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AUTOMORPHISMS OF A$\mathbb{T}$ ALGEBRAS WITH THE ROHLIN PROPERTY

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Abstract. We consider automorphisms of a unital simple A$\mathbb{T}$ algebra of real rank zero with unique tracial state and give several conditions equivalent to the Rohlin property, partially extending similar results in the UHF algebra case. The conditions on an automorphism include that the crossed product has a unique tracial state and that the crossed product has real rank zero.

Keywords: *Automorphisms, Rohlin property,* A$\mathbb{T}$ *algebras, real rank zero, crossed products, tracial states.*

MSC (2000): Primary 46L40; Secondary 46L35, 46L55.

1. INTRODUCTION

A$\mathbb{T}$ algebras are C^*-algebras obtained as inductive limits of $\mathbb{T}$ algebras; $\mathbb{T}$ algebras are direct sums of matrix algebras over $C(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

The class of A$\mathbb{T}$ algebras of real rank zero is shown by Elliott ([10]) to be classified by K-theoretic data, which, in the unital simple case, are $K_0(A)$, $[1]_0$, and $K_1(A)$, and this class includes AF algebras, irrational rotation C^*-algebras ([11]), and most of their higher-dimensional analogues – non-commutative tori ([12], [13], [20], [2]).

We may conjecture that this class is closed under the operation of $\mathbb{Z}$-crossed product as far as the action of $\mathbb{Z}$ is sufficiently outer and, at the same time, is approximately inner (as a strong condition which ensures that the crossed product remains finite). The crossed product of an AF algebra by an approximately inner automorphism with the Rohlin property is unique up to isomorphism and is an A$\mathbb{T}$

algebra of real rank zero ([16], [14]). Thus we expect that an appropriate notion of the outerness in this case is the Rohlin property.

We are still far from proving the above conjecture. But we give various characterizations of the Rohlin property for approximately inner automorphisms of unital simple A$\mathbb{T}$ algebras of real rank zero with unique tracial state, extending similar results in the UHF case ([11]). The conditions include that the crossed product has real rank zero (see Theorem 2.1).

We first recall the Rohlin property. The Rohlin property in ergodic theory was adopted to the context of von Neumann algebras by A. Connes ([7]) and then to the context of some C^*-algebras by Herman and Ocneanu ([15]). The property was used to prove the so-called stability for the automorphism (appropriately formulated in each case), which was what we actually needed to study the conjugacy and outer conjugacy problems (at least in the von Neumann algebra case).

In the C^*-algebra case a non-trivial example of an automorphism with the Rohlin property was first obtained in [5]: the shift automorphism σ of the UHF algebra $\otimes_{\mathbb{Z}} M_2$ (the infinite tensor product of copies of the two by two matrices M_2 indexed by the integers $\mathbb{Z}$) satisfies the property that for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there are projections $e_0, e_1, \ldots, e_{2^n-1}$ such that

$$\sum_{i=0}^{2^n-1} e_i = 1, \quad \|\sigma(e_i) - e_{i+1}\| < \varepsilon$$

for $i = 0, 1, \ldots, 2^n - 1$ with $e_{2^n} = e_0$. Note that for any $k \in \mathbb{N}$, the projections $\sigma^k(e_0), \ldots, \sigma^k(e_{2^n-1})$ satisfy the same properties as $e_0, \ldots, e_{2^n-1}$ and that $(\sigma^k(e_i))_k$ forms a central sequence, which is also a necessary property for proving the above-mentioned stability. Having 2^n projections which almost cyclically permute under σ certainly depends on a particular property of $\otimes_{\mathbb{Z}} M_2$ (or rather of $K_0(\otimes_{\mathbb{Z}} M_2) \cong \mathbb{Z}[1/2]$) and we cannot expect this in general. But, thanks to the above example, we now expect that this property is not so stringent as it may look if properly formulated. An appropriate property (strong enough to show the stability and possibly valid in general case) is given by the properties in the following proposition, which we will state without proof (cf. [4], [17]):

PROPOSITION 1.1. *Let A be a unital* AF *algebra and α an approximately inner automorphism of A. Then the following conditions are equivalent:*

(i) *For any $k \in \mathbb{N}$ there are positive integers $k_1, \ldots, k_m \geqslant k$ satisfying the following condition: For any finite subset F of A and $\varepsilon > 0$ there are projections e_{lj}, $l = 1, \ldots, m$, $j = 0, \ldots, k_l - 1$ in A such that*

$$\sum_{l=1}^{m} \sum_{j=0}^{k_l - 1} e_{lj} = 1,$$

$$\|\alpha(e_{lj}) - e_{l,j+1}\| < \varepsilon,$$

$$\| [x, e_{lj}] \| < \varepsilon$$

for $l = 1, \ldots, m$, $j = 0, \ldots, k_l - 1$ and $x \in F$, where $e_{lk_l} = e_{l0}$.

(ii) *For any $k \in \mathbb{N}$ there are positive integers $k_1, \ldots, k_m \geqslant k$ satisfying the following condition: For any finite subset F of A and $\varepsilon > 0$ there are matrix units $e_{l,ij}$, $i, j = 0, \ldots, k_l - 1$ in A for each $l = 1, \ldots, m$ such that*

$$\sum_{l=1}^{m} \sum_{j=0}^{k_l - 1} e_{l,jj} = 1,$$

$$\|\alpha(e_{l,ij}) - e_{l,i+1\,j+1}\| < \varepsilon,$$

$$\| [x, e_{l,ij}] \| < \varepsilon$$

for $l = 1, \ldots, m$, $i, j = 0, \ldots, k_l - 1$ and $x \in F$, where the indices i, j in $e_{l,ij}$ are taken modulo k_l for each l.

(iii) (i) *holds for $\{k, k+1\}$ in place of $\{k_1, \ldots, k_m\}$.*

(iv) (ii) *holds for $\{k, k+1\}$ in place of $\{k_1, \ldots, k_m\}$.*

Note that the implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) hold trivially for any unital C^*-algebra A. It is not difficult to see (ii) $\Rightarrow$ (iv) (cf. [17]). We use the fact that A is AF only for proving (i) $\Rightarrow$ (ii). (Here we use the stability for automorphisms with the Rohlin property ([15]); if A is not AF, it is not clear how to arrange the situation where the stability is applicable.) We were unable to prove this implication for unital A$\mathbb{T}$ algebras of real rank zero. We call the condition (i) above the *Rohlin* property (see [14] for the non-unital case). We recall that an automorphism α of a unital C^*-algebra A is *uniformly outer* if for any $a \in A$, any projection $p \in A$, and any $\varepsilon > 0$, there are finite number of projections $p_1, \ldots, p_n$ in A such that $\sum_i p_i = p$ and $\|p_i a \alpha(p_i)\| < \varepsilon$, $i = 1, \ldots, n$ and that if α has the Rohlin property then α is uniformly outer ([17]).

