J. OPERATOR THEORY
40(1998), 323–337

WEAKLY COMPACTLY GENERATED BANACH ALGEBRAS ASSOCIATED TO LOCALLY COMPACT GROUPS

E. KANIUTH, A.T. LAU and G. SCHLICHTING

Communicated by Norberto Salinas

Abstract. Let G be a locally compact group, and let X be any one of the Banach algebras $C^*(G)$, $C_0(G)$, $B(G)$ or $A(G)$. We characterize the property that X, as a Banach space, is weakly compactly generated in terms of conditions on G and its dual space.

Keywords: *Weakly compactly generated Banach space, locally compact group, C^*-algebra, Fourier- and Fourier-Stieltjes algebra, dual space.*

MSC (2000): Primary 46B20, 43A30; Secondary 46L05, 22D25.

INTRODUCTION

A Banach space X is called *weakly compactly generated* (WCG) if there exists a weakly compact subset C of X such that the linear span of C is dense in X. Much of the work on the geometry of Banach spaces was related to WCG spaces, and there are various applications to operator theory, renorming theorems, the structure of weakly compact sets in general Banach spaces and the theory of vector valued measures (see [8], [9] and [10]).

The most obvious examples of WCG Banach spaces are the reflexive and the separable Banach spaces. For any set $\Gamma, c_0(\Gamma)$ is WCG, whereas $l^1(\Gamma)$ is WCG precisely when Γ is countable. If Ω is a compact Hausdorff space, then $C(\Omega)$ is WCG if and only if Ω is homeomorphic to a weakly compact subset of some Banach space.

In this note we investigate weakly compactly generatedness of certain Banach algebras associated to a locally compact group G, such as the group C^*-algebra

$C^*(G)$, the Fourier and Fourier-Stieltjes algebras $A(G)$ and $B(G)$, and $C_0(G)$. The main results are as follows. In Theorem 2.3 we prove that for a C^*-algebra A, the dual Banach space A^* is WCG if and only if A is separable and $\widehat{A}$, the set of equivalence classes of irreducible $*$-representations of A, is countable. This applies to $B(G)$ and other coefficient function spaces determined by unitary representations of G. The group C^*-algebra $C^*(G)$ turns out to be WCG exactly when G is σ-compact (Theorem 3.1), and $A(G)$ and $C_0(G)$ are WCG if and only if G is first countable (Theorem 3.2). We also comment on how σ-compactness and first countability of G are related to topological properties of the dual space $\widehat{G}$ of G.

Several other geometric properties, like the Dunford-Pettis property, the Schur property, the Radon-Nikodym property and flatness, for the Banach spaces $B(G)$ and $A(G)$ have been investigated in [19] (and also [4] and [21]).

1. NOTATIONS AND PRELIMINARIES

Throughout this paper, G denotes a locally compact group with fixed left Haar measure, $L^1(G)$ the convolution algebra of integrable functions on G and $C^*(G)$ the group C^*-algebra of G. Let $P(G)$ be the set of all continuous positive definite functions on G, and set $P^1(G) = \{\varphi \in P(G) : \varphi(e) = 1\}$. The linear span $B(G)$ of $P(G)$ is an algebra, the Fourier-Stieltjes algebra of G, and can be identified with the Banach space dual of $C^*(G)$. Under this identification, the functions in $P(G)$ correspond to the positive linear functionals on $C^*(G)$.

Let λ denote the left regular representation of G, that is, $\lambda(x)f(y) = f(x^{-1}y)$ for $f \in L^2(G)$ and $x, y \in G$. Let $\mathrm{VN}(G)$ be the von Neumann algebra in $B(L^2(G))$ generated by the operators $\lambda(x), x \in G$. The Fourier algebra of $G, A(G)$, consists of all functions $\varphi \in C_0(G)$ of the form $\varphi(x) = \langle \lambda(x)f, g \rangle, f, g \in L^2(G)$. $A(G)$ can be identified with the predual of $\mathrm{VN}(G)$. Concerning the basic properties of $A(G)$ and $B(G)$, we refer the reader to Eymard's fundamental paper ([12]).

More generally, for an arbitrary unitary representation π of G in the Hilbert space H_π, let $A_\pi(G)$ be the closed linear span in $B(G)$ of all coefficient functions of π, that is, functions $x \to \langle \pi(x)\xi, \eta\rangle$, $\xi, \eta \in H_\pi$. These Fourier spaces $A_\pi(G)$ have been introduced and studied by Arsac ([2]). $B_\pi(G)$ will denote the weak* closure of $A_\pi(G)$ in $B(G)$. Then $B_\pi(G)$ is the dual of $\pi(C^*(G))$ and can be realized as the space of all coefficient functions determined by representations that are weakly contained in π ([2]).

