J. OPERATOR THEORY
41(1999), 175–197

FOURIER-STIELTJES ALGEBRAS OF r-DISCRETE GROUPOIDS

KARLA J. OTY

Communicated by Norberto Salinas

Abstract. The Fourier-Stieltjes algebra, $\mathcal{B}(G)$, of a groupoid G has recently been defined using suitably defined positive definite functions. In this paper we prove various properties of positive definite functions including that continuous positive definite functions separate the points of G. We also show that in certain cases the continuous elements of $\mathcal{B}(G)$ (denoted $B(G)$) and the space of complex-valued bounded continuous functions (denoted $C(G)$) are topologically isomorphic as Banach algebras but not as ordered or $*$-Banach algebras. The same is shown to be true for $\mathcal{B}(G)$ and the space of complex-valued bounded Borel functions (denoted $M(G)$). We explore various conditions including that of an ordering map that one can place on groupoids and their connections, and we show that if G has such an ordering map then $C_c(G) \subseteq B(G)$ and $M_c(G) \subseteq \mathcal{B}(G)$.

Keywords: *Fourier-Stieltjes algebras, positive definite functions, groupoids.*

MSC (2000): 43A32.

INTRODUCTION

G.W. Mackey introduced the notions of virtual group and measured groupoid as a tool for studying certain problems in analysis and geometry ([7], [8]). One motivation was to extend his theory of unitary representations of group extensions. P. Hahn in [5] and [6] developed the theory of Haar measure, convolution of functions, and the regular representation on Mackey's measured groupoids. This allowed him to interpret Murray and von Neumann's constructions of non-type I factors in the context of convolution algebras of groupoids.

Groupoids have also been used in the study of operator algebras and much work has been done in this area. J. Renault studied C^*-algebras of locally compact groupoids with a fixed Haar system ([15]). Since A. Ramsay showed in [12] that measured groupoids with a quasiinvariant measure have an inessential reduction that has a locally compact topology, not much structure is lost by adapting Renault's point of view.

Since duality theory has played such an important role in the theory of groups, it would be hoped that duality theory could play a similar role in the study of groupoids. Until [14] and [16] however, no work had been done in the duality theory of groupoids. This paper is intended to extend the work in this area. The outline of this paper is as follows. Section 1 is devoted to background material on groupoids. Section 2 deals with various conditions one can place on groupoids and their connections. We will study the Fourier-Stieltjes algebras of groupoids with some of these properties in Section 4. Additionally, in Section 2 we use a result from [12] to show that in some sense we are not losing anything by restricting our attention to groupoids that satisfy certain properties. In Section 3 we prove various properties of positive definite functions including the fact that the continuous positive definite functions on G separate the points in G. Section 4 discusses the Fourier-Stieltjes algebras for two special cases of groupoids. When G is a locally compact topological groupoid with uniformly bounded fibers, we show that $B(G)$ and $C(G)$ are topologically isomorphic as Banach algebras. The same is shown to be true for $\mathcal{B}(G)$ and $M(G)$. However, in Section 5 we show that the natural order and natural involutions do distinguish these algebras. When G has an ordering map with certain properties, we show that $C_{\rm c}(G) \subseteq B(G)$ and that $M_{\rm c}(G) \subseteq \mathcal{B}(G)$.

1. PRELIMINARIES

This section is intended to give a brief introduction to the author's definitions and notation that will be used throughout this paper. For more details of groupoids, including complete definitions, see [5], [6], [10], [11], [12], [13], [14], [15] and [16].

Suppose G and X are sets, i is a map from X to G with the image of x denoted by i_x, and that r and s are maps from G onto X. Let

$$G^2 = \{(\gamma, \gamma_1) \in G \times G \mid r(\gamma_1) = s(\gamma)\}$$

which can be thought of as the set of *multipliable* or *composable* pairs. The set G will denote a groupoid as defined in [15]. Then r will be called the *range map* and s will be called the *source map.*

Because of the map i, we can think of X as being a subset of G, and we denote the image of X under i by G^0. Let xG and Gx denote the sets $r^{-1}(\{x\})$ and $s^{-1}(\{x\})$ respectively, and call xG a *fiber* of G.

A *topological groupoid* is a groupoid G with a Hausdorff, second countable topology such that G^0 is closed in G and such that r, s, multiplication, and the inverse map are all continuous. An *r-discrete groupoid* is a topological groupoid such that G^0 is open in G. We assume all groupoids in this paper are locally compact topological groupoids admitting a left Haar system $\{\lambda^x \mid x \in X\}$ unless otherwise stated.

Let $M(G)$ (respectively $C(G)$) denote the space of complex-valued bounded Borel (respectively continuous) functions, let $M_{\mathrm{c}}(G)$ (respectively $C_{\mathrm{c}}(G)$) denote the subspace of $M(G)$ (respectively $C(G)$) of functions with compact support, and let $C_0(G)$ denote the space of complex-valued continuous functions which vanish at infinity.

A measure μ on G is *quasisymmetric* if μ and μ^{-1} have the same null sets. In general, let $[\mu]$ denote the measure class of μ. If $\lambda \in [\mu]$, we will say that λ and μ are *equivalent*.

If λ is a left Haar system for G and μ is a probability measure on X, we form a new measure ν on G defined by

$$\nu(E) = \int\limits_X \lambda^x(E)\,\mathrm{d}\mu(x).$$

Integration against this measure for a Borel function f is defined by

$$\int f(\gamma)\,\mathrm{d}\nu(\gamma) = \int\int f(\gamma)\,\mathrm{d}\lambda^x(\gamma)\,\mathrm{d}\mu(x)$$

and we write λ^μ for ν. We say that μ is *quasiinvariant* if λ^μ is quasisymmetric ([14]).

Let $\mathcal{Q}$ denote the set of all quasiinvariant probability measures on X.

A Borel set $N \subseteq X$ is *$\mathcal{Q}$-null* if $\mu(N) = 0$ for all $\mu \in \mathcal{Q}$. For a Borel set $N \subseteq G$, N is *$\lambda^{\mathcal{Q}}$-null* if $\lambda^\mu(N) = 0$ whenever $\mu \in \mathcal{Q}$. A function f on X is *$\mathcal{Q}$-essentially bounded* if and only if the restriction of f to the complement of some $\mathcal{Q}$-null set is bounded.

Define

$$\|f\|_\infty = \inf\{B \mid |f| \leqslant B\ \mu\text{-a.e. } \forall \mu \in \mathcal{Q}\}$$

and denote the space of all $\mathcal{Q}$-essentially bounded functions by $L^\infty(\mathcal{Q})$. We define $\lambda^{\mathcal{Q}}$-essentially bounded functions and their norms similarily ([14]).

Recall that a unitary representation is given by a Hilbert bundle K over X and a Borel homomorphism π of G such that for all $\gamma \in G$, $\pi(\gamma)$ mapping $K(s(\gamma))$ to $K(r(\gamma))$ is unitary. The left regular representation, L, is a unitary representation given by the Hilbert bundle $K(x) = L^2(\lambda^x)$, and where for all $\gamma \in G$, we define $L_\gamma : L^2(\lambda^{s(\gamma)}) \to L^2(\lambda^{r(\gamma)})$ by $(L_\gamma f)(\gamma_0) = f(\gamma^{-1}\gamma_0)$. Note that the left regular representation is faithful.

2. TOPOLOGICAL PROPERTIES AND RELATIONS

In this section we mainly explore some topological properties of groupoids and the relationships among these properties. We are especially interested in property (v) of Proposition 2.2 as we will study the Fourier-Stieltjes algebras of groupoids with this property in Section 4. After proving Proposition 2.2, we discuss some examples of groupoids that satisfy property (ii) but not (i). We also show that if a locally compact topological groupoid G has a left Haar system, a quasiinvariant measure μ, and satisfies property (iv) or (v) of Proposition 2.2, then we have an inessential reduction of G such that (iii) is satisfied. Thus, in some sense, we can always assume property (iii) when we have a locally compact topological groupoid with a Borel ordering map, a quasiinvariant measure, and a left Haar system. Finally, we show that properties (i), (ii), (iii), (v), and (vii), the ones we are most interested in, are preserved under the operations of direct sums, direct products, and inductive limits.