2. MAIN RESULT

THEOREM 2.1. *Let $\mathcal{A}$ be a unital simple* A$\mathbb{T}$ *algebra of real rank zero with unique tracial state τ and let α be an approximately inner automorphism of $\mathcal{A}$. Then the following conditions are equivalent:*

(i) *α has the Rohlin property;*
(ii) *α^m is uniformly outer for any $m \neq 0$;*
(iii) *α^m is not weakly inner in π_τ for any $m \neq 0$;*
(iv) *$\mathcal{A} \times_\alpha \mathbb{Z}$ has a unique tracial state;*
(v) *$\mathcal{A} \times_\alpha \mathbb{Z}$ has real rank zero.*

We recall that a unital C^*-algebra has *real rank zero* if and only if the set of elements with finite spectra in $A_{\mathrm{sa}} = \{h \in A \mid h = h^*\}$ is dense in A_{sa} ([6]).

If α is an automorphism of a simple C^*-algebra A, the crossed product $A \times_\alpha \mathbb{Z}$ is simple if and only if all non-zero powers of α are outer. Hence the above $\mathcal{A} \times_\alpha \mathbb{Z}$ is simple if the conditions are satisfied.

Note that (i) $\Rightarrow$ (ii) follows trivially from the definitions and (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) follow easily (see 4.3 and 4.4 of [17]). The proofs of (v) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) are essentially the same as in the UHF case ([16]). But we will present these results in a slightly general form below. We will give the proof of (i) $\Rightarrow$ (v) in Section 3 using that $\mathcal{A}$ is a unital simple A$\mathbb{T}$ algebra of real rank zero and then the proof of (iii) $\Rightarrow$ (i) in Section 4 using the full assumption on $\mathcal{A}$.

Let A be a unital C^*-algebra and $T(A)$ the compact convex set of tracial states of A. If α is an automorphism of A, $T^\alpha(A)$ denotes the α-invariant tracial states, which is again a compact convex set. (If α is approximately inner, then $T^\alpha(A) = T(A)$.) We define an affine mapping r of $T(A \times_\alpha \mathbb{Z})$ into $T^\alpha(A)$ by the restriction $r(\psi) = \psi | A$.

PROPOSITION 2.2. *In the situation as above, if the linear span of projections in $A \times_\alpha \mathbb{Z}$ and elements in A is dense in $A \times_\alpha \mathbb{Z}$, then $T(A \times_\alpha \mathbb{Z})$ is isomorphic with $T^\alpha(A)$ under r.*

Proof. Denote by $\mathcal{E}$ the canonical projection of $A \times_\alpha \mathbb{Z}$ onto A. If $\varphi \in T^\alpha(A)$, it follows that $\varphi \circ \mathcal{E} \in T(A \times_\alpha \mathbb{Z})$ since for $a, b \in A$ and $m, n \in \mathbb{N}$,

$$\varphi \circ \mathcal{E}(aU^m bU^n) = \delta_{m+n,0}\varphi(a\alpha^m(b)) = \delta_{m+n,0}\varphi(b\alpha^n(a)) = \varphi \circ \mathcal{E}(bU^n aU^m)$$

where U is the canonical unitary in $A \times_\alpha \mathbb{Z}$. Thus r is surjective.

Denote by $\widehat{\alpha}$ the dual action of $\mathbb{T}$ on $A \times_\alpha \mathbb{Z}$. Let $\psi \in T(A \times_\alpha \mathbb{Z})$. If p is a projection in $A \times_\alpha \mathbb{Z}$, then $\psi \circ \widehat{\alpha}_t(p)$ is constant in t (cf. [18]). If $a \in A$, then $\widehat{\alpha}_t(a) = a$. Thus $\psi \circ \widehat{\alpha}_t$ is equal to ψ on the linear span of projections and A. Hence $\psi \circ \widehat{\alpha}_t = \psi$ for $t \in \mathbb{T}$, which implies that $\psi = r(\psi) \circ \mathcal{E}$. Thus r is injective. ∎

Note that if a C^*-algebra has real rank zero, then the linear span of projections is dense ([6]). Hence we have that (v) $\Rightarrow$ (iv) by this proposition.

PROPOSITION 2.3. *In the situation as above if $T(A\times_\alpha \mathbb{Z})$ is isomorphic with $T^\alpha(A)$ under r, then for any extreme point φ of $T^\alpha(A)$ and any non-zero integer m, α^m is not weakly inner in π_φ.*

Proof. Suppose that for some extreme $\varphi \in T^\alpha(A)$ and some $m > 0$, α^m is weakly inner in π_φ. Thus there is a unitary V in $\pi_\varphi(A)''$ such that

$$V\pi_\varphi(a)V^* = \pi_\varphi \circ \alpha^m(a), \quad a \in A.$$

Define a unitary W on the GNS representation space H_φ by

$$W\pi_\varphi(a)\Omega_\varphi = \pi_\varphi \circ \alpha(a)\Omega_\varphi, \quad a \in A.$$

Since φ is extreme in $T^\alpha(A)$, $\overline{\alpha} = \mathrm{Ad}\, W$ on $\pi_\varphi(A)''$ acts ergodically on its center Z_φ. Since $\overline{\alpha}^m$ is trivial on Z_φ, it follows that $Z_\varphi \cong \mathbb{C}^n$ for some n with $n|m$ and $\overline{\alpha}$ acts on the spectrum of Z_φ as a cyclical permutation.

Since $\mathrm{Ad}\,\overline{\alpha}(V) = \mathrm{Ad}\, V$ on $\pi_\varphi(A)''$ and $\mathrm{Ad}\, W^m(V) = V$, it follows that $\overline{\alpha}(V) = \gamma V$ for some unitary $\gamma \in Z_\varphi$ which satisfies

$$\overline{\alpha}^{m-1}(\gamma)\overline{\alpha}^{m-2}(\gamma)\cdots\overline{\alpha}(\gamma)\gamma = 1.$$

Then, by an easy computation, replacing m by a multiple of m and choosing V suitably one can assume that $\gamma = 1$.

Since $Q = W^m V^* \in \pi_\varphi(A)'$, $[W, Q] = 0$, and $[V, Q] = 0$, one finds a unitary $Q_1 \in \pi_\varphi(A)'$ (as a function of Q) such that $Q_1^m = Q$, $[W, Q_1] = 0$, and $[V, Q_1] = 0$. Then one defines a representation ρ of $A \times_\alpha \mathbb{Z}$ on H_φ by

$$\rho(a) = \pi_\varphi(a), \quad a \in A, \quad \rho(U) = WQ_1^*,$$

and a state ϕ of $A \times_\alpha \mathbb{Z}$ by

$$\phi(x) = \frac{1}{m}\sum_{k=0}^{m-1}(\rho \circ \widehat{\alpha}_{k/m}(x)\Omega_\varphi, \Omega_\varphi).$$

Regarding $A \times_{\alpha^m} \mathbb{Z}$ as a C^*-subalgebra of $A \times_\alpha \mathbb{Z}$, $\phi | A \times_{\alpha^m} \mathbb{Z}$ is a tracial state because $\rho(A \times_{\alpha^m} \mathbb{Z})'' = \pi_\varphi(A)''$ and

$$\phi(x) = (\rho(x)\Omega_\varphi, \Omega_\varphi), \quad x \in A \times_{\alpha^m} \mathbb{Z}.$$