2. Duals of C^*-algebras

Let A be a C^*-algebra, A^* its dual and A^{**} the bidual of A, which is a W^*-algebra. By duality, both A and A^{**} operate on A^* from the left and from the right. The norm closed two-sided A-invariant linear subspaces of A^* correspond to the $\sigma(A^{**}, A^*)$-closed ideals of A^{**}, and these ideals are generated by central projections (see Chapter III.2 of [20]).

For a representation π of A let A^*_π denote the norm closed, two-sided A-invariant subspace of A^* generated by the positive linear functionals on A associated to π, that is, functionals of the form $a \to \langle \pi(a)\xi, \xi\rangle, \xi \in H_\pi$. $\widehat{A}$ will denote the set of equivalence classes of irreducible $*$-representations of A.

Lemma 2.1. *If A^* is weakly compactly generated, then A^*_π is weakly compactly generated for every representation π of A.*

Proof. By [20], Theorem III. 2.7 there exists a central projection e_π in A^{**} such that $A^*_\pi = e_\pi A^*$. Thus A^*_π is a complemented subspace of A^*. In particular, there is a continuous linear mapping from A^* onto A^*_π. Since A^* is WCG it follows that A^*_π is WCG. ∎

The following lemma is the main step towards establishing Theorem 2.3 below.

Lemma 2.2. *Suppose that A^* is weakly compactly generated. Then A is separable and $\widehat{A}$ is countable.*

Proof. We first show that if π is an irreducible representation of A, then H_π, the Hilbert space of π, is separable. To that end let $\mathrm{TC}(H_\pi)$denote the space of trace class operators in H_π. Each $T \in \mathrm{TC}(H_\pi)$ defines an element φ_T of A^*_π by $\varphi_T(x) = \mathrm{tr}(\pi(x)T), x \in A$. The mapping

$$\phi : \mathrm{TC}(H_\pi) \to A^*_\pi, \quad T \to \varphi_T$$

is linear and onto, and ϕ is isometric because π is irreducible. Indeed,

$$(A^*_\pi)^* = \pi(G)'' = B(H_\pi) = \mathrm{TC}(H_\pi)^*.$$

Now fix an orthonormal basis $(\xi_i)_{i\in I}$ of H_π. This basis defines a continuous linear mapping

$$T \to (\langle T\xi_i, \xi_i\rangle)_{i\in I}$$

from $\mathrm{TC}(H_\pi)$ onto $l^1(I)$. Since A^*_π is WCG (Lemma 2.1), we conclude that $\mathrm{TC}(H_\pi)$ is WCG and hence so is $l^1(I)$. However, this implies that I is countable.

Next we apply Lemma 2.1 to the direct sum of all $\sigma \in \widehat{A}$. Note that $A_\sigma^* = A_\tau^*$ and $e_\sigma = e_\tau$ if σ and τ are equivalent. Let e_σ be the central projection in A^{**} such that $A_\sigma^* = e_\sigma A^*$. The projections $e_\sigma, \sigma \in \widehat{A}$, are pairwise orthogonal, their sum is $\sigma(A^{**}, A^*)$-convergent and

$$\sum_{\sigma \in \widehat{A}} e_\sigma = e_\pi,$$

where π is the direct sum of all $\sigma \in \widehat{A}$. This gives rise to a decomposition of A_π^* into the subspaces $A_\sigma^*, \sigma \in \widehat{A}$. Indeed, for $\varphi \in A_\pi^*$,

$$\varphi = e_\pi \varphi = \sum_{\sigma \in \widehat{A}} e_\sigma \varphi \quad \text{and} \quad \|\varphi\| = \sum_{\sigma \in \widehat{A}} \|e_\sigma \varphi\|,$$

and conversely, if $\varphi_\sigma \in A_\sigma^*$ are such that $\sum_{\sigma \in \widehat{A}} \|\varphi_\sigma\| < \infty$, then

$$\varphi = \sum_{\sigma \in \widehat{A}} \varphi_\sigma \in A_\pi \quad \text{and} \quad \|\varphi\| = \sum_{\sigma \in \widehat{A}} \|\varphi_\sigma\|.$$

If A is non-unital we adjoin a unit e to A and extend functionals on A in the usual way to functionals on $A \oplus \mathbb{C}e$. Then, since $\|\varphi\| = \varphi(e)$ for every positive functional $\varphi \in A^*$, it follows that

$$\varphi \to \left(e_\sigma \varphi(e)\right)_{\sigma \in \widehat{A}}$$

is a continuous linear mapping from A_π^* onto $l^1(\widehat{A})$. Thus $l^1(\widehat{A})$ is WCG, and hence $\widehat{A}$ must be countable.

We have seen that A is a C^*-algebra with countable dual and that H_π is separable for every irreducible representation π of A. Lemma 1.5 of [22] now shows A is separable. ∎

In passing we remind the reader that there are several equivalent formulations of the *Radon-Nikodym property* for Banach spaces, which is of particular importance in the theory of vector valued integration. However, we refrain from presenting any of these formulations because in the following theorem the Radon-Nikodym property is only required for dual Banach spaces. For further information we refer the reader to [5] and [10].

