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THE STABLE RANK OF TENSOR PRODUCTS OF FREE PRODUCT C^*-ALGEBRAS

KENNETH J. DYKEMA

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Abstract. Let A be the minimal tensor product of C^*-algebras, $A^{(j)}$, which are reduced free products with respect to traces of C^*-algebras that are not too small in a specific sense. Then the stable rank of A is 1.

Keywords: *Stable rank, tensor product, free product.*

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1. INTRODUCTION

The (topological) stable rank, $\mathrm{sr}(A)$, of a Banach algebra, A, was invented by Rieffel ([4]) and is intimately related to "non-stable" K-theory. The case $\mathrm{sr}(A) = 1$ has been of particular interest; by definition, $\mathrm{sr}(A) = 1$ if and only if the invertible elements of A are dense in A. Recently, Villadsen ([6]) constructed the first examples of finite, simple C^*-algebras whose stable rank is greater than 1.

In [3], it was shown that if

$$(A, \tau) = (A_1, \tau_1) * (A_2, \tau_2)$$

is the reduced free product of C^*-algebras with respect to traces τ_1 and τ_2, then $\mathrm{sr}(A) = 1$, provided that the Avitzour conditions are satisfied, namely, that there are unitaries $x \in A_1$ and $y, z \in A_2$ such that

$$\tau_1(x) = 0 = \tau_2(y) = \tau_2(z) = \tau_2(z^*y).$$

(See [7], [8] and [1] for the definition of reduced free products.) In [2], more classes of reduced free products were shown to have stable rank 1.

It should be mentioned that it is not known if it is possible to find out about $\mathrm{sr}(A \otimes B)$ for simple C^*-algebras knowing only $\mathrm{sr}(A)$ and $\mathrm{sr}(B)$, or even knowing $\mathrm{sr}(A) = 1 = \mathrm{sr}(B)$. In this note, we show that minimal tensor products of reduced free product C^*-algebras have stable rank 1, provided that the Avitzour conditions are satisfied in each free product. The proof is a generalization of the proof of [3], 3.8.

2. ON TENSOR PRODUCTS OF FREE PRODUCTS

Consider a C^*-algebra, A, which is a minimal tensor product,

$$A = \bigotimes_{j \in J} A^{(j)},$$

of C^*-algebras $A^{(j)}$ which are in turn reduced free products of C^*-algebras with respect to tracial states,

$$(A^{(j)}, \tau^{(j)}) = \mathop{*}_{\iota \in I^{(j)}} (A_\iota^{(j)}, \tau_\iota^{(j)}). \tag{2.1}$$

We also let τ be the tensor product trace on A,

$$\tau = \bigotimes_{j \in J} \tau^{(j)},$$

and we work with the inner product $\langle c, d \rangle = \tau(d^* c)$ on A. Here J is nonempty and each $I^{(j)}$ is a set with at least two elements.

Let $X_\iota^{(j)}$ be a standard orthonormal basis for $(A_\iota^{(j)}, \tau_\iota^{(j)})$ and let

$$Y^{(j)} = \mathop{*}_{\iota \in I^{(j)}} X_\iota^{(j)}.$$

(See [3], Section 2 for definitions.) Thus, $Y = \bigcup_{k=0}^{\infty} Y_k^{(j)}$ where for $k \geqslant 1$, $Y_k^{(j)}$ is the set of reduced words in the family $\big((X_\iota^{(j)})^{\mathrm{o}}\big)_{\iota \in I^{(j)}}$ having length k, while $Y_0^{(j)} = \{1\}$. Let $E_k^{(j)}$ denote the orthogonal projection of $\operatorname{span} Y^{(j)}$ onto $\operatorname{span} Y_k^{(j)}$. Let

$$K = \{k : J \to \mathbb{N} \cup \{0\} \mid k(j) = 0 \text{ for all but finitely many } j \in J\}.$$

