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CONTINUITY OF THE DRAZIN INVERSE

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ABSTRACT. In this paper we investigate the continuity of the Drazin inverse of a bounded linear operator on Banach space. Then as a corollary, among other things, we get the well known result of Campbell and Meyer ([1]) for the continuity of the Drazin inverse of square matrix.

KEYWORDS: *Drazin inverse, continuity of the Drazin inverse, bounded linear operator.*

MSC (2000): 47A05, 47A53, 15A09.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to give some characterizations of the continuity of the Drazin inverse of a bounded linear operator on Banach space.

Let us recall that if S is an algebraic semigroup (or associative ring), then an element $a \in S$ is said to have a *Drazin* inverse ([4]) if there exists $x \in S$ such that

$$a^m = a^{m+1}x \quad \text{for some non-negative integer } m, \tag{1.1}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{1.2}$$

If a has Drazin inverse, then the smallest non-negative integer m in (1.1) above is called the *index* $i(a)$ of a. It is well known that there is at most one x such that equations (1.1) and (1.2) hold. The unique x is denoted by a^{D} and is called the *Drazin* inverse of a. Recall that if a has Drazin inverse, then a^{D} also has Drazin inverse, $i(a^{\mathrm{D}}) \leqslant 1$, $(a^{\mathrm{D}})^{\mathrm{D}} = a^2 a^{\mathrm{D}}$ and $((a^{\mathrm{D}})^{\mathrm{D}})^{\mathrm{D}} = a^{\mathrm{D}}$ ([4]). If S is an associative ring and $a \in S$ has Drazin inverse then a may always be written as

$$a = c + n, \tag{1.3}$$

where $c, n \in S$, c has Drazin inverse, $i(c) \leqslant 1$, $cn = nc = 0$, and $n^{i(a)} = 0$. The elements c, n are unique. Then c is called the *core* of a, and n the *nilpotent* part of a. Let us mention that in this case

$$c = a^2 a^{\mathrm{D}} \quad \text{and} \quad n = a - a^2 a^{\mathrm{D}}. \tag{1.4}$$

We shall refer to $c + n$ as the *core nilpotent* decomposition of a ([1], [4]).

Recall that a square matrix always has Drazin inverse, and that the Drazin inverse of a matrix is not necessarily a continuous function of the elements of the matrix ([1], [2]).

Let X be an infinite-dimensional complex Banach space and denote the set of bounded linear operators on X by $B(X)$. For T in $B(X)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. Let $\sigma(T)$ and $\rho(T)$ be the spectrum and the resolvent set of T, respectively. If $z \in \rho(T)$, the resolvent $R(z,T) = (zI - T)^{-1} \in B(X)$. Recall that $a(T)$ $(d(T))$, the *ascent (descent)* of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ $(R(T^n) = R(T^{n+1}))$. If no such n exists, then $a(T) = \infty$ $(d(T) = \infty)$. It is well known that an operator $T \in B(X)$ has a Drazin inverse T^{D} if and only if it has finite ascent and descent. In this case, the index of T is equal to the ascent of T ([3], [12]).

The *minimal angle* $\varphi(Y, Z)$ $(0 \leqslant \varphi \leqslant \pi/2)$ between two nonzero subspaces Y, Z of a Banach space is defined by

$$\sin \varphi(Y, Z) = \inf_{\substack{y \in Y,\, z \in Z \\ \max\{\|y\|, \|z\|\} = 1}} \|y + z\|. \tag{1.5}$$

The following result ([6], Lemma 1) is valid:

Let Y and Z be nonzero subspaces of a Banach space. Then $Y \cap Z = \{0\}$ and $Y + Z$ is closed if and only if $\varphi(Y, Z) > 0$.

Also, let us remark that if $Y \cap Z = \{0\}$, then

$$\sin \varphi(Y, Z) = \frac{1}{\max\{\|P\|, \|I - P\|\}}, \tag{1.6}$$

where P is the projector from $Y + Z$ onto Y along Z.

If M and N are two closed subspaces of the Banach space X, we set

$$\delta(M,N)=\sup\{\text{dist}\,(u,N):u\in M,\|u\|=1\} \tag{1.7}$$

and

$$\widehat{\delta}(M,N)=\max[\delta(M,N),\delta(N,M)]. \tag{1.8}$$

Then $\widehat{\delta}$ is called the *gap* (or *opening*) between the M and N ([11], p. 197). For an element T in $B(X)$ the *reduced minimum modulus* of T, $\gamma(T)$, is defined by

$$\gamma(T)=\inf\{\|Tz\|/\text{dist}\,(z,N(T)):\text{dist}\,(z,N(T))>0\}. \tag{1.9}$$

Recall that $R(T)$ is closed if and only if $\gamma(T)>0$ ([11], p. 251). If there is S in $B(X)$, such that $TST=T$, then $R(T)$ is closed and $\gamma(T)\geqslant 1/\|S\|$ ([5], Lemma 4). Let us remark that $1/\gamma(T)=k(T)$, where $k(T)=\sup\{\inf\{\|z\|:Tz=y\}:y\in R(T),\|y\|=1\}$. For the convenience of the reader, recall the following well known result of A.S. Markus ([13], Theorem 2 and Remark 1).