Since $\rho(U)\Omega_\varphi = Q_1^*\Omega_\varphi$, it also follows that $\phi \circ \mathrm{Ad}\, U = \phi$ on $A \times_{\alpha^m} \mathbb{Z}$. Then if $a, b \in A$ and $k, l \in \mathbb{N}$, then $\phi(aU^k bU^l) = \phi(a\alpha^k(b)U^{k+l})$ is equal to $0{=}\phi(bU^l aU^k)$ if $k + l \notin m\mathbb{Z}$ and otherwise, to $\phi(\alpha^k(b)U^{k+l}a) = \phi(bU^{k+l}\alpha^{-k}(a)) = \phi(bU^l aU^k)$. Thus ϕ is a tracial state. But since $\phi|AU^m \neq 0$, ϕ is not $\widehat{\alpha}$-invariant, which implies that the restriction map r is not injective. ∎

Remark 2.4. When A is a unital separable C^*-algebra and α is an automorphism of A, let us consider the following conditions:

(i) α^m is uniformly outer for any $m \neq 0$;

(ii) $T(A \times_\alpha \mathbb{Z})$ is isomorphic with $T^\alpha(A)$ under the mapping $r : \psi \to \psi|A$;

(iii) For any $\varphi \in T^\alpha(A)$ and $m \neq 0$, α^m is not weakly inner in π_φ.

Then it follows that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) (and hence that (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) for Theorem 2.1). (Since A is separable, the conclusion of 2.3 holds for any $\varphi \in T^\alpha(A)$.) See 4.3 of [17] for (i) $\Rightarrow$ (ii). Note that if $T(A) = \emptyset$ then the conditions (ii) and (iii) hold trivially.

3. PROOF OF (i) $\Rightarrow$ (v)

Recall that for a pair u, v of unitaries in a C^*-algebra A with $\|\,[u,v]\,\|$ sufficiently small one can define a *Bott* element $B(u,v) \in K_0(A)$ ([21]). When the spectrum of u is finite, this may be defined as follows: Let t_1, t_2, t_3, t_4 be a string of points in $\mathbb{T}$ in counter-clockwise order with mutual distances bigger than some constant and let Q (resp. E) be the spectral projection of u corresponding to $(t_1, t_3]$ (resp. $(t_2, t_4]$). Then vQv^*E is close to a projection whose K_0 class will be denoted by $[vQv^*E]_0$ and we set $B(u,v) = [vQv^*E]_0 - [QE]_0$. If $B(u,v) = 0$ and $\mathrm{Spec}\,(u)$ is finite, then vuv^* and u are connected by a path of isospectral unitaries in a small neighbourhood of u ([3]). Note that $B(1,v) = 0$, $B(u,v) = -B(v,u)$, $B(u_1u_2, v) = B(u_1, v) + B(u_2, v)$ if all the terms are well-defined, $B(u,v) = B(wuw^*, wvw^*)$ for any unitary w, and that $B(u,v)$ is continuous in u, v as far as it is well-defined. We quote the following result from [3].

Theorem 3.1. *If $\mathcal{A}$ is a unital simple* A$\mathbb{T}$ *algebra of real rank zero and $u, v \in \mathcal{A}$ are unitaries such that $[u,v] \approx 0$, $[u]_1 = 0$, $B(u,v) = 0$, and* $\mathrm{Spec}\,(v)$ *is almost dense in $\mathbb{T}$, then there is a rectifiable path u_t of unitaries in $\mathcal{A}$ of length less than a universal constant such that $u_0 = 1$, $u_1 = u$, and $[u_t, v] \approx 0$.*

We briefly indicate how to prove this. First suppose that $[v]_1 = 0$. Then we can suppose that $\mathrm{Spec}\,(v)$ is finite (but is almost dense in $\mathbb{T}$) and connect uvu^* and v by a path of isospectral unitaries of length less than a universal constant in a small neighbourhood of v. Since the spectrum is finite and constant along this path, the path is given as $w_t uvu^* w_t^*$; thus we have a path $w_t u$ from u to $w_1 u$ which commutes exactly with v. Its length is bounded by a universal constant. Since $[w_1 u]_1 = 0$ and $\mathrm{Spec}\,(v)$ is almost dense, one can assume (after a continuous deformation in a small neighbourhood of $w_1 u$) that the $w_1 u$ restricted to each eigen projection of v has zero K_1 class. Then $w_1 u$ can be connected to 1 in the commutant of v.

In the case $[v]_1 \neq 0$, we find unitaries $u_1, v_1 \in \mathcal{A}$ and a projection $e \in \mathcal{A}$ such that $u \approx u_1$, $v \approx v_1$, $[u_1, e] = 0$, $[v_1, e] = 0$, $u_1 v_1 e = v_1 u_1 e$, $\mathrm{Spec}\,(u_1)$ is finite, and $\mathrm{Spec}\,(v_1 e)$ is finite and almost dense in $\mathbb{T}$. Then there is a path u_t of unitaries in $e\mathcal{A}e$ from $u_1 e$ to e of length $\leqslant \pi$ in the commutant of $v_1 e$. Thus it suffices to show the assertion for $u = e + u_1(1-e)$, and $v = v_1 e + v_1(1-e)$. We then find two partial unitaries $v_2, v_3 \in \mathcal{A}$ with $e_i = v_i^* v_i$ such that $e_2 + e_3 = e$, $[v_2] = -[v_3] = [v]$, and $v_2 + v_3 \approx v_1 e$. Then we consider two almost commuting unitaries $U = e_3 + u_1(1-e)$, $V = v_3 + v_1(1-e)$ in $(1-e_2)\mathcal{A}(1-e_2)$. Since $[U]_1 = 0 = [V]_1$ and $B(U, V) = 0$, we are reduced to the above case.

We call a unital subalgebra $\mathcal{B}$ of an A$\mathbb{T}$ algebra $\mathcal{A}$ *a local algebra* if $\mathcal{B} \cong B \otimes C(\mathbb{T})$ with B finite-dimensional. Let $\mathcal{B}_i = B_i \otimes C(\mathbb{T})$ for $i = 1, 2$ with B_i finite-dimensional and z_i the canonical unitary of $\mathcal{B}_i$: $z_i(z) = 1 \otimes z$. We call an embedding φ of $\mathcal{B}_1$ into $\mathcal{B}_2$ *of standard form* if $\varphi(z_1)$ is a direct sum of elements of the form:

$$\begin{pmatrix} 0 & \cdot & & & z_2^{\natural} \\ 1 & 0 & & & \\ & 1 & \cdot & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

with $z_2^{\natural} = z_2$ or z_2^* up to unitary equivalence.