DEFINITION 2.1. A Borel map O which maps G to $\mathbb{Z}^+$ and which is one-to-one on every fiber of G will be called an *ordering map*. A set V such that r restricted to V is one-to-one will be called an *r-set* and a set S such that both r and s restricted to S are one-to-one will be called a *G-set*.

PROPOSITION 2.2. *Suppose that G is a locally compact topological groupoid. Consider the following statements:*

(i) *G admits a cover of compact open G-sets;*

(ii) *G admits a cover of disjoint open r-sets;*

(iii) *there exists an ordering map, O, such that O is continuous with respect to the topology on G;*

(iv) *there exists an ordering map O on G such that the function f mapping X to $\mathbb{Z}^+ \cup \{\infty\}$ defined by $f(x) = \sup\{O(\gamma) \mid \gamma \in xG\}$ is upper semicontinuous;*

(v) *there exists an ordering map O on G that is bounded on compact subsets of G;*

(vi) *r is a covering map;*

(vii) *G is r-discrete and admits a left Haar system which is essentially the counting measure.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v)*,* (vi) $\Rightarrow$ (vii)*, and* (ii) $\Rightarrow$ (vii)*. If in (iii) we also have that the fibers have constant cardinality, then* (iii) $\Rightarrow$ (vi)*. Thus, if G admits a cover of compact open G-sets and the fibers of G have constant cardinality, then G satisfies all seven properties.*

Proof. (i) $\Rightarrow$ (ii) This follows from the definitions and from making the open compact G-sets disjoint in the usual manner.

(ii) $\Rightarrow$ (iii) Suppose that G admits a cover of disjoint open r-sets. This cover is countable, possibly finite, and will be denoted by $\{V_n\}_{n=1}^{\infty}$. Let $O(\gamma) = n$ for $\gamma \in V_n$. O is well defined, continuous, and is one-to-one on the fibers of G since V_n is an r-set. Thus O is a continuous ordering map.

(iii) $\Rightarrow$ (ii) Suppose that O is a continuous ordering map of G. Let $V_n = O^{-1}(\{n\})$. Then $\{V_n\}$ is a disjoint collection of open r-sets which cover G.

(iii) $\Rightarrow$ (iv) Suppose that O is a continuous ordering map. Define f from X into $\mathbb{Z}^+ \cup \{\infty\}$ by $f(x) = \sup\{O(\gamma) \mid \gamma \in xG\}$. Suppose that $\alpha \in \mathbb{R}$. If $\alpha \leqslant 1$, then $\{x \in X \mid f(x) < \alpha\} = \emptyset$. If $\alpha > 1$, let

$$A = \{x \in X \mid f(x) < \alpha\} = \{x \in X \mid O(\gamma) < \alpha \, \forall \gamma \in xG\}.$$

Suppose that $x \in A$ and let U be the open set $r(O^{-1}([1, \alpha)))$ which contains x. Since $U \subseteq A$, A is open for all $\alpha \in \mathbb{R}$.

(iv) $\Rightarrow$ (v) Suppose there exists an ordering map, O, on G given by property (iv) and suppose that F is a compact subset of G. Since f is upper semicontinuous and has domain $r(F)$, f is bounded and achieves a maximum on $r(F)$. In other words, there exists an $N \in \mathbb{Z}^+$ such that for all $x \in r(F)$ and all $\gamma \in xG$, we have that $O(\gamma) \leqslant N$.

(vi) $\Rightarrow$ (vii) Since r is a covering map, r is a local homeomorphism. In [15], Renault shows that this is equivalent to G being r-discrete with a left Haar system.

(ii) $\Rightarrow$ (vii) Suppose that G admits a cover of disjoint open r-sets V_n. Since G is second countable, we know that there exists a countable basis, $\{U_m\}$, for the topology of G. Let $\mathcal{A} = \{V_k \cap U_l \mid l, k \in \mathbb{Z}^+\}$, which is a basis of open r-sets. Renault shows in [15] that if G has a basis of open G-sets, then G is r-discrete and admits a left Haar system. The same proof works here if you replace G-sets by r-sets.

We now suppose that $|xG|$ is constant for all $x \in X$.

(iii) $\Rightarrow$ (vi) Suppose that O is continuous, $x \in X$, and that U is any open set containing x. Let $V_n = O^{-1}(\{n\}) \cap r^{-1}(U)$ and let $\Lambda = \{n \in \mathbb{Z}^+ \mid V_n \neq \emptyset\}$.

Then $\{V_n \mid n \in \Lambda\}$ is a collection of nonempty, pairwise disjoint sets and $r^{-1}(U) = \bigcup_{n\in\Lambda} V_n$.

If $\gamma, \gamma' \in V_n$ with $r(\gamma) = r(\gamma') = x$, then $O(\gamma) = O(\gamma') = n$. Since O is one-to-one on xG, we have that $\gamma = \gamma'$. To see that r restricted to V_n is onto U, suppose that $x \in U$. Since $V_n \neq \emptyset$, there exists $\gamma \in G$ such that $O(\gamma) = n$ and $r(\gamma) \in U$. As the fibers have constant cardinality, $|r(\gamma)G| = |xG|$ and $r(\gamma)G$ has a nth element. Thus there exists a $\gamma' \in V_n$ such that $r(\gamma') = x$. Hence r is a covering map. ∎

Note that we have actually shown the stronger statement that for any open set $U \subseteq X$, $r^{-1}(U)$ can be written as $\bigcup_{n\in\Lambda} V_n$ where the V_n's are open, pairwise disjoint, and such that r restricted to V_n is a homeomorphism of V_n onto U for each n. The hypothesis that $|xG|$ is constant for all $x \in X$ was only used to show that r restricted to V_n was onto U. Thus if the cardinality of the fibers is not constant, then r is an open, continuous map such that for any open set U in X, $r^{-1}(U)$ can be written as the union of open, pairwise disjoint r-sets.

Many common examples of r-discrete groupoids such as transformation groups, where the group is countable and discrete and the space X is compact, compact spaces with the equivalence relation $x \sim y$ if and only if $x = y$, and the principal, transitive groupoid on n elements, satisfy all seven of the above properties. These examples all have fibers with constant cardinality as well. The next two examples show that property (ii) does not imply property (i) in the above proposition.

EXAMPLE 2.3. Let $G = X \times H$ where H is a countable discrete group acting on X and X is a locally compact, connected, Hausdorff, but not compact space. Then G has a covering of disjoint, open r-sets given by $X \times \{h_n\}$, $h_n \in H$, but G does not have a covering of compact open G-sets as X is connected.

EXAMPLE 2.4. Let G be a locally compact topological groupoid that admits a covering of compact open G-sets and let A be a connected, locally compact Hausdorff group which is not compact. Let $c : G \to A$ be a continuous cocycle. Then the skew product $G(c)$ admits a covering of disjoint open r-sets V_n, but does not admit a covering of compact open G-sets.

If G satisfies (iv) or (v), then G automatically has a Borel ordering map, O. The problem is that O may or may not be continuous. However, if G is a measured groupoid, then there is a process described in [12] by which we can imbed G into a Polish groupoid to get an inessential reduction G_0 which has a locally compact metric topology in which it is a locally compact topological groupoid. We can define an ordering map on G_0 and by prearranging the topology on X we will have that this ordering map on G_0 is continuous.

We need the following two facts about Polish topologies ([18]).

Proposition 2.5. *Let $\tau_1, \tau_2, \ldots$ be Polish topologies on a space Y, and suppose $\tau_0 \subseteq \bigcap_{n=1}^{\infty} \tau_n$ is a Hausdorff topology on Y. Then the topology τ generated by $\bigcup_{n=1}^{\infty} \tau_n$ is Polish. If $\tau_0, \tau_1, \tau_2, \ldots$ all generate the same Borel sets on Y, then τ generates the same family of Borel sets.*

Proposition 2.6. *Let A be a Borel subset of a Polish space Y with topology τ. Then there exists a Polish topology τ_A on Y such that:*

(i) $\tau \subseteq \tau_A$;

(ii) τ_A *generates the same Borel sets as* τ;

(iii) A *and* $Y \setminus A$ *are in* τ_A.

