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ALGEBRAIC REDUCTION FOR HARDY SUBMODULES OVER POLYDISK ALGEBRAS

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ABSTRACT. For a Hardy submodule M of $H^2(\mathbb{D}^n)$, assume that $M \cap \mathcal{C}$ (or $M \cap \mathcal{R}$) is dense in M, where $\mathcal{C}$ (or $\mathcal{R}$) is the ring of all polynomials (or $\mathcal{R}$ is a Noetherian subring of $\operatorname{Hol}(\overline{\mathbb{D}}^n)$ containing $\mathcal{C}$). We describe those finite codimensional submodules of M by considering zero varieties. The codimension formulas related to zero varieties, and some algebraic reduction theorems are obtained. These results can be regarded as generalizations of the result of Ahern-Clark ([2]). Finally, we point out that the results in this paper extend with essentially no change to any reproducing Hilbert $A(\Omega)$-module H which satisfies certain technical hypotheses.

KEYWORDS: *Hardy submodules, ideals, charactistic space.*

MSC (2000): Primary 46H25; Secondary 47D06.

1. INTRODUCTION

The starting point for the present paper is the remarkable algebraic reduction of Ahern-Clark ([2]) for finite codimensional submodules of Hardy module $H^2(\mathbb{D}^n)$ over the polydisk algebra $A(\mathbb{D}^n)$. In the following, we will use $\mathcal{C}$ to denote the ring of all polynomials on $\mathbb{C}^n$.

THEOREM 1.1. ([2]) *Suppose M is a submodule of $H^2(\mathbb{D}^n)$ of codimension $k < \infty$. Then $\mathcal{C} \cap M$ is an ideal in the ring $\mathcal{C}$ such that:*

(i) *$\mathcal{C} \cap M$ is dense in M;*

(ii) *$\dim \mathcal{C}/\mathcal{C} \cap M = k$;*

(iii) *$Z(\mathcal{C} \cap M)$ is finite and lies in $\mathbb{D}^n$.*

Conversely, if I is an ideal in $\mathcal{C}$ with $Z(I)$ being finite, and $Z(I) \subset \mathbb{D}^n$, then $[I]$, the closure of I in $H^2(\mathbb{D}^n)$, is a submodule of the same codimension and $[I] \cap \mathcal{C} = I$.

Also arising from this motivation, R.G. Douglas, V.I. Paulsen, C.H. Sah and K. Yan ([8]), S. Axler and P. Bourdon ([4]), O.P. Agrawal and N. Salinas ([1]), M. Putinar ([9]), X.M. Chen and R.G. Douglas ([5]) have developed a series of techniques to investigate algebraic reduction and rigidity for Hilbert modules over function algebras. In the present paper, we focus on Theorem 1.1 of [2].

Given submodules M_1, M_2 of $H^2(\mathbb{D}^n)$, with $M_1 \supseteq M_2$, we can define a canonical module homomorphism over $\mathcal{C}$

$$\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$$

by $\tau(\widetilde{p}) = \widetilde{p}$. If $M_1 \cap \mathcal{C}$ is dense in M_1, and M_2 is finite codimensional in M_1, then it is not difficult to verify the following proposition.

PROPOSITION 1.2. *Under the above assumption, we have:*

(i) *$M_2 \cap \mathcal{C}$ is dense in M_2;*

(ii) *the canonical homomorphism $\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$ is an isomorphism.*

Focusing on the above Theorem 1.1, we will be concerned with the following problems.

(A) How do we describe the structure of $Z(M_2 \cap \mathcal{C})$ related to $Z(M_1 \cap \mathcal{C})$?

(B) How is the submodule M_2 represented by M_1 and the zeros of M_2 via considering mulitiplicity?

(C) How is codimension $\dim M_1/M_2$ related to the zeros and their mulitiplicity of M_1, M_2 (or $M_1 \cap \mathcal{C}, M_2 \cap \mathcal{C}$)?

(D) Conversely, suppose I_1, I_2 are ideals of $\mathcal{C}$, $I_1 \supseteq I_2$ and the boundary of $Z(I_2)$ related to $Z(I_1)$ is finite. We want to know how $[I_2]$ is related to $[I_1]$.