PROPOSITION 3.2. *Let $\mathcal{A}$ be a unital simple* A$\mathbb{T}$ *algebra of real rank zero and α an approximately inner automorphism of $\mathcal{A}$. If α has the Rohlin property, then there is a sequence $\{U_n\}$ of unitaries in $\mathcal{A}$ such that*

$$\lim \|\alpha(U_n) - U_n\| = 0, \quad \lim \mathrm{Ad}\, U_n = \alpha.$$

Proof. There exists a sequence $\{V_n\}$ of unitaries in $\mathcal{A}$ such that $\lim \mathrm{Ad}\, V_n = \alpha$. Since

$$\lim \mathrm{Ad}\, V_n^* = \alpha^{-1}, \quad \lim \mathrm{Ad}\, \alpha(V_n^*) = \alpha^{-1}, \text{ etc.},$$

it follows that $\lim \mathrm{Ad}\, \alpha(V_n^*) V_n = \mathrm{id}$.

Let $\mathcal{B}$ be a local algebra of $\mathcal{A}$ and let B be a finite-dimensional C^*-algebra and u a unitary in $\mathcal{A} \cap B'$ such that B and u generate $\mathcal{B}$. Let $e_1, \ldots, e_r$ be the set of minimal central projections of $\mathcal{B}$.

Let $N \in \mathbb{N}$. For any sufficiently large n, we have that $\|(\mathrm{Ad}\, V_n - \alpha)|B\| \approx 0$ and $\|\mathrm{Ad}\, V_n(u) - \alpha(u)\| \approx 0$ and further for $U_n = \alpha(V_n^*) V_n$ that $\|(\mathrm{Ad}\, \alpha^k(U_n) - \mathrm{id}\,)|B\| \approx 0$ and $\|\mathrm{Ad}\, \alpha^k(U_n)(u) - u\| \approx 0$ for $k = 0, 1, \ldots, N$. Hence we have a unitary $W \in \mathcal{A}$ such that $W \approx 1$, $U_0 = WU_n\alpha(U_n)\cdots\alpha^N(U_n) \in B'$, and $[U_0, u] \approx 0$. By the following lemmas we obtain that $[U_0e_i]_1 = 0$ and $B(U_0, u_i) = 0$ where $u_i = ue_i + 1 - e_i$. Thus it follows that $[U_0]_1 = \sum_i [U_0e_i]_1 = 0$ in $K_1(\mathcal{A} \cap B')$ and that

$$B_{\mathcal{A}\cap B'}(U_0(1-e_i)+e_i, u_i) = 0,$$

where the Bott element is computed in $\mathcal{A} \cap B'$. Since

$$B(U_0, u_i) = B(U_0e_i + 1 - e_i, u_i) + B(U_0(1-e_i)+e_i, u_i),$$

it follows that

$$B(U_0e_i + 1 - e_i, u_i) = 0.$$

Since $\mathcal{A} \cap B'e_i \hookrightarrow \mathcal{A}$ induces an isomorphism of $K_0(\mathcal{A} \cap B'e_i)$ onto $K_0(\mathcal{A})$, it further follows that

$$B_{\mathcal{A}\cap B'e_i}(U_0e_i, u_i) = 0.$$

Hence we have that

$$\begin{aligned} B_{\mathcal{A}\cap B'}(U_0, u) &= \sum_i B_{\mathcal{A}\cap B'}(U_0, u_i) \\ &= \sum_i B_{\mathcal{A}\cap B'}(U_0e_i + 1 - e_i, u_i) + \sum_i B_{\mathcal{A}\cap B'}(U_0(1-e_i)+e_i, u_i) = 0. \end{aligned}$$

Thus there is a rectifiable path $U(t)$ of unitaries in $\mathcal{A} \cap B' = \bigoplus_{i=1}^r \mathcal{A} \cap B'e_i$ of length less than a universal constant such that $U(0) = 1$, $U(1) = U_0$, and $[U(t), u] \approx 0$. By using such paths and the stability for α (which comes from the Rohlin property for α), we obtain a unitary $U \in \mathcal{A} \cap B'$ such that $[u, U] \approx 0$ and $W\alpha(V_n^*)V_n \approx \alpha(U)U^*$. Then $\alpha(V_nU) \approx V_nU$ and $\operatorname{Ad} V_nU|\mathcal{B} \approx \alpha|\mathcal{B}$. We apply this process repeatedly. ∎

LEMMA 3.3. *If* $[\cdots]_1$ *denotes the* K_1 *class of an invertible element,* $[e_i\alpha(V_n)V_n^*e_i]_1 = 0$ *for any sufficiently large* n.

Proof. We have a k such that $\|(\operatorname{Ad} V_k - \alpha)|B\| \approx 0$ and $\|\operatorname{Ad} V_k(u) - \alpha(u)\| \approx 0$. We have $m > n (> k)$ such that $\alpha(V_n) \approx V_mV_nV_m^*$, $[V_mV_n^*, e_i] \approx 0$, $[V_m^*V_n, e_i] \approx 0$, and $[V_mV_n^*, V_k] \approx 0$. Then we obtain that

$$\begin{aligned} [e_iV_mV_n^*e_i]_1 &= [\alpha(e_i)V_kV_mV_n^*V_k^*\alpha(e_i)]_1 = [\alpha(e_i)V_mV_n^*\alpha(e_i)]_1 \\ &= [e_iV_m^*V_mV_n^*V_me_i]_1 = [e_iV_n^*V_me_i]_1 = -[e_iV_m^*V_ne_i]_1 \end{aligned}$$

in $K_0(\mathcal{A})$. Hence we conclude that

$$[e_i\alpha(V_n^*)V_ne_i]_1 = [e_iV_mV_n^*V_m^*V_ne_i]_1 = [e_iV_mV_n^*e_i]_1 + [e_iV_m^*V_ne_i]_1 = 0. \quad ∎$$

LEMMA 3.4. *$B(\alpha(V_n^*)V_n, u) = 0$ for any sufficiently large n.*

Proof. We choose $m > n > k$ as in the proof of the previous lemma. Assuming $[V_m V_n^*, u] \approx 0$ and $\mathrm{Ad}\, V_m^* \alpha(u) \approx u$ as we may, we obtain that

$$\begin{aligned} B(V_m V_n^*, u) &= B(V_k V_m V_n^* V_k, V_k u V_k) = B(V_m V_n^*, \alpha(u)) \\ &= B(V_m^* V_m V_n^* V_m, V_m^* \alpha(u) V_m) = B(V_n^* V_m, u) = -B(V_m^* V_n, u). \end{aligned}$$

Hence

$$B(\alpha(V_n^*)V_n, u) = B(V_m V_n^* V_m^* V_n, u) = B(V_m V_n^*, u) + B(V_m^* V_n, u) = 0. \quad \blacksquare$$

LEMMA 3.5. *Let K be the compact operators on $\ell^2(\mathbb{Z})$ and σ the automorphism of K implemented by the right shift on $\ell^2(\mathbb{Z})$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists an $N_0 \in \mathbb{N}$ with the following property: For any $N \geqslant N_0$ there exists a set $\{e_{ij} \mid i, j = 0, \ldots, n-1\}$ of matrix units in K such that*

$$\sum_{i=0}^{n-1} e_{ii} \leqslant \sum_{i=0}^{N-1} P_i,$$

$$\|\sigma(e_{ij}) - e_{ij}\| < \varepsilon,$$

$$n\,\mathrm{rank}\,(e_{00}) > (1-\varepsilon)N,$$

where P_i is the projection onto $\ell^2(\{i\})$.