Recall that a *reduction* of G by a Borel set $Y \subseteq X$ is the set $r^{-1}(Y) \cap s^{-1}(Y)$. If ν is a finite Borel measure on G, we will call this reduction *inessential* if $r(\nu)(X \setminus Y) = 0$. For $\gamma \in G$, let T_γ be the left translation map from $s(\gamma)G$ to $r(\gamma)G$ defined by $T_\gamma(\gamma_1) = \gamma\gamma_1$. We say that ν is *left quasiinvariant* if it has a decomposition for which there is an inessential reduction such that $T_\gamma(\nu^{s(\gamma)})$ and $\nu^{r(\gamma)}$ have the same null sets whenever $r(\gamma)$ and $s(\gamma)$ are in Y. We will call ν *quasiinvariant in the sense of measured groupoids* if ν is left quasiinvariant and is quasisymmetric. If G is an analytic Borel groupoid with a measure ν which is quasiinvariant in the sense of measured groupoids, (G, ν) is called a *measured groupoid.*

Suppose that G is a locally compact topological groupoid with a left Haar system λ and a quasiinvariant measure μ. Then G can be given a complete metric and hence can be given a Polish topology; X has a Polish topology as well. Since every measure class contains a Borel probability measure ([14]), we first take a measure α^x equivalent to λ^x such that $\alpha^x(xG) = 1$ for all x. Let ν denote the measure α^μ. If A is a Borel set in X, we have that

$$\begin{aligned} r(\nu)(A) &= \int_X \alpha^x(r^{-1}(A))\,\mathrm{d}\mu(x) = \int_X \alpha^x(xG)\chi_A(x)\,\mathrm{d}\mu(x) \\ &= \int_X \chi_A(x)\,\mathrm{d}\mu(x) = \mu(A). \end{aligned}$$

By the definition of ν we have that for all Borel functions g,

$$\int g(\gamma)\,\mathrm{d}\nu(\gamma) = \int\int g(\gamma)\,\mathrm{d}\alpha^x(\gamma)\,\mathrm{d}\mu(x).$$

Also, for all $f \in C_c(G)$ we know that $x \mapsto \int f(\gamma)\,\mathrm{d}\alpha^x(\gamma)$ is Borel. Thus $x \mapsto \alpha^x$ is a decomposition of ν with respect to r.

Since the left Haar system was assumed to be left invariant everywhere, ν is left quasiinvariant with no reduction. If μ is quasiinvariant, then ν is quasisymmetric by definition. Thus ν is quasiinvariant in the measured groupoid sense and if G is a locally compact topological groupoid with a left Haar system and a quasiinvariant measure μ, then (G,ν) is a measured groupoid.

Now suppose that G also satisfies property (iv) or (v). We know that $r(O^{-1}(\{n\})) = A_n$ is a Borel set in X for all n. As X has a Polish topology, denoted τ, by Proposition 2.6 there exists a Polish topology τ_n on X such that τ_n generates the same Borel sets as τ, $\tau \subseteq \tau_n$, and such that A_n and $X \setminus A_n$ are in τ_n. Let $A = \bigcup\limits_{n=1}^{\infty} A_n$ and let τ_A be the topology generated by $\bigcup\limits_{n=1}^{\infty} \tau_n$. Since $\tau \subseteq \bigcap\limits_{n=1}^{\infty} \tau_n$ is a Hausdorff topology, by Proposition 2.5 the topology τ_A is Polish. Since $A_n \in \tau_n$ for all n, we have that $A \in \tau_A$. Adjoin $X \setminus A$ to τ_A and generate a Polish topology τ_A^0 which gives the same Borel sets on X as τ_A. Since τ_n generates the same Borel sets on X as τ for every n, by Proposition 2.5 τ_A generates the same Borel sets on X as τ. Thus τ_A^0 is a Polish topology on X generating the same Borel sets on X as τ and such that $A_n \in \tau_A^0$ for all n. This says that with respect to this Polish topology on X, $r(O^{-1}(\{n\}))$ is open for all n.

We now follow the construction in Theorem 2.5 of [12]. Let H be the Hilbert bundle associated with the left regular representation using the measures α^x. By a reduction argument in [12] we can assume that the spaces $H(x)$ have the same dimension. There exist a Hilbert space $\mathcal{K}$, a bundle isomorphism $V : X \times \mathcal{K} \to X * H$, unitary composition operators $R(\gamma) : H(s(\gamma)) \to H(r(\gamma))$, and a Borel homomorphism $U : G \to \mathcal{U}(\mathcal{K}) = \mathcal{U}$, defined by $U(\gamma) = V(r(\gamma))^{-1}R(\gamma)V(s(\gamma))$. Let $\mathcal{W} = X \times \mathcal{U} \times X$ and give it its usual Polish groupoid structure ([12]). With the product topology, $r(O^{-1}(\{n\})) \times \mathcal{U} \times X$ is open in $\mathcal{W}$ for all n. Define $w : G \to \mathcal{W}$ by $w(\gamma) = (r(\gamma), U(\gamma), s(\gamma))$. Then the argument in Theorem 2.5 shows that w is a Borel homomorphism and that $G \cong w(G)$. Give G the topology it inherits from $w(G)$ by this identification.

To get an ordering map on $w(G)$, define $O_w(\beta) = O(w^{-1}(\beta))$ for $\beta \in w(G)$. O_w will be a Borel map and since O is one-to-one on the fibers on G and w^{-1} is one-to-one, O_w will be one-to-one on the fibers of $w(G)$. Then $(r(O^{-1}(\{n\})) \times \mathcal{U} \times X) \cap w(G)$ is open in $w(G)$ which implies that $O_w^{-1}(\{n\})$ is open in $w(G)$ for all n.

The proof of Theorem 4.1 in [12] gives a process by which to construct an inessential reduction, G_0, that has a locally compact metric topology in which it is a topological groupoid. We note that if U is an open set in the inherited topology on G, then $U \cap G_0$ will be open in G_0. We have the following generalization of Theorem 4.1 in [12].

Theorem 2.7. *Let G be a locally compact topological groupoid with a left Haar system, a quasiinvariant measure μ, and an ordering map O. Then G has an inessential reduction G_0 which has a locally compact metric topology in which it is a topological groupoid and such that O is a continuous ordering map.*

We note that we could have just as easily let our ordering map take G to $\mathbb{Z}$ instead of to the positive integers. The advantage to using $\mathbb{Z}$ instead of $\mathbb{Z}^+$ is that in the case of a transformation group when the group is $\mathbb{Z}$, the ordering map defined by $O(x, n) = n$ is a continuous cocycle such that $O^{-1}(\{0\}) = G^0$. Muhly, Qui, and Xia showed in [9] that in this case, there is a $\mathbb{R}$-invariant, faithful conditional expectation from $C^*(G)$ onto $C_0(G^0)$ and that the K-theory of the closure of $C_c(P)$ in $C^*(G)$ is isomorphic to that of $C_0(G^0)$, where $P = O^{-1}([0, +\infty))$.

The next proposition shows that performing the operations of taking direct sums, direct products, and inductive limits preserve some of the seven properties of Proposition 2.2. Hence using examples that satisfy these properties, and in particular property (v), we can construct groupoids that satisfy all of the properties by any of these three construction methods.

Moreover, since the compact spaces corresponding to the equivalence relation $u \sim v$ if and only if $u = v$ have a covering of compact open G-sets and the transitive principal groupoids on n elements have a covering of compact open G-sets, so do elementary groupoids of type n, elementary groupoids, approximately elementary groupoids, and approximately finite groupoids as defined in [15]. In particular, these types of groupoids have an ordering map that satisfies property (v) of Proposition 2.2, and hence we will be able to say something about their Fourier-Stieltjes algebras in Section 4.

Proposition 2.8. *Properties* (i)*,* (ii)*,* (iii)*,* (v)*, and* (vii) *of Proposition* 2.2 *are preserved under the operations of direct product, direct sum, and inductive limit of groupoids.*

Proof. Let G_{-1} and G_0 be arbitrary locally compact topological groupoids with unit spaces X_{-1} and X_0 respectively, and let G_n, $n \in \mathbb{Z}^+$, be an increasing sequence of locally compact topological groupoids all having the same unit space X and such that G_n is open in G_{n+1} with the topology on G_n induced by G_{n+1}.

