) = 0$. Therefore $\mathrm{K}_1(M(A)/J_2) = 0$. This shows that $\mathrm{K}_1(J_1/J_2) = 0$.

Let $P = \{p^{(m)}\}$ be a projection in $M(A)$ such that $\pi(\{p^{(m)}\}) = p$. Let

$$T = \{\{a_m\} \,|\, a_m = \tau_m(d^{(m)}) - \tau_m(g^{(m)}) \text{ for some projections } \{d^{(m)}\}, \{g^{(m)}\} \in M(A)\}$$

and

$$N = \Big\{ \{b_m\} \in T \,\Big|\, |b_m| \leqslant K|\tau_m(p^{(m)})| + c_m,$$
$$\text{where } K \text{ is a positive number and } c_m \to 0 \Big\}.$$

Similar to 1.11 in [39], one easily computes that

$$\mathrm{K}_0(M(A)/J_2) = T/N.$$

It is clear that T/N is torsion free. Therefore $\mathrm{K}_0(J_1/J_2)$ is torsion free. ∎

The following is a key lemma which is a generalized form of Lemma 2.14 in [52]. A proof is also given in [39].

3.7. LEMMA. (We keep the notation in 3.5) *Let Y be a compact metric space. Suppose that $\psi : C(Y) \to (M(A)/A)$ is a homomorphism and $p_{k,n}$ is as in* 3.4. *If $(\tau_m(p_{k,n}^m))_m$ converges for every (k, n), where $\pi(\{p_{k,n}^m\}) = p_{k,n}$ (τ_m as in* 3.3), *then $\pi_1 \circ \psi$ can be approximated by homomorphism from $C(Y)$ into $(M(A)/A)/I_1$ with finite dimensional range.*

3.8. COROLLARY. *Let p, I, J, π_I and π_J be as in* 3.5. *Let $\psi : C_0(Y) \to J(\subset M(A)/A)$ and $p_{k,n}$ be as in* 3.4. *If $(\tau_m(p_{k,n}^m))_m$ converges for every (k, n), where $\pi(\{p_{k,n}^m\}) = p_{k,n}$ (τ_m as in T), then $\pi_I \circ \psi$ is approximated by homomorphisms from $C_0(Y) \to J/I$ with finite dimensional range.*

Proof. Consider a finite subset $\mathcal{F}$ of $C_0(Y)$ and $\varepsilon > 0$. There is a compact subset $Y_1 \subset Y$ and a function g with support in Y_1 such that

$$\|gf - f\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$. So, without loss of generality, we may assume that every $f \in \mathcal{F}$ has support in Y_1. Let Ω be an open subsets of Y such that

$$Y_1 \subset \Omega$$

and $\overline{\Omega}$ is compact. By [6], there is a projection $q \in M(A)/A$ such that

$$p_\Omega \leqslant q \leqslant p_{\overline{\Omega}}.$$

Suppose that $\pi(\{q^{(m)}\}) = q$, where $q^{(m)}$ are projections in B_m. Since $q \in J$, from 3.3, qIq is generated by projections e with

$$\lim_{m\to\infty} \tau_m^q(e_m) = 0,$$

where $\pi(\{e_m\}) = e$. Set $B = \bigoplus_{m=1}^{\infty} q^{(m)} B_m q^{(m)}$. Let D be the C^*-subalgebra of $C_0(Y)$ generated by functions with support in Y_1. Then $\mathcal{F} \subset D$. Define $\psi' : \widetilde{D} \to M(B)/qIq$ be the unital homomorphism induced by ψ, where $\widetilde{D}$ is the unitization of D. Then by Lemma 3.7, ψ' is approximated by homomorphisms with finite dimensional range. So the corollary follows. ∎

Now we are ready to complete the proof of the main theorem.

3.9. *Proof of Theorem* 3.1. We now construct a finite tower of ideals which satisfy the conditions in Lemma 1.2.

Step 2. Let τ_m be the normalized quasitrace on A_m. There exists a subset $\mathbb{N}_1 \subset \mathbb{N}$ such that $\tau_m(p_{k,n}^m)$ converges ($m \in \mathbb{N}_1$) for each k and n, where $\pi(\{p_{k,n}^j\}) = p_{k,n}$. For the sake of notation, we may assume, without loss of generality, that $\mathbb{N}_1 = \mathbb{N}$. Let I_1 be the ideal generated by those projections $e \in M(A)/A$ which satisfy

$$\lim \tau_m(e_m) = 0,$$

where $\pi(\{e_m\}) = e$. So there are projection $e \in A$ such that

$$p \oplus p \oplus \cdots \oplus p \lesssim e$$

for any number of copies of any projections $p \in I_1$.

Let $\pi_1 : M(A)/A \to (M(A)/A)/I_1$ and φ_1 be the monomorphism from $C(X_1) \to M(A)/I_1$ induced by φ, where $X_1 = \mathrm{sp}\,(\pi \circ \varphi)$ is a compact subset of X. Suppose that $O_{k,n} \cap X_1 \neq \emptyset$. Then

$$\lim \tau_m(p_{k,n}^m) > 0.$$

Therefore, by 3.3,

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_1$. It then follows that condition (a) in Lemma 1.2 is satisfied for $i = 1$. By Lemma 3.7, φ_1 satisfies Condition (A).

Let $\alpha = \delta_{\mathrm{c}}(\varepsilon/4, \mathcal{F})/2$. For each $a > 0$, let

$$O_a^1 = \{\xi \in X \mid \mathrm{dist}\,(\xi, X_1) > a\}.$$

For any (Borel) subset S, we denote by p_S the spectral projection of φ in $(M(A)/A)^{**}$ corresponding to S.

If $O_\alpha^1 = \emptyset$, we will stop the construction and let $I_2 = 0$. Note by Lemma 2.4 that $\Gamma(\varphi_1) = 0$ and $\Gamma(\psi_1) = 0$, where $\psi_1 : C_0(Y_2) \to I_1$ is the monomorphism induced by φ, where $Y_2 = X \setminus X_1$. (Here $X = X_2$. We also let $Y_1 = X_1$.)

So we may assume that $O_\alpha^1 \neq \emptyset$.

Step 3. Let F be a closed subset such that $O_\alpha^1 \subset F \subset O_{3\alpha/4}^1$. Suppose that there exists a projection $p_2 \in I_1$ such that

$$p_{O_\alpha^1} \leqslant p_2 \leqslant p_F$$

and p_2 satisfies the following:

(3a) There is an infinite subset of $\mathbb{N}_2 \subset \mathbb{N}$ such that the sequence $(\tau_m^{(2)}(p_{k,n}^m))_m$ converges ($m \in \mathbb{N}_2$) (to a finite number, or infinity) for each (k, n), where $\tau_m^{(2)} = \tau_m/\tau_m(p_2^m)$ and $\pi(\{p_2^m\}) = p_2$.