Proof. We can prove this by using the method employed to prove 2.1 of [16]. Let $\{E_{ij}\}_{i,j\in\mathbb{Z}}$ be matrix units in K such that $\sigma(E_{ij}) = E_{i+1,j+1}$. For $k, l \in \mathbb{N}$ with $1 < k < l$ we define

$$\begin{aligned} e_{00} = \sum_{m=1}^{k-1} \biggl\{ &\frac{m}{k} E_{mm} + \frac{k-m}{k} E_{m+k+l,m+k+l} \\ &+ \frac{\sqrt{m(k-m)}}{k} \bigl(E_{m,m+k+l} + E_{m+k+l,m}\bigr) \biggr\} + \sum_{m=k}^{k+l} E_{mm} \end{aligned}$$

which is a projection with rank $k+l$ such that $\|\sigma(e_{00}) - e_{00}\|$ is order of $1/\sqrt{k}$. Let V be the shift operator: $V = \sum_i E_{i+1,i}$. Define

$$e_{ij} = V^{(2k+l)i} e_{00} V^{-(2k+l)j}$$

for $i, j = 0, \ldots, k-1$. Then $\{e_{ij}\}$ forms matrix units and satisfies that $\|\sigma(e_{ij}) - e_{ij}\| = \|\sigma(e_{00}) - e_{00}\|$. Since $\sum_{i=0}^{n-1} e_{ii} \leqslant \sum_{i=0}^{N-1} P_i$ with $N = (2k+l)n$, we obtain that

$$\frac{n\,\mathrm{rank}\,(e_{00})}{N} = \frac{k+l}{2k+l} = 1 - \frac{k}{2k+l}.$$

Thus we first choose k sufficiently large and then choose l so that $(k+1)/(2k+l)$ is sufficiently small. Set $N_0 = (2k+l)n$. Since we can increase l while retaining the same k for $N \geqslant N_0$, this completes the proof. $\blacksquare$

Lemma 3.6. *Let $\{e_l\}$ and $\{f_l\}$ be central sequences of projections. If there is an increasing sequence $\{m_l\}$ of positive integers such that $m_l \to \infty$ and $m_l[f_l] \leqslant [e_l]$, then there exists a central sequence $\{v_l\}$ of partial isometries such that $v_l^* v_l = f_l$ and $v_l v_l^* \leqslant e_l$ for all sufficiently large l.*

Proof. Let $\{\mathcal{A}_n\}$ be an increasing sequence of local algebras such that $\bigcup_n \mathcal{A}_n$ is dense in $\mathcal{A}$ and the inclusions $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ are of standard form, which is possible by Elliott's classification theory ([10]). We shall find a partial isometry v_m for any sufficiently large m such that v_m almost commutes with $\mathcal{A}_1$, $v_m^* v_m = f_m$, and $v_m v_m^* \leqslant e_m$.

Let $\mathcal{A}_n = A_n \otimes C(\mathbb{T})$ with A_n finite-dimensional, $\{P_{ni} \mid \ i = 1, \ldots, K_n\}$ the set of minimal central projections in $\mathcal{A}_n$, and z_n the canonical unitary of $\mathcal{A}_n$. Fix a large $N \in \mathbb{N}$. We find a non-zero projection $q_i \in A_n \cap A_1' P_{1i}$ for some n such that

$$\operatorname{Ad} z_1^k(q_i), \quad k = 0, \ldots, N$$

are mutually orthogonal. (Note that the embedding $\mathcal{A}_1 \subset \mathcal{A}_n$ is also of standard form.) Now we consider $\mathcal{A} \cap A_1' P_{1i}$ instead of $\mathcal{A}$ and $e_l P_{1i}$, $f_l P_{1i}$ instead of e_l, f_l respectively. We may and do assume that $e_l P_{1i}, f_l P_{1i}$ are projections. Also we have an increasing sequence $\{m_l\}$ of positive integers (which may be different from the above) such that the same condition is satisfied for $e_l P_{1i}, f_l P_{1i}$ instead of e_l, f_l. We will now denote $\mathcal{A} \cap A_1' P_{1i}, q_i, e_l P_{1i}$, etc. by $\mathcal{A}, q, e_l$, etc. respectively.

Since $\mathcal{A}$ is simple, we have, for a sufficiently large $m > n$, a $k \in \mathbb{N}$ such that $k[qP_{mj}] \geqslant [P_{mj}]$ for any j. Choose l so large that e_l almost commutes with A_m and f_l almost commutes with z_1. Then $k[e_l q] \geqslant [e_l]$, where $[e_l q]$ denotes the equivalence class of a projection close to $e_l q$. Hence if $m_l > k$, we obtain that

$$[f_l] \leqslant [e_l q].$$

Let u be a partial isometry in $\mathcal{A}$ such that $u^* u = f_l$ and uu^* is a projection dominated by a projection close to $q e_l q$. Then define $v \in \mathcal{A}$ by

$$v = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \operatorname{Ad} z_1^j(u).$$

Then

$$\|\operatorname{Ad} z_1(v) - v\| \leqslant \frac{2}{\sqrt{N}},$$

$$v^* v = \frac{1}{N} \sum_{ij} \operatorname{Ad} z_1^i(u^*) \operatorname{Ad} z_1^j(u) = \frac{1}{N} \sum_j \operatorname{Ad} z_1^j(u^* u) \approx f_l,$$

$$e_l v f_l \approx v.$$

Thus by the polar decomposition of $e_l v f_l$ we obtain a partial isometry w such that $w^*w = f_l$, $ww^* \leqslant e_l$, and $\mathrm{Ad}\, z_1(w) \approx w$ up to the order of $1/\sqrt{N}$. Going back to the original situation, denote w by $w_i \in \mathcal{A} \cap A_1' P_{1i}$ and take the sum of w_i over i, which produces the desired partial isometry in $\mathcal{A} \cap A_1'$ almost commuting with z_1. ∎

LEMMA 3.7. *Let $\{e_l\}$ and $\{f_l\}$ be central sequences of projections such that $\|\alpha(e_l) - e_l\| \to 0$ and $\|\alpha(f_l) - f_l\| \to 0$. If there is an increasing sequence $\{m_l\}$ of positive integers such that $m_l \to \infty$ and $m_l[f_l] \leqslant [e_l]$, then there exists a central sequence $\{v_l\}$ of partial isometries such that $v_l^* v_l = f_l$, $v_l v_l^* \leqslant e_l$, and $\|\alpha(v_l) - v_l\| \to 0$ as $l \to \infty$.*