Property (i). Suppose that there exists a cover, $\{V_{m,\alpha} \mid \alpha \in \Lambda\}$, of compact open G_m-sets for $m \in \{-1, 0, 1, \ldots\}$. Then $\{V_{-1,\beta} \times V_{0,\alpha} \mid \alpha, \beta \in \Lambda\}$, $\{V_{j,\alpha} \mid j \in$

$\{-1,0\}$, $\alpha \in \Lambda\}$, and $\{V_{j,\alpha} \mid j \in \mathbb{Z}^+, \alpha \in \Lambda\}$ are covers of compact open G-sets for $G_{-1} \times G_0$, $G_{-1} \oplus G_0$, and $\bigcup_n G_n$ respectively.

Property (iii). Suppose there exist ordering maps $O_j : G_j \to \mathbb{Z}^+$ such that O_j is continuous with respect to the topology on G_j for $j \in \{-1,0,1,\ldots\}$. For each operation, we define an ordering map that will be continuous for the respective topologies.

For $G_{-1} \times G_0$, define

$$O(\gamma_{-1}, \gamma_0) = 2^{O_{-1}(\gamma_{-1})} 3^{O_0(\gamma_0)};$$

for $G_{-1} \oplus G_0$, define

$$O(\gamma) = \begin{cases} 2^{O_{-1}(\gamma)} & \text{if } \gamma \in G_{-1}, \\ 3^{O_0(\gamma)} & \text{if } \gamma \in G_0; \end{cases}$$

and for $\bigcup_n G_n$, define $O(\gamma) = p_n^{O_n(\gamma)}$ where G_n is the first groupoid in the sequence such that $\gamma \in G_n$ and p_n is the nth prime. Each of these maps can be shown to be well-defined, continuous, and one-to-one on the appropriate fibers, and hence will be ordering maps.

Property (ii). Since the operations of direct product, direct sum, and inductive limit of groupoids preserve property (iii), it follows from the equivalence of properties (ii) and (iii) that these operations preserve property (ii) as well.

Property (v). Suppose that G_n satisfies property (v) for $n \in \{-1,0,1,\ldots\}$. For $G_{-1} \times G_0$ let O be the ordering as defined in the proof of property (iii) and let F be a compact set in $G_{-1} \times G_0$. Then $\pi_j(F)$ is a compact set of G_j, where π_j is the projection map onto G_j, $j \in \{-1,0\}$. By property (v), there exists N_j such that for all $\gamma \in \pi_j(F)$, $O_j(\gamma) \leqslant N_j$. Let $N = 2^{N_{-1}} 3^{N_0}$ and suppose that $(\gamma_{-1}, \gamma_0) \in F$. Then as $\gamma_j \in \pi_j(F)$, we have that

$$O(\gamma_{-1}, \gamma_0) = 2^{O_{-1}(\gamma_{-1})} 3^{O_0(\gamma_0)} \leqslant 2^{N_{-1}} 3^{N_0} = N.$$

For $G_{-1} \oplus G_0$, let O be the ordering map as in the proof of property (iii) and suppose that $F \subseteq G_{-1} \oplus G_0$ is a compact set. Since $G_{-1} \oplus G_0$ has the disjoint union topology, $F \cap G_j$ is compact in G_j for $j \in \{-1,0\}$. By property (v), there exists $N_j \in \mathbb{Z}^+$ such that for all $\gamma \in F \cap G_j$, $O_j(\gamma) \leqslant N_j$. Let $N = 2^{N_{-1}} 3^{N_0}$. If $\gamma \in F$, then either $\gamma \in F \cap G_{-1}$ or $\gamma \in F \cap G_0$. Suppose first that $\gamma \in F \cap G_{-1}$. Then

$$O(\gamma) = 2^{O_{-1}(\gamma)} \leqslant 2^{N_{-1}} \leqslant N.$$

The case where $\gamma \in F \cap G_0$ is similar and thus the disjoint union operation preserves property (v).

Finally, let O be the ordering map as in the proof of property (iii) for the inductive limit and suppose that $F \subseteq \bigcup_n G_n$ is a compact set. Since $\bigcup_n G_n$ is an open covering of F, there exists a finite subcovering of F, $\{G_{n_j}\}_{j=1}^{k}$, with $n_1 < n_2 < \cdots < n_k$. Since G_{n_j} satisfies property (v), there exists a N_{n_j} such that for all $\gamma \in F \cap G_{n_j}$, $O_{n_j}(\gamma) \leqslant N_{n_j}$. Let $N = \max\{p_{n_1}^{N_{n_1}}, p_{n_2}^{N_{n_2}}, \ldots, p_{n_k}^{N_{n_k}}\}$ where p_m is the mth prime. Suppose that $\gamma \in F$ and let G_{n_j} be the first groupoid in the finite subcovering such that $\gamma \in G_{n_j}$. Then $O(\gamma) = p_n^{O_n(\gamma)} \leqslant N$ and hence $\bigcup_n G_n$ satisfies property (v).

Property (vii). In [15], Renault shows that the operations of direct product, direct sum and inductive limits of topological groupoids with a left Haar system result in a topological groupoid with a left Haar system. If G_n, $n \in \{-1, 0, 1, \ldots\}$, is r-discrete as well, then $G_{-1} \times G_0$, $G_{-1} \oplus G_0$, and $G = \bigcup_n G_n$ will be all r-discrete. ∎

3. POSITIVE DEFINITE FUNCTIONS

Since groupoids are generalizations of groups, we would like to extend the definitions and results of positive definite functions on groups to groupoids. In [14], the following definition is given for positive definite functions on locally compact topological groupoids with a left Haar system λ.

Definition 3.1. Suppose that $p : G \to \mathbb{C}$ is a bounded Borel function. Then p is *positive definite* if and only if for all $x \in X$ and for all $f \in C_{\mathrm{c}}(G)$

$$\int\int f(\gamma_1)\overline{f(\gamma_2)}p(\gamma_2^{-1}\gamma_1)\,\mathrm{d}\lambda^x(\gamma_1)\,\mathrm{d}\lambda^x(\gamma_2) \geqslant 0.$$

Positive definite functions p_1 and p_2 will be called *equivalent* if they agree $\lambda^{\mathcal{Q}}$-almost everywhere. The set of all such (equivalence classes of) functions p will be denoted by $\mathcal{P}(G)$. The set of all continuous functions that satisfy Definition 3.1 will be denoted $P(G)$. Let $\mathcal{B}(G)$ denote the set of (equivalence classes of) functions of the form $p_1 - p_2 + \mathrm{i}(p_3 - p_4)$, where $p_j \in \mathcal{P}(G)$, and let $B(G)$ denote the continuous functions of $\mathcal{B}(G)$. $\mathcal{B}(G)$ is called the *Fourier-Stieltjes algebra* of G.

In [14] $\mathcal{B}(G)$ is shown to be a commutative Banach algebra under pointwise products. $\mathcal{B}(G)$ is provided a normed algebra structure by representing $\mathcal{B}(G)$ as an algebra of completely bounded operators on $M^*(G)$, the completion of $M_{\mathrm{c}}(G)$ under the universal representation, ω, of G. The norm of $b \in \mathcal{B}(G)$ is then given by the completely bounded norm of the extension of the operator T_b taking $\omega(f)$ to $\omega(bf)$ for $f \in M_c(G)$. This Fourier-Stieltjes norm, denoted $\|\cdot\|$, is shown to be complete and that $\|b\| \geqslant \|b\|_\infty$ for all $b \in \mathcal{B}(G)$.

Note also that $B(G)$ is defined to be the continuous elements in $\mathcal{B}(G)$ and not the linear combination of continuous elements in $P(G)$. Let $B_1(G)$ denote the set of (equivalence classes of) elements of the form $p_1 - p_2 + \mathrm{i}(p_3 - p_4)$, where $p_j \in P(G)$. Then $B_1(G) \subseteq B(G)$, but they are not necessarily equal. For an example of a groupoid in which $B_1(G) \neq B(G)$ see Section 7 of [14]. However, we show in Section 4 that when G is an r-discrete groupoid with uniformly bounded fibers, then $B_1(G) = B(G)$.