THEOREM 2.3. *For a C^*-algebra A the following conditions are equivalent:*

(i) *A^* is weakly compactly generated;*

(ii) *A is separable and $\widehat{A}$ is countable;*

(iii) *A is separable and A^* has the Radon-Nikodym property;*

(iv) *A^* is separable.*

Proof. (i) $\Rightarrow$ (ii) is the preceding lemma. Assume that (ii) holds. Then A is a scattered C^*-algebra (see [21], Theorem 4.5), that is, every positive functional on A is the sum of a sequence of pure functionals. By Theorem 3 of [6], A^* has the Radon-Nikodym property if (and only if) A is scattered. Thus (ii) $\Rightarrow$ (iii).

Next, by a theorem of Stegall (see [5], Corollary 4.1.7 and [9], Chapter VI, Section 6, Corollary 1) a dual Banach space X^* has the Radon-Nikodym property (if and) only if every separable subspace of X has a separable dual. Hence (iii) $\Rightarrow$ (iv).

Finally, (iv) implies (i) since every separable Banach space is WCG. ∎

Before applying Theorem 2.3 to dual Banach spaces of C^*-algebras arising from unitary representations of locally compact groups, we have to introduce some more notation and recall a few basics from representation theory. We shall always use the same letter, say π, for a unitary representation of the locally compact group G and the corresponding $*$-representation of $C^*(G)$. Recall that $B_\pi(G) \subseteq B(G)$ is identified with the Banach space dual of $\pi(C^*(G))$. If S and T are sets of unitary representations of G, then S is weakly contained in T ($S \prec T$) if

$$\bigcap_{\sigma\in S} \ker\sigma \supseteq \bigcap_{\tau\in T} \ker\tau,$$

where $\ker\pi$ denotes the C^*-kernel of π. Equivalently, every coefficient function of any $\sigma \in S$ can be approximated uniformly on compact subsets of G by sums of coefficient functions associated to representations from T. S and T are called weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$. For all this see [11] and [13].

The dual space $\widehat{G}$ of G consists of all equivalence classes of irreducible unitary representations of G and is topologized so that $\tau \in \widehat{G}$ belongs to the closure of a subset S of $\widehat{G}$ if and only if $\tau \prec S$. For an arbitrary representation π of G, the support of π, $\operatorname{supp}\pi$, is the closed subset of all $\tau \in \widehat{G}$ such that $\tau \prec \pi$. In particular, the support of the regular representation is the so-called reduced dual $\widehat{G}_{\mathrm{r}}$. Of course, $\widehat{G}$ is identified with $\widehat{C^*(G)}$ and $\widehat{G}_{\mathrm{r}}$ with the dual space of $\lambda(C^*(G))$. Finally, if N is a closed normal subgroup of G, then $\widehat{G/N}$ is considered as a closed subset of $\widehat{G}$ in the obvious way.

Corollary 2.4. *For an arbitrary unitary representation π of G the following conditions are equivalent:*

(i) $B_\pi(G)$ *is weakly compactly generated;*
(ii) $\operatorname{supp}\pi$ *is countable and* $\pi(C^*(G))$ *is separable;*
(iii) $B_\pi(G)$ *is separable.*

We now specialize to π the left regular representation λ and the universal representation ω. Note that $B_\omega(G) = B(G)$ and that separability of either $B(G)$ or $B_\lambda(G)$ implies that $A(G)$ is separable, which in turn implies that G is second countable ([15], Corollary 6.9). From Corollary 2.4 we then conclude the next two corollaries.

Corollary 2.5. *For any locally compact group G the following are equivalent:*

(i) $B(G)$ *is weakly compactly generated;*
(ii) G *is second countable and* $\widehat{G}$ *is countable;*
(iii) $B(G)$ *is separable.*

Corollary 2.6. *The following conditions are equivalent:*

(i) $B_\lambda(G)$ *is weakly compactly generated;*
(ii) G *is second countable and the reduced dual* $\widehat{G}_{\mathrm{r}}$ *is countable;*
(iii) $B_\lambda(G)$ *is separable.*

Clearly, if G is a second countable compact group, then $\widehat{G}$ is countable (equivalently, $B(G) = B_\lambda(G)$ is WCG). As to the converse, however, for general second countable locally compact groups G countability of $\widehat{G}$ does not force G to be compact. The first such counterexample was Fell's group, presented in [3]. For a fixed prime p, Fell's group is the natural semi-direct product of the compact group of p-adic units with the p-adic numbers. Nevertheless, the next corollary, which is an application of results from [3], shows that under a certain mild structural condition on G, countability of $\widehat{G}$ indeed implies that G is compact.

Corollary 2.7. *Suppose that G contains an almost connected open normal subgroup. Then the following are equivalent:*

(i) $B(G)$ *is weakly compactly generated;*
(ii) $B_\lambda(G)$ *is weakly compactly generated;*
(iii) G *is compact and second countable.*

Proof. In view of Corollaries 2.5 and 2.6 it only remains to verify that a second countable locally compact group G must be compact provided that $\widehat{G}_{\mathrm{r}}$ is countable and G has an almost connected open normal subgroup N.