Furthermore, if we replace ideals of polynomials in the above problems by ideals of Noetherian ring $\mathcal{R}$, where $\mathcal{R}$ consists of some holomorphic functions defined on neighborhoods of the closure of $\mathbb{D}^n$, and $\mathcal{R}$ contains $\mathcal{C}$, then, how are these related to the above problems? In order to proceed with our discussion about the answers to the above problems, some basic concepts and terminologies are introduceed in Section 2. We give the basic analysis for ideals of polynomials, as carried out in Section 2, as preliminaries for Sections 3 and 4. In Section 3 we give complete answers to problems (A), (B), (C), (D) in the case of polynomials. In Section 4, we proceed to discuss the case of a Noetherian ring $\mathcal{R}$. Since a

Noetherian ring $\mathcal{R}$ has the same algebraic properties as the ring of polynomials, we essentially obtain similar results to those of Section 3. Finally, we point out that the techniques in this paper are also available for Hardy submodules and Bergman submodules on a bounded connected domain Ω which satisfies some technical conditions; for exemple, we may assume that all polynomials are dense in the Hardy module $H^2(\Omega)$ (Bergman module $L^2_{\mathrm{a}}(\Omega)$), and for $\lambda \notin \Omega$, there exists a polynomial q such that $q(\lambda) = 1$, and $|q(z)| < 1$ for all $z \in \overline{\Omega} \setminus \{\lambda\}$ $(z \in \Omega)$.

2. ANALYSIS OF IDEALS OF POLYNOMIALS

Let $q = \sum a_{m_1 \cdots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ be in $\mathcal{C}$, and let $q(D)$ denote the linear partial differential operator $\sum a_{m_1 \cdots m_n} (\partial^{m_1+m_2+\cdots+m_n} / \partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n})$. If I is an ideal of $\mathcal{C}$ and $\lambda \in \mathbb{C}^n$, set

$$I_\lambda = \{q \in \mathcal{C} \mid q(D)f|_\lambda = 0, \, \forall f \in I\}$$

where $q(D)f|_\lambda$ denotes $(q(D)f)(\lambda)$. From the Leibniz rule, for any polynomial q and any analytic function f, the following holds

$$q(D)(z_j f)|_\lambda = \lambda_j q(D) f|_\lambda + \frac{\partial q}{\partial z_j}(D) f|_\lambda \qquad j = 1, 2, \ldots, n.$$

We thus conclude that I_λ is invariant under the action of the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, \ldots, \partial/\partial z_n\}$, and I_λ is called the characteristic space of I at λ. Clearly, if $\lambda \in Z(I)$, then $1 \in I_\lambda$, and if $\lambda \notin Z(I)$, then $I_\lambda = 0$, where $Z(I)$ is the zero variety of I, that is, $Z(I) = \{z \in \mathbb{C}^n \mid f(z) = 0, \, \forall f \in I\}$. Also arising from the observation for polynomials of one variable, we define the multiplicity of I at λ by the dimension of its characteristic space, $\dim I_\lambda$. Of course, we allow the case where the multiplicity is infinite. Let I_1, I_2 be ideals of polynomials, and $\lambda \in \mathbb{C}^n$. We say that I_1 and I_2 have the same mulitiplicity at λ if $I_{1\lambda} = I_{2\lambda}$, and we use the symbol $Z(I_2) \setminus Z(I_1)$ to denote the set $\{\lambda \in Z(I_2) \mid I_{2\lambda} \neq I_{1\lambda}\}$. If $I_1 \supseteq I_2$, and $\lambda \in Z(I_2) \setminus Z(I_1)$, the mulitiplicity of I_2 related to I_1 at λ is defined by $\dim I_{2\lambda}/I_{1\lambda}$. In this way, the cardinality of $Z(I_2) \setminus Z(I_1)$ is defined by $\sum\limits_{\lambda \in Z(I_2) \setminus Z(I_1)} \dim I_{2\lambda}/I_{1\lambda}$ (counting multiplicity), and is denoted by $\operatorname{card}(Z(I_2) \setminus Z(I_1))$.

Theorem 2.1. *Let I_1, I_2 be ideals in $\mathcal{C}$. If for any $\lambda \in \mathbb{C}^n$, $I_{1\lambda} = I_{2\lambda}$, then $I_1 = I_2$.*

Proof. There are three main steps in the proof of Theorem 2.1. Firstly, for every finite codimensional ideal O in $\mathcal{C}$, we claim that $I_1 + O = I_2 + O$.

Obviously, the finite codimensional ideals $I_1 + O$, $I_2 + O$ have the same zero set. Let $\lambda \in Z(I_1 + O)$ and $q \in (I_1 + O)_\lambda$; then $q \in I_{1\lambda}$, and $q \in O_\lambda$. It follows that q is in $I_{2\lambda}$ by the assumption. We thus obtain that q is in $(I_2 + O)_\lambda$, that is, $(I_1 + O)_\lambda \subseteq (I_2 + O)_\lambda$. Similarly, we have that $(I_2 + O)_\lambda \subseteq (I_1 + O)_\lambda$. That is, they have the same multiplicity at every zero point. Some basic analysis for ideals of polynomials implies that $I_1 + O = I_2 + O$.