We let $\mathcal{A}(G)$ denote the Fourier-Stieltjes closure of $\mathcal{B}(G) \cap M_c(G)$ and let $A(G)$ denote the Fourier-Stieltjes closure of $B(G) \cap C_c(G)$. For locally compact Hausdorff groups, $A(G)$ can be shown to be equal to the set of functions of the form $f * g^\flat$ where f and g are in $L^2(G)$ ([3]). Although we have not shown the analogous result for locally compact topological groupoids, we will still call $A(G)$ the *Fourier algebra* of G.

PROPOSITION 3.2. *The following are properties of positive definite functions:*

(i) *$\mathcal{P}(G)$ and $P(G)$ are convex sets containing the nonnegative constant functions and which are closed under pointwise products, sums, and complex conjugation.*

(ii) *If p is a continuous positive definite function, then for all $x \in X$, $p(i_x) \geqslant 0$.*

(iii) *If p is a positive definite function, then for every $\mu \in \mathcal{Q}$, we have that $p^\flat(\gamma) = p(\gamma)$ λ^μ-a.e. where $p^\flat(\gamma) = \overline{p(\gamma^{-1})}$.*

(iv) *If p is a positive definite function, then so are $\mathrm{Re}\,(p)$ and $|p|^2$. We also have that $\mathrm{Im}\,(p) \in \mathcal{B}(G)$.*

(v) *If $f : X \to \mathbb{C}$ is any bounded continuous function, then $p(\gamma) = f(s(\gamma))\overline{f(r(\gamma))}$ is in $P(G)$.*

Proof. The proofs of properties (i), (iii), and (iv) are straightforward and follow from Definition 3.1 and results from [14] involving unitary representations of groupoids and positive definite functions.

For the proof of property (ii), since p is a continuous positive definite function, we know from [14] that there exists a unitary groupoid representation, π_p, and a continuous section, ξ_p, such that

$$p(\gamma) = (\pi_p(\gamma)\xi_p(s(\gamma)) \mid \xi_p(r(\gamma)))\,.$$

For $x \in X$, we have that

$$p(i_x) = \|\pi_p(i_x)\xi_p(x)\|^2 .$$

For the proof of property (v), to see that p is positive definite, let $\pi(\gamma) : \mathbb{C} \to \mathbb{C}$ be the identity function and let $\xi(x) = f(x)$ for all $x \in X$. Then π is a unitary representation for G and ξ is a section. Define

$$p(\gamma) = (\pi(\gamma)\xi(s(\gamma)) \mid \xi(r(\gamma))) = f(s(\gamma))\overline{f(r(\gamma))}$$

which is a positive definite function by Lemma 3.2 in [14]. If f has compact support, then so will p. ∎

Proposition 3.3. *If $f, h \in M_c(G)$, then $f * f^\flat \in \mathcal{P}(G)$, $f^\flat * f \in \mathcal{P}(G)$, and $f * h \in \mathcal{A}(G)$. If $f, h \in C_c(G)$, then $f * f^\flat \in P(G)$, $f^\flat * f \in P(G)$, and $f * h \in A(G)$.*

Proof. Let $\eta(x) = f\,|xG$ and define $p(\gamma) = (L_\gamma \eta(s(\gamma)) \mid \eta(r(\gamma)))$. Then $p(\gamma)$ is equal to $\overline{f * f^\flat}(\gamma)$ λ^μ-a.e., and hence $f * f^\flat$ is positive definite. If, in addition, f is continuous with compact support, $f * f^\flat$ is continuous with compact support ([15]).

We now define four positive definite functions. Letting $g_1 = (f^\flat + h)^\flat * (f^\flat + h)$, $g_2 = (f^\flat - h)^\flat * (f^\flat - h)$, $g_3 = (f^\flat - \mathrm{i}h)^\flat * (f^\flat - \mathrm{i}h)$, and $g_4 = (f^\flat + \mathrm{i}h)^\flat * (f^\flat + \mathrm{i}h)$, we can write $f * h$ as $1/4\,[g_1 - g_2 + \mathrm{i}g_3 - \mathrm{i}g_4]$. Thus, if $f, h \in M_c(G)$, $f * h \in \mathcal{A}(G)$ and if $f, h \in C_c(G)$, $f * h \in A(G)$. ∎

For the group case, it is known that if G is a locally compact Hausdorff group, then $B(G)$ and hence $P(G)$ separate the points of G ([3]). The proof relies on the fact that if f and h are two Borel functions of compact support, then their convolution is a continuous function. Unfortunately, this is not necessarily true for the convolution of two Borel functions with compact support on a groupoid. Thus, we must construct suitable continuous functions with compact support and use their convolution to separate points.

Proposition 3.4. *$P(G)$ separates the points of G.*

Proof. Suppose γ_1 and γ_2 are in G with $\gamma_1 \neq \gamma_2$. There is a compact set, C, such that $\gamma_1 \in \mathrm{int}(C)$ and $\gamma_2 \notin C$. Thus there exists an open symmetric neighborhood, V, of $i_{s(\gamma_1)}$. Since $\overline{\gamma_2^{-1}C}$ and $\{i_{s(\gamma_1)}\}$ are disjoint and G is regular, there exists an open set U containing $i_{s(\gamma_1)}$ such that $\overline{U} \cap \overline{\gamma_2^{-1}C}$ is the empty set. Let $W = U \cap V$. Then W is an open and symmetric neighborhood of $i_{s(\gamma_1)}$ disjoint from $\gamma_2^{-1}C$. Because G is locally compact, we can choose W such that $\overline{W}$ is compact.

By Urysohn's lemma, there exists a continuous function f mapping G to $[0,1]$ such that $f(\gamma_1)=1$ and such that $f(\gamma)=0$ for all $\gamma \in G \setminus \operatorname{int}(C)$. Likewise, there exists a continuous function h mapping G to $[0,1]$ such that $h(i_{s(\gamma_1)})=1$ and such that $h(\gamma)=0$ for all $\gamma \in G \setminus W$. Since $\operatorname{int}(C) \cap \gamma_2 W = \emptyset$, $(f*h)(\gamma_2)$ which is by definition equal to

$$\int f(\gamma) h(\gamma^{-1}\gamma_2)\, \mathrm{d}\lambda^{r(\gamma_2)}(\gamma),$$

will be equal to 0.

To see that $(f*h)(\gamma_1) > 0$, we first note that $f(\gamma)h(\gamma^{-1}\gamma_1) \neq 0$ if and only if $\gamma \in \operatorname{int}(C) \cap \gamma_1 W$. As $\gamma_1 W$ is open in $r(\gamma_1)G$ and since $\gamma_1 \in \gamma_1 W \cap \operatorname{int}(C) \cap r(\gamma_1)G$, we have that $\gamma_1 W \cap \operatorname{int}(C)$ is an open nonempty set in $r(\gamma_1)G$ such that $f(\gamma)h(\gamma^{-1}\gamma_1) \geqslant 0$. Finally we know that $f(\gamma_1)h(\gamma_1^{-1}\gamma_1)=1$, so that $(f*h)(\gamma_1) > 0$.

Since f and h are in $C_{\mathrm{c}}(G)$, by Proposition 3.3, $f*g \in A(G)$ and we can write $f*h$ as $1/4(p_1 - p_2 + \mathrm{i}p_3 - \mathrm{i}p_4)$ with $p_j \in P(G)$. Since $(f*h)(\gamma_1) \neq (f*h)(\gamma_2)$, one of the p_j's must be such that $p_j(\gamma_1) \neq p_j(\gamma_2)$. Thus $P(G)$ separates the points in G. ∎

COROLLARY 3.5. *If G is a compact topological groupoid, then the sup-norm closure of $B(G)$ is $C(G)$.*

The following corollary is proven for locally compact Hausdorff groups in [3].

COROLLARY 3.6. *If G is a locally compact topological groupoid, then:*
(i) *if $p \in A(G)$, then $p \in C_0(G)$;*
(ii) *if $p \in C_0(G)$, then p is the uniform limit of functions belonging to $A(G)$.*

Proof. (i) This follows from the fact that if p is in $B(G)$, then $\|p\| \geqslant \|p\|_\infty$.