(3b) $\mathrm{sp}\,(\pi_2(\overline{Q}_{\mathbb{N}_2}) \cdot \pi_2 \circ \varphi) \cap O^1_{3\alpha/4} \neq \mathrm{sp}\,(\pi_3(\overline{Q}_{\mathbb{N}_2}) \cdot \pi_3 \circ \varphi) \cap O^1_{3\alpha/4}$, where I_2 is the ideal generated by p_2 and I_3 is the ideal of $M(A)/A$ generated by those projections $e \in M(A)/A$ such that

$$\lim \tau_m^{(2)}(e_m) = 0, \quad m \in \mathbb{N}_2,$$

where $\pi(\{e_m\}) = e$, and $\pi_2 : M(A)/A \to (M(A)/A)/I_2$ and $\pi_3 : M(A)/A \to (M(A)/A)/I_3$ are the quotient maps. Denote $X_2 = \mathrm{sp}\,(\pi_2 \circ \varphi)$, $X_3 = \mathrm{sp}\,(\pi_3 \circ \varphi)$, and let $\varphi_2 : C(X_2) \to M(A)/I_2$ and $\varphi_3 : C(X_3) \to M(A)/I_3$ be the monomorphisms induced by φ.

One should note that, by [6], there always exists a projection $p_2 \in I_1$ such that $p_{O^1_\alpha} \leqslant p_2 \leqslant p_F$ and also condition (3a) is always satisfied.

To save notation, without loss of generality, we may assume that $\mathbb{N}_2 = \mathbb{N}$.

If $\xi \in \mathrm{sp}\,(\varphi_2)$ and $\xi \in O_{k,n}$, then by 3.3,

$$\lim \tau_m^{(2)}(p_{k,n}^m) = \infty.$$

Thus

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_2$. Similarly, if $\xi \in \mathrm{sp}\,(\varphi_3)$ and $\xi \in O_{k,n}$, then,

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_3$.

Since $p_2 \in I_2$,

$$X_2 \subset \{\xi \mid \mathrm{dist}\,(\xi, X_1) \leqslant \alpha\}.$$

Let $Y_2 = X_2 \setminus X_1, Y_3 = X_3 \setminus X_2$, $\psi_2 = \varphi_2|C_0(Y_2)$ and $\psi_3 = \varphi_3|C_0(Y_3)$. It follows from Lemma 2.4 that $\Gamma(\varphi_2) = \Gamma(\varphi_3) = 0$ and $\Gamma(\psi_2) = 0$, and it follows from Corollary 3.8 that ψ_3 satisfies Condition (A).

So, in this case, we constructed I_2 and I_3. Both satisfy (a) and (b) in Lemma 1.2. I_2 satisfies (d′) and I_3 satisfies (d).

Moreover, since $(X_3 \setminus X_2) \cap O^1_{3\alpha/4} \neq \emptyset$,

$$\sup_{\xi \in X_3} \mathrm{dist}\,(X_1, \xi) \geqslant \frac{3\alpha}{4}.$$

Step 4. Suppose that, for any projection $e \in M(A)/A$ with

$$p^1_{O_a} \leqslant e \leqslant p_F$$

e does not satisfy (3a) and (3b) at the same time, where $O^1_a \subset F \subset O^1_{3\alpha/4}$ and F is closed.

By [6] (see also 1.4 in [52]), there is a projection $p_2 \in M(A)/A$ such that

$$p_{O_{7\alpha/8}} \leqslant p_2 \leqslant p_{\overline{O}_{7\alpha/8}}.$$

Notice that $p_2 \in I_1$. By the assumption on projections e, since (3a) always holds, by passing to a subsequence, if necessary, we may assume that

(3a) The sequence $(\tau^{(2)}_m(p^m_{k,n}))_m$ converges (to a finite number, or to infinity), where $\tau^{(2)}_m = \tau_m/\tau_m(p^m_2)$ and $\pi(\{p^m_2\}) = p_2$.

(4b) $\mathrm{sp}\,(\pi'_2 \circ \varphi) \cap O^1_{3\alpha/4} = \mathrm{sp}\,(\pi_2 \circ \varphi) \cap O^1_{3\alpha/4}$, where J_2 is the ideal generated by p_2, $\pi'_2 : M(A)/A \to (M(A)/A)/J_2$ is the quotient map, I_2 is the ideal generated by those projections $p \in M(A)/A$ such that

$$\lim \tau^{(2)}_m(p_m) = 0,$$

where $\pi(\{p_m\}) = p$, and $\pi_2 : M(A)/A \to (M(A)/A)/I_2$ is the quotient map. Let $X'_2 = \mathrm{sp}\,(\pi'_2 \circ \varphi)$, $X_2 = \mathrm{sp}\,(\pi_2 \circ \varphi)$ and $\varphi_2 : C(X_2) \to M(A)/I_2$ be the monomorphism induced by φ. It follows from Lemma 2.4 that $\Gamma(\varphi_2) = 0$ and $\Gamma(\psi_2) = 0$.

For any $\xi \in X_2$, if $\xi \in O_{k,n}$

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_2$.

We also have (by (5) of 3.5)

$$X'_2 \subset \Big\{\xi \mid \mathrm{dist}\,(\xi, X_1) \leqslant \frac{7\alpha}{8}\Big\}$$

and

$$X_2 \subset \Big\{\xi \mid \mathrm{dist}\,(\xi, X_1) \leqslant \frac{7\alpha}{8}\Big\}.$$

So

$$\mathrm{dist}\,(X_1, \xi) < \frac{\delta_{\mathrm{c}}\big(\frac{\varepsilon}{4}, \mathcal{F}\big)}{2}$$

for all $\xi \in X_2$.

In other words, in this case, we constructed I_2 which satisfies (a), (b) and (d′).

Furthermore, by (4) of 3.5,

$$[X_2 \setminus X_1] \cap O^1_{7\alpha/8} \neq \emptyset \quad \text{and} \quad \operatorname{dist}(X_1, X_2) \geqslant \frac{3\alpha}{4}.$$

Step 5. Let $i = 3$ if I_2 and I_3 have constructed as in Step 3, and let $i = 2$ if I_2 has been constructed as in Step 4.

For any $a > 0$, set

$$O^2_a = \{\xi \in X \mid \operatorname{dist}(\xi, X_i)) > a\}.$$

If $O^2_\alpha = \emptyset$, then we stop the construction and let $I_{i+1} = 0$. Note that $\Gamma(\varphi_{i+1}) = 0$ and $\Gamma(\psi_{i+1}) = 0$. Otherwise, we continue to construct I_{i+1} (and I_{i+2}) as in Step 3 and Step 4. Denote $\pi_i : M(A) \to M(A)/I_i$ the quotient map.

Step 6. Let F be a closed subset such that $O^2_\alpha \subset F \subset O^2_{3\alpha/4}$. Suppose that there exists a projection $p_3 \in I_i$ such that

$$p_{O^2_\alpha} \leqslant p_3 \leqslant p_F$$

and p_3 satisfies the following: Let $\tau^{(3)}_m = \tau_m/\tau_m(p^m_3)$ and $\pi(\{p^m_3\}) = p_3$.