Secondly, set

$$I = \bigcap_O (I_1 + O) = \bigcap_O (I_2 + O)$$

where O runs over all finite codimensional ideals in $\mathcal{C}$. Then $I \supseteq I_1$ and $Z(I) = Z(I_1)$. We also claim that for every $\lambda \in Z(I)$, $I_\lambda = I_{1\lambda}$. In fact, it is obvious that $I_\lambda \subseteq I_{1\lambda}$. Let p be in $I_{1\lambda}$ and $p \neq 0$. Use $\mathcal{P}$ to denote the linear space generated by p which is invariant under the action by $\{\partial/\partial z_1, \partial/\partial z_2, \ldots, \partial/\partial z_n\}$. Put

$$O_{\mathcal{P}} = \{q \in \mathcal{C} \mid f(D)q|_\lambda = 0, \forall f \in \mathcal{P}\}.$$

From the Leibniz rule, we see that $O_{\mathcal{P}}$ is an ideal in $\mathcal{C}$, and is finite codimensional because $\mathcal{P}$ is finite dimensional. For any $g \in I$, write $g = g_1 + g_2$, where $g_1 \in I_1$ and $g_2 \in O_{\mathcal{P}}$. We see that $p(D)g|_\lambda = p(D)g_1|_\lambda + p(D)g_2|_\lambda = 0$. Thus p is in I_λ. This shows that $I_{1\lambda} = I_\lambda$.

Finally, our task is to prove that $I = I_1$ under the assumption that $I \supseteq I_1$ and $I_\lambda = I_{1\lambda}$ for every $\lambda \in \mathbb{C}^n$. Below, the technique which we use is essentially due to Douglas and Paulsen ([7], Chapter 6). For each $\lambda \in \mathbb{C}^n$, use $\mathcal{U}_\lambda$ to denote the maximal ideal of polynomials that vanish at λ, that is, $\mathcal{U}_\lambda = \{q \in \mathcal{C} \mid q(\lambda) = 0\}$. Obviously,

$$I = I_1 + \mathcal{U}_\lambda^j \cap I$$

for all $j \geqslant 1$ because $\mathcal{U}_\lambda^j$ is finite codimensional. By the Artin-Rees Lemma ([3], Corollary 10.10), there is an integer k such that for $j \geqslant k$, $\mathcal{U}_\lambda^j \cap I = \mathcal{U}_\lambda^{j-k}(\mathcal{U}_\lambda^k \cap I)$. Setting $j = k+1$, we have that $I = I_1 + \mathcal{U}_\lambda I$ for all $\lambda \in \mathbb{C}^n$. Using the method of localization ([3], Chapter 3), let $S_\lambda = \mathcal{C} \setminus \mathcal{U}_\lambda$; then S_λ is a multipliciatively closed set. Consider the quotient ring $S_\lambda^{-1}(\mathcal{C}) = \{p/q \mid p \in \mathcal{C}, q \in S_\lambda\}$. Since $S_\lambda^{-1}(\mathcal{U}_\lambda)$ is the unique maximal ideal in $S_\lambda^{-1}(\mathcal{C})$, and $S_\lambda^{-1}(I) = S_\lambda^{-1}(I_1) + S_\lambda^{-1}(\mathcal{U}_\lambda) S_\lambda^{-1}(I)$, we can apply Nakayama Lemma ([3], Corollary 2.7) to deduce that $S_\lambda^{-1}(I) = S_\lambda^{-1}(I_1)$ for any $\lambda \in \mathbb{C}^n$. Let $I_1 = \bigcap_{j=1}^m I_j'$ be an irredundant primary decomposition of I_1, where I_j' is a primary ideal with associated prime ideal P_j ([3], Chapter 4). For every $\lambda \in Z(P_j)$, from [3], Proposition 4.8, we have that

$$I \subseteq S_\lambda^{-1}(I_1) \cap \mathcal{C} \subseteq S_\lambda^{-1}(I_j') \cap \mathcal{C} = I_j'$$

for $j = 1, \dots, m$. This leads to $I \subseteq \bigcap_{j=1}^{m} I'_j = I_1$. It follows that $I = I_1$. Similarly we have that $I = I_2$. We thus conclude that $I_1 = I_2$. The proof of Theorem 2.1 is complete. ∎

Notice that the proof of Theorem 2.1 yields the following corollary.

COROLLARY 2.2. *Let I be an ideal of polynomials. Then I is equal to the intersection of all finite codimensional ideals containing I.*

From the proof of Theorem 2.1, we can obtain the following important conclusion.