Proof. We have assumed that α has the Rohlin property. It is not difficult to see that for any N there is a sequence $\{e_{l1}, \ldots, e_{lN}\}$ of projections such that

$$\sum_{i=1}^{N} e_{li} \leqslant e_l,$$
$$\|\alpha(e_{li}) - e_{l,i+1}\| \to 0,$$
$$\frac{[e_l] - N[e_{l1}]}{[e_l]} \to 0,$$

where the last condition means that there is an increasing sequence $\{n_l\}$ of positive integers such that $n_l \to \infty$ and $n_l([e_l] - N[e_{l1}]) \leqslant [e_l]$. Then it follows that $[f_l]/[e_{l1}] \to 0$. By the previous lemma we obtain a central sequence $\{w_l\}$ of partial isometries such that $w_l^* w_l = f_l$ and $w_l w_l^* \leqslant e_{l1}$ for any sufficiently large l. Set

$$v_l = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \alpha^j(w_l).$$

Then $\{v_l\}$ is a central sequence, $\|\alpha(v_l) - v_l\| \leqslant 2/\sqrt{N}$, and $v_l^* v_l \approx f_l$ and $e_l v_l \approx v_l$. Then by the polar decomposition of $e_l v_l f_l$ we obtain a partial isometry with initial projection f_l and final projection dominated by e_l such that $\|\alpha(v_l) - v_l\|$ is of order of $1/\sqrt{N}$. ∎

A (separable) unital C^*-algebra is *approximately divisible* if for any finite subset F of $\mathcal{A}$ and $\varepsilon > 0$ there is a finite-dimensional C^*-sublagebra $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ has no abelian central projections and $\|\,[x, y]\,\| < \varepsilon$ for any $x \in F$ and any $y \in \mathcal{B}$ with $\|y\| \leqslant 1$ ([1]). Note that any unital simple A$\mathbb{T}$ algebra of real rank zero is approximately divisible.

Proposition 3.8. *Let $\mathcal{A}$ be a unital simple* $\mathrm{A}\mathbb{T}$ *algebra of real rank zero and α an approximately inner automorphism of $\mathcal{A}$. If α has the Rohlin property, then $\mathcal{A} \times_\alpha \mathbb{Z}$ is approximately divisible.*

Proof. Let $n \in \mathbb{Z}$ with $n \geqslant 2$ and $\varepsilon > 0$. We choose $N \in \mathbb{N}$ as in Lemma 3.5. By the Rohlin property we obtain $N_1, \dots, N_m \geqslant N+1$ such that there is a central sequence $\{E^l_{ij} \mid \ i = 1, \dots, m,\ j = 0, \dots, N_i - 1\}$ of projections with

$$\sum_i \sum_j E^l_{ij} = 1,$$

$$\|\alpha(E^l_{ij}) - E^l_{i,j+1}\| \to 0.$$

By Proposition 3.2 there is a sequence $\{U_n\}$ of unitaries in $\mathcal{A}$ such that $\operatorname{Ad} U_n \to \alpha$ and $\alpha(U_n) - U_n \to 0$.

Let $\mathcal{B} = B \otimes C(\mathbb{T})$ be a local algebra of $\mathcal{A}$. We first choose U_k such that $\alpha^{-1}|\mathcal{B} \approx \operatorname{Ad} U_k^*|\mathcal{B}$ and $\alpha(U_k) \approx U_k$, and then choose l such that $[U_k, E^l_{ij}] \approx 0$, $\alpha(E^l_{ij}) \approx E^l_{i,j+1}$ and E^l_{ij} almost commutes with $\mathcal{B}$. Then we choose U_n such that $\operatorname{Ad} U_n(E^l_{ij}) \approx E^l_{i,j+1}$, $\operatorname{Ad} U_n|\alpha^{-1}(\mathcal{B}) \approx \alpha|\alpha^{-1}(\mathcal{B})$, and $\alpha(U_n) \approx U_n$. Then we obtain a unitary W in a small neighbourhood of $U_n U_k^*$ such that

$$\begin{gathered} \operatorname{Ad} W(E^l_{ij}) = E^l_{i,j+1}, \\ \operatorname{Ad} W|\mathcal{B} \approx \operatorname{id}|\mathcal{B}, \\ \alpha(W) \approx W. \end{gathered}$$

Set

$$E_{k;ij} = W^i E^l_{k0} W^{-j}.$$

Then $\{E_{k;ij}\}$ forms matrix units for each k such that $\alpha(E_{k;ij}) \approx E_{k;i+1,j+1}$ for $i, j = 0, \dots, N_k - 2$ and $E_{k;ij}$'s almost commute with $\mathcal{B}$, and there is a unitary U such that $U \approx 1$ and $\operatorname{Ad} U \circ \alpha(E_{k;ij}) = E_{k;i+1,j+1}$ for $i, j \leqslant N_k - 2$. By applying Lemma 3.5 to $\{E_{k;ij}\}$ for each k, we obtain matrix units $\{e_{ij}\}_{i,j=0}^{n-1}$ such that $\|\alpha(e_{ij}) - e_{ij}\| < \varepsilon$, and $[1] - n[e_{00}] \leqslant \varepsilon[1]$. In this way we obtain a central sequence $\{e^l_{ij} \mid i, j = 0, \dots, n-1\}$ of matrix units such that

$$\begin{gathered} \alpha(e^l_{ij}) - e^l_{ij} \to 0, \\ \frac{[1] - n[e^l_{00}]}{[1]} \to 0. \end{gathered}$$

Applying Lemma 3.7 to the central sequences $\{e^l_{00}\}$ and $\{f^l\}$ with $f^l = 1 - \sum_i e^l_{ii}$, we obtain a central sequence $\{v_l\}$ of partial isometries such that $v_l^* v_l = f^l$, $v_l v_l^* \leqslant$

e^l_{00}, and $\alpha(v_l) - v_l \to 0$. Define $w_0 = f^l$ and $w_i = e_{i-1,0}v_l$ for $i \geqslant 1$ and let $f^l_{ij} = w_i w_j^*$ and $g^l_{ij} = e^l_{i0}(1 - f^l_{11})e^l_{0j}$. Then $\{f^l_{ij}\}_{ij=0}^n$ and $\{g^l_{ij}\}_{ij=0}^{n-1}$ form central sequences of matrix units such that

$$\alpha(f^l_{ij}) - f^l_{ij} \to 0, \quad \alpha(g^l_{ij}) - g^l_{ij} \to 0, \quad \sum_i f^l_{ii} + \sum_i g^l_{ii} = 1.$$

Thus, since $\mathcal{A} \subset \mathcal{A} \times_\alpha \mathbb{Z}$, we obtain a central sequence $\{\mathcal{B}_l\}$ of unital C^*-subalgebras of $\mathcal{A} \times_\alpha \mathbb{Z}$ such that $\mathcal{B}_l \cong M_n \oplus M_{n+1}$, which implies that $\mathcal{A} \times_\alpha \mathbb{Z}$ is approximately divisible. ∎

COROLLARY 3.9. *Let $\mathcal{A}$ be a unital simple* A$\mathbb{T}$ *algebra of real rank zero and α an approximately inner automorphism of $\mathcal{A}$. If α has the Rohlin property, then $\mathcal{A} \times_\alpha \mathbb{Z}$ has real rank zero.*