THEOREM 1.1. *Suppose that $A, A_n\in B(X)$, $R(A)$ and $R(A_n)$ are closed, $n=1,2,\ldots$, and let $A_n\to A$. Then the following conditions are equivalent:*

$$\sup_n k(A_n)<\infty; \tag{1.10}$$

$$\lim_{n\to\infty} k(A_n)=k(A); \tag{1.11}$$

$$\lim_{n\to\infty}\widehat{\delta}(N(A_n),N(A))=0; \tag{1.12}$$

$$\lim_{n\to\infty}\widehat{\delta}(R(A_n),R(A))=0. \tag{1.13}$$

In this paper we study the continuity of the Drazin inverse of a bounded linear operator on a Banach space, i.e., the continuity of the map $A\mapsto A^{\mathrm{D}}$, $A\in B(X)$. Then, among other things, as a corollary we get the well known result of Campbell and Meyer for the continuity of the Drazin inverse of a square matrices ([1], [2]). Let us mention that Campbell and Meyer in their proof used the continuity of the Moore-Penrose inverse of matrix. It seems that our proof of Theorem 2.2 (or Corollary 3.5) is more natural, because it does not invoke the Moore-Penrose inverse and the definition of Drazin inverse does not clearly involve the notation of the Moore-Penrose inverse (see Remark 3.6).

2. THE MAIN RESULTS

In this section we first prove an auxiliary result and then the main theorem.

LEMMA 2.1. *Suppose that $A \in B(X)$ has Drazin inverse. If $A = C + N$ is the core nilpotent decomposition of A, then*

$$\{\lambda \in \mathbb{C} : 0 < |\lambda| < \gamma(C)\} \subset \rho(A). \tag{2.1}$$

Proof. Since $a(C) = d(C) \leqslant 1$ we have $X = R(C) \oplus N(C)$ and $R(C)$ is a closed subspace of X. Set $X_0 = R(C)$, $X_1 = N(C)$, and let C_0 and C_1 be the restrictions of C to X_0 and X_1, respectively. Now, C_0 is invertible, and for $|\lambda| < \gamma(C) = \|C_0^{-1}\|^{-1}$ we have $C_0 - \lambda$ invertible. Since C_1 is nilpotent, it follows that $C_1 - \lambda$ is invertible for all $\lambda \neq 0$. Thus we have that $C - \lambda$ is invertible for $0 < |\lambda| < \gamma(C)$. Finally, to prove (2.1), let us remark that since N is nilpotent and commutes with C we have $\sigma(A) = \sigma(C)$. This completes the proof. ∎

Now we prove the main result of this paper.

THEOREM 2.2. *Let $\{A_n\}$ be a sequence in $B(X)$, and let $A_n \to A \in B(X)$. Suppose that A and A_n, $n = 1, 2, \ldots$, have Drazin inverses A^{D} and A_n^{D}, $n = 1, 2, \ldots$, respectively, and let $A = C + N$ and $A_n = C_n + N_n$, $n = 1, 2, \ldots$, be the core nilpotent decompositions of A and A_n, $n = 1, 2, \ldots$, respectively. Then the following conditions are equivalent:*

$$A_n^{\mathrm{D}} \to A^{\mathrm{D}}; \tag{2.2}$$

$$A_n^{\mathrm{D}} A_n \to A^{\mathrm{D}} A; \tag{2.3}$$

$$\sup_n \|A_n^{\mathrm{D}}\| < \infty; \tag{2.4}$$

$$\widehat{\delta}(N(C_n), N(C)) \to 0 \quad \text{and} \quad \widehat{\delta}(R(C_n), R(C)) \to 0; \tag{2.5}$$

$$C_n \to C \quad \text{and} \quad \widehat{\delta}(N(C_n), N(C)) \to 0; \tag{2.6}$$

$$C_n \to C \quad \text{and} \quad \widehat{\delta}(R(C_n), R(C)) \to 0; \tag{2.7}$$

$$C_n \to C \quad \text{and} \quad \sup_n k(C_n) < \infty; \tag{2.8}$$

$$\sup_n k(C_n) < \infty. \tag{2.9}$$

Proof. It is clear that (2.2) $\Rightarrow$ (2.3).