COROLLARY 2.3. *Let I_1, I_2 be ideals in $\mathcal{C}$, and $I_1 \supseteq I_2$. If the set $Z(I_2) \backslash Z(I_1)$ is bounded, then $\dim I_1/I_2 < \infty$, that is, I_2 is finite codimensional in I_1.*

Proof. Let $I_2 = \bigcap_{j=1}^{m} I'_j$ be an irredundant primary decomposition of I_2, where I'_j is a primary ideal with associated prime ideal P_j. We can assume that $I'_1, \dots, I'_{m_0}$ are finite codimensiomal, and $I'_{m_0+1}, \dots, I'_m$ are infinite codimensional. Since $Z(P_j) = Z(I'_j)$, $j = 1, \dots, m$, we see that $Z(I_2) = \bigcup_{j=1}^{m} Z(P_j)$. It is well known that $\bigcup_{j=1}^{m_0} Z(P_j)$ is bounded, and $Z(P_j)$ is unbounded for $j \geqslant m_0 + 1$. Let $\bigcup_{j=1}^{m_0} Z(P_j)$ and $Z(I_2) \setminus Z(I_1)$ are contained in the ball $B_l = \left\{(z_1, \dots, z_n) \mid \sum_{i=1}^{n} |z_i|^2 \leqslant l^2\right\}$, where l is some positive real number. Let $j \geqslant m_0 + 1$ and s be any natural number. Then for every $\lambda \in Z(P_j) \cap (\mathbb{C}^n \setminus B_l)$, it is easy to see that finite codimensional ideal $I_2 + \mathcal{U}_\lambda^s$ has only a zero point λ, and

$$I_2 + \mathcal{U}_\lambda^s = \{q \in \mathcal{C} \mid p(D)q|_\lambda = 0,\, p \in (I_2 + \mathcal{U}_\lambda^s)_\lambda\}.$$

Since for such λ,

$$(I_2 + \mathcal{U}_\lambda^s)_\lambda \subseteq I_{2\lambda} = I_{1\lambda}$$

it follows that

$$I_1 \subseteq I_2 + \mathcal{U}_\lambda^s$$

that is, $I_1 = I_2 + \mathcal{U}_\lambda^s \cap I_1$. Using the last part of the proof of Theorem 2.1, we obtain that $I_1 \subseteq I'_j$, $j \geqslant m_0 + 1$. This implies that $I_1 \subseteq \bigcap_{j=m_0+1}^{m} I'_j$. Since $\bigcap_{j=1}^{m_0} I'_j$ is finite codimensional and $\bigcap_{j=m_0+1}^{m} I'_j$ is finitely generated, it follows that $\bigcap_{j=1}^{m} I'_j$ is finite codimensional in $\bigcap_{j=m_0+1}^{m} I'_j$. We conclude thus that $I_2 \left(= \bigcap_{j=1}^{m} I'_j\right)$ is finite codimensional in I_1. This completes the proof of Corollary 2.3. ∎

Now suppose that $I_1 \supseteq I_2$. Let $\{M_{z_1}, M_{z_2}, \dots, M_{z_n}\}$ be the n-tuple of operators which are defined on the quotient ring I_1/I_2 by $M_{z_i}\widetilde{f} = \widetilde{(z_i f)}$ for $i = 1, \dots, n$. We use $\sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$ to denote the joint eigenvalues for $\{M_{z_1}, M_{z_2}, \dots, M_{z_n}\}$, that is, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$ if and only if there exists a $q \in I_1$ and $q \notin I_2$ such that $(z_i - \lambda_i)q \in I_2$, $i = 1, 2, \dots, n$.

THEOREM 2.4. *Let I_1, I_2 be ideals in $\mathcal{C}$, $I_1 \supseteq I_2$ and $\dim I_1/I_2 = k < \infty$. Then we have:*

(i) $Z(I_2) \setminus Z(I_1) = \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$;

(ii) $I_2 = \{q \in I_1 \mid p(D)q|_\lambda = 0,\ p \in I_{2\lambda},\ \lambda \in Z(I_2) \setminus Z(I_1)\}$;

(iii) $\dim I_1/I_2 = \sum_{\lambda \in Z(I_2) \setminus Z(I_1)} \dim I_{2\lambda}/I_{1\lambda} = \operatorname{card}(Z(I_2) \setminus Z(I_1))$.

It is worth noticing that (iii) of Theorem 2.4 says that the codimension $\dim I_1/I_2$ of I_2 in I_1 is equal to the cardinality of zeros of I_2 related to I_1. In this way, the equality (iii) is an interesting codimension formula whose left side is an algebraic invariant, while the right side is a geometric invariant.

Proof. (i) Write

$$I_1 = I_2 \dot{+} R$$

where R is a linear space of polynomials with $\dim R = \dim I_1/I_2$. We may regard $\{M_{z_1}, M_{z_2}, \dots, M_{z_n}\}$ being defined on R by $M_{z_i}q =$ decomposition of $z_i q$ on R for any $q \in R$. By [6], they can be simultaneously triangularized as

$$M_{z_i} = \begin{pmatrix} \lambda_i^{(1)} & & * \\ & \ddots & \\ & & \lambda_i^{(k)} \end{pmatrix};$$

here $i = 1, 2, \dots, n$, and $k = \dim I_1/I_2$, so that $\sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$ is equal to $\{\lambda^{(1)}, \dots, \lambda^{(k)}\}$. Then we have

$$\mathcal{U}_{\lambda^{(k)}} \mathcal{U}_{\lambda^{(k-1)}} \cdots \mathcal{U}_{\lambda^{(1)}} I_1 \subseteq I_2 \subseteq I_1.$$