Given $k \in K$, let

$$
\begin{aligned}
Y_k &= \Big\{ \bigotimes_{j\in J} v(j) \mid v(j) \in Y^{(j)}_{k(j)} \Big\} \\
Y &= \bigcup_{k\in K} Y_k.
\end{aligned} \tag{2.2}
$$

Then Y is a standard orthonormal basis for (A,τ). Let E_k denote the orthogonal projection of $\operatorname{span} Y$ onto $\operatorname{span} Y_k$. Given elements $v = \bigotimes_{j\in J} v(j)$ and $w = \bigotimes_{j\in J} w(j)$ of Y, we say that vw is *reduced* if, for each $j \in J$ the word $v(j)w(j)$ of $Y^{(j)}$ is reduced, i.e. $v(j)$ ends with an element of $(X^{(j)}_\iota)^{\text{o}}$ and $w(j)$ starts with an element of $(X^{(j)}_{\iota'})^{\text{o}}$ with $\iota \neq \iota'$.

Let $a \in \operatorname{span} Y$. We define the *support* of a to be the set of all $w \in Y$ such that $\langle w, a\rangle \neq 0$. Given $j_0 \in J$ and $\iota \in I^{(j_0)}$ let $F^{(j_0)}_\iota(a)$ be the set of all $x \in (X^{(j_0)}_\iota)^{\text{o}}$ such that there is $w = \bigotimes_{j\in J} w(j)$ in the support of a and with x appearing as a letter in $w(j_0)$. Note that $F^{(j_0)}_\iota(a)$ is always finite and is empty for all but finitely many pairs $(j_0,\iota) \in J \times \bigcup_{j\in J} I^{(j)}$. Let

$$
I = \Big\{ i : J \to \bigcup_{j\in J} I^{(j)} \mid i(j) \in I^{(j)} \text{ for every } j \in J \Big\}.
$$

Given $i \in I$ and a finite subset $J' \subseteq J$, let

$$
F^{(J')}_i(a) = \Big\{ x = \bigotimes_{j\in J} x(j) \mid x(j) \in F^{(j)}_{i(j)}(a) \text{ if } j \in J',\ x(j) = 1 \text{ if } j \notin J' \Big\}
$$

and let

$$
M^{(J')}_i(a) = \Bigg(\sum_{x \in F^{(J')}_i(a)} \|x\|^2 \Bigg)^{\frac{1}{2}},
$$

with the convention that $M^{(J')}_i(a) = 0$ if $F^{(J')}_i(a)$ is empty. Let

$$
M(a) = \max\{ M^{(J')}_i(a) \mid i \in I,\ J' \text{ a finite subset of } J\}.
$$

Note that $M(a) < \infty$.

LEMMA 2.1. *Let $k, l, n \in K$, let $a \in Y_k$ and $b \in Y_l$. If $n(j) < |k(j) - l(j)|$ or $n(j) > k(j) + l(j)$ for some $j \in J$ then $E_n(ab) = 0$. Otherwise*

$$
\|E_n(ab)\|_2 \leqslant M(a)\|a\|_2\|b\|_2.
$$

Proof. If $n(j_0) < |k(j_0) - l(j_0)|$ or $n(j_0) > k(j_0) + l(j_0)$ for some $j_0 \in J$ then for every $v = \bigotimes_{j\in J} v(j)$ in the support of a and every $w = \bigotimes_{j\in J} w(j)$ in the support of b we have $E^{(j_0)}_{n(j_0)}(v(j_0)w(j_0)) = 0$, so $E_n(ab) = 0$. Now suppose $|k(j) - l(j)| \leqslant n(j) \leqslant k(j) + l(j)$ for every $j \in J$. Let

$$\begin{aligned} J_{\mathrm{e}} &= \{j \in J \mid k(j) + l(j) - n(j) \text{ even}\} \\ J_{\mathrm{o}} &= \{j \in J \mid k(j) + l(j) - n(j) \text{ odd}\}. \end{aligned}$$

Let $q \in K$ be such that

$$k(j) + l(j) - n(j) = \begin{cases} 2q(j) & \text{if } j \in J_{\mathrm{e}}; \\ 2q(j) + 1 & \text{if } j \in J_{\mathrm{o}}. \end{cases}$$