By Theorem 2.7 of [3] there exists a compact open subgroup of G. Therefore the connected component of the identity of G has to be compact, and hence N is compact. In particular, $(\widehat{G/N})_{\mathrm{r}} \subset \widehat{G}_{\mathrm{r}}$ since N is amenable. Thus G/N is a countable discrete group with countable reduced dual, and as such G/N has to be finite ([3], Proposition 1.5). This proves that G is compact. ∎

Besides the regular representation, one of the most interesting representations of a locally compact group G is the conjugation representation γ. Denoting by Δ the modular function of G, γ is the representation on $L^2(G)$ defined by

$$\gamma(x)f(y) = \Delta(x)^{\frac{1}{2}} f(x^{-1}yx),$$

$f \in L^2(G), x, y \in G$. We finish this section by applying Corollaries 2.4 and 2.5 and a result from [18] to the C^*-algebra generated by γ. Notice, however, that in contrast to $B_\lambda(G)$, in general $B_\gamma(G)$ fails to be a subalgebra of $B(G)$.

Corollary 2.8. *For a connected group G the following two conditions are equivalent:*

(i) $B_\gamma(G) = \gamma(C^*(G))^*$ *is weakly compactly generated;*

(ii) *G is a direct product of a vector group and a compact group K such that the quotient group of K by its centre is second countable.*

Proof. By Corollary 2.4, (i) implies that γ has a countable support. Since G is connected, Theorem 3 of [18] shows that G has the structure stated in (ii).

Conversely, suppose G is as in (ii) and let $Z(G)$ and $Z(K)$ denote the centre of G and K, respectively. Then $G/Z(G) = K/Z(K)$ is second countable. Hence $C^*(G/Z(G))$ is separable and, since γ is trivial on $Z(G)$, it follows that $\gamma(C^*(G))$ is separable. Moreover $\operatorname{supp}\gamma \subseteq (K/Z(K))\widehat{\ }$, which is countable. Corollary 2.4 now shows that $B_\gamma(G)$ is WCG. ∎

3. $C^*(G)$, $A(G)$ and $C_0(G)$

The purpose of this section is to characterize, for a locally compact group G, weakly compactly generatedness of $C^*(G)$, $A(G)$ and $C_0(G)$ in terms of topological properties of G. We are going to establish the following two theorems which show that rarely any of these Banach spaces fails to be weakly compactly generated.

Theorem 3.1. *$C^*(G)$ is weakly compactly generated if and only if G is σ-compact.*

Theorem 3.2. *For a locally compact group G the following conditions are equivalent:*

(i) *$A(G)$ is weakly compactly generated;*

(ii) *$C_0(G)$ is weakly compactly generated;*

(iii) *G is first countable.*

In the sequel we shall several times use the following simple fact. Let E and F be Banach spaces, and suppose that E is WCG and that there exists a continuous linear mapping from E into F with dense range. Then F is WCG.

There are some comments in order before we proceed to prove Theorems 3.1 and 3.2. Firstly, recall that a compact Hausdorff space Ω is said to be Eberlein compact whenever Ω is homeomorphic to a weakly compact subset of some Banach space. By a theorem of Amir and Lindenstrauss (see [1], Theorem 2 and [9], Chapter V, Section 2, Theorem 4), $C(\Omega)$ is WCG if and only if Ω is Eberlein compact. Although there is an intrinsic, purely topological characterization of Eberlein compacts ([9], Chapter V, Section 3, Theorem 1), this does not seem to yield a result like the equivalence of (ii) and (iii) of Theorem 3.2.

Secondly, suppose that G is a locally compact abelian group. Then the equivalence of (ii) and (iii) in Theorem 3.2 follows from Theorem 3.1, and vice versa. Indeed, $C_0(G)$ is then isomorphic to $C^*(\widehat{G})$ and $\widehat{G}$ is σ-compact if and only if G is first countable, as can easily be shown by using Pontrjagin's duality theory for locally compact abelian groups. Moreover, the equivalence of (i) and (iii) follows from the three facts that (i) $A(G)$ is isomorphic to $L^1(\widehat{G})$, (ii) $L^1(X,\mu)$ is WCG exactly when the measure μ is σ-finite ([9], p. 143), and (iii) Haar measure on a locally compact group H is σ-finite if and only if H is σ-compact.

For general locally compact groups, however, due to the lack of duality theory, Theorems 3.1 and 3.2 appear to be not related at all. Incidentically, the proofs require the application of quite different deep theorems from the theory of WCG Banach spaces.

In what follows $\mathcal{H}$ will denote the set of all open subgroups of G, and for $H \in \mathcal{H}$ let χ_H denote the characteristic function of H.

Lemma 3.3. *The set* $A = \{c\chi_H : H \in \mathcal{H}, 0 \leqslant c \leqslant 1\}$ *is* w*-*closed in* $B(G)$.