This implies that $Z(I_2) \setminus Z(I_1) \subseteq \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$. Let $\lambda \in \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$. Since λ is a joint eigenvalue of $\{M_{z_1}, M_{z_2}, \dots, M_{z_n}\}$, there is a polynomial q in R such that $q\mathcal{U}_\lambda \subseteq I_2$. Defining $I_2^\dagger$ to be the ideal generated by I_2 and q, then for any $\lambda' \neq \lambda$, there holds that $(I_2^\dagger)_{\lambda'} = I_{2\lambda'}$. Therefore, by Theorem 2.1, we have that $(I_2^\dagger)_\lambda \subsetneqq I_{2\lambda}$. Since $(I_2^\dagger)_\lambda \supseteq I_{1\lambda}$, we see that $I_{1\lambda} \subsetneqq I_{2\lambda}$, that is, λ is in $Z(I_2) \setminus Z(I_1)$. Combining the above discussion, we thus conclude that $Z(I_2) \setminus Z(I_1) = \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n})$. This completes the proof of (i).

(ii) Let $I_2^\natural = \{q \in I_1 \mid p(D)q|_\lambda = 0,\, p \in I_{2\lambda}, \lambda \in Z(I_2) \setminus Z(I_1)\}$. Then $I_2^\natural$ is an ideal which contains I_2. It follows that $(I_2^\natural)_\lambda \subseteq I_{2\lambda}$ for all $\lambda \in \mathbb{C}^n$. By the representation of $I^\natural$, we have that $(I_2^\natural)_\lambda \supseteq I_{2\lambda}$ for all $\lambda \in \mathbb{C}^n$. According to Theorem 2.1, we obtain that $I_2^\natural = I_2$. The proof of (ii) is complete.

(iii) The proof is by induction on number of points in $Z(I_2) \setminus Z(I_1)$. If $Z(I_2) \setminus Z(I_1)$ contains only a point λ, then by (ii) I_2 can be written as

$$I_2 = \{q \in I_1 \mid p(D)q|_\lambda = 0,\, p \in I_{2\lambda}\}.$$

We define the pairing

$$[-,-] : I_{2\lambda}/I_{1\lambda} \times I_1/I_2 \to \mathbb{C}$$

by $[\widetilde{p}, \widetilde{q}] = p(D)q|_\lambda$. Clearly, this is well-defined. From this pairing and the representation of I_2, it is not difficult to see that $\dim I_1/I_2 = \dim I_{2\lambda}/I_{1\lambda} = \operatorname{card}(Z(I_2) \setminus Z(I_1))$.

Now let $l > 1$, and assume that (iii) has been proved for $Z(I_2) \setminus Z(I_1)$ containing different points less than l. Let $Z(I_2) \setminus Z(I_1) = \{\lambda_1, \ldots, \lambda_l\}$ here $\lambda_i \neq \lambda_j$ for $i \neq j$. Writting

$$I_2^\star = \{q \in I_1 \mid p(D)q|_{\lambda_1} = 0,\, p \in I_{2\lambda_1}\}$$

then $I_2^\star$ is an ideal, and $(I_2^\star)_{\lambda_1} = I_{2\lambda_1}$. Similarly to the preceding proof, we have that

$$\dim I_1/I_2^\star = \dim I_{2\lambda_1}/I_{1\lambda_1}.$$

Set $I_{2\lambda_1} = I_{1\lambda_1} \dot{+} R$ with $\dim R = \dim I_{2\lambda_1}/I_{1\lambda_1}$. Let $\sharp R$ denote the linear space of polynomials generated by R which is invariant under the action of $\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}$. Put

$$\mathcal{Q}_\mathcal{R} = \{p \in \mathcal{C} \mid q(D)p|_{\lambda_1} = 0,\, q \in \sharp R\}.$$

Then it is easily verified that $\mathcal{Q}_\mathcal{R}$ is a finite codimensional ideal of $\mathcal{C}$ with only zero point λ_1 because $\sharp R$ is finite dimensional. Thus

$$\mathcal{Q}_\mathcal{R} I_1 \subseteq I_2^\star \subseteq I_1.$$

From the above relation, we see that for $\lambda \neq \lambda_1$, $I_{1\lambda} = (I_2^\star)_\lambda$. Therefore

$$Z(I_2) \setminus Z(I_2^\star) = \{\lambda_2, \ldots, \lambda_l\}.$$

By the induction hypothesis, we have

$$\dim I_2^\star/I_2 = \sum_{j=2}^{l} \dim I_{2\lambda_j}/(I_2^\star)_{\lambda_j} = \sum_{j=2}^{l} \dim I_{2\lambda_j}/I_{1\lambda_j}.$$