Let $q' \in K$ be

$$q'(j) = \begin{cases} q(j) & \text{if } j \in J_{\mathrm{e}}; \\ q(j) + 1 & \text{if } j \in J_{\mathrm{o}}. \end{cases}$$

Let $k - q' \in K$ be $(k - q')(j) = k(j) - q'(j)$ and similarly for $l - q' \in K$. Given $i \in I$ and a finite subset J' of J, let

$$Z(i, J') = \Big\{x = \bigotimes_{j\in J} x(j) \mid x(j) \in (X^{(j)}_{i(j)})^{\mathrm{o}} \text{ if } j \in J',\ x(j) = 1 \text{ if } j \notin J'\Big\}.$$

Then we may write

$$\begin{aligned} a &= \sum_{i\in I} \sum_{v_1,x,v_2} \alpha_{v_1xv_2} v_1xv_2 \\ b &= \sum_{i\in I} \sum_{w_2,y,w_1} \beta_{w_2yw_1} w_2yw_1 \end{aligned}$$

where $\alpha_{v_1xv_2}, \beta_{w_2yw_1} \in \mathbb{C}$ and where the sums are over all $x, y \in Z(i, J_{\mathrm{o}})$ and all $v_1 \in Y_{k-q'}$, $v_2 \in Y_q$, $w_2 \in Y_q$ and $w_1 \in Y_{l-q'}$ such that $v_1xv_2 \in Y_k$ and $w_2yw_1 \in Y_l$. Then, writing $v_1 = \bigotimes_{j\in J} v_1(j)$, etc., we have

$$\begin{aligned} &E^{(j)}_{n(j)}(v_1(j)x(j)v_2(j)w_2(j)y(j)w_1(j)) \\ &\quad = \begin{cases} \langle v_2(j)w_2(j), 1\rangle v_1(j)w_1(j) & \text{if } j \in J_{\mathrm{e}}; \\ \displaystyle\sum_{u\in (X^{(j)}_{i(j)})^{\mathrm{o}}} \langle v_2(j)w_2(j), u\rangle v_1(j)uw_1(j) & \text{if } j \in J_{\mathrm{o}}. \end{cases} \end{aligned}$$

So

$$E_n(ab) = \sum_{v_1,w_1} \sum_{i\in I} \sum_{u} \left(\sum_{x,y} \sum_{v_2,w_2} \alpha_{v_1xv_2}\beta_{w_2yw_1} \langle v_2w_2, 1\rangle \langle xy, u\rangle \right) v_1uw_1,$$

where the sums are over all $v_1 \in Y_{k-q'}$, all $w_1 \in Y_{l-q'}$ and all $u \in Z(i, J_\mathrm{o})$ such that $v_1uw_1 \in Y_n$ and over all $x, y \in Z(i, J_\mathrm{o})$ and all $v_2, w_2 \in Y_q$ such that $v_1xv_2 \in Y_k$ and $w_2yw_1 \in Y_l$. Thus

$$\|E_n(ab)\|_2 = \sum_{v_1,w_1}\sum_{i\in I}\sum_{u}\left|\sum_{x,y}\sum_{v_2,w_2}\alpha_{v_1xv_2}\beta_{w_2yw_1}\langle v_2w_2,1\rangle\langle xy,u\rangle\right|^2.$$

For fixed v_1, w_1 and $i \in I$ set

$$z = \sum_{x,y\in Z(i,J_\mathrm{o})}\left\langle \sum_{w_2}\beta_{w_2yw_1}w_2, \sum_{v_2}\overline{\alpha_{v_1xv_2}}v_2^*\right\rangle xy.$$

Hence

$$\left|\sum_{x,y}\sum_{v_2,w_2}\alpha_{v_1xv_2}\beta_{w_2yw_1}\langle v_2w_2,1\rangle\langle xy,u\rangle\right|^2 = |\langle z,u\rangle|^2.$$