Proof. Let $(c_\alpha \chi_{H_\alpha})_\alpha$ be a net in A which is w*-convergent to some $\psi \in B(G)$. We can assume that $\psi \neq 0$ and, after passing to a subnet if necessary, that $c_\alpha \to c$ for some $c \in [0,1]$. Then $c \neq 0$ since $\psi \neq 0$, and

$$\chi_{H_\alpha} \to \varphi = \frac{1}{c}\psi$$

in the w*-topology of $B(G)$. Clearly, φ is positive definite.

Let $\mathcal{S}(G)$ denote the set of all closed subgroups of G, endowed with Fell's topology (see Section 2 of [13]) which makes $\mathcal{S}(G)$ a compact (Hausdorff) space. By passing to a further subnet if necessary, we can assume that $H_\alpha \to H$ in $\mathcal{S}(G)$. By a theorem of Glimm ([14]) there exists a continuous choice of Haar measures on $\mathcal{S}(G)$, that is, a mapping $K \to m_K$ assigning to each $K \in \mathcal{S}(G)$ a left Haar measure of K such that

$$K \to \int_K f(x)\,\mathrm{d}m_K(x)$$

is a continuous function on $\mathcal{S}(G)$ for every $f \in C_{\mathrm{c}}(G)$. Obviously, for $K \in \mathcal{H}, m_K$ is a positive multiple of the restriction of m_G to K, i.e. $c(K)m_K = m_G|K$ for some $c(K) > 0$. It follows that for every $f \in C_{\mathrm{c}}(G)$,

$$\int_{H_\alpha} f(x)\,\mathrm{d}m_{H_\alpha}(x) \to \int_H f(x)\,\mathrm{d}m_H(x) \tag{3.1}$$

and

$$c(H_\alpha)\int_{H_\alpha} f(x)\,\mathrm{d}m_{H_\alpha}(x) = \langle f, \chi_{H_\alpha}\rangle \to \langle f, \varphi\rangle = \int_G f(x)\varphi(x)\,\mathrm{d}m_G(x). \tag{3.2}$$

Now, take any $f \in C_{\mathrm{c}}(G)$ such that $H \cap \operatorname{supp} f = \emptyset$. Then $H_\alpha \cap \operatorname{supp} f = \emptyset$ eventually, and hence (3.2) implies that $\langle f, \varphi\rangle = 0$. Thus φ vanishes outside of H. As φ is a non-zero continuous function, H must be open in G.

We are going to show that $\varphi = c\chi_H$ for some $c > 0$. To that end, notice first that $c(H_\alpha) \to d$ for some $d \geqslant 0$. Indeed, this follows by choosing $f \in C_{\mathrm{c}}^+(H), f \neq 0$, and combining (3.1) and (3.2). Finally, applying (3.1) and (3.2) again and using that $\varphi|G \setminus H = 0$ and $c(H_\alpha) \to d$, we obtain

$$\begin{aligned} d\int_H f(x)\,\mathrm{d}m_H(x) &= \lim_\alpha c(H_\alpha)\int_{H_\alpha} f(x)\,\mathrm{d}m_{H_\alpha}(x) = \lim_\alpha \langle f, \chi_{H_\alpha}\rangle \\ &= \langle f, \varphi\rangle = \int_H f(x)\varphi(x)\,\mathrm{d}m_G(x) = c(H)\int_H f(x)\varphi(x)\,\mathrm{d}m_H(x) \end{aligned}$$

for all $f \in C_{\mathrm{c}}(G)$. This implies that $\varphi(x) = c(H)^{-1}d$ for all $x \in H$, as required. ∎

Lemma 3.4. *Retain the previous notations and let $\varphi = c\chi_H \in A, c \neq 0$. If φ has a countable neighbourhood basis in A in the* w*-*topology of $B(G)$, then H is σ-compact.*

Proof. For a finite subset F of $L^1(G)$ and $\varepsilon > 0$, let

$$U(F,\varepsilon) = \{\psi \in B(G) : |\langle f,\psi\rangle - \langle f,\varphi\rangle| < \varepsilon \text{ for all } f \in F\}.$$

Since $L^1(G)$ is dense in $C^*(G)$, these sets $U(F,\varepsilon)$ form a neighbourhood basis of φ in A. Thus, by hypothesis, there exist $f_n \in L^1(G)$, $n \in \mathbb{N}$, such that if $\psi \in A$ satisfies $\langle f_n, \psi\rangle = \langle f_n, \varphi\rangle$ for all n, then $\psi = \varphi$.