(3a) there is a subsequence $\mathbb{N}'$ of $\mathbb{N}$ such that the sequence $(\tau^{(3)}_m(p^m_{k,n}))_m$ converges ($m \in \mathbb{N}'$) (to a finite number, or infinity) for each (k, n), where $\tau^{(3)}_m = \tau_m/\tau_m(p^m_3)$ and $\pi(\{p^m_3\}) = p_3$ and

(3b) $\operatorname{sp}(\pi_{i+2}(\overline{Q}_{\mathbb{N}'}) \cdot \pi_{i+2} \circ \varphi) \cap O^2_{3\alpha/4} \neq \operatorname{sp}(\pi_{i+1}(\overline{Q}_{\mathbb{N}'}) \cdot \pi_{i+1} \circ \varphi) \cap O^2_{3\alpha/4}$, where I_{i+1} is the ideal generated by p_3 and I_{i+2} is the ideal of $M(A)/A$ generated by those projections $e \in M(A)/A$ such that

$$\lim \tau^{(2)}_m(e_m) = 0,$$

and $\pi_{i+1} : M(A)/A \to (M(A)/A)/I_{i+1}$, $\pi_{i+2} : M(A)/A \to (M(A)/A)/I_{i+2}$ are the quotient maps.

To save notation, without loss of generality, we may assume that $\mathbb{N}' = \mathbb{N}$. Denote $X_{i+1} = \operatorname{sp}(\pi_{i+1} \circ \varphi)$ and $X_{i+2} = \operatorname{sp}(\pi_{i+2} \circ \varphi)$, and let $\varphi_{i+1} : C(X_{i+1}) \to M(A)/I_{i+1}$ and $\varphi_{i+2} : C(X_{i+2}) \to M(A)/I_{i+2}$ be monomorphisms induced by φ.

If $\xi \in X_{i+1}$ and $\xi \in O_{k,n}$, then

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_{i+1}$ and, if $\xi \in X_{i+2}$ and $\xi \in O_{k,n}$, then

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_{i+2}$.

Let $Y_{i+1} = X_{i+1} \setminus X_i$, $Y_{i+2} = X_{i+2} \setminus X_{i+1}$, $\psi_{i+1} = \varphi_{i+1}|C_0(Y_{i+1})$ and $\psi_{i+2} = \varphi_{i+2}|C_0(Y_{i+2})$. It follows from Lemma 2.4 that $\Gamma(\varphi_{i+1}) = \Gamma(\varphi_{i+2}) = 0$ and $\Gamma(\psi_{i+1}) = 0$, and it follows from Corollary 3.8 that ψ_{i+2} satisfies Condition (A).

Since $p_3 \in I_i$,

$$X_{i+1} \subset \{\xi \mid \operatorname{dist}(\xi, X_i) < \alpha\}.$$

So in this case, we constructed I_{i+1} and I_{i+2} which satisfy (a) and (b) in Lemma 1.2. I_{i+1} satisfies (d′) and I_{i+2} satisfies (d). Furthermore

$$(X_{i+2} \setminus X_i) \cap O^2_{3\alpha/4} \neq \emptyset \quad \text{and} \quad \sup_{\xi \in X_{i+2}} \operatorname{dist}(X_i, \xi) \geqslant \frac{3\alpha}{4}.$$

Step 7. Suppose that, for any projection $e \in M(A)/A$ with

$$p_{O_a{}^1} \leqslant e \leqslant p_F$$

e does not satisfy (3a) and (3b) at the same time, where $O^2_a \subset F \subset O^1_{3\alpha/4}$ and F is closed.

By [6] (see also 1.4 in [48]), there is a projection $p_3 \in M(A)/A$ such that

$$p_{O^2_{7\alpha/8}} \leqslant p_3 \leqslant p_{\overline{O}^2_{7\alpha/8}}.$$

Notice that $p_3 \in I_i$. By the assumption on projections e, since (3a) always holds, by passing to a subsequence, if necessary, we may assume that:

(3a) the sequence $(\tau^{(3)}_m(p^m_{k,n}))_m$ converges (to a finite number, or to infinity), where $\tau^{(3)}_m = \tau_m/\tau_m(p^m_3)$ and $\pi(\{p^m_3\}) = p_3$;

(4b) $\operatorname{sp}(\pi'_3 \circ \varphi) \cap O^2_{3\alpha/4} = \operatorname{sp}(\pi_3 \circ \varphi) \cap O^3_{3\alpha/4}$, where J_{i+1} is the ideal generated by p_3, $\pi'_{i+1} : M(A)/A \to (M(A)/A)/J_{i+1}$ is the quotient map, I_{i+1} is the ideal generated by those projections $p \in M(A)/A$ such that

$$\lim \tau^{(3)}_m(p_m) = 0,$$

where $\pi(\{p_m\}) = p$, and $\pi_{i+1} : M(A)/A \to (M(A)/A)/I_{i+1}$ is the quotient map. Let $X'_{i+1} = \operatorname{sp}(\pi'_{i+1} \circ \varphi)$, $X_{i+1} = \operatorname{sp}(\pi_{i+1} \circ \varphi)$ and $\varphi_{i+1} : C(X_{i+1}) \to M(A)/I_{i+1}$ be the monomorphism induced by φ. It follows from Lemma 2.4 that $\Gamma(\varphi_{i+1}) = 0$ and $\Gamma(\psi_{i+1}) = 0$.

For any $\xi \in X_{i+1}$, if $\xi \in O_{k,n}$

$$e \oplus e \oplus \cdots \oplus e \lesssim p_{k,n}$$

for any number of copies of $e \in I_{i+1}$.

We also have (by (5) of 3.5)

$$X'_{i+1} \subset \left\{\xi \mid \mathrm{dist}\,(\xi, X_i) \leqslant \frac{7\alpha}{8}\right\}$$

and

$$X_{i+1} \subset \left\{\xi \mid \mathrm{dist}\,(\xi, X_i) \leqslant \frac{7/\alpha}{8}\right\}.$$

So

$$\mathrm{dist}\,(X_i, \xi) < \frac{\delta_{\mathrm{c}}\left(\frac{\varepsilon}{4}, \mathcal{F}\right)}{2}$$

for all $\xi \in X_{i+1}$.

In other words, in this case, we constructed I_{i+1} which satisfies (a), (b) and (d$'$).