Now since $\alpha_{v_1xv_2} = 0$ if $x \notin F_i^{(J_\mathrm{o})}(a)$, we have

$$\begin{aligned}
\|z\|^2 &= \left\|\sum_{x\in F_i^{(J_\mathrm{o})}(a)} x \sum_{y\in Z(i,J_\mathrm{o})}\left\langle\sum_{w_2}\beta_{w_2yw_1}w_2,\sum_{v_2}\overline{\alpha_{v_1xv_2}}v_2^*\right\rangle y\right\|_2^2\\
&\leqslant \left(\sum_{x\in F_i^{(J_\mathrm{o})}(a)}\|x\|\cdot\left\|\sum_{y\in Z(i,J_\mathrm{o})}\left\langle\sum_{w_2}\beta_{w_2yw_1}w_2,\sum_{v_2}\overline{\alpha_{v_1xv_2}}v_2^*\right\rangle y\right\|_2\right)^2\\
&\leqslant \left(\sum_{x\in F_i^{(J_\mathrm{o})}(a)}\|x\|^2\right)\cdot\left(\sum_{x\in F_i^{(J_\mathrm{o})}(a)}\left\|\sum_{y\in Z(i,J_\mathrm{o})}\left\langle\sum_{w_2}\beta_{w_2yw_1}w_2,\sum_{v_2}\overline{\alpha_{v_1xv_2}}v_2^*\right\rangle y\right\|_2^2\right)\\
&\leqslant M(a)^2\sum_{x\in F_i^{(J_\mathrm{o})}(a)}\sum_{y\in Z(i,J_\mathrm{o})}\left|\left\langle\sum_{w_2}\beta_{w_2yw_1}w_2,\sum_{v_2}\overline{\alpha_{v_1xv_2}}v_2^*\right\rangle\right|^2\\
&\leqslant M(a)^2\sum_{x\in F_i^{(J_\mathrm{o})}(a)}\sum_{y\in Z(i,J_\mathrm{o})}\left\|\sum_{w_2}\beta_{w_2yw_1}w_2\right\|_2^2\cdot\left\|\sum_{v_2}\overline{\alpha_{v_1xv_2}}v_2^*\right\|_2^2\\
&= M(a)^2\sum_{x\in F_i^{(J_\mathrm{o})}(a)}\sum_{y\in Z(i,J_\mathrm{o})}\sum_{w_2}|\beta_{w_2yw_1}|^2\cdot\sum_{v_2}|\alpha_{v_1xv_2}|^2\\
&= M(a)^2\sum_{x,v_2}|\alpha_{v_1xv_2}|^2\cdot\sum_{w_2,y}|\beta_{w_2yw_1}|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{u\in Z(i,J_\mathrm{o})}\left|\sum_{x,y\in Z(i,J_\mathrm{o})}\sum_{v_2,w_2}\alpha_{v_1xv_2}\beta_{w_2yw_1}\langle v_2w_2,1\rangle\langle xy,u\rangle\right|^2\\
&\qquad = \sum_{u\in Z(i,J_\mathrm{o})}|\langle z,u\rangle|^2 \leqslant \|z\|_2^2 \leqslant M(a)^2\sum_{x,v_2}|\alpha_{v_1xv_2}|^2\cdot\sum_{w_2,y}|\beta_{w_2yw_1}|^2.
\end{aligned}$$

Finally, this shows that

$$\begin{aligned}\|E_n(ab)\|_2^2 &\leqslant \sum_{v_1,w_1}\sum_{i\in I} M(a)^2 \sum_{x,v_2} |\alpha_{v_1xv_2}|^2 \cdot \sum_{w_2,y} |\beta_{w_2yw_1}|^2 \\ &= M(a)^2\bigg(\sum_{i\in I}\sum_{v_1,x,v_2} |\alpha_{v_1xv_2}|^2\bigg)\bigg(\sum_{i\in I}\sum_{w_1,y,w_2} |\beta_{w_2yw_1}|^2\bigg) \\ &= M(a)^2\|a\|_2^2\|b\|_2^2. \quad \blacksquare\end{aligned}$$