(2.3) $\Rightarrow$ (2.2) Since $C_n = A_n^2 A_n^{\mathrm{D}}$ and $C = A^2 A^{\mathrm{D}}$, it is clear that (2.3) implies $C_n \to C$. Since $a(C) = d(C) \leqslant 1$, we have $X = R(C) \oplus N(C)$ and $R(C)$ is a closed subspace of X. Further, since A has Drazin inverse, there is an integer n_0 such that $a(A) = d(A) = n_0$. Now,

$$R(A^{n_0}) = R(AA^{\mathrm{D}}) = R(C) \quad \text{and} \quad N(A^{n_0}) = N(AA^{\mathrm{D}}) = N(C). \tag{2.10}$$

Recall that

$$A^{\mathrm{D}} = C^{\mathrm{D}} = \begin{cases} (C_{|R(C)})^{-1} & \text{on } R(C), \\ 0 & \text{on } N(C). \end{cases} \tag{2.11}$$

Since CC^{D} is the projector onto $R(C)$ along $N(C)$, by [15], (13) and (1.6) we have

$$\|C^{\mathrm{D}}\| \leqslant k(C)[\max\{\|CC^{\mathrm{D}}\|, \|I - CC^{\mathrm{D}}\|\}]^2. \tag{2.12}$$

Clearly, from (2.12) we get

$$\|C_n^{\mathrm{D}}\| \leqslant k(C_n)[\max\{\|C_n C_n^{\mathrm{D}}\|, \|I - C_n C_n^{\mathrm{D}}\|\}]^2, \quad n = 1, 2, \ldots. \tag{2.13}$$

Let us remark that by (2.3) we have

$$\begin{aligned} \widehat{\delta}(R(C_n), R(C)) &= \widehat{\delta}(R(A_n A_n^{\mathrm{D}}), R(AA^{\mathrm{D}})) \\ &\leqslant \max\{\|(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})AA^{\mathrm{D}}\|, \|(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})A_n A_n^{\mathrm{D}}\|\} \\ &\to 0, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\widehat{\delta}(N(C_n), N(C)) = \widehat{\delta}(N(A_n A_n^{\mathrm{D}}), N(AA^{\mathrm{D}})) \\ &\leqslant \max\{\|(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})(I - AA^{\mathrm{D}})\|, \|(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})(I - A_n A_n^{\mathrm{D}})\|\} \\ &\to 0. \end{aligned} \tag{2.15}$$

Hence, by Theorem 1.1 and (2.14), it follows that

$$\sup_n k(C_n) < \infty. \tag{2.16}$$

Further, by (2.14) and (2.15) we have (see [15], p. 271 and [6], Lemma 2)

$$\begin{aligned} &|\sin\varphi^{(m)}(N(C_n), R(C_n)) - \sin\varphi^{(m)}(N(C), R(C))| \\ &\qquad \leqslant 2\widehat{\delta}(N(C_n), N(C)) + 2\widehat{\delta}(R(C_n), R(C)) \to 0. \end{aligned} \tag{2.17}$$

Now, by (2.13), (1.6) and (2.16) we get

$$\sup_n \|C_n^{\mathrm{D}}\| \leqslant k(C_n)[\max\{\|C_n C_n^{\mathrm{D}}\|, \|I - C_n C_n^{\mathrm{D}}\|\}]^2 < \infty. \tag{2.18}$$

Finally, since $A_n^{\mathrm{D}} = C_n^{\mathrm{D}}$, it is clear that

$$A_n^{\mathrm{D}} - A^{\mathrm{D}} = A_n^{\mathrm{D}}(A_n A_n^{\mathrm{D}} - AA^{\mathrm{D}}) + (A_n A_n^{\mathrm{D}} - AA^{\mathrm{D}})A^{\mathrm{D}} + A_n^{\mathrm{D}}(A_n - A)A^{\mathrm{D}} \to 0. \tag{2.19}$$

Hence, we have that (2.3) implies (2.2).

(2.3) $\Rightarrow$ (2.4) This is (2.18).

(2.4) $\Rightarrow$ (2.3) Let $A = C + N$ and $A_n = C_n + N_n$ be the core nilpotent decompositions of A and A_n, $n = 1, 2, \ldots$, respectively. Now, $A^{\mathrm{D}} = C^{\mathrm{D}}$, and $CC^{\mathrm{D}}C = C$. Hence, $\|C^{\mathrm{D}}\| \geqslant 1/\gamma(C) = k(C)$. Thus, we have that

$$\|C_n^{\mathrm{D}}\| \geqslant 1/\gamma(C_n) = k(C_n), \quad n = 1, 2, \ldots \tag{2.20}$$

and (2.4) implies

$$\inf_n \gamma(C_n) = \gamma > 0. \tag{2.21}$$

Set $\delta = \min\{\gamma, \gamma(C)\}$ and $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = \delta/2\}$. By Lemma 2.1 we have

$$\{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta\} \subset \Big(\bigcap_{n=1}^{\infty} \rho(A_n)\Big) \cap \rho(A). \tag{2.22}$$