(ii) Note that $A(G)$ is an algebra, is closed under complex conjugation, and separates points of G. If $\gamma \in G$ is such that $r(\gamma) = s(\gamma)$, then there exists $f \in C_{\mathrm{c}}(X)$ such that $f(r(\gamma)) = 1 = f(s(\gamma))$. Then defining p by

$$p(\gamma') = f(s(\gamma'))\overline{f(r(\gamma'))},$$

gives a function p in $A(G)$ with $p(\gamma) = 1$. If $r(\gamma) \neq s(\gamma)$, then there exists $f \in C_{\mathrm{c}}(X)$, such that $f(r(\gamma)) = 1$ and $f(s(\gamma)) = 2$. Defining p as above, $p \in A(G)$ and $p(\gamma) = 2$. Thus the sup-norm closure of $A(G)$ is $C_0(G)$. ∎

4. r-DISCRETE GROUPOIDS

We will need the following lemma which shows that we do not need to work with equivalence classes of functions when G is an r-discrete groupoid as we have to in the general case. Specifically, this means that we can evaluate p at specific points in r-discrete groupoids.

Lemma 4.1. *Suppose G is a r-discrete groupoid and that $N \subseteq G$ is a $\lambda^{\mathcal{Q}}$-null set. Then N is the empty set.*

Proof. We have the following equivalent statements:

$$\begin{aligned}\lambda^{\mathcal{Q}}(N) = 0 &\Leftrightarrow \lambda^{\mu}(N) = 0, \quad \forall \mu \in \mathcal{Q}\\ &\Leftrightarrow \int\limits_{xG} \lambda^{x}(N)\,\mathrm{d}\mu(x) = 0, \quad \forall \mu \in \mathcal{Q}\\ &\Leftrightarrow \lambda^{x}(N) = 0 \ \mu\text{-a.e. } x, \quad \forall \mu \in \mathcal{Q}.\end{aligned}$$

Let $E = \{x \in X \mid \lambda^{x}(N) \neq 0\}$ which is a $\mathcal{Q}$-null set. From [14], E is a $\mathcal{Q}$-null set if and only if $\lambda^{x}(GE) = 0$ for every x, where $GE = \{\gamma \in G \mid s(\gamma) \in E\}$. Since λ^{x} is the counting measure for every x, we have that $GE \cap xG = \emptyset$. In other words, for all $x \in X$, $\{\gamma \in G \mid s(\gamma) = x \in E\} = \emptyset$ which implies that $E = \emptyset$. Thus, for all $x \in X$, $\lambda^{x}(N) = 0$, or for all $x \in X$, $N \cap xG = \emptyset$. Since this is true for all $x \in X$, we must have that $N = \emptyset$. ∎

Since λ^{x} is the counting measure on xG, we can rewrite the integral in the definition of positive definite functions using sums. Furthermore, for fixed $x \in X$ and for $f \in C_{\mathrm{c}}(G)$, $f(\gamma) \neq 0$ for only a finite number of $\gamma \in xG$. We can thus reformulate Definition 3.1 by saying a bounded Borel function $p : G \to \mathbb{C}$ is *positive definite*, where G is an r-discrete groupoid, if and only if for all $x \in X$, for all $n \in \mathbb{Z}^{+}$, for all choices $c_j \in \mathbb{C}$, $1 \leqslant j \leqslant n$, and for all choices $\gamma_j \in xG$, $1 \leqslant j \leqslant n$,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} c_j \overline{c}_k p(\gamma_k^{-1}\gamma_j) \geqslant 0.$$

We note that it is sufficient to pick distinct elements $\gamma_j \in xG$.

For fixed $x \in X$, $n \in \mathbb{Z}^{+}$, and distinct elements $\gamma_j \in xG$, we get n^2 distinct elements for $\gamma_k^{-1}\gamma_j$. We can write a “multiplication table” for part of G by listing the elements γ_j, $1 \leqslant j \leqslant n$, across the top and γ_k^{-1}, $1 \leqslant k \leqslant n$, down the side. The n^2 elements that we are interested in will appear as entries once and only once in this multiplication table. The multiplication table is written in this manner so that the “identities” of the groupoid appear along the diagonal. Thus to determine whether or not p is positive definite for $x \in X$, it is sufficient to apply p to each entry and check whether or not the resulting matrix is positive definite.

In addition to the properties of positive definite functions from the previous section, we have the following properties for r-discrete groupoids.

PROPOSITION 4.2. *If p is a positive definite function, then for all $x \in X$, $p(i_x) \geqslant 0$ and p is a self-adjoint function. Additionally, p satisfies the following inequalities for all $x \in X$ and all $\gamma_1, \gamma_2 \in xG$:*

(i) $p(\gamma_1^{-1}\gamma_1) \pm 2\,\mathrm{Re}(p(\gamma_2^{-1}\gamma_1)) + p(\gamma_2^{-1}\gamma_2) \geqslant 0$;

(ii) $p(\gamma_1^{-1}\gamma_1) \pm 2\,\mathrm{Im}(p(\gamma_2^{-1}\gamma_1)) + p(\gamma_2^{-1}\gamma_2) \geqslant 0$;

(iii) $p(\gamma_1^{-1}\gamma_1)p(\gamma_2^{-1}\gamma_2) \geqslant |p(\gamma_1^{-1}\gamma_2)|^2$.

These inequalites are necessary but not sufficient. For example, let G be the transformation group of $\mathbb{Z}_3$ acting on the circle by rotation. Define $p(z,0) = 1$, $p(z,1) = -.435954$, and $p(z,2) = -.3313$ for all $z \in \mathbb{T}$. Then p satisfies the above inequalities, but p is not positive definite.

PROPOSITION 4.3. *Suppose p is a real-valued, self-adjoint bounded Borel function. Then p is positive definite if and only if for all $x \in X$, for all $n \in \mathbb{Z}^+$, for all choices $d_j \in \mathbb{R}$, $1 \leqslant j \leqslant n$, and for all choices $\gamma_j \in xG$,*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_j d_k p(\gamma_k^{-1}\gamma_j) \geqslant 0.$$

The following property is a generalization of Proposition 3.2 (i).

PROPOSITION 4.4. *$\mathcal{P}(G)$ is a convex cone closed under pointwise products and closed under pointwise convergence.*

Proof. Suppose that p and $-p$ are in $\mathcal{P}(G)$. Then for all $x \in X$, $p(i_x) \geqslant 0$ and $-p(i_x) \geqslant 0$. So, $p(i_x) = 0$ for all $x \in X$. By Proposition 4.2 for all $\gamma \in G$

$$p(\gamma^{-1}\gamma)p(i_{r(\gamma)}^{-1} i_{r(\gamma)}) \geqslant |p(i_{r(\gamma)}\gamma)|^2$$

and hence $p(\gamma) = 0$ for all $\gamma \in G$.

Now suppose that $\{p_n\}$ is a sequence in $\mathcal{P}(G)$ that converges to p pointwise. Then for all $x \in X$, for all suitable choices n_x, for all $c_j \in \mathbb{C}$, and for all distinct choices $\gamma_j \in xG$, $1 \leqslant j \leqslant n_x$, we have that

$$\sum_{j=1}^{n_x}\sum_{k=1}^{n_x} c_j \bar{c}_k p(\gamma_k^{-1}\gamma_j) = \lim_{n\to+\infty} \sum_{j=1}^{n_x}\sum_{k=1}^{n_x} c_j \bar{c}_k p_n(\gamma_k^{-1}\gamma_j) \geqslant 0.$$

Hence p is in $\mathcal{P}(G)$. ∎

PROPOSITION 4.5. *If p is a positive definite function on G strictly bounded by ρ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a holomorphic function in $\{z \in \mathbb{C} \,\big|\, |z| < \rho\}$ with $a_n \geqslant 0$, then $f \circ p$ is positive definite. In particular, if p is positive definite on G, then so is $\exp(p)$, and if p is a positive definite function bounded by 1 and α is any fixed number strictly between 0 and 1, then $(1 - \alpha p)^{-1}$ is a positive definite function.*

Proof. We know from Proposition 4.4 that for all $N \in \mathbb{Z}^+$ that $\varphi_N = \sum_{n=0}^{N} a_n p^n$ is positive definite. For all $\gamma \in G$, $\sum_{n=0}^{N} a_n p^n(\gamma)$ converges in $\mathbb{C}$ so we can define $(f \circ p)(\gamma) = \sum_{n=0}^{\infty} a_n p^n(\gamma)$ for all $\gamma \in G$. Since $f \circ p$ is the pointwise limit of φ_N, $f \circ p$ is positive definite.