There exists an open, σ-compact subgroup K of G such that, for every n, $f_n = 0$ almost everywhere outside of K. Since, for each n, we have

$$\langle f_n, c\chi_H\rangle = c\int\limits_H f_n(x)\,\mathrm{d}m_G(x) = c\int\limits_{K\cap H} f_n(x)\,\mathrm{d}m_G(x) = \langle f_n, c\chi_{K\cap H}\rangle,$$

it follows that $c\chi_H = c\chi_{K\cap H}$. Because $c \neq 0$, we then have $H = K\cap H$, whence H is σ-compact. ∎

Proof of Theorem 3.1. Of course, if G is σ-compact then the Haar measure of G is σ-finite and hence $L^1(G)$ is WCG. Since $L^1(G)$ is dense in $C^*(G)$, it follows that $C^*(G)$ is WCG.

Conversely, suppose that $C^*(G)$ is WCG. By a theorem of Amir and Lindenstrauss ([1], Proposition 2; see also [9], Chapter V, Section 2, Theorem 2) there exist a Banach space E and an injective linear mapping $\phi : B(G) \to E$ which is continuous for the w*-topology on $B(G)$ and the weak topology on E (in fact, E can be chosen to be $c_0(\Gamma)$ for a certain set Γ).

Let A be as in the preceding lemmas. By Lemma 3.3, A is a w*-closed subset of $B(G)_1$, the unit ball of $B(G)$. Let $B = \phi(A)$ and $K = \overline{\mathrm{co}}(B)$, the norm closure (equivalently, the weak closure) of the convex hull $\mathrm{co}(B)$ of B. Recall that if D is a bounded subset of a Banach space X, then an element $x_0 \in D$ is called exposed point of D if there exists $x^* \in X^*$ such that $x^*(x_0) > x^*(x)$ for all $x \in D, x \neq x_0$. Now, by another result of Amir and Lindenstrauss ([1], Theorem 4; [9], Chapter V, Section 6, Theorem 2), K is the closed convex hull of its set $\exp K$ of exposed points, i.e. $K = \overline{\mathrm{co}}(\exp K)$. By definition, each such exposed point has a countable neighbourhood basis for the weak topology of K. Moreover, by the converse to the Krein-Milman theorem, $\exp K \subseteq B$. Thus every point in K is the norm limit of convex combinations of exposed points.

It follows that for every $n \in \mathbb{N}$, we find $b_{n,j} \in \exp K$ and $\lambda_{n,j} \geqslant 0$, $1 \leqslant j \leqslant m_n$, such that $\sum_{j=1}^{m_n} \lambda_{n,j} = 1$ and

$$\Big\|\phi(\chi_G) - \sum_{j=1}^{m_n} \lambda_{n,j} b_{n,j}\Big\| \leqslant \frac{1}{n},$$

and every $b_{n,j}$ has a countable neighbourhood basis in K.

Now, let $a_{n,j} = \phi^{-1}(b_{n,j}) \in A$, and set $a_n = \sum_{j=1}^{m_n} \lambda_{n,j} a_{n,j}$. Since ϕ is linear and a homeomorphism between $\phi^{-1}(K)$, the w*-closed convex hull of A, and K, it follows that $a_n \to \chi_G$ in the weak*-topology on $B(G)$.

Each $a_{n,j}$ has a countable neighbourhood basis in $\phi^{-1}(K)$ and $a_{n,j} \in A$. It follows from Lemma 3.4 that every a_n is supported on some σ-compact open subgroup of G. Hence there exists a σ-compact open subgroup H of G such that $a_n = 0$ on $G \setminus H$ for all n. Finally, since $a_n|H \to \chi_H$ in $B(H)$ and hence

$$a_n = a_n \chi_H \to \chi_H \text{ in } B(G),$$

we get that $\chi_H = \chi_G$. This proves that $G = H$, which is σ-compact. ∎

Proof of Theorem 3.2. (i) implies (ii) since $A(G)$ is norm dense in $C_0(G)$.

Suppose that (ii) holds, and let G_∞ denote the one point compactification of G. Then $C(G_\infty)$ is WCG since $C(G_\infty) = C_0(G) \oplus \mathbb{C}$. This implies, by the theorem of Amir and Lindenstrauss alluded to above, that G_∞ is Eberlein compact. Thus there exist a Banach space E, a weakly compact subset K of E and a homeomorphism $\phi : G_\infty \to K$.

Let C denote the closed convex hull of K in E. Then C is the closed convex hull of its set of exposed points, and all these exposed points are contained in K. Hence there exists $x \in G$ such that $\phi(x)$ is exposed. Now, taking $g \in E^*$ that peaks at $\phi(x)$, $f = g \circ \phi$ is a continuous function on G_∞ such that $f(y) \neq f(x)$ for all $y \in G_\infty, y \neq x$. This proves that x has a countable neighbourhood basis, and hence G is first countable.