Now

$$I - AA^{\mathrm{D}} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} R(z, A)\, \mathrm{d}z$$

and

$$I - A_n A_n^{\mathrm{D}} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} R(z, A_n)\, \mathrm{d}z, \quad n = 1, 2, \ldots.$$

Hence

$$\begin{aligned} (I - AA^{\mathrm{D}}) - (I - A_n A_n^{\mathrm{D}}) &= A_n A_n^{\mathrm{D}} - AA^{\mathrm{D}} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} [R(z, A) - R(z, A_n)]\, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} R(z, A_n)(A - A_n)R(z, A)\, \mathrm{d}z. \end{aligned} \tag{2.23}$$

Since $R(z, A_n) \to R(z, A)$ uniformly on Γ, then for some n_0

$$\sup_{n \geqslant n_0} \{\|R(z, A_n)\| : z \in \Gamma\} < \infty.$$

Now from (2.23) it follows that for some $M > 0$

$$\|AA_n^{\mathrm{D}} - AA^{\mathrm{D}}\| \leqslant M\|A_n - A\| \to 0.$$

Hence (2.4) $\Rightarrow$ (2.3).

That (2.3) $\Rightarrow$ (2.5) is clear from (2.14) and (2.15).

(2.5) $\Rightarrow$ (2.3) Since A_n, $n = 1, 2, \ldots$, has Drazin inverse, we know that

$$X = R(A_n^{i(A_n)}) \oplus N(A_n^{i(A_n)}), \quad n = 1, 2, \ldots. \tag{2.24}$$

Thus, if $x \in X$, then there are $y_n \in R(A_n^{i(A_n)})$ and $z_n \in N(A_n^{i(A_n)})$, $n = 1, 2, \ldots$, such that $x = y_n + z_n$, $n = 1, 2, \ldots$. Hence

$$(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})x = AA^{\mathrm{D}} z_n - (I - AA^{\mathrm{D}})y_n, \quad n = 1, 2, \ldots. \tag{2.25}$$

Since $AA^{\mathrm{D}} z_n = AA^{\mathrm{D}}(z_n - u)$, $u \in N(A^{i(A)})$, we have

$$\begin{aligned} \|AA^{\mathrm{D}} z_n\| &\leqslant \|AA^{\mathrm{D}}\| \,\mathrm{dist}\, (z_n, N(A^{i(A)})) \\ &\leqslant \|AA^{\mathrm{D}}\| \,\widehat{\delta}(N(A_n^{i(A)}), N(A^{i(A)}))\, \|z_n\| \\ &\leqslant \|AA^{\mathrm{D}}\| \,\widehat{\delta}(N(A_n^{i(A)}), N(A^{i(A)}))\, \|I - A_n A_n^{\mathrm{D}}\| \,\|x\|. \end{aligned} \tag{2.26}$$

In a similar way we could prove

$$\|(I - AA^{\mathrm{D}})y_n\| \leqslant \|I - AA^{\mathrm{D}}\| \,\widehat{\delta}(R(A_n^{i(A)}), R(A^{i(A)}))\, \|A_n A_n^{\mathrm{D}}\| \,\|x\|. \tag{2.27}$$

Hence

$$\begin{aligned} \|(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})x\| \leqslant (&\|AA^{\mathrm{D}}\| \,\widehat{\delta}(N(A_n^{i(A)}), N(A^{i(A)}))\, \|I - A_n A_n^{\mathrm{D}}\| \\ &+ \|I - AA^{\mathrm{D}}\| \,\widehat{\delta}(R(A_n^{i(A)}), R(A^{i(A)}))\, \|A_n A_n^{\mathrm{D}}\|)\|x\|, \end{aligned}$$

and

$$\begin{aligned} \|(AA^{\mathrm{D}} - A_n A_n^{\mathrm{D}})\| \leqslant{} & [\widehat{\delta}(N(A_n^{i(A_n)}), N(A^{i(A)})) + \widehat{\delta}(R(A_n^{i(A_n)}), R(A^{i(A)})] \\ & \times \max\{\|AA^{\mathrm{D}}\|, \|I - AA^{\mathrm{D}}\|\} \max\{\|A_n A_n^{\mathrm{D}}\|, \|I - A_n A_n^{\mathrm{D}}\|\}. \end{aligned} \tag{2.28}$$

Now, by (1.6), (2.17) and (2.28) we have that (2.5) $\Rightarrow$ (2.3).

Let us remark that by Theorem 1.1 the conditions (2.6), (2.7) and (2.8) are equivalent. Clearly (2.6) and (2.7) imply (2.5), and from the proof of (2.3) $\Rightarrow$ (2.2) we have that (2.3) implies (2.6).