Given $k,n\in K$, define $n+k$, $|n-k|\in K$ by

$$\begin{aligned}(n+k)(j) &= n(j)+k(j)\\ |n-k|(j) &= |n(j)-k(j)|\end{aligned}$$

and write $k\leqslant n$ if $k(j)\leqslant n(j)$ for every $j\in J$. Simliarly, given finitely many $l_1,\ldots,l_m\in K$ we define $\max(l_1,\ldots,l_m)\in K$ by

$$\max(l_1,\ldots,l_m)(j)=\max(l_1(j),\ldots,l_m(j)).$$

Finally, for $k\in K$ let

$$\rho(k)=\prod_{j\in J}(2k(j)+1).$$

LEMMA 2.2. *Let $k\in K$ and $a\in \operatorname{span} Y_k$. Then*

$$\|a\|\leqslant \rho(k)M(a)\|a\|_2.$$

Proof. It suffices to show that

$$\|ab\|_2\leqslant \rho(k)M(a)\|a\|_2\|b\|_2$$

for every $b\in\operatorname{span} Y$. For $l\in K$ let $b_l=E_l(b)$. Then for each $n\in K$, using Lemma 2.1 we have

$$\begin{aligned}\|E_n(ab)\|_2 &= \bigg\|\sum_{\substack{l\in K\\ |n-k|\leqslant l\leqslant n+k}} E_n(ab_l)\bigg\|_2 \leqslant \sum_{\substack{l\in K\\ |n-k|\leqslant l\leqslant n+k}} \|E_n(ab_l)\|_2 \\ &\leqslant \sum_{\substack{l\in K\\ |n-k|\leqslant l\leqslant n+k}} M(a)\|a\|_2\|b_l\|_2 \leqslant M(a)\|a\|_2\rho(k)^{\frac12}\bigg(\sum_{\substack{l\in K\\ |n-k|\leqslant l\leqslant n+k}}\|b_l\|_2^2\bigg)^{\frac12}.\end{aligned}$$

This last inequality follows from the fact that the number of $l\in K$ satisfying $|n-k|\leqslant l\leqslant n+k$ is bounded above by $\rho(k)$. Hence

$$\begin{aligned}\|ab\|_2^2 &= \sum_{n\in K}\|E_n(ab)\|_2^2 \leqslant \rho(k)M(a)^2\|a\|_2^2\sum_{n\in K}\sum_{\substack{l\in K\\ |n-k|\leqslant l\leqslant n+k}}\|b_l\|_2^2 \\ &= \rho(k)M(a)^2\|a\|_2^2\sum_{l\in K}\sum_{\substack{n\in K\\ |l-k|\leqslant n\leqslant l+k}}\|b_l\|_2^2 \\ &\leqslant \rho(k)^2M(a)^2\|a\|_2^2\sum_{l\in K}\|b_l\|_2^2 = \rho(k)^2M(a)^2\|a\|_2^2\|b\|_2^2. \quad \blacksquare\end{aligned}$$

Given $a \in \operatorname{span} Y$ define

$$\operatorname{supp}_K(a) = \{k \in K \mid Y_k \text{ meets the support of } a\}$$
$$\max_K(a) = \max\{k \in K \mid k \in \operatorname{supp}_K(a)\}.$$

LEMMA 2.3. *Let $a \in \operatorname{span} Y$. Then*

$$\|a\| \leqslant \rho(\max_K(a))^{\frac{3}{2}} M(a) \|a\|_2.$$

Proof. For $k \in K$ let $a_k = E_k(a)$. Note that $M(a_k) \leqslant M(a)$, and for every $k \in \operatorname{supp}_K(a)$, $\rho(k) \leqslant \rho(\max_K(a))$. Furthermore

$$|\operatorname{supp}_K(a)| \leqslant \prod_{j \in J} (\max_K(a)(j) + 1) \leqslant \rho(\max_K(a)).$$