Furthermore, by (4) of 3.5,

$$[X_{i+1} \setminus X_i] \cap O^1_{7\alpha/8} \neq \emptyset \quad \text{and} \quad \sup_{\xi \in X_{i+1}} \mathrm{dist}\,(X_i, \xi) \geqslant \frac{3\alpha}{4}.$$

Step 8. If we continue, we obtain a sequence of ideals

$$M(A) = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k \cdots$$

where each I_i satisfies (a), (b), and either (d) or (d$'$). We also note that there are projections $e \in I_{i-1}$ such that

$$p \oplus p \oplus \cdots \oplus p \lesssim e$$

for any number of copies of any projection $p \in I_i$. Since we also have

$$\sup_{\xi \in X_{i+2}} \mathrm{dist}\,(X_i, \xi) \geqslant \frac{3\alpha}{4},$$

and X is compact, this construction has to stop after a finite number of steps, say n. We note that if I_i is not generated by a projection p_i, then by the first part of Lemma 3.6, $\mathrm{K}_1(I_{i-1}/I_i) = 0$, and $\mathrm{K}_0(I_{i-1}/I_i)$ is torsion free. Suppose that I_j is generated by a projection p_j and I_i is also generated by a projection p_i and $i > j$. Then, from our construction, $p_i \leqslant p_j$. Choose the smallest of those projections, say p_k. Suppose that $\pi(\{p_k^{(m)}\}) = p_k$ and $p_k^{(m)} \in B_m$ are not zero if $m \in \mathbb{N}'$. So, by passing to another subsequence (the last one, we promise!), we may assume that $p_i^{(m)} \neq 0$ for all m and all those i's. Thus, by Lemma 3.6, $\mathrm{K}_1(I_i/I_{i+1}) = 0$ for $0 \leqslant i \leqslant n-1$. Similarly, by applying Lemma 3.6, $\mathrm{K}_0(I_i/I_{i+1})$ are torsion free for $0 \leqslant i \leqslant n-1$.

Note also that

$$\sup_{\xi \in X} \{\mathrm{dist}\,(\xi, X_n)\} < \alpha.$$

Thus Lemma 1.2 applies. ∎

3.12. *Proof of Corollary* M1. Note that, without σ-injective condition, in the proof of Theorem 3.1, the homomorphism φ may not be injective. Denote by ψ the induced monomorphism from $C(Y) \to M(A)/A$, where Y is a compact subset of X ($\varphi = \psi \circ s$, where $s : C(X) \to C(Y)$ is surjective). As in the proof of Theorem 3.1, we need to show that ψ can be approximated by homomorphisms with finite dimensional ranges. To do this, with the proof of Theorem 3.1, we only need to verify that

$$\Gamma(\psi) \in \mathcal{N}.$$

Note that (in the proof of the main theorem)

$$\mathrm{K}_0(M(A)/A) = \prod_n \mathrm{K}_0(B_n)/\oplus \mathrm{K}_0(B_n).$$

We see that, if each $\mathrm{K}_0(B_n)$ is torsion free, so is $\mathrm{K}_0(M(A)/A)$. Furthermore, since $\mathrm{K}_1(B_n) = 0$ for all n, $\mathrm{K}_1(M(A)/A) = 0$. We then compute that $\mathrm{K}L(C(Y), M(A)/A) = \mathrm{Hom}(\mathrm{K}_0(C(Y)), \mathrm{K}_0(M(A)/A)$. By Lemma 2.2 and the following commutative diagram

$$\begin{array}{ccc} \mathrm{K}_0(C(X)) & \longrightarrow & \mathrm{K}_0(M(A)/A) \\ \downarrow & \nearrow & \\ \mathrm{K}_0(C(Y)) & & \end{array}$$

we conclude that $\Gamma(\psi) \in \mathcal{N}$. ∎

3.13. *Proof of the Main Theorem.* The only difference between 3.1 and the Main Theorem is about σ-injective. It is quite clear that the Main Theorem follows from 3.1 easily. The following proof gives us a choice of σ. Please see Remark 3.14.

Let $\sigma = 1/2\delta_{\mathrm{c}}(\varepsilon/3, \mathcal{F})$ be as in 1.1. By the proof of Corollary M1, we may assume that A is not elementary.

Suppose that ψ is σ-injective with respect to δ and $\mathcal{F}$. Given any $\varepsilon_1 > 0$, and $\mathcal{G}_1$, with sufficiently small δ and sufficiently large $\mathcal{G}$, by Lemma 1.5 in [58], without loss of generality, we may write that

$$\psi(f) = \sum_{i=1}^{m} f(\zeta_i)p_i \oplus \psi_1(f)$$

for all $f \in C(X)$, where $\{\zeta_i\}$ is σ-dense in X and ψ_1 is δ_1-$\mathcal{G}_1$-multiplicative contractive positive linear morphism. Since we now assume that A is not elementary and simple, there exists a nonzero projection $e \in A$ such that $e \lesssim p_i$ for each i. Again, for any $\sigma_1 > 0$, since eAe is nonelementary and simple, there exists a homomorphism $h_0 : C(X) \to eAe$ which is σ_1-$\mathcal{G}_1$-injective. Now we apply Theorem 3.1 to the map $\psi_1 \oplus h_0$ (with sufficiently small δ_1 and sufficiently large $\mathcal{G}_1$). We obtain a homomorphism $h_1 : C(X) \to QM_2(A)Q$ (with $Q = \operatorname{diag}\Big(1 - \sum_{i=1}^{m} p_i, e\Big)$) with finite spectrum such that

$$\|\psi_1(f) \oplus h_0(f) - h_1(f)\| < \frac{\varepsilon}{3}$$

for all $f \in \mathcal{F}$. Since $\{\zeta_i\}$ is σ-dense in X, by changing h_0 slightly, we obtain a homomorphism $h_2(f) = \sum_{i=1}^{m} f(\zeta_i)q_i$, where $\{q_i\}$ are mutually orthogonal projections in eAe such that

$$\|\psi_1(f) \oplus h_2(f) - h_1(f)\| < \frac{2\varepsilon}{3}$$

for all $f \in \mathcal{F}$. Note that $q_i \lesssim p_i$ for each i. The absorption argument that we use several times in this paper shows that

$$\|\psi(f) - h_3(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$ for some homomorphism $h_3 : C(X) \to A$ with finite dimensional range. ∎

3.14. REMARK. The proof in 3.13 actually shows that in the Main Theorem σ can be chosen to be $\delta_c(\varepsilon/3, \mathcal{F})$ as in 1.1. Note that, from the proof in 3.13, we see that if $Y \subset X$, the same σ works for the subset $s(\mathcal{F})$, where $s : C(X) \to C(F)$ is the quotient map. If we assume that all compact metric spaces of dimension no more than two in this paper are compact subsets of the unit ball B^5 of $\mathbb{R}^5$, then σ does not depend on X. It certainly does depend on ε and $\mathcal{F}$. But $\mathcal{F}$ can be thought an image of a finite set of $C(B^5)$.

The reader might wonder what we can say when φ is not σ-injective. The point is that if φ is not sufficiently injective, the condition on $\mathcal{P}$ might be meaningless, in general. For example, a compact subset X of the plane is always a compact subset of a disk $\mathbb{D}$. A positive linear map $\varphi : C(X) \to A$ can always be viewed as a positive linear map from $C(\mathbb{D}) \to A$ by mapping $C(\mathbb{D})$ into $C(X)$ then mapping to A. Then, of course, $\varphi_*(\mathcal{P})$ is always in $\mathcal{N}$ since $\mathbb{D}$ is contractive. On the other hand, one can always use the following lemma to replace it by a σ-injective morphism first. However, if we do not require the homomorphism to have finite dimensional range, σ-injectivity may be removed, for many cases at least. The proof of 3.19 probably explains more about this.