It follows that we obtain

$$\dim I_1/I_2 = \dim I_1/I_2^\star + \dim I_2^\star/I_2 = \sum_{j=1}^{l} \dim I_{2\lambda_j}/I_{1\lambda_j} = \operatorname{card}(Z(I_2) \setminus Z(I_1)).$$

The proof of Theorem 2.4 is thus completed. ∎

From Theorem 2.4 and Corollary 2.3 we have

COROLLARY 2.5. *Let I_1, I_2 be ideals in $\mathcal{C}$, and $I_1 \supseteq I_2$. Then I_2 is finite codimensional in I_1 if and only if $Z(I_2) \setminus Z(I_1)$ is bounded, if and only if $Z(I_2) \setminus Z(I_1)$ is finite, and codimension* $\dim I_2/I_1 = \operatorname{card}(Z(I_2) \setminus Z(I_1))$.

3. ALGEBRAIC REDUCTION FOR HARDY SUBMODULES

In this section we will completely answer the problems raised in Section 1. Similarly to Section 2, the following concepts are useful. Let M be a submodule of $H^2(\mathbb{D}^n)$ and let $\lambda \in \mathbb{D}^n$; set

$$M_\lambda = \{q \in \mathcal{C} \mid q(D)f|_\lambda = 0, \forall f \in M\}.$$

Then M_λ is invariant under the action of the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, \dots, \partial/\partial z_n\}$, and M_λ is called the characteristic space of M at λ. Clearly, if $\lambda \in Z(M)$, then $1 \in M_\lambda$, and if $\lambda \notin Z(M)$, then $M_\lambda = 0$, where $Z(M)$ is zero set of M, that is, $Z(M) = \{z \in \mathbb{D}^n \mid f(z) = 0, \forall f \in M\}$. We define the multiplicity of M at λ by the dimension of characteristic space, $\dim M_\lambda$. Let M_1, M_2 be Hardy submodules and $\lambda \in \mathbb{D}^n$. We say that M_1 and M_2 have the same multiplicity at λ if $M_{1\lambda} = M_{2\lambda}$. The symbol $Z(M_2) \setminus Z(M_1)$ denotes the set $\{\lambda \in Z(M_2) \mid M_{2\lambda} \neq M_{1\lambda}\}$. If $M_1 \supseteq M_2$ and $\lambda \in Z(M_2) \setminus Z(M_1)$, the multiplicity of M_2 related to M_1 at λ is defined by $\dim M_{2\lambda}/M_{1\lambda}$. In this way, the cardinality of $Z(M_2) \setminus Z(M_1)$ is defined by $\sum\limits_{\lambda \in Z(M_2)\setminus Z(M_1)} \dim M_{2\lambda}/M_{1\lambda}$ by counting multiplicity, and is denoted by $\operatorname{card}(Z(M_2) \setminus Z(M_1))$. The following is our main result in this section.

THEOREM 3.1. *Suppose M_2 is finite codimensional in M_1 and $M_1 \cap \mathcal{C}$ is dense in M_1. Then we have:*

(i) *$M_2 \cap \mathcal{C}$ is dense in M_2;*

(ii) *The canonical homomorphism $\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$ is an isomorphism;*

(iii) $Z(M_2 \cap \mathcal{C}) \setminus Z(M_1 \cap \mathcal{C}) = Z(M_2) \setminus Z(M_1) = \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \dots, M_{z_n}) \subset \mathbb{D}^n$;

(iv) $M_2 = \{f \in M_1 \mid p(D)f|_\lambda = 0, p \in M_{2\lambda}, \lambda \in Z(M_2) \setminus Z(M_1)\}$;

(v) $\dim M_1/M_2 = \dim M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} = \operatorname{card}(Z(M_2 \cap \mathcal{C}) \setminus Z(M_1 \cap \mathcal{C})) = \operatorname{card}(Z(M_2) \setminus Z(M_1))$.

Conversely, let I_1, I_2 be ideals in $\mathcal{C}, I_1 \supseteq I_2$ and $Z(I_2) \setminus Z(I_1) \subset \mathbb{D}^n$. Then $\dim[I_1]/[I_2] = \dim I_1/I_2$, *that is, the canonical homomorphism $\tau : I_1/I_2 \to [I_1]/[I_2]$ is an isomorphism.*