Using Lemma 2.2 we now have

$$\begin{aligned}
\|a\| = \Big\| \sum_{k \in \operatorname{supp}_K(a)} a_k \Big\| &\leqslant \sum_{k \in \operatorname{supp}_K(a)} \|a_k\| \leqslant \sum_{k \in \operatorname{supp}_K(a)} \rho(k) M(a_k) \|a_k\|_2 \\
&\leqslant \rho(\max_K(a)) M(a) \sum_{k \in \operatorname{supp}_K(a)} \|a_k\|_2 \\
&\leqslant \rho(\max_K(a)) M(a) |\operatorname{supp}_K(a)|^{\frac{1}{2}} \Big(\sum_{k \in \operatorname{supp}_K(a)} \|a_k\|_2^2 \Big)^{\frac{1}{2}} \\
&= \rho(\max_K(a)) M(a) |\operatorname{supp}_K(a)|^{\frac{1}{2}} \|a\|_2 \leqslant \rho(\max_K(a))^{\frac{3}{2}} M(a) \|a\|_2. \quad \blacksquare
\end{aligned}$$

LEMMA 2.4. *Suppose that for every $j \in J$ there are $i_1(j), i_2(j) \in I^{(j)}$ such that there are at least two unitary elements in $(X^{(j)}_{i_2(j)})^{\mathrm{O}}$ and at least one unitary element in $(X^{(j)}_{i_1(j)})^{\mathrm{O}}$. Then for each $a \in \operatorname{span} Y$ there are unitaries $u, v \in \operatorname{span} Y$ and a constant $M < \infty$ such that*

$$\big\|(uav)^n\big\|_2 = \|a\|_2, \quad M\big((uav)^n\big) \leqslant M$$

for every $n \geqslant 1$.

Proof. Let $y(j), z(j) \in (X^{(j)}_{i_2(j)})^{\mathrm{O}}$ and $x(j) \in (X^{(j)}_{i_1(j)})^{\mathrm{O}}$ be distinct unitary elements. Let $m = \max_K(a) \in K$. Fix for the moment $j \in J$. Let $l(j) \in \mathbb{N}$ be such that $l(j) \geqslant (m(j) + 3)/2$ and set

$$u_0(j) = (x(j)y(j)^*)^{l(j)}, \quad v_0(j) = (x(j)z(j))^{l(j)}.$$

As in the proof of [3], Lemma 3.7, it then follows that if $w \in \bigcup_{i=0}^{m(j)} Y_i^{(j)}$, then $u_0(j)wv_0(j)$ is a linear combination of reduced words belonging to $Y^{(j)}$ which start with $x(j)$ and end with $z(j)$. Note also that every such $u_0(j)wv_0(j)$ belongs to span $\bigcup_{i=0}^{4l(j)+m(j)} Y_i^{(j)}$. Let $p(j) \in \mathbb{N}$ be such that $p(j) \geqslant (4l(j)+m(j)+1)/2$ and let

$$r(j) = \big(x(j)y(j)\big)\big(x(j)z(j)\big)^{p(j)}\big(x(j)y(j)\big).$$

Thus, whenever $n \in \mathbb{N}$ and $w_1, \ldots, w_n, w_1', \ldots, w_n' \in \bigcup_{i=0}^{4l(j)+m(j)} Y_i^{(j)}$ are words each starting with $x(j)$ and ending with $z(j)$, then each $r(j)w_j$ is a reduced word in $Y^{(j)}$, as is $r(j)w_1r(j)w_2 \cdots r(j)w_n$, and if

$$r(j)w_1r(j)w_2 \cdots r(j)w_n = r(j)w_1'r(j)w_2' \cdots r(j)w_n'$$

then $w_1 = w_1'$, $w_2 = w_2'$, …, $w_n = w_n'$.