However sometime, we do not need to worry about this problem as we see in Corollary M1.

3.15. LEMMA. (Lemma 1.17 in [53]) *Let X be a compact metric space. For any $\varepsilon > 0$, $\sigma > 0$, $\eta > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that whenever A is a unital C^*-algebra and whenever $\psi : C(X) \to A$ is a unital contractive positive δ-$\mathcal{G}$-multiplicative linear map, then there is an ε-$h(\mathcal{F})$-multiplicative contractive positive linear map $\psi : C(F) \to A$ which is σ-injective with respect to ε and $h(\mathcal{F})$ such that*

$$\|\varphi(f) - \psi \circ h(f)\| < \eta$$

for all $f \in \mathcal{F}$, where F is a compact subset of X and $h : C(X) \to C(F)$ is the quotient map (from $C(X) \to C(X)/I \cong C(F)$, $I = \{f \in C(X) \mid f(x) = 0$ for $x \in F\}$).

3.16. DEFINITION. Let A be a simple C^*-algebra of real rank zero, X be a compact metric space and $\alpha \in \mathrm{K}L(C(X), A)$ such that $\gamma(\alpha)$ in $\mathrm{Hom}(\mathrm{K}_*(C(X)), \mathrm{K}_*(A))$ preserves the order of $\mathrm{K}_0(C(X))$. A is said to satisfy *condition $B(X, \alpha)$*, if for any nonzero projection $e \in A$, there is a homomorphism $h : C(X) \to eAe$ such that

$$\alpha - [h] \in \mathcal{N}.$$

3.17. THEOREM. *Let X be a compact metric space with dimension no more than two. For any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset C(X)$ there is $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{G} \subset C(X)$ and finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$ satisfying the following:*

Let $A \in \mathbb{A}$ and a contractive positive linear morphism $\psi : C(X) \to A$ which is δ-$\mathcal{G}$-multiplicative and σ-$\mathcal{F}$-injective with

$$\psi_*(\overline{\mathcal{P}}) = \alpha(\overline{\mathcal{P}})$$

for some $\alpha \in \mathrm{K}L(C(X), A)$ such that $\gamma(\alpha)$ preserves the order of $\mathrm{K}_0(C(X))$. If A satisfies the condition ($B_{X,\alpha}$), then there exists a homomorphism $h : C(X) \to A$ such that

$$\|\psi(f) - h(f)\| < \varepsilon$$

for all $f \in C(X)$.

Proof. We first would like to point out that, in the case that A is an elementary C^*-algebra, then Theorem 3.17 follows from the Main Theorem directly. This is because $\mathrm{K}_1(A) = 0$, $\mathrm{K}_0(A)$ has no infinitesimal element and free, and $\mathrm{K}_1(C(X))$ is torsion free. Thus $\mathrm{K}L(C(X), A) = \mathrm{Hom}(\mathrm{K}_0(C(X)), \mathrm{K}_0(A))$. It is clear that if $\gamma(\alpha)$ preserves order, and A satisfies the condition $B(X, \alpha)$, then $\alpha \in \mathcal{N}$.

So now we assume that A is a nonelementary C^*-algebra among other conditions.

Suppose that ψ is a contractive positive linear morphism from $C(X)$ into A which is δ-$\mathcal{G}$-multiplicative and σ-$\mathcal{G}$-injective with

$$\psi_*(\mathcal{P}) = \alpha(\overline{\mathcal{P}})$$

for some $\alpha \in \mathrm{K}L(C(X), A)$ such that $\gamma(\alpha)$ preserves the order on K_0, where $\mathcal{P}$ is a finite subset in $\mathbf{P}(C(X))$.

For any $\sigma > 0$, by Lemma 1.5 in [58], without loss of generality, we may write that

$$\psi(f) = \sum_{i=1}^{m} f(\zeta_i) p_i \oplus \psi_1(f)$$

for all $f \in C(X)$, where $\{\zeta_i\}$ is σ-dense in X and $\psi_1(f)$ is δ-$\mathcal{G}$-multiplicative. Since A is a nonelementary simple C^*-algebra of real rank zero, there is a projection $e \in A$ such that

$$e \oplus e \oplus e \oplus e \oplus e \lesssim p_i$$

for each i. By the assumption, for any given $\mathcal{P}$, with small enough δ and large enough $\mathcal{G}$, there is a homomorphism $\varphi : C(X) \to eAe$, such that

$$(\psi_1)_* - \varphi_* : \mathcal{P} \to \underline{\mathrm{K}}(A) \in \mathcal{N}.$$

By 2.9 in [26] (note that $\dim(X) \leqslant 2$), there is a homomorphism $\overline{\varphi} : C(X) \to M_4(eAe)$ such that

$$(\varphi \oplus \overline{\varphi})_* \in \mathcal{N}.$$

Note that, since A is a non-elementary simple C^*-algebra, $\overline{\varphi}$ can always be chosen so that it is σ-injective. Applying the Main Theorem, with sufficiently small δ and sufficiently large $\mathcal{G}$, there are homomorphisms $h_1 : C(X) \to QM_5(A)Q$ (with $Q = \mathrm{diag}\,(1 - \sum_i p_i, e, e, e, e)$) and $h_2 : C(X) \to M_5(eAe)$ both with finite dimensional range such that

$$\|\psi_1(f) \oplus \overline{\varphi}(f) - h_1(f)\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|\varphi(f) \oplus \overline{\varphi}(f) - h_2(f)\| < \frac{\varepsilon}{4}$$

for all $f \in \mathcal{F}$. Without loss of generality (with sufficiently small σ), we may write $h_2(f) = \sum_{i=1}^{n} f(\zeta_i)d_i$, where $\{d_i\}$ are mutually orthogonal projections in $M_5(eAe)$. There is a unitary $U \in M_6(A)$ such that

$$U^* d_i U \leqslant p_i \quad \text{and} \quad U\Big(1 - \sum_{i=1}^{n} p_i\Big) = \Big(1 - \sum_{i=1}^{n} p_i\Big)U = \Big(1 - \sum_{i=1}^{n} p_i\Big)$$

for $i = 1, 2, \ldots, n$. We estimate that

$$\begin{aligned}
&\Big\|\psi(f) - \Big[\sum_{i=1}^{n} f(\zeta_i)(p_i - U^* d_i U) \oplus U^*(h_1(f) \oplus \varphi(f))U\Big]\Big\| \\
&\quad \leqslant \Big\|\psi(f) - \Big[\sum_{i=1}^{n} f(\zeta_i)(p_i - U^* d_i U) \oplus \varphi_1(f) \oplus U^* h_2(f) U\Big]\Big\| \\
&\qquad + \Big\|\sum_{i=1}^{n} f(\zeta_i)(p_i - U^* d_i U) \oplus \varphi_1(f) \oplus U^* h_2(f) U - U^*(h_1 \oplus \varphi(f))U\Big\| \\
&\quad < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

for all $f \in \mathcal{F}$. ∎

3.18. Remark. We see that in 3.17 σ can be chosen to be $1/2\sigma_{\mathrm{c}}(\varepsilon/6, \mathcal{F})$ as remarked in Remark 3.14 and the same σ works for compact subset $Y \subset X$ which also satisfies the required conditions in the theorem if the finite subset is $s(\mathcal{F})$, where $s : C(X) \to C(Y)$ is the quotient map.