Proof. By the above proposition $\mathcal{A} \times_\alpha \mathbb{Z}$ is approximately divisible. Note that $\mathcal{A}$ and $\mathcal{A} \times_\alpha \mathbb{Z}$ are nuclear. By [1] we only have to show that the projections in $\mathcal{A} \times_\alpha \mathbb{Z}$ separate the space $T = T(\mathcal{A} \times_\alpha \mathbb{Z})$ of tracial states. Since α has the Rohlin property and so α^m is uniformly outer for any $m > 0$, it follows that all the tracial states of $\mathcal{A} \times_\alpha \mathbb{Z}$ are invariant under $\widehat{\alpha}$ (4.3 of [17]). Hence $\mathcal{A}$ already separates T. Since $\mathcal{A}$ has real rank zero, the projections in $\mathcal{A}$ separates T, which completes the proof. ∎

4. PROOF OF (iii) ⇒ (i)

Let $\{\mathcal{A}_n\}$ be an increasing sequence of local algebras of $\mathcal{A}$ such that the union $\bigcup_n \mathcal{A}_n$ is dense in $\mathcal{A}$. We let $\mathcal{A}_n = A_n \otimes C(\mathbb{T})$ and $A_n = \bigoplus_{i=1}^{K_n} A_{n,i}$ where $A_{n,i}$'s are full matrix algebras. Let z_n be the canonical unitary of $1 \otimes C(\mathbb{T}) \subset \mathcal{A}_n$. We assume that the inclusions $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ are of standard form. In particular z_n is of the form $a_1 + a_2 z_{n+1} + a_3 z^*_{n+1}$, where $a_i \in A_{n+1}$ and $\|a_i\| \leqslant 1$. For each $k \in \mathbb{N},\ n \in \mathbb{N}$, and $\varepsilon > 0$ we will choose projections

$$e_{1,0}, \ldots, e_{1,k-1}; \quad e_{2,0}, \ldots, e_{2,k}$$

in $\mathcal{A}$ such that

(i) $\sum_i \sum_j e_{ij} = 1$,
(ii) $\|\alpha(e_{ij}) - e_{i,j+1}\| < \varepsilon$,
(iii) $e_{ij} \in A_n'$,
(iv) $\big\| [e_{ij}, z_n] \big\| < \varepsilon$,

where $e_{1k} = e_{10}$ and $e_{2,k+1} = e_{20}$. The main problem different from the AF case [16] is to handle the condition (iv).

We recall that τ is the unique tracial state of $\mathcal{A}$.

LEMMA 4.1. *Let u be a unitary in $\mathcal{A}$ and $\varepsilon > 0$. Then there exist projections e_0, e_1 in $\mathcal{A}$ and unitaries $u_i \in e_i\mathcal{A}e_i$ such that $e_0 + e_1 = 1$ and $\tau(e_0) > 1 - \varepsilon$, $\|u - u_0 - u_1\| < \varepsilon$, $\mathrm{Spec}\,(u_0)$ is finite.*

For the proof see e.g. [19], [9], [3].

Now we shall try to construct a set of projections satisfying the conditions (i)–(iv). Note that we may replace the conditions (iii) and (iv) by the weaker conditions:

$$e_{i0} \in A'_n, \quad \big\|\, [e_{i0}, z_n] \,\big\| < \varepsilon.$$

By the assumption the automorphism $\widetilde{\alpha}$ of $\mathcal{R} = \pi_\tau(\mathcal{A})''$, which is an AFD type II_1 factor, defined by $\widetilde{\alpha} \circ \pi_\tau = \pi_\tau \circ \alpha$, is aperiodic; hence by Connes ([7]), $\widetilde{\alpha}$ satisfies the Rohlin property in the von Neumann algebra sense. Hence for any $l \in \mathbb{N}$ there is a central sequence $\{E_{\nu i} \mid \ i = 0, \dots, l-1\}$ of families of projections in $\mathcal{R}$ such that

$$\sum_{i=0}^{l-1} E_{\nu i} = 1,$$
$$\|\widetilde{\alpha}(E_{\nu i}) - E_{\nu\, i+1}\|_2 \to 0 \quad (\text{as } \nu \to \infty),$$

where $E_{\nu l} = E_{\nu 0}$ and $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in \mathcal{R}$ with τ the unique tracial state of $\mathcal{R}$.

Let $n \in \mathbb{N}$. By Lemma 4.1 we choose a projection $p_n \in \mathcal{A} \cap A'_n$ and partial unitaries $u_n, v_n \in \mathcal{A} \cap A'_n$ such that $u_n u_n^* = p_n,\ v_n v_n^* = 1 - p_n,\ \tau(p_n) > 1 - 1/n,\ \|z_n - u_n - v_n\| < 1/n$, and $\mathrm{Spec}\,(u_n)$ is finite. Let B_n be the (abelian finite-dimensional) C^*-subalgebra generated by u_n. Then we can find a projection $e_\nu \in \mathcal{A}$ such that $e_\nu \in \mathcal{A} \cap A'_{n(\nu)} \cap B'_{n(\nu)},\ e_\nu \leqslant p_{n(\nu)}$, and $\|E_\nu - \pi_\tau(e_\nu)\|_2 \to 0$ for some $n(\nu)$ with $n(\nu) \to \infty$ as $\nu \to \infty$. Then, since $\big\|\, [e_\nu, z_{n(\nu)}] \,\big\| < 2/n(\nu)$, we have that $\big\|\, [e_\nu, z_n] \,\big\| \to 0$ as $\nu \to \infty$ for any n. Thus we can conclude that $\{e_\nu\}$ is a central sequence in A, and that $\{\alpha^k(e_\nu)\}$ is also a central sequence for any k. Then we find projections $e_\nu^{(k)}, k = 0, 1, \dots, l-1$ in $\mathcal{A}$ such that $e_\nu^{(k)} \in A'_{m(\nu)} \cap B'_{m(\nu)}$ for some $m(\nu)$ with $m(\nu) \to \infty$, and $\|e_\nu^{(k)} - \alpha^k(e_\nu)\| \to 0$ as $\nu \to \infty$. Since

$$\Big\| e_\nu^{(0)} \Big(\sum_{k=1}^{l-1} e_\nu^{(k)} \Big) e_\nu^{(0)} - e_\nu \Big(\sum_{k=1}^{l-1} \alpha^k(e_\nu) \Big) e_\nu \Big\| \to 0 \quad \text{and} \quad \tau\Big(e_\nu \Big(\sum_{k=1}^{l-1} \alpha^k(e_\nu) \Big) e_\nu \Big) \to 0$$

we obtain that $\varepsilon_\nu = \tau(x_\nu) \to 0$ where

$$x_\nu = e_\nu^{(0)} \Big(\sum_{k=1}^{l-1} e_\nu^{(k)} \Big) e_\nu^{(0)} \in e_\nu^{(0)} \mathcal{A} \cap A'_{m(\nu)} \cap B'_{m(\nu)} e_\nu^{(0)} = D_{m(\nu)}.$$