Let $u = \bigotimes_{j\in J} u(j)$ and $v = \bigotimes_{j\in J} v(j)$ where

$$u(j) = \begin{cases} r(j)u_0(j) & \text{if } m(j) > 0, \\ 1 & \text{if } m(j) = 0; \end{cases}$$

$$v(j) = \begin{cases} v_0(j) & \text{if } m(j) > 0, \\ 1 & \text{if } m(j) = 0. \end{cases}$$

What we have shown above implies that

$$uav = \sum_{i=1}^{N} \alpha_i w_i$$

where $\alpha_i \in \mathbb{C}$ and $w_1, w_2, \ldots, w_N$ are distinct elements of Y, and that for every $n \in \mathbb{N}$

$$(uav)^n = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_n=1}^{N} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n} w_{i_1}w_{i_2}\cdots w_{i_n},$$

with the words $w_{i_1}w_{i_2}\cdots w_{i_n}$ being reduced words and distinct elements of Y. This implies that for every $n \in \mathbb{N}$,

$$M\big((uav)^n\big) = M(uav)$$

and

$$\|(uav)^n\|_2 = \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_n=1}^N |\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n}|^2$$
$$= \sum_{i_1=1}^N |\alpha_{i_1}|^2 \sum_{i_2=1}^N |\alpha_{i_2}|^2 \cdots \sum_{i_n=1}^N |\alpha_{i_n}|^2 = \|uav\|_2^n = \|a\|_2^n. \qquad \blacksquare$$

In a unital C^*-algebra A, let $\mathrm{U}(A)$ denote the group of unitaries of A and let $\mathrm{GL}(A)$ denote the group of invertible elements of A. For $a \in A$, let $r(a)$ denote spectral radius of a. As in [3], we will use that

$$\mathrm{dist}(a, \mathrm{GL}(A)) \leqslant \inf_{u,v\in\mathrm{U}(A)} r(uav). \tag{2.3}$$

Theorem 2.5. *Let J be a nonempty set, and for each $j \in J$ let $I^{(j)}$ be a set. For every $j \in J$ and $\iota \in I^{(j)}$, let $A_\iota^{(j)}$ be a unital C^*-algebra with a faithful, tracial state $\tau_\iota^{(j)}$. Assume that for every $j \in J$ there are distinct indices $\iota_1(j), \iota_2(j) \in I^{(j)}$ and unitary elements $x(j) \in A_{\iota_1(j)}^{(j)}$ and $y(j), z(j) \in A_{\iota_2(j)}^{(j)}$ such that*

$$\tau_{\iota_1(j)}^{(j)}(x(j)) = 0 = \tau_{\iota_2(j)}^{(j)}(y(j)) = \tau_{\iota_2(j)}^{(j)}(z(j)) = \tau_{\iota_2(j)}^{(j)}(z(j)^*y(j)).$$

Let

$$(A^{(j)}, \tau^{(j)}) = \mathop{*}_{\iota\in I^{(j)}} (A_\iota^{(j)}, \tau_\iota^{(j)})$$

be the reduced free product of C^*-algebras and let

$$A = \bigotimes_{j\in J} A^{(j)}$$

be the minimal tensor product of C^*-algebras. Then A has stable rank one.

Proof. Since any element of A belongs to a subalgebra which is the tensor product of countably many algebras $B^{(j)}$ where $B^{(j)} = \mathop{*}_{\iota\in G^{(j)}} B_\iota^{(j)}$, where $G^{(j)} \subseteq I^{(j)}$ is countable and $B_\iota^{(j)} \subseteq A_\iota^{(j)}$ are separable C^*-subalgebras, we may assume without loss of generality that J and each $I^{(j)}$ is countable and that each $A_\iota^{(j)}$ is separable.