For many X, $A \in \mathbb{A}$, and many order preserving α, A satisfies the condition $(B(X, \alpha))$. These are two examples.

3.19. Theorem. *Let X be a finite* CW*-complex of dimension no more than two, and let $A \in \mathbb{A}$ such that $\mathrm{K}_0(A)$ is a dimension group and $\mathrm{K}_1(A) = 0$. Then, for any $\alpha \in \mathrm{K}L(C(X), A)$ which preserves the order of K_0, A satisfies the condition $B(X, \alpha)$.*

Proof. By 2.9 in [44], there is a C^*-subalgebra $B \subset A$ which is isomorphic to a simple AF-algebra and the inclusion induces an isomorphism from $\mathrm{K}_0(B)$ onto $\mathrm{K}_0(A)$.

A result in [43] says that any element in $\alpha \in \mathrm{K}L(C(X), C)$ which preserves the order of K_0, and satisfies that $\alpha_*([1_{C(X)}]) = [1_C]$, where C is any unital simple AF-algebra, can be realized by a unital homomorphism from $C(X)$ into C. Then the theorem follows. ∎

3.20. If X is a finite CW-complex in the plane, then for any $A \in \mathbb{A}$ and any $\alpha \in \mathrm{K}L(C(X), A)$, A satisfies the condition $B(X, \alpha)$. It is clear, since X is a finite CW-complex, that when δ is sufficiently small and $\mathcal{G}$ is sufficiently large, ψ_* induces a homomorphism $\alpha : \mathrm{K}_1(C(X)) \to \mathrm{K}_1(A)$. Thus it is sufficient to show that there is homomorphism $h : C(X) \to eAe$ such that $h_* = \alpha$ on $\mathrm{K}_1(C(X))$. Let $g_1, g_2, \ldots, g_n$ be the generators of $\mathrm{K}_1(C(X))$ corresponding to the bounded connected components

$$\Delta_1, \Delta_2, \ldots, \Delta_n \quad \text{of } \mathbb{C} \setminus X.$$

Suppose that $z_i = \alpha(g_i)$, $i = 1, 2, \ldots, n$. Let $p_1, p_2, \ldots, p_n$ be nonzero mutually orthogonal projections in eAe (they exist because A is assumed to be a non-elementary simple C^*-algebra of real rank zero). There are unitaries $v_i \in p_iAp_i$ such that $[v_i] = z_i$ in $\mathrm{K}_i(A)$. There is a continuous function $f_i : \mathbb{S}^1 \to \partial\Delta_i$, the boundary of Δ_i, for each i. Set

$$x = \sum_{i=1}^{n} f_i(v_i).$$

It is easy to see now that the homomorphism from $C(X)$ into eAe induced by the normal element x satisfies the requirement.

Now suppose that F is a proper compact subset of a compact connected manifold X of dimension no more than two and F itself is a finite simplicial complex. There is a compact subset $Y \subset F$ which is a retraction of F and is homeomorphic to an one-dimensional finite simplicial complex. So Y is homeomorphic to a compact subset of the plane. Thus, for such F, and for any $A \in \mathbb{A}$ and any $\alpha \in \mathrm{K}L(C(X), A)$, from above, A satisfies the condition $B(F, \alpha)$.

3.21. COROLLARY. *Let X be a compact manifold with dimension no more than two and let $\mathcal{F}$ be a finite subset of (the unit ball of) $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\mathbf{P}(C(X))$, $\delta > 0$, and a finite subset $\mathcal{G}$ of (the unit ball of) $C(X)$ satisfying the following: for any $A \in \mathbb{A}$ and any $\psi : C(X) \to A$ which is a contractive unital positive linear map and δ-$\mathcal{G}$-multiplicative, if $\psi_*(\overline{\mathcal{P}}) \in \mathcal{N}$ then there exists a unital homomorphism $\varphi : C(X) \to A$ such that*

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. Let $\sigma = 1/2\delta_{\mathrm{c}}(\varepsilon/6, \mathcal{F})$ (see the above remark). We may assume that X is connected. There are finitely many subsets $F_1, F_2, \ldots, F_l$ of X which are finite simplicial complexes and, for any compact subset $F \subset X$, there is an F_j such that $F \subset F_j$ and

$$\sup\{\mathrm{dist}\,(x, F_j) + \mathrm{dist}\,(F, y) \mid x \in F, y \in F_j\} < \frac{\sigma}{2}.$$

Note that F_i satisfies the condition that we discussed in the third part of Remark 3.18. Let $\delta_i = \delta(\varepsilon/3, \mathcal{F})$ and $\mathcal{G}_i = \mathcal{G}(\varepsilon/3, \mathcal{F})$ be as in 3.17 for $X = F_i$, $i = 1, 2, \ldots, l$, if F_i is a proper subset, or $\delta_i = \delta(\varepsilon/3, \mathcal{F})$ and $\mathcal{G}_i = \mathcal{G}(\varepsilon/3, \mathcal{F})$ be as in the Main Theorem. Set $\delta' = \min\{\delta_i \mid i = 1, 2, \ldots, l\}$ and $\mathcal{G}' = \bigcup_i H_i$, H_i is a finite subset of $C(X)$ such that $s_i(H_i) = \mathcal{G}_i$, where $s_i : C(X) \to C(F_i)$ is the quotient map. By Lemma 3.15, with sufficiently small δ and sufficiently small $\mathcal{G}$, there is a compact subset $F \subset X$ and a contractive positive linear morphism $L' : C(F) \to A$ which is $\delta'/2$-$\mathcal{G}'$-multiplicative and $\sigma/2$-$s'(\mathcal{G}')$-injective, where $s' : C(X) \to C(F)$ is the quotient map, such that

$$\|\psi(f) - L' \circ s'(f)\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{G}'$. Choose F_j above so that $F \subset F_j$ and

$$\sup\{\mathrm{dist}\,(x, F_j) + \mathrm{dist}\,(F, y) \mid x \in F, y \in F_j\} < \frac{\sigma}{2}.$$

Let $s_0 : C(F_j) \to C(F)$ be the quotient map and $L = L' \circ s_0 : C(F_j) \to A$. Then L is $\delta/2$-$s_j(\mathcal{G}')$-multiplicative and σ-$s_j(\mathcal{G}')$-injective. By applying 3.17 and the second part of 3.20, there is a homomorphism $h_1 : C(F_j) \to A$ such that

$$\|L(f) - h_1(f)\| < \varepsilon/2$$

for all $f \in s_j(\mathcal{F})$. Note that $L' \circ s_0 \circ s_j = L \circ s'$. We have

$$\|\varphi(f) - h_1 \circ s_j(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. Take $\varphi = h_1 \circ s_j$. ∎

3.22. *Proof of Corollary* M2. We consider a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism $\varphi : C(\mathbb{D}) \to A$, where $\mathbb{D}$ is the unit disk. It suffices to show that such a contractive positive linear morphism is close to a homomorphism, provided that δ is small enough. But this now follows from Corollary 3.21 immediately.