Define a continuous function g on $\mathbb{R}$ by

$$g(t) = \begin{cases} 1 & t \geqslant 1/4, \\ 4t & 0 < t < 1/4, \\ 0 & t \leqslant 0; \end{cases}$$

and let $a_{\nu j} = g(j/4 - x_\nu/\sqrt{\varepsilon_\nu})$ for $j = 2, 3, 4$. Then since $a_{\nu 4}a_{\nu 3} = a_{\nu 3}$, $a_{\nu 3}a_{\nu 2} = a_{\nu 2}$, and $D_{m(\nu)}$ has real rank zero, we have a projection f_ν in $D_{m(\nu)}$ such that $a_{\nu 4}f_\nu = f_\nu$ and $\|f_\nu a_{\nu 2} - a_{\nu 2}\| \to 0$. (We approximate $a_{\nu 3}$ by a self-adjoint element b_ν with finite spectrum in the closure of $a_{\nu 3}D_{m(\nu)}a_{\nu 3}$ and then take the spectral projection of b_ν corresponding to $[1/2, 1]$.) Since $x_\nu^{1/2}(1 - a_{\nu 2})x_\nu^{1/2} \geqslant \sqrt{\varepsilon_\nu}(e_\nu^{(0)} - a_{\nu 2})/4$ and $\tau(x_\nu^{1/2}(1 - a_{\nu 2})x_\nu^{1/2}) \leqslant \varepsilon_\nu$, it follows that $\tau(a_{\nu 2}) \to 1/l$, which implies that $\tau(f_\nu) \to 1/l$ as $\nu \to \infty$. Since $\|f_\nu x_\nu f_\nu\| \leqslant \|a_{\nu 4}x_\nu a_{\nu 4}\| \leqslant \sqrt{\varepsilon_\nu}$, it follows that $\|f_\nu e_\nu^{(k)} f_\nu\| < \sqrt{\varepsilon_\nu}$ for $k = 1, \ldots, l-1$, which implies that $\|f_\nu \alpha^k(f_\nu)\| \to 0$ for $k = 1, \ldots, l-1$. Replacing f_ν by $f_\nu p_{m(\nu)}$, we still have that

$$\tau(f_\nu) \to \frac{1}{l}, \quad \|f_\nu \alpha^k(f_\nu)\| \to 0, \quad k = 1, \ldots, l-1$$

as $\nu \to \infty$. Since $f_\nu \leqslant p_{m(\nu)}$, $f_\nu \in A'_{m(\nu)} \cap B'_{m(\nu)}$ and $m(\nu) \to \infty$, we can conclude that $\{f_\nu\}$ is a central sequence. Then for any sufficiently large ν we find an automorphism α_ν of $\mathcal{A}$ by perturbing α by an inner automorphism such that $\alpha_\nu^k(f_\nu)$, $k = 0, 1, \ldots, l-1$, are mutually orthogonal and $\|\alpha_\nu - \alpha\| \to 0$.

As in the proof of Proposition 3.8 we obtain (by passing to a subsequence of $\{f_\nu\}$) a central sequence $\{W_\nu\}$ of unitaries in $\mathcal{A}$ such that

$$\begin{aligned} &\operatorname{Ad} W_\nu^k(f_\nu) = \alpha_\nu^k(f_\nu), \quad k = 0, \ldots, l-1, \\ &\alpha_\nu(W_\nu) - W_\nu \to 0. \end{aligned}$$

Thus we find a central sequence $\{W_\nu^k f_\nu W_\nu^{-j}\}_{k,j=0}^{l-1}$ of matrix units satisfying

$$\alpha_\nu(W_\nu^k f_\nu W_\nu^{-j}) \approx W_\nu^{k+1} f_\nu W_\nu^{-j-1}$$

for $k, j < l - 1$. Then as in [16] for any $k \in \mathbb{N}$ we construct a central sequence $\{e_{mi} \mid i = 0, \ldots, k-1\}$ of Rohlin towers and a sequence $\{\alpha_m\}$ of automorphisms:

$$\begin{aligned} &\alpha_m(e_{mi}) = e_{m,i+1}, \quad \text{with } e_{mk} = e_{m0}, \\ &e_m = \sum_{i=0}^{k-1} e_{mi} \quad \text{is a projection}, \\ &\tau(e_m) \to 1, \\ &\|\alpha_m - \alpha\| \to 0, \end{aligned}$$

(by using central sequences of matrix units obtained above for various l). This is the *approximate* Rohlin property; we have to show how to get the genuine Rohlin property.

We use the method in [16] to derive the genuine Rohlin property. Note that e_{m0} is left invariant under α_m^k and that $\alpha_m^k|e_{m0}\mathcal{A}e_{m0}$ has also the approximate Rohlin property. In particular one finds, for any $N \in \mathbb{N}$, projections f_{mj}; $j = 1, \dots, N$ in $e_{m0}\mathcal{A}e_{m0}$ and automorphisms β_m of $e_{m0}\mathcal{A}e_{m0}$ such that

$$\begin{aligned} &f_m = \sum_j f_{mj} \quad \text{is a projection,} \\ &\tau(f_{m1}) \to \frac{1}{k}N, \\ &\beta_m(f_{mj}) = f_{m,j+1}, \quad j < N, \\ &\|\beta_m - \alpha_m^k\| \to 0. \end{aligned}$$

Since the order of the simple dimension group $K_0(\mathcal{A})$ is determined by the trace τ (4.2 of [8]), we have that $[1 - e_m]/[f_{m1}] \to 0$. By Lemma 3.6 applied to $\{f_{m1}\}$ and $\{1 - e_m\}$ we obtain a central sequence $\{v_m\}$ of partial isometries such that

$$v_m^* v_m = 1 - e_m; \quad v_m v_m^* \leqslant f_{m1}.$$

Define a partial isometry V_m by

$$V_m = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \beta^j(v_m).$$

Then the algebra generated by

$$\alpha_m^j(V_m), \quad j = 0, \dots, k-1$$

is an almost α_m-invariant $(k+1) \times (k+1)$ matrix algebra and as $m \to \infty$ forms a central sequence.

We apply the above procedure for a big multiple of k, say Mk, instead of k. Then we obtain a central sequence D_n of almost α-invariant subalgebras which are isomorphic to the $(Mk+1)$ by $(Mk+1)$ matrices. Then from the spectral property for α restricted to each D_n, we can obtain projections $g_0, \dots, g_{k-1}; h_0, \dots, h_k$ with sum the identity of D_n such that $\{g_i\}$ and $\{h_i\}$ cyclically permute under α up to the order of $1/M$ ([16]). Subtracting the identity of D_n from the original Rohlin towers, we still obtain Mk projections which almost cyclically permute under α, from which we obtain the Rohlin tower $e_0, \dots, e_{k-1}$ consisting of k projections. Thus we obtain the Rohlin towers $e_0 + g_0, \dots, e_{k-1} + g_{k-1}$; $h_0, \dots, h_k$ with sum the identity of $\mathcal{A}$. This completes the proof; see [16], [17] for details. ∎

Note added in proof. I can now extend Theorem 2.1 for some (KK-trivial) approximately inner automorphisms α by adding another condition that $A \times_\alpha \mathbb{Z}$ is a unital simple A$\mathbb{T}$ algebra of real rank zero. See 6.4 of my paper *Unbounded derivations in AT algebras*, which will appear in J. Funct. Anal.

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