By [3], 2.1 there is for every $j \in J$ and $\iota \in I^{(j)}$ a standard orthonormal basis $X_\iota^{(j)}$ for $(A_\iota^{(j)}, \tau_\iota^{(j)})$ such that $x(j) \in X_{\iota_1(j)}^{(j)}$ and $y(j), z(j) \in X_{\iota_2(j)}^{(j)}$. Let $Y^{(j)} = \mathop{*}_{\iota\in I^{(j)}} X_\iota^{(j)}$ and let Y be the standard orthonormal basis for (A, τ) defined in equation (2.2). We will show that

$$\inf_{u,v\in\mathrm{U}(A)} r(uav) \leqslant \|a\|_2 \quad (= \tau(a^*a)^{\frac{1}{2}}) \tag{2.4}$$

whenever $a \in \operatorname{span} Y$. Indeed, let $M > 0$ and unitaries $u, v \in \operatorname{span} Y$ be as found in Lemma 2.4. Let $m = \max_K(uav) \in K$. Let $p < \infty$ be the number of $j \in J$ such that $m(j) \neq 0$. Then for every $n \in \mathbb{N}$,

$$\max_K\big((uav)^n\big) \leqslant n \cdot m,$$

where, naturally, $n \cdot m \in K$ is $(n \cdot m)(j) = n \cdot m(j)$, and hence

$$\rho\big(\max_K\big((uav)^n\big)\big) \leqslant n^p \rho(m).$$

Lemmas 2.3 and 2.4 give

$$\|(uav)^n\| \leqslant (n^p\rho(m))^{\frac{3}{2}} M \|(uav)^n\|_2 = (n^p\rho(m))^{\frac{3}{2}} M \|a\|_2^n.$$

Therefore

$$\begin{aligned}\inf_{u,v\in \mathrm{U}(A)} r(uav) \leqslant r(uav) &= \liminf_{n\to\infty} \|(uav)^n\|^{\frac{1}{n}} \\ &\leqslant \liminf_{n\to\infty} (n^p\rho(m))^{\frac{3}{2n}} M^{\frac{1}{n}} \|a\|_2 = \|a\|_2.\end{aligned}$$

Now, the proof that $\operatorname{sr}(A) = 1$ follows by the exactly same argument as in the proof of [3], 3.8, which we briefly review here. Suppose for contradiction that $\operatorname{sr}(A) > 1$. Then by Rørdam's result [5], 2.6, there is $b \in A$ having norm 1 and whose distance to $\mathrm{GL}(A)$ is 1. But b is a norm limit, $b = \lim_{n\to\infty} a_n$, where each $a_n \in \operatorname{span} Y$. Using (2.3) and (2.4), we have

$$\operatorname{dist}(a_n, \mathrm{GL}(A)) \leqslant \|a_n\|_2,$$

and hence

$$\operatorname{dist}(b, \mathrm{GL}(A)) \leqslant \|b\|_2.$$

But this implies that $\|b\| = \|b\|_2 = 1$, hence that b is unitary, which contradicts that $\operatorname{dist}(b, \mathrm{GL}(A)) = 1$. ∎

COROLLARY 2.6. *Let J be a nonempty set and let G be a group which is the (restricted) direct sum*

$$G = \bigoplus_{j\in J} G^{(j)}$$

where for each $j \in J$, $G^{(j)}$ is the free product of groups

$$G^{(j)} = G_1^{(j)} * G_2^{(j)}$$

with $|G_1^{(j)}| \geqslant 2$ *and* $|G_2^{(j)}| \geqslant 3$. *Then the reduced group* C^**-algebra* $C_{\rm r}^*(G)$ *has stable rank one.*

Proof.

$$C_{\rm r}^*(G) = \bigotimes_{j\in J} C_{\rm r}^*(G^{(j)})$$

is the minimal tensor product of C^*-algebras and, letting τ_H denote the canonical trace on $C_{\rm r}^*(H)$,

$$(C_{\rm r}^*(G^{(j)}), \tau_{G^{(j)}}) = (C_{\rm r}^*(G_1^{(j)}), \tau_{G_1^{(j)}}) * (C_{\rm r}^*(G_2^{(j)}), \tau_{G_2^{(j)}}).$$

Now the theorem applies. ∎

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KENNETH J. DYKEMA
Department of Mathematics
and Computer Science
Odense Universitet, Campusvej 55
DK–5230 Odense M
DENMARK

E-mail: dykema@imada.ou.dk

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