3.23. *Proof of Corollary* M3. Define two homomorphisms $h_1, h_2 : C(\mathbb{S}^1) \to A$ by the unitaries u and v in A. It follows from Lemma 2.1 in [58] that, for any $\varepsilon_1 > 0$ and any finite subset $\mathcal{F}_1 \in C(\mathbb{T}^2)$ there exists a contractive positive linear map $L : C(\mathbb{T}^2) \to A$ which is ε_1-$\mathcal{F}_1$-multiplicative such that

$$\|L(z_1) - u\| < \varepsilon \quad \text{and} \quad \|L(z_2) - v\| < \varepsilon,$$

where z_1 and z_2 are standard unitaries generators of $C(\mathbb{T}^2)$. So, without loss of generality, we may assume that $L(z_1) = u$ and $L(z_2) = v$. Therefore the first part of M3 follows from 3.19.

To obtain the second part, we apply 3.17. Note, if $\alpha \in \mathrm{K}L(C(X), A)$ such that $L_*(\mathcal{P}) = \alpha(\mathcal{P})$, then, since $\tau(\kappa(u, v)) = 0$, $\gamma(\alpha)$ preserves the order on K_0. It is proved in [28] that, if B is a unital simple AF-algebra, then for any order preserving homomorphism, $\beta \in \mathrm{Hom}(\mathrm{K}_0(C(\mathbb{T}^2)), B)$ which also preserves the identity, there exists a homomorphism $h' : C(\mathbb{T}^2) \to B$ such that $h'_* = \beta$. By the additional assumption that $\mathrm{K}_0(A)$ is a dimension group, for any nonzero projection $e \in A$, it follows from 2.9 in [44] that there is a unital simple AF-algebra B which can be injectively mapped into eAe. ∎

4. HIGHER DIMENSION CASES

4.1. THEOREM. *Let $\mathbb{D}^3$ be the three-dimensional unit solid ball. There are contractive positive linear maps $\Lambda_n : C(\mathbb{D}^3) \to M_{n^3}$ which are σ_n-injective with $\sigma_n \to 0$ and which satisfy*

$$\|\Lambda_n(fg) - \Lambda_n(f)\Lambda_n(g)\| \to 0$$

for all $f, g \in C(\mathbb{D}^3)$, as $n \to \infty$ and $a > 0$ such that

$$\inf_{\psi_n}\{\sup\{\|\Lambda_n(f) - \psi_n(f)\| \mid f \in \mathcal{F}\}\} \geqslant a,$$

where $\mathcal{F}$ is a given set of finite generators and the infimum is taken from all homomorphisms $\psi_n : C(\mathbb{D}^3) \to M_{n^3}$ for each n.

Proof. We start with a known example ([60], [31], [32] and [75]). There are two sequences of unitaries $u_n, v_n \in M_n$ with

$$\|u_n v_n - v_n u_n\| \to 0$$

but $\dim[e(u_n, v_n)] = n - 1$. Denote $p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q_n = e(u_n, v_n)$ in $M_2(M_n)$ (see [60] for $e(u_n, v_n)$). There are partial isometries $w_n \in M_2(M_n)$ such that $p_n - w_n^* q_n w_n$ is a rank one projection in M_n.

Let $B' = \bigoplus_n M_n$ be as a C^*-algebra. Then $M(B') = \prod_n M_n$. Let $U = \{u_n\}$ and $V = \{v_n\}$ and $\pi' : M(B') \to M(B')/B'$ be the quotient map. Then $\pi'(U)$ commutes with $\pi(V)$. Thus we obtain a homomorphism $\overline{\Psi} : C(\mathbb{T}^2) \to M(B')/B'$. By [11], there is a contractive positive linear map $\Psi : C(\mathbb{T}^2) \to M(B')$ such that $\pi' \circ \Psi = \overline{\Psi}$. Write $\Psi = \{\varphi_n\}$; then

$$\|\varphi_n(u) - u_n\| \to 0 \quad \text{and} \quad \|\varphi_n(v) - v_n\| \to 0$$

as $n \to \infty$, where u and v are standard unitaries generators of $C(\mathbb{T}^2)$. Since $\overline{\Psi}$ is a homomorphism,

$$\|\varphi_n(fg) - \varphi_n(f)\varphi_n(g)\| \to 0$$

as $n \to \infty$ for all $f \in C(\mathbb{T}^2)$.

Now let $h_n : C(\mathbb{D}^3) \to M_n$ be $4/n^{1/3}$-injective homomorphisms. Such a homomorphism is found easily. Now define a contractive positive linear map $\Lambda_n : C(\mathbb{D}^3) \to M_{n^3}$ as follows:

$$\Lambda_n(f) = \operatorname{diag}(h_n, \varphi_n(f|C(\mathbb{T}^2)), \varphi_n(f|C(\mathbb{T}^2)), \ldots, \varphi_n(f|C(\mathbb{T}^2)))$$

for $f \in C(\mathbb{D}^3)$, where we view $\mathbb{T}^2$ as a compact subset of $\mathbb{D}^3$ and there are $n^2 - 1$ copies of $\varphi_n(f|C(\mathbb{T}^2))$. Clearly $\Lambda_n \mid C(\mathbb{D}^3)$ is a contractive positive linear map. We also have

$$\|\Lambda_n(fg) - \Lambda_n(f)\Lambda_n(g)\| \to 0$$

for all $f \in C(\mathbb{D}^3)$ as $n \to \infty$. Furthermore, for any (finite) subset $\mathcal{F}_n \in C(\mathbb{D}^3)$, Λ_n is $4/n^{1/3}$-injective.

Consider $B = \bigoplus_n M_{n^3}$ be as a C^*-algebra. Then $M(B) = \prod_n M_{n^3}$. Let $\pi : M(B) \to M(B)/B$ be the quotient map. Denote by d_n a projection in M_{n^3} with rank n, $d = \{d_n\}$ and let I_d be the ideal of $M(B)$ generated by d. Let $\pi_I : M(B) \to M(B)/I_d$ be the quotient map. Set

$$p = \operatorname{diag}(0, \{p_n\}, \{p_n\}, \ldots, \{p_n\}), \quad q = \operatorname{diag}(0, \{q_n\}, \{q_n\}, \ldots, \{q_n\})$$

and

$$w = \operatorname{diag}(0, \{w_n\}, \{w_n\}, \ldots, \{w_n\})$$

(in $M(B)$). Since

$$\operatorname{diag}(0, p_n - w_n^* q_n w_n, p_n - w_n^* q_n w_n, \ldots, p_n - w_n^* q_n w_n)$$

has rank n^2-1, by 3.3, $p-w^*qw \notin I_d$. Let $\Lambda = \{\Lambda_n\}$. Then $\Lambda : C(\mathbb{D}^3) \to M(B)$ is a contractive positive linear map and $\pi\circ\Lambda : C(\mathbb{D}^3) \to M(B)/B$ is a monomorphism. However, $\operatorname{sp}(\pi_I \circ \Lambda) = \mathbb{T}^2$ and $\pi_I \circ \Lambda$ induces a homomorphism $C(\mathbb{T}^2)$ from into $M(B)/I_d$. Furthermore this homomorphism is the same as

$$H = \operatorname{diag}(\overline{\Phi}, \overline{\Phi}, \ldots, \overline{\Phi}).$$