Applying this result to the function $f(z) = (1-z)^{-1}$, we get that $(1-\alpha p)^{-1}$ is a positive definite function. ∎

The final property in our list requires an additional assumption on the size of the sets xG for $x \in X$.

PROPOSITION 4.6. *Suppose G is an r-discrete groupoid whose fibers all have finite constant cardinality of $N > 1$. If p is a real-valued, nonnegative positive definite function in $\mathcal{P}(G)$, then for all real $\alpha \geqslant N-2$, p^α is a positive definite function. If $0 < \alpha < N-2$ and α is not an integer, then there exists a real valued positive definite function with positive values such that p^α is not positive definite.*

Proof. In [4], Fitzgerald and Horn proved that if N is an integer greater than 1 and $A = (a_{kj})$ is a $N \times N$ real symmetric positive definite matrix with nonnegative entries and if $\alpha \geqslant N-2$, then $A^\alpha = (a_{kj}^\alpha)$ is positive definite. Thus, for the groupoid G, for all $x \in X$ we let $p(\gamma_k^{-1}\gamma_j) = a_{kj}$ for $\gamma_k, \gamma_j \in xG$. We assume that the cardinality of xG is constant N, so that for all $x \in X$ we are applying p to a $N \times N$ matrix. Then p is a positive definite function on G.

If $0 < \alpha < N-2$ and α is not an integer, Fitzgerald and Horn constructed a matrix $A_\varepsilon = (1 + \varepsilon kj)$ which is positive definite for all $\varepsilon > 0$. They then showed that A_ε^α fails to be positive definite for sufficiently small ε (that depends on α). Again, for all $x \in X$, we let $p(\gamma_k^{-1}\gamma_j) = 1 + \varepsilon kj$ for all γ_k, γ_j in xG. Since A_ε^α fails to be positive definite, p also will not be positive definite. ∎

In particular, note that if $N = 2$, then the above shows that if p is a positive definite function, then $p^{1/2}$ is a positive definite function. However, if $N > 2$, then there will always exist a positive definite function, p, such that $p^{1/2}$ is not positive definite.

We will need to use the following theorem, attributed to Gershgorin, to compute the Fourier-Stieltjes algebras of groupoids which satisfy property (v) of Proposition 2.2.

THEOREM 4.7. (Gershgorin) *The union of all discs*

$$K_j = \left\{ \mu \in \mathbb{C} \mid |\mu - a_{jj}| \leqslant \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{jk}| \right\}$$

contains all eigenvalues of the $n \times n$ matrix $A = (a_{jk})$.

COROLLARY 4.8. *Let A be a self-adjoint $n \times n$ matrix such that $a_{jj} \geqslant 0$ for $1 \leqslant j \leqslant n$. If*

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{jk}| \leqslant a_{jj}$$

for all j, $1 \leqslant j \leqslant n$, then A is positive definite.

If A is a positive definite matrix, then A is a self-adjoint matrix with non-negative eigenvalues. However, the inequality in Corollary 4.8 does not necessarily have to be true. For example, consider the $n \times n$ matrix $A = [1]_{j,k=1}^{n}$, i.e., the matrix whose entries are all ones. A is positive definite but does not satisfy the inequality in Corollary 4.8 if n is greater than two.

As an interesting fact we include the following theorem on a nonstandard way to write any matrix as a linear combination of positive definite matrices.

THEOREM 4.9. *Suppose that B is any $n \times n$ matrix with complex entries. Then B can be written as $P_1 - P_2 + \mathrm{i}(P_3 - P_4)$, where P_j is a positive definite matrix for $j = 1, 2, 3, 4$.*

Proof. We may assume without loss of generality that B is a self-adjoint matrix with entries $(b_{jk})_{j,k=1}^{n}$. If $b_{jj} \neq 0$, let m_j be the first integer such that

$$m_j |b_{jj}| \geqslant \sum_{\substack{k=1 \\ k \neq j}}^{n} |b_{jk}|.$$

If $b_{jj} = 0$, then let m_j be the first integer such that

$$m_j \geqslant \sum_{\substack{k=1 \\ k \neq j}}^{n} |b_{jk}|.$$

Let $\operatorname{sgn}(b_{jj}) = 1$ if $b_{jj} > 0$ and $\operatorname{sgn}(b_{jj}) = -1$ if $b_{jj} < 0$ and let P be the $n \times n$ matrix such that

$$p_{jk} = \begin{cases} m_j|b_{jj}| & \text{if } j = k \text{ and } b_{jj} \neq 0, \\ m_j & \text{if } j = k \text{ and } b_{jj} = 0, \\ b_{jk} & \text{if } j \neq k. \end{cases}$$

Let Q be the $n \times n$ matrix such that

$$q_{jk} = \begin{cases} |b_{jj}|(m_j - \operatorname{sgn}(b_{jj})) & \text{if } j = k \text{ and } b_{jj} \neq 0, \\ m_j & \text{if } j = k \text{ and } b_{jj} = 0, \\ 0 & \text{if } j \neq k. \end{cases}$$

Then $P - Q = B$ and P and Q are positive definite matrices by the corollary to Gershgorin's theorem. ∎

We now suppose that there exists an $N \in \mathbb{Z}^+$ such that for all $x \in X$, $|xG| \leqslant N$. If G has this property, we say that G has *uniformly bounded fibers.* Suppose that b is a bounded and Borel (respectively continuous) function on G. We write b as the sum of self-adjoint, bounded and Borel (respectively continuous) functions and construct p and q for a self-adjoint bounded Borel (respectively continuous) function. Let

$$p(\gamma) = \begin{cases} N\|b\|_\infty & \text{if } \gamma \in G^0, \\ b(\gamma) & \text{if } \gamma \in G \setminus G^0; \end{cases}$$

and let

$$q(\gamma) = \begin{cases} N\|b\|_\infty - b(\gamma) & \text{if } \gamma \in G^0, \\ 0 & \text{if } \gamma \in G \setminus G^0. \end{cases}$$

Then p and q are bounded, Borel (respectively continuous) functions. The only possible points of discontinuity for p and q would be if there exist a sequence $\{\gamma_n\} \subset G \setminus G^0$ that converge to an element $\gamma \in G^0$. Since G is an r-discrete groupoid, however, this can not happen as $G \setminus G^0$ is closed.

We use the matrix definition of positive definiteness to show that p and q are positive definite. Suppose $x \in X$ is fixed and the elements in xG are ordered in some fixed way. For fixed k, if $j \neq k$, then $\gamma_k^{-1}\gamma_j \in G \setminus G^0$. We have that

$$\sum_{\substack{j=1 \\ j \neq k}}^{N} |p(\gamma_k^{-1}\gamma_j)| = \sum_{\substack{j=1 \\ j \neq k}}^{N} |b(\gamma_k^{-1}\gamma_j)| \leqslant (N-1)\|b\|_\infty < p(\gamma_k^{-1}\gamma_k).$$

Thus, p is positive definite. Likewise, for fixed k, $N\|b\|_\infty - b(\gamma_k^{-1}\gamma_k) \geqslant 0$, so that q is a positive definite function.

Furthermore, from [14] Theorem 4.1, if $p \in \mathcal{P}(G)$ we have that $\|p\| = \operatorname{ess\,sup}\{p(x) \mid x \in X\}$ which implies that $\|p\| = N\|b\|_\infty$. Since $q(x) = N\|b\|_\infty - b(x)$ for all $x \in X$ and $N\|b\|_\infty - b(x) \leqslant (N+1)\|b\|_\infty$, we have that $\|q\| \leqslant (N+1)\|b\|_\infty$. So, if b is a self-adjoint, Borel function which is decomposed as above into $p - q$, then $\|b\| \leqslant (2N+1)\|b\|_\infty$. This gives the following theorem.