Finally, suppose that G is first countable. Then by Theorem 2.6 of [16], the left regular representation λ is cyclic. Since λ is unitarily equivalent to the right regular representation ρ, there is a cyclic vector for ρ. Now, every cyclic vector for a von Neumann algebra is a separating vector for its commutant. Since $\mathrm{VN}(G) = \lambda(G)''$ is the commutant of $\rho(G)''$, there is a separating vector for $\mathrm{VN}(G)$ in $L^2(G)$. It follows that $\mathrm{VN}(G)$ is σ-finite (see [20], Proposition 3.19), and Chu's theorem ([7]) shows that $A(G)$, the predual of $\mathrm{VN}(G)$, is WCG. ∎

4. COMPLEMENTS

As mentioned earlier, a locally compact abelian group G is σ-compact if and only if $\widehat{G}$ is first countable, and by duality G is first countable if and only if $\widehat{G}$ is σ-compact. We shall now prove that essentially the same results hold true for general locally compact groups, thus showing that the topological conditions on G in Theorems 3.1 and 3.2 can be replaced by topological conditions on the dual spaces.

Lemma 4.1. *G is σ-compact if and only if $\widehat{G}$ is first countable.*

Proof. Suppose first that G is σ-compact. Let π be an irreducible representation of G, choose a unit vector ξ in H_π and consider the coefficient function $\varphi(x) = \langle \pi(x)\xi, \xi\rangle$. By a theorem of Kakutani and Kodaira (see [17], Satz 6) there exists a compact normal subgroup K of G such that G/K is second countable and φ is constant on cosets of K. Thus π is the pullback of an irreducible representation of G/K. Now $\widehat{G/K}$ is second countable ([11], Proposition 3.3.4), and also open in $\widehat{G}$ since K is compact. It follows that π has a countable neighbourhood basis in $\widehat{G}$.

Conversely, let $(U_n)_{n\in\mathbb{N}}$ be a countable neighbourhood basis of 1_G, the trivial representation of G, in $\widehat{G}$. Let

$$\phi : \operatorname{ex} P^1(G) \to \widehat{G},\ \varphi \to \pi_\varphi$$

denote the mapping given by the GNS-construction. Since the w*-topology on $\operatorname{ex} P^1(G)$ agrees with the topology of uniform convergence on compact subsets of G ([11], Theorem 13.5.2) and since ϕ is continuous ([11], Theorem 3.4.11), for each $n \in \mathbb{N}$ there exists a compact subset K_n of G with the following property: If $\varphi \in \operatorname{ex} P^1(G)$ satisfies $|\varphi(x) - 1| < 1/n$ for all $x \in K_n$, then $\pi_\varphi \in U_n$. Let H be a σ-compact open subgroup of G containing K_n for all n. Now, if $\varphi \in \operatorname{ex} P^1(G)$ is such that $\varphi|H = 1$, then

$$\pi_\varphi \in \bigcap_{n=1}^{\infty} U_n = \{1_G\}.$$

We claim that $H = G$. To verify this let C denote the set of all $\psi \in P^1(G)$ such that $\psi|H = 1$. C is convex and w*-compact, and hence the w*-closed convex hull of $\operatorname{ex} C$. Now it is easy to check that every $\psi \in \operatorname{ex} C$ actually belongs to $\operatorname{ex} P^1(G)$. It follows that $\operatorname{ex} C = \{\chi_G\}$, and this implies that $G = H$, which is σ-compact. ∎

Lemma 4.2. (i) *If G is first countable, then $\widehat{G}$ is σ-compact.*

(ii) *If G is σ-compact and $\widehat{G}$ is σ-compact, then G is first countable.*

Proof. (i) Let $(V_n)_{n\in\mathbb{N}}$ be a neighbourhood basis of e in G, and for $n \in \mathbb{N}$ let $f_n = |V_n|^{-1}\chi_{V_n}$. Then $(f_n)_{n\in\mathbb{N}}$ is an approximate identity for $L^1(G)$ and hence for $C^*(G)$. For $n, m \in \mathbb{N}$ let

$$C_{n,m} = \Big\{\pi \in \widehat{G} : \|\pi(f_n)\| \geqslant \frac{1}{m}\Big\}.$$

Then $C_{n,m}$ is compact by the analogue of the Riemann-Lebesgue lemma ([11], Proposition 3.3.7), and

$$\widehat{G} = \bigcup\{C_{n,m} : n, m \in \mathbb{N}\}.$$

Indeed, if $\pi \in \widehat{G}$ does not belong to $C_{n,m}$ for all n and m, then $\|\pi(f_n)\| < 1/m$ for all m, so that $\pi(f_n) = 0$ and hence

$$\pi(f) = \lim_{n\to\infty} \pi(f)\pi(f_n) = 0$$

for every $f \in C^*(G)$, a contradiction.