Proof. Firstly, we claim that M_1 can be expressed as

$$M_1 = M_2 \dot{+} R$$

where R is a linear space of polynomials with $\dim R = \dim M_1/M_2$. In fact, since $M_1 \cap \mathcal{C}$ is dense in M_1, there exists a polynomial q in $M_1 \cap \mathcal{C}$, $q \notin M_2$. Let Σ be the collection $\{L \mid L$ is a linear space of polynomials, $L \subseteq M_1 \cap \mathcal{C}$, and $L \cap M_2 = \{0\}\}$. We thus see that Σ is not empty. If $\cdots \subseteq \Phi_\alpha \subseteq \Phi_\beta \subseteq \Phi_\gamma \subseteq \cdots$ is an ascending chain in Σ, then $\bigcup_\alpha \Phi_\alpha$ is a linear space of polynomials, and $\bigcup_\alpha \Phi_\alpha$ is in Σ. It follows that there exists a maximal element R in Σ such that $M_2 \cap R = \{0\}$. Since $M_2 \dot{+} R \subseteq M_1$, R is finite dimensional. So $M_2 \dot{+} R$ is closed. If $M_2 \dot{+} R \neq M_1$, then there is a polynomial $p \in M_1$, such that $p \notin M_2 \dot{+} R$. This induces that the linear space $\{R, p\}$ generated by R and p satisfies that $\{R, p\} \cap M_2 = 0$. This is impossible. Therefore, we conclude that $M_1 = M_2 \dot{+} R$ with $\dim R = \dim M_1/M_2$. From this assertion we immediately obtain

$$M_1 \cap \mathcal{C} = M_2 \cap \mathcal{C} \dot{+} R.$$

The above argument tells us that $M_2 \cap \mathcal{C}$ is dense in M_2, and the canonical homomorphism $\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$ is an isomorphism. This completes the proof of (i) and (ii).

To prove (iii), pick any $\lambda \in Z(M_1 \cap \mathcal{C}) \setminus Z(M_2 \cap \mathcal{C})$. By Theorem 2.4 (i), we see that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a joint eigenvalue of $\{M_{z_1}, M_{z_2}, \ldots, M_{z_n}\}$ on $M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C}$, that is, there is $q \in M_1 \cap \mathcal{C}$, $q \notin M_2 \cap \mathcal{C}$ such that $(z_i - \lambda_i)q \in M_2 \cap \mathcal{C}$ for $i = 1, 2, \ldots, n$. If for some i $|\lambda_i| \geqslant 1$, then $z_i - \lambda_i$ is an outer function of one variable. It follows that there exist polynomials $\{q_n(z_i)\}_n$ such that $q_n(z_i)(z_i - \lambda_i)$ converges to 1 as $n \to \infty$. This implies that $q \in \overline{M_2 \cap \mathcal{C}} = M_2$. Thus $q \in M_2 \cap \mathcal{C}$. This contradiction says that $|\lambda_i| < 1$, $i = 1, 2, \ldots, n$. So

$$Z(M_2 \cap \mathcal{C}) \setminus Z(M_1 \cap \mathcal{C}) \subset \mathbb{D}^n.$$

By (i), (ii) and Theorem 2.4 (i), we immediately obtain

$$Z(M_2 \cap \mathcal{C}) \setminus Z(M_1 \cap \mathcal{C}) = Z(M_2) \setminus Z(M_1) = \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \ldots, M_{z_n}) \subset \mathbb{D}^n.$$

The proof of (iii) is complete.

We notice that (iv) is from (i) and Theorem 2.4 (ii), and (v) is from (ii), (iii) and Theorem 2.4 (iii).

Let I_1, I_2 be ideals in $\mathcal{C}$, with $I_1 \supseteq I_2$ and $Z(I_2) \setminus Z(I_1) \subset \mathbb{D}^n$. Then by Corollary 2.3, I_2 is finite codimensional in I_1, and by Theorem 2.4 (ii),

$$I_2 = \{q \in I_1 \mid p(D)q|_\lambda = 0,\, p \in I_{2\lambda},\, \lambda \in Z(I_2) \setminus Z(I_1) \subset \mathbb{D}^n\}.$$

Set

$$I_1 = I_2 \dot{+} R$$

where R is a linear space of polynomials with $\dim R = \dim I_1/I_2$. Since each function f in $[I_2]$ satisfies that $p(D)f|_\lambda = 0$, $p \in I_{2\lambda}$, $\lambda \in Z(I_2) \setminus Z(I_1)$, it follows that $[I_2] \cap R = \{0\}$. By the fact that $[I_2] \dot{+} R$ contains I_1, and $[I_2] \dot{+} R$ is closed, we obtain that $[I_2] \dot{+} R = [I_1]$. Therefore, it holds that $I_1/I_2 \cong [I_1]/[I_2] \cong R$. The proof of Theorem 3.1 is complete. ∎

4. SOME FURTHER RESULTS AND REMARKS

We denote by $\mathrm{Hol}\,(\overline{\mathbb{D}}^n)$ the ring of holomorphic functions defined on neighborhoods of the closure of $\mathbb{D}^n$. Now let $\mathcal{R}$ be a Noetherian subring of $\mathrm{Hol}\,(\overline{\mathbb{D}}^n)$ containing $\mathcal{C}$. For exemple, the ring of all rational functions with poles off the closure $\overline{\mathbb{D}}^n$ of $\mathbb{D}^n$ is such a ring ([8]). Since the rings $\mathcal{R}$ and $\mathcal{C}$ have the same algebraic properties, the techniques in Sections 2 and 3 are also completely available in the case of the Noetherian ring $\mathcal{R}$. Let I be an ideal of $\mathcal{R}$. We use $Z(I)$ to denote $\{z \in \overline{\mathbb{D}}^n \mid f(z) = 0, \forall f \in I\}$. Let $I_1 \supseteq I_2$, the definitions of $Z(I_2) \setminus Z(I_1)$ and $\mathrm{card}\,(Z(I_2) \setminus Z(I_1))$ are completely similar to that of Sections 2 and 3. We thus have the following theorem.