It is clear that (2.8) $\Rightarrow$ (2.9), and from (2.21) it follows that (2.9) $\Rightarrow$ (2.3). The proof of the theorem is complete. ∎

3. APPLICATIONS

In this section we prove several corollaries of Theorem 2.2. In a particular case we get the main result of Campbell and Meyer ([1], Theorem 2; see also [2], Theorem 10.7.1). We also study the pointwise continuity of the Drazin inverse.

Let us recall that if $A \in B(X)$ has Drazin inverse, then A^k has Drazin inverse for any positive integer k, and $(A^k)^{\mathrm{D}} = (A^{\mathrm{D}})^k$ ([4], Theorem 2). We continue with the following auxiliary result.

LEMMA 3.1. *Let A_n, A, C_n, C, N_n and N be as in Theorem 2.2 above. Then the following conditions are equivalent:*

(3.1) $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$;

(3.2) *there is $m \in \mathbb{N}$ such that* $(A_n^m)^{\mathrm{D}} \to (A^m)^{\mathrm{D}}$;

(3.3) *for any $m \in \mathbb{N}$ we have* $(A_n^m)^{\mathrm{D}} \to (A^m)^{\mathrm{D}}$.

Proof. Since $(A^{\mathrm{D}})^m = (A^m)^{\mathrm{D}}$, it is clear that (3.1) implies (3.3), and it is enough to prove (3.2) implies (3.1). Suppose that $m > 1$. Now from (3.2) we have $(A_n^m)^{\mathrm{D}} \to (A^m)^{\mathrm{D}}$, and

$$A_n^{\mathrm{D}} = (A_n^{\mathrm{D}})^m A_n^{m-1} = (A_n^m)^{\mathrm{D}} A_n^{m-1} \to (A^m)^{\mathrm{D}} A^{m-1} = (A^{\mathrm{D}})^m A^{m-1} = A^{\mathrm{D}}.$$

The proof is complete. ∎

Now, using Theorem 2.2 and Lemma 3.1 we can prove

COROLLARY 3.2. *Let A_n, A, C_n, C, N_n and N be as above in Theorem 2.2. Then the following conditions are equivalent:*

(3.4) $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$;

(3.5) *there are $m, k \in \mathbb{N}$ such that* $C_n^m \to C^m$ *and* $\widehat{\delta}(N(C_n^k), N(C^k)) \to 0$;

(3.6) *there are $m, k \in \mathbb{N}$ such that* $C_n^m \to C^m$ *and* $\widehat{\delta}(R(C_n^k), R(C^k)) \to 0$;

(3.7) *there are $m, k \in \mathbb{N}$ such that* $C_n^m \to C^m$ *and* $\sup_n k(C_n^k) < \infty$;

(3.8) *there is $m \in \mathbb{N}$ such that* $\sup_n k(C_n^m) < \infty$;

(3.9) *for any $m, k \in \mathbb{N}$ we have* $C_n^m \to C^m$ *and* $\widehat{\delta}(N(C_n^k), N(C^k)) \to 0$;

(3.10) *for any $m, k \in \mathbb{N}$ we have* $C_n^m \to C^m$ *and* $\widehat{\delta}(R(C_n^k), R(C^k)) \to 0$;

(3.11) *for any $m, k \in \mathbb{N}$ we have* $C_n^m \to C^m$ *and* $\sup_n k(C_n^k) < \infty$;

(3.12) *for any $m \in \mathbb{N}$ we have* $\sup_n k(C_n^m) < \infty$.

Proof. Since $a(C) = d(C) \leqslant 1$, and $a(C_n) = d(C_n) \leqslant 1$, $n = 1, 2, \ldots$, the proof follows by Theorem 2.2, Lemma 3.1 and Theorem 1.1. ∎

COROLLARY 3.3. *Let A_n, A, C_n, C, N_n and N be as above in Theorem* 2.2. *If the index of A_n, $n = 1, 2, \ldots$, is bounded, i.e, if $\sup_n i(A_n) < \infty$, then the following conditions are equivalent:*

$$A_n^{\mathrm{D}} \to A^{\mathrm{D}}, \tag{3.13}$$

$$\widehat{\delta}(N(C_n), N(C)) \to 0, \tag{3.14}$$

$$\widehat{\delta}(R(C_n), R(C)) \to 0. \tag{3.15}$$

Proof. Set $m_0 = \max\{\sup_n i(A_n), i(A)\}$. Then $A^{m_0} = C^{m_0} + N^{m_0} = C^{m_0}$ and $A_n^{m_0} = C_n^{m_0} + N_n^{m_0} = C_n^{m_0}$, $n = 1, 2, \ldots$. Hence $A_n \to A$ implies $C_n^{m_0} \to C^{m_0}$, and the proof follows by Corollary 3.2, (3.5) and (3.6). ∎