(ii) By the theorem of Kakutani and Kodaira referred to earlier, there exists a compact normal subgroup K of G such that G/K is second countable. For $\gamma \in \widehat{K}$, the G-orbit $G(\gamma)$ of γ in $\widehat{K}$ is countable since G is σ-compact and $\widehat{K}$ is discrete. Now, for any G-orbit Γ in $\widehat{K}$, let

$$\widehat{G}_\Gamma = \{\pi \in \widehat{G} : \pi|K \sim \Gamma\}.$$

The compactness of K implies that each $\widehat{G}_\Gamma$ is open in $\widehat{G}$. Moreover, $\widehat{G}$ is the disjoint union of all subsets $\widehat{G}_\Gamma$, where Γ runs through the G-orbits in $\widehat{K}$. In fact, if π is an irreducible representation of G, then $\pi|K \sim G(\gamma)$ for every irreducible subrepresentation γ of $\pi|K$, and conversely every $\gamma \in \widehat{K}$ occurs in some $\pi|K$. Since $\widehat{G}$ is σ-compact, every open cover of $\widehat{G}$ contains a countable subcover. Thus there are only countably many Γ. Hence $\widehat{K}$ is countable, and this implies that K is second countable. Since G/K is first countable, we conclude that G is first countable. ∎

Corollary 4.3. *For a locally compact group G the following conditions are equivalent:*

(i) *Every closed subgroup H of G has a σ-compact dual $\widehat{H}$;*

(ii) *There exists an open compactly generated subgroup H of G such that $\widehat{H}$ is σ-compact;*

(iii) *G is first countable.*

Proof. (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow from parts (ii) and (i) of Lemma 4.2, respectively. ∎

Acknowledgements. The authors have been supported by NATO collaborative research grant 940184.

REFERENCES

1. D. Amir, J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math.(2)* **88**(1968), 35–46.
2. G. Arsac, Sur l'espace de Banach engendré par la coefficients d'une représentation unitaire, *Publ. Dép. Math. Nouvell Sér.*, vol. 13, Univ. Claude-Bernard, Lyon, 1976, pp. 1–101.
3. L. Baggett, A separable group having a discrete dual space is compact, *J. Funct. Anal.* **10**(1972), 131–148.
4. A. Bélanger, B. Forrest, Geometric properties of coefficient function spaces determined by unitary representations of a locally compact group, *J. Math. Anal. Appl.* **193**(1995), 390–405.
5. R. Bourgin, Geometric aspects of convex sets with the Radon-Nikodym property, *Lecture Notes in Math.*, vol. 993, Springer, 1983.
6. C.H. Chu, A note on scattered C^*-algebras and the Radon-Nikodym property, *J. London Math. Soc.(2)* **24**(1981), 533–536.
7. C.H. Chu, Von Neumann algebras which are second dual spaces, *Proc. Amer. Math. Soc.* **112**(1991), 999–1000.
8. H.M. Day, *Normed Linear Spaces*, Springer, 1973.
9. J. Diestel, Geometry of Banach spaces - selected topics, *Lecture Notes in Math.*, vol. 485, Springer, 1975.
10. J. Diestel, J.J. Uhl, Vector measures, *Math. Surveys Monogr.*, vol. 15, Amer. Math. Soc., 1977.
11. J. Dixmier, *C^*-Algebras*, North-Holland, 1977.
12. P. Eymard, L'algèbre de Fourier d'une groupe localement compact, *Bull. Soc. Math. France* **92**(1964), 181–236.
13. J.M.G. Fell, Weak containment and induced representations. II, *Trans. Amer. Math. Soc.* **110**(1964), 424–447.
14. J. Glimm, Families of induced representations, *Pacific J. Math.* **12**(1962), 885–911.
15. C.C. Graham, A.T. Lau, M. Leinert, Separable translation invariant subspaces of $M(G)$ and other dual spaces on locally compact groups, *Colloq. Math.* **55**(1988), 131–145.
16. F. Greenleaf, M. Moskowitz, Cyclic vectors for representations of locally compact groups, *Math. Ann.* **190**(1971), 265–288.
17. S. Kakutani, K. Kodaira, Über das Haarsche Mass in der lokal bikompakten Gruppe, in *Selected Papers of S. Kakutani*, vol. I, Birkhäuser, 1986, pp. 68–74.
18. E. Kaniuth, A. Markfort, Locally compact groups whose conjugation representations satisfy a Kazhdan type property or have countable support, *Math. Proc. Cambridge Philos. Soc.* **116**(1994), 79–97.
19. A.T. Lau, A. Ülger, Some geometric properties of the Fourier and Fourier-Stieltjes algebras of locally compact groups, Arens regularity, and related problems, *Trans. Amer. Math. Soc.* **377**(1993), 321–359.
20. M. Takesaki, *Theory of Operator Algebras. I*, Springer, 1979.

21. K.F. Taylor, Geometry of the Fourier algebras and locally compact groups with atomic unitary representations, *Math. Ann.* **262**(1983), 183–190.
22. S.P. Wang, On isolated points in the dual spaces of locally compact groups, *Math. Ann.* **218**(1975), 19–34.

E. KANIUTH
Fachbereich Mathematik/Informatik
Universität Paderborn
D-33095, Paderborn
GERMANY

A.T. LAU
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta, T6G 2G1
CANADA

G. SCHLICHTING
Mathematisches Institut
Technische Universität München
D-80290 München
GERMANY

Received March 29, 1997; revised June 21,1997.