THEOREM 4.1. *Suppose M_2 is finite codimensional in M_1 and $M_1 \cap \mathcal{R}$ is dense in M_1. Then we have:*

(i) *$M_2 \cap \mathcal{R}$ is dense in M_2;*

(ii) *the canonical homomorphism $\tau : M_1 \cap \mathcal{R}/M_2 \cap \mathcal{R} \to M_1/M_2$ is an isomorphism;*

(iii) *$Z(M_2 \cap \mathcal{R}) \setminus Z(M_1 \cap \mathcal{R}) = Z(M_2) \setminus Z(M_1) = \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \ldots, M_{z_n}) \subset \mathbb{D}^n$;*

(iv) *$M_2 = \{f \in M_1 \mid p(D)f|_\lambda = 0,\, p \in M_{2\lambda},\, \lambda \in Z(M_2) \setminus Z(M_1)\}$;*

(v) $\dim M_1/M_2 = \dim M_1 \cap \mathcal{R}/M_2 \cap \mathcal{R} = \mathrm{card}\,(Z(M_2 \cap \mathcal{R}) \setminus Z(M_1 \cap \mathcal{R})) = \mathrm{card}\,(Z(M_2) \setminus Z(M_1))$.

Conversely, if I_1, I_2 are ideals in $\mathcal{R}$, $I_1 \supseteq I_2$ and $Z(I_2) \setminus Z(I_1) \subset \mathbb{D}^n$. Then $\dim [I_1]/[I_2] = \dim I_1/I_2$, *that is, the canonical homomorphism $\tau : I_1/I_2 \to [I_1]/[I_2]$ is an isomorphism.*

Let $\mathcal{Q}$ be a subring of $\mathrm{Hol}(\mathbb{D}^n)$ containing $\mathcal{C}$, where $\mathrm{Hol}(\mathbb{D}^n)$ is the ring of all holomorphic functions on $\mathbb{D}^n$. Using the techniques in this paper, we can prove the following theorem.

THEOREM 4.2. *Suppose M_2 is finite codimensional in M_1, and $M_1 \cap \mathcal{Q}$ is dense in M_1. Then we have:*

(i) *$M_2 \cap \mathcal{Q}$ is dense in M_2;*

(ii) *the canonical homomorphism $\tau : M_1 \cap \mathcal{Q}/M_2 \cap \mathcal{Q} \to M_1/M_2$ is an isomorphism;*

(iii) *$Z(M_2 \cap \mathcal{Q}) \cap \mathbb{D}^n \setminus Z(M_1 \cap \mathcal{Q}) \cap \mathbb{D}^n = Z(M_2) \setminus Z(M_1) \subseteq \sigma_{\mathrm{p}}(M_{z_1}, M_{z_2}, \ldots, M_{z_n}) \subset \mathbb{D}^n$;*

(iv) *$M_2 \subseteq \{f \in M_1 \mid p(D)f|_\lambda = 0,\, p \in M_{2\lambda},\, \lambda \in Z(M_2) \setminus Z(M_1)\}$;*

(v) *$\dim M_1/M_2 = \dim M_1 \cap \mathcal{Q}/M_2 \cap \mathcal{Q} \geqslant \mathrm{card}\,(Z(M_2 \cap \mathcal{Q}) \cap \mathbb{D}^n \setminus Z(M_1 \cap \mathcal{Q}) \cap \mathbb{D}^n) = \mathrm{card}\,(Z(M_2) \setminus Z(M_1))$.*

Finally, we point out that techniques in this paper are also available for Hardy submodules and Bergman submodules on bounded connected domain Ω which satisfies some additional conditions, for exemple, we may assume that all polynomials are dense in Hardy module $H^2(\Omega)$ (Bergman module $L^2_a(\Omega)$), and for $\lambda \notin \Omega$, there exists a polynomial q such that $q(\lambda) = 1$, and $|q(z)| < 1$ for all $z \in \overline{\Omega} \setminus \{\lambda\}$ ($z \in \Omega$). Furthermore, if a reproducing Hilbert $A(\Omega)$-module H satisfies certain technical hypotheses, then the results in this paper extend with essentially no change to H.

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