Now we study the continuity of the Drazin inverse when the operators have finite dimensional null spaces and when their ranges have finite codimension. For T in $B(X)$ set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X)$ is *semi-Fredholm* if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. Let $\Phi_+(X)$ ($\Phi_-(X)$) denote the set of *upper (lower) semi-Fredholm* operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$ ($\beta(T) < \infty$). An operator T is *Fredholm* if it is both upper semi-Fredholm and lower semi-Fredholm ([7], [11]). Let $\Phi(X)$ denote the set of Fredholm operators, i.e., $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. It is clear that if T is semi-Fredholm and has Drazin inverse, then T is Fredholm. Let us remark that the convergence problem of Drazin inverses is connected with perturbation problems (see [19] and [20] for recent results on perturbations of semi-Fredholm operators with finite ascent or descent).

COROLLARY 3.4. *Let A_n, A, C_n, C, N_n and N be as above in Theorem* 2.2. *If the index of A_n, $n = 1, 2, \ldots$, is bounded and $C, C_n \in \Phi(X)$, $n = 1, 2, \ldots$, then the following conditions are equivalent:*

$$A_n^{\mathrm{D}} \to A^{\mathrm{D}}; \tag{3.16}$$

$$\widehat{\delta}(N(C_n), N(C)) \to 0; \tag{3.17}$$

$$\widehat{\delta}(R(C_n), R(C)) \to 0; \tag{3.18}$$

$$\textit{there is } n_0 \in \mathbb{N} \textit{ such that } \alpha(C_n) = \alpha(C), \textit{ for } n \geqslant n_0; \tag{3.19}$$

$$\textit{there is } n_0 \in \mathbb{N} \textit{ such that } \beta(C_n) = \beta(C), \textit{ for } n \geqslant n_0. \tag{3.20}$$

Proof. From the proof of Corollary 3.3 we know that there exists m_0 such that $C_n^{m_0} \to C^{m_0}$. Now, since $\alpha(C_n^{m_0}) = \alpha(C_n)$, $n = 1, 2, \ldots$, and $\alpha(C^{m_0}) = \alpha(C)$, the proof follows by [15], Theorem 2 and Corollary 3.3. ∎

Now, as a corollary, we get the main result of Campbell and Meyer ([1], Theorem 2; see also [2], Theorem 10.7.1). Our formulation of that result is somehow different from that of Campbell and Meyer's, but resembles that of Theorem 2.2.

COROLLARY 3.5. (Campbell and Meyer) *Suppose that A_j and A are $n \times n$ complex matrices such that $A_j \to A$. Then $A_j^{\mathrm{D}} \to A^{\mathrm{D}}$ if and only if there exists j_0 such that* $\operatorname{rank}(C_j) = \operatorname{rank}(C)$ *for $j \geqslant j_0$, where C and C_j are the core of A and A_j, respectively, $j = 1, 2, \ldots$.*

Proof. It is clear that the index of A_j, $j = 1, 2, \ldots$, is bounded by n, and that the matrices A, A_j, as operators, are Fredholm. Now the proof follows by Corollary 3.4. ∎

REMARK 3.6. Recall that if X is a complex Hilbert space and $T \in B(X)$, then $R(T)$ is closed if and only if there exists a unique operator $T^\dagger \in B(X)$ satisfying the following four Penrose identities (see e.g., [2], [3], [8], [10], [16]):

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad \text{and} \quad (T^\dagger T)^* = T^\dagger T. \tag{3.21}$$

The operator $T^\dagger$ is called the *Moore-Penrose* inverse of T. Let us remark that in the proof of [1], Theorem 2, Campbell and Meyer used the continuity properties of the Moore-Penrose inverse of a matrices. It seems that our proof of Theorem 2.2 (or Corollary 3.5) is more natural, because it does not invoke the Moore-Penrose inverse and the definition of Drazin inverse does not clearly involve the notion of the Moore-Penrose inverse.

Let X and Y be Banach spaces and let A_n ($n = 1, 2, \ldots$) and A be operators in $B(X, Y)$. We then write $A_n \xrightarrow{s} A$ if the sequence A_n converges to A strongly. We next show an equivalent condition for the strong convergence of Drazin inverses, which is to be compared with Theorem 2.2.

THEOREM 3.6. *Let $\{A_n\}$ be a sequence in $B(X)$, and let $A_n \xrightarrow{s} A \in B(X)$. Suppose that A and A_n, $n = 1, 2, \ldots$, have Drazin inverses A^{D} and A_n^{D}, $n = 1, 2, \ldots$, respectively. Then the following conditions are equivalent:*

$$A_n^{\mathrm{D}} \xrightarrow{s} A^{\mathrm{D}}, \tag{3.22}$$

$$\sup_n \|A_n^{\mathrm{D}}\| < \infty \quad \text{and} \quad A_n^{\mathrm{D}} A_n \xrightarrow{s} A^{\mathrm{D}} A. \tag{3.23}$$

Proof. If we assume (3.22), then the inequality in (3.23) is obtained from the Banach-Steinhaus Theorem and the other assertation in (3.23) is easily seen by the uniform boundedness of $\{A_n^{\mathrm{D}}\}$.