(there are $n^2 - 1$ many copies). Since $\pi_I(p - v^*qv)$ is a nonzero projection in $M(B)/I_d$, H can not be approximated by homomorphisms from $C(\mathbb{T}^2)$ into $M(B)/I_d$ with finite dimensional range. This implies that $\pi \circ \Lambda$ can not be approximated by homomorphisms from $C(\mathbb{D}^3)$ into $M(B)/B$ with finite dimensional range. This, in turn, implies that Λ can not be approximated by homomorphisms with finite dimensional range. Since every homomorphism from $C(\mathbb{D}^3)$ into $M(B)$ ($M(B)$ is a W*-algebra) can be approximated by homomorphisms with finite dimensional range, we conclude that Λ is bounded away from homomorphisms, whence, $\{\Lambda_n\}$ is bounded away from homomorphisms. ∎

4.2. THEOREM. *Let X be a finite* CW*-complex with* $\dim(X) \geqslant 3$ *and* $A \in \mathbb{A}$ *be nonelementary. There are contractive positive linear maps* $\Lambda_n : C(X) \to A$ *which are σ_n-injective with $\sigma_n \to 0$ and which satisfy*

$$\|\Lambda_n(fg) - \Lambda_n(f)\Lambda_n(g)\| \to 0$$

for all $f, g \in C(X)$, as $n \to \infty$,

$$(\Lambda_n)_*(\overline{\mathcal{P}}) \in \mathcal{N}$$

for any finite subset $\mathcal{P} \in \overline{P}(C(X))$ when n is sufficiently large, and $a > 0$ such that

$$\inf_{\psi_n}\{\sup\{\|\Lambda_n(f) - \psi_n(f)\| \mid \|f\| \leqslant 1\}\} \geqslant a,$$

where the infimum is taken from all homomorphisms $\psi_n : C(X) \to A$ for each n.

Proof. Let τ be the unique normalized quasitrace. By [78], there are projections $e_n \in A$ such that $\tau(e_n) \leqslant 1/n^3$ and $(1 - e_n)A(1 - e_n) = M_{n^3}(B_n)$, where B_n is a unital hereditary C^*-subalgebra of A. Let C_n be the C^*-subalgebra of A which is isomorphic to M_{n^3}. We may further assume that $1-1_{C_n} \neq 0$. With p and

q being as in Theorem 4.1; as in the proof of Theorem 4.1, we obtain a sequence of unital contractive positive linear morphisms $\psi_n : C(\mathbb{T}^2) \to C_n$ with

$$\|\psi_n(fg) - \psi_n(f)\psi_n(g)\| \to 0$$

for all $f, g \in C(\mathbb{T}^2)$ and

$$n[\tau(\psi_n(p)) - \tau(\psi_n(q))] \to 1$$

as $n \to \infty$. Since $\dim(X) \geqslant 3$, there is a subset $Y \subset X$ such that Y is homeomorphic to $\mathbb{D}^3$. Thus there is a compact subset $Y_1 \subset Y$ such that Y_1 is homeomorphic to $\mathbb{T}^2$. Without loss of generality, we may assume that $Y_1 = \mathbb{T}^2$. Since e_nAe_n is a nonelementary simple C^*-algebra, there is a unital monomorphism $h_n : C(X) \to e_nAe_n$. Now define

$$\Lambda_n(f) = h_n(f) \oplus \psi_n(f|Y_1).$$

Clearly $(\Lambda_n)_*(\mathcal{P}) \in \mathcal{N}$ when n is large enough. Moreover,

$$\frac{\tau(e_n)}{\tau(\psi_n(p) - \psi_n(q))} = \mathrm{O}\Big(\frac{1}{n^2}\Big).$$

Note that, by 4.5 in [27], every monomorphism from $C(\mathbb{D}^3)$ into A can be approximated by homomorphisms with finite dimensional range. The proof of Theorem 4.1 shows that Λ_n is bounded away from homomorphisms (choose $\{d_n\} \in \prod A$ with each $d_n \in A$ and $1/n^2 - 1/n^4 \leqslant \tau(d_n) \leqslant 1/n^2 + 1/n^4$). ∎

4.3. Let A be a unital separable simple C^*-algebra of real rank zero, stable rank one with weakly unperforated $\mathrm{K}_0(A)$ and unique (normalized) quasi-trace τ, and let X be a compact metric space. Suppose that $\psi : C(X) \to A$ is a unital positive linear map. Then $\tau \circ \psi$ is a state for $C(X)$. Fix a Borel measure m on X with the property that every open ball O of X with positive radius has positive measure. Such a measure will be called *strictly positive*. Let m_ψ be the (Borel) measure on X induced by $\tau \circ \psi$. For any $\varepsilon > 0$ and $\sigma > 0$, a positive linear map ψ is said to be m-ε-σ-injective if there is an ε-net $\{x_1, x_2, \dots, x_m\} \subset X$ (i.e. for any $x \in X$ there is $x_i \in \{x_1, x_2, \dots, x_m\}$ such that $\mathrm{dist}\,(x, x_i) < \varepsilon$) such that

$$m_\psi(O_i) \geqslant \sigma m(O_i), \quad i = 1, 2, \dots, m,$$

where $O_i = \{x \in X \mid \mathrm{dist}\,(x, x_i) < \eta\}$ for any $\eta > 0$.

For a general space X, we have proved (see [39]) the following positive result:

4.4. THEOREM. *Let X be a compact metric space. For any $\varepsilon > 0$, $\sigma > 0$, a strictly positive Borel measure m on X and a finite subset $\mathcal{F} \subset C(X)$ there exist a finite subset $\mathcal{P}$ of projections in $\bigcup_{m=1}^{\infty} M_\infty(C(X) \otimes C(C_m \times \mathbb{S}^1))$, $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following:*

If $A \in \mathbb{A}$ and $\psi : C(X) \to A$ is a δ-$\mathcal{G}$-multiplicative contractive positive linear map which is m-δ-σ-injective and which satisfies that $\psi_* : \overline{\mathcal{P}} \to \underline{\mathrm{K}}(B)$ lies in $\mathcal{N}$, then there exist a homomorphism $\varphi : C(X) \to A$ with finite dimensional range such that

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

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GUIHUA GONG
Department of Mathematics
University of Puerto Rico, Rio Piedras
San Juan, PR 00931
U.S.A.

HUAXIN LIN
Department of Mathematics
University of Oregon
Eugene, Oregon 97403–1222
U.S.A.

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