For the converse, (3.23) $\Rightarrow$ (3.22), it is enough to note that

$$A_n^{\mathrm{D}} - A^{\mathrm{D}} = A_n^{\mathrm{D}}(A_nA_n^{\mathrm{D}} - AA^{\mathrm{D}}) + (A_nA_n^{\mathrm{D}} - AA^{\mathrm{D}})A^{\mathrm{D}} + A_n^{\mathrm{D}}(A_n - A)A^{\mathrm{D}}. \quad (3.24)$$ ∎

REMARK 3.7. In contradiction to the case of uniform convergence, we cannot deduce the inequality $\sup_n \|A_n^{\mathrm{D}}\| < \infty$ from $A_n^{\mathrm{D}}A_n \xrightarrow{s} A^{\mathrm{D}}A$. For example (see [10] for other applications of the following example), let $A = I$ on $X = l_2$, and let

$$A_n = \operatorname{diag}\Big\{\overbrace{1,\ldots,1}^{n}, \frac{1}{n}, \frac{1}{n}, \ldots\Big\} \quad \text{on } X = l_2,\ n = 1, 2, \ldots.$$

Clearly, $A_n \xrightarrow{s} A$, and

$$A_n^D = \operatorname{diag}\{\overbrace{1,\ldots,1}^{n}, n, n, \ldots\}, \quad n = 1, 2, \ldots.$$

Now $A_n^{\mathrm{D}}A_n \xrightarrow{s} A^{\mathrm{D}}A$, but $\|A_n^{\mathrm{D}}\| = n$, $\quad n = 1, 2, \ldots$, i.e., $\sup_n \|A_n^{\mathrm{D}}\| = \infty$.

4. DRAZIN INVERSE IN BANACH ALGEBRAS

Finally, in this section we prove that some of the above results could be presented in general Banach algebras. Let us recall some notation and results.

Let $\mathcal{A}$ denote a complex Banach algebra with identity 1. An element $a \in \mathcal{A}$ is (*von Neumann*) *regular* if $a \in a\mathcal{A}a$. That is, there exists a solution of the equation $axa = a$. These solutions are usually called *inner* or 1-*inverses* of a. The set of all regular elements in $\mathcal{A}$ will be denoted by $\widehat{\mathcal{A}}$ and it obviously includes the invertible group of $\mathcal{A}$. Recall that an element a in $\mathcal{A}$ is hermitian if $\|\exp(ita)\| = 1$ for all real t ([21]). In connection with the Moore-Penrose generalized inverse, we have studied ([16], [17]) the set of elements a in $\mathcal{A}$ for which there exists an x in $\mathcal{A}$ satisfying the following conditions:

$$axa = a, \quad xax = x, \quad ax \text{ and } xa \text{ are hermitian.} \quad (4.1)$$

By [16], Lemma 2.1 there is at most one x such that the four above equations hold. The unique x is denoted by $a^\dagger$ and is called the *Moore-Penrose inverse* of a. Let $\mathcal{A}^\dagger$ denote the set of all elements in $\mathcal{A}$ which have Moore-Penrose inverses.

Given an element a in $\mathcal{A}$ let $L_a \in B(\mathcal{A})$ be the left regular representation of a, i.e. $L_ax = ax$, $x \in \mathcal{A}$. Further, for $a \in \mathcal{A}^\dagger$ it is known that $\|a^\dagger\| = 1/\gamma(L_a)$ ([16], Theorem 2.3). Recall that if $\mathcal{A}$ is a C^*-algebra then $a \in \mathcal{A}$ is hermitian if and only if $a^* = a$, and by the Harte-Mbekhta Theorem ([8], Theorem 6) in that case $\mathcal{A}^\dagger = \widehat{\mathcal{A}}$. Now by Theorem 2.2 we have

THEOREM 4.1. *Let $\mathcal{A}$ denote a complex Banach algebra with identity* 1. *Let* $\{a_m\}$ *be a sequence in* $\mathcal{A}$, *with* $a_m \to a \in \mathcal{A}$. *Suppose that* a *and* a_m, $m = 1, 2, \ldots$, *have Drazin inverses* a^{D} *and* a_m^{D}, $m = 1, 2, \ldots$, *respectively, and let* $a = c + n$ *and* $a_m = c_m + n_m$, $m = 1, 2, \ldots$, *be the core nilpotent decompositions of* a *and* a_m, $m = 1, 2, \ldots$, *respectively. Then the following conditions are equivalent:*

$$a_m^{\mathrm{D}} \to a^{\mathrm{D}}; \tag{4.2}$$

$$a_m^{\mathrm{D}} a_m \to a^{\mathrm{D}} a; \tag{4.3}$$

$$\sup_m \|a_m^{\mathrm{D}}\| < \infty; \tag{4.4}$$

$$\widehat{\delta}(N(L_{c_m}), N(L_c)) \to 0 \quad \textit{and} \quad \widehat{\delta}(R(L_{c_m}), R(L_c)) \to 0; \tag{4.5}$$

$$c_m \to c \quad \textit{and} \quad \widehat{\delta}(N(L_{c_m}), N(L_c)) \to 0; \tag{4.6}$$

$$c_m \to c \quad \textit{and} \quad \widehat{\delta}(R(L_{c_m}), R(L_c)) \to 0; \tag{4.7}$$

$$c_m \to c \quad \textit{and} \quad \sup_m k(L_{c_m}) < \infty; \tag{4.8}$$

$$\sup_m k(L_{c_m}) < \infty. \tag{4.9}$$

THEOREM 4.2. *Let $\mathcal{A}$ denote a complex Banach algebra with identity* 1. *Suppose that all the assumptions from Theorem* 4.1 *are valid, and that in addition* c *and* c_m *are in* $\mathcal{A}^\dagger$, $m = 1, 2, \ldots$. *Then the following conditions are equivalent:*

$$a_m^{\mathrm{D}} \to a^{\mathrm{D}},\ \widehat{\delta}(N(L_{c_m^\dagger}), N(L_{c^\dagger})) \to 0,\ \textit{and}\ \widehat{\delta}(R(L_{c_m^\dagger}), R(L_{c^\dagger})) \to 0; \tag{4.10}$$

$$c_m^\dagger \to c^\dagger. \tag{4.11}$$

Proof. If we assume (4.10), then by (4.9) $\sup_m k(L_{c_m}) < \infty$, and by [16], Theorem 2.3 we have $\sup_m \|c_m^\dagger\| < \infty$. Now (4.11) follows by [17], Theorem 2.5.

For the converse, (4.11) $\Rightarrow$ (4.10), it is enough to note that by [17], Theorem 2.5 (ii), (4.11) implies

$$\sup_m \|c_m^\dagger\| < \infty, \quad \widehat{\delta}(N(L_{c_m^\dagger}), N(L_{c^\dagger})) \to 0, \quad \text{and} \quad \widehat{\delta}(R(L_{c_m^\dagger}), R(L_{c^\dagger})) \to 0.$$

Again, by [16], Theorem 2.3 we have $\sup_m k(L_{c_m}) < \infty$, and (4.9) implies $a_m^{\mathrm{D}} \to a^{\mathrm{D}}$. The proof of the theorem is complete. ∎

Let us remark that if $\mathcal{A}$ is a C^*–algebra, then the previous results could be presented in a simplier form.

THEOREM 4.3. *Let $\mathcal{A}$ be a C^*-algebra, and suppose that all the assumptions from Theorem* 4.1 *are valid. Then c and c_m are in $\mathcal{A}^\dagger$, $m = 1, 2, \ldots$, and the following conditions are equivalent:*

$$a_m^{\mathrm{D}} \to a^{\mathrm{D}}, \tag{4.12}$$

$$c_m^\dagger \to c^\dagger. \tag{4.13}$$

Proof. By the Harte-Mbekhta Theorem ([8], Theorem 6) we know that c and c_m are in $\mathcal{A}^\dagger$, $m = 1, 2, \ldots$. Now the conditions (4.12) and (4.13) are equivalent by (4.9), [16], Theorem 2.3 and [9], Theorem 6 or [18], Theorem 2.2. ∎

REMARK 4.4. Recently Koliha ([13]) has introduced and investigated a generalized inverse (he calls it a *generalized Drazin inverse*) in associative rings and Banach algebras; namely, if A is a complex unital Banach algebra, then an element $a \in A$ is said to have a *generalized Drazin inverse* if there exists $x \in A$ such that

$$a - a^2 x \quad \text{is quasinilpotent;} \tag{4.14}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{4.15}$$

If a has generalized Drazin inverse, then there is at most one x such that equations (4.14) and (4.15) hold. In our opinion it is interesting to study the continuity of the generalized Drazin inverse.

REMARK 4.5. Finally, let us remark that Labrousse and Mbekhta ([14]) have investigated the continuity properties of the Moore-Penrose inverse of closed densely defined linear operators on Hilbert space. In our opinion it is worth investigating the continuity properties of the Drazin inverse of closed densely defined linear operators on Banach (or Hilbert) space.

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