THEOREM 4.10. *Let G be an r-discrete groupoid with uniformly bounded fibers. If b is any bounded Borel function on G, then $b = p_1 - p_2 + \mathrm{i}(p_3 - p_4)$ where p_j is positive definite, bounded, and Borel for $j = 1, 2, 3, 4$. Moreover, if b is continuous, then the p_j's are continuous.*

THEOREM 4.11. *Let G be an r-discrete groupoid having uniformly bounded fibers. Then we have that $B(G) = B_1(G) \cong C(G)$ and $\mathcal{B}(G) \cong \mathcal{M}(G)$ as topological spaces and as Banach algebras.*

Proof. From the definitions, we have that $B_1(G) \subseteq B(G) \subseteq C(G)$ and $\mathcal{B}(G) \subseteq \mathcal{M}(G)$. By Theorem 4.10, $B_1(G) = B(G) = C(G)$ and that $M(G) = \mathcal{B}(G)$ as sets. Since for any $b \in \mathcal{B}(G)$ we have that $\|b\|_\infty \leqslant \|b\| \leqslant 2(2N+1)\|b\|_\infty$, where N is the uniform bound of the fibers, we know that the topologies on $\mathcal{B}(G)$ and $B(G)$ given by the sup-norm and the Fourier-Stieltjes norm are equivalent. Thus $B(G) \cong C(G)$ and that $\mathcal{B}(G) \cong \mathcal{M}(G)$ as Banach algebras. ∎

Instead of assuming that G has uniformly bounded fibers, we now assume that G has an ordering map that is bounded on compact sets. In this case, the above can be modified to show that the Fourier-Stieltjes closure of $C_c(G)$ is $A(G)$ and that the Fourier-Stieltjes closure of $M_c(G)$ is $\mathcal{A}(G)$.

Suppose that b is a bounded Borel (respectively continuous) function with compact support written as a sum of self-adjoint, bounded Borel (respectively continuous) functions with compact support. We thus construct p and q for a self-adjoint bounded Borel (respectively continuous) function with compact support. Let F be the compact set of G on which b is supported. Since O is bounded on F, there exists $N \in \mathbb{Z}^+$ such that if $\gamma \in F$, then $O(\gamma) \leqslant N$. We define p and q basically as above. That is, define

$$p(\gamma) = \begin{cases} \|b\|_\infty & \text{if } \gamma \in G^0, \\ b(\gamma) & \text{if } \gamma \in G \setminus G^0; \end{cases}$$

and define

$$q(\gamma) = \begin{cases} N\|b\|_\infty - b(\gamma) & \text{if } \gamma \in G^0, \\ 0 & \text{if } \gamma \in G \setminus G^0. \end{cases}$$

Then p and q are bounded, Borel (respectively continuous) functions. Again, we use the matrix definition of positive definiteness to show that p and q are positive definite functions. In this case, however, since the fibers are possibly infinite, we must check that for any $x \in X$ and for any $n \in \mathbb{Z}^+$ that the $n \times n$ matrix is positive definite. Suppose that $\gamma_1, \ldots, \gamma_n$ are distinct elements in xG. Then, for fixed k, if $j \neq k$, $\gamma_k^{-1}\gamma_j \in G \setminus G^0$. We have that

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} |p(\gamma_k^{-1}\gamma_j)| = \sum_{\substack{j=1 \\ j\neq k}}^{n} |b(\gamma_k^{-1}\gamma_j)|.$$

Since k is fixed, as j varies we get distinct elements $\gamma_k^{-1}\gamma_j \in r(\gamma_k^{-1})G$. We know that, at most, b is nonzero on N of these elements as O is 1-1 on $r(\gamma_k^{-1})G$. Thus, we can make the following estimate:

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} |b(\gamma_k^{-1}\gamma_j)| \leqslant N\|b\|_\infty.$$

So, again by the corollary to Gershgorin's theorem, we have that p is positive definite. That q is positive definite follows exactly as before.

Theorem 4.12. *Let G be an r-discrete groupoid having an ordering map that is bounded on compact sets of G. If b is any bounded Borel function with compact support on G, then $b = p_1 - p_2 + \mathrm{i}(p_3 - p_4)$ where p_j is positive definite, bounded, and Borel for $j = 1, 2, 3, 4$. Moreover, if b is continuous, then the p_j's are continuous.*

Theorem 4.13. *Let G be an r-discrete groupoid having an ordering map that is bounded on compact sets of G. Then $M_{\mathrm{c}}(G) \subseteq \mathcal{A}(G) \subseteq \mathcal{B}(G)$ and $C_{\mathrm{c}}(G) \subseteq A(G) \subseteq B(G)$.*

5. INVOLUTION AND ORDER

In the previous section, we saw that if G is an r-discrete groupoid with uniformly bounded fibers, then $C(G) \cong B(G)$ as topological Banach algebras. It can be shown for topological groupoids that $\flat$ is an involution on $\mathcal{B}(G)$ and on $B(G)$ such that $\|b^\flat\| = \|b\|$. Thus, $B(G)$ and $\mathcal{B}(G)$ can be viewed as Banach $*$-algebras. For $B(G)$ this involution is different from the typical involution defined on $C(G)$ by $f^*(\gamma) = \overline{f(\gamma)}$ and thus $B(G)$ and $C(G)$ are not isomorphic as Banach $*$-algebras when G is an r-discrete groupoid with uniformly bounded fibers. $B(G)$ and $C(G)$ also have different natural order structures which we now discuss briefly.

Since $P(G)$ is a cone in $B(G)$, we can use $P(G)$ to define a partial ordering on $B(G)$. Likewise the set of nonnegative real-valued functions is a cone in $C(G)$ and can also be used to define a partial ordering on $C(G)$ and hence on $B(G)$ when

G is an r-discrete groupoid with uniformly bounded fibers. We want to show that these two sets are unequal when $N \geqslant 2$.

Suppose that G is an r-discrete groupoid such that for all $x \in X$, $|xG| \leqslant N$. Let $p(i_x) = N$ for all $x \in X$ and let $p(\gamma) = -1$ for all $\gamma \in G \setminus G^0$. Then p is a self-adjoint function and by Corollary 4.8, p is positive definite. Since G is an r-discrete groupoid, p as defined above is continuous. Thus, there is a function in $P(G)$ that is not in the cone for $C(G)$. To construct a function that is in the cone for $C(G)$ which is not in $P(G)$, we consider two cases.

Suppose first that G has a uniform bound of N where $N > 2$. By Proposition 4.6, we know that there always exists a real-valued, nonnegative positive definite function p such that $p^{1/2}$ is not positive definite. Since p is real-valued and nonnegative, $p^{1/2}$ will also be real-valued and nonnegative and hence will be in the cone for $C(G)$ but not in $P(G)$.

If $N = 2$, then we can define a function that is in the cone of $C(G)$ and is not in $P(G)$ as follows. For $\gamma \in G^0$ define $p(\gamma) = 1$ and for $\gamma \in G \setminus G^0$, let $p(\gamma) = 2$. Then p is continuous because G is an r-discrete groupoid and thus p is in the cone of $C(G)$. To see that p is not positive definite, fix an x in X such that $|xG| = 2$. Let γ denote the element in xG that is not i_x. If p is positive definite, then by Proposition 4.2, p must satisfy

$$p(i_x)p(i_x) \geqslant |p(\gamma)|^2,$$

which is obviously not satisfied by p, and thus the function constructed must not be positive definite.

If $N = 1$, then $G = G^0$. In this case $P(G)$ and the nonnegative real-valued continuous functions on G are equal.

For locally compact groups, it is true that $B(G_1)$ and $B(G_2)$ are isometrically isomorphic as Banach algebras if and only if G_1 and G_2 are topologically isomorphic as groups. We would like that this property hold for locally compact topological groupoids as well, but as yet, we have not seriously considered this property in the groupoid case. However, the following proposition does state that in one specific case, we do have this duality property.

PROPOSITION 5.1. *If $n \neq m$, then $B(\mathbb{T} \times \mathbb{Z}_m)$ and $B(\mathbb{T} \times \mathbb{Z}_n)$ are not isomorphic.*

Proof. Let $\mathbb{T} \times \mathbb{Z}_n$ be the groupoid obtained by letting $\mathbb{Z}_n$ act on $\mathbb{T}$ by rotation by $e^{2\pi \mathrm{i}/n}$. Using a standard result from analysis, $C(\mathbb{T} \times \mathbb{Z}_n) \cong C(\mathbb{T} \times \mathbb{Z}_m)$ if and only if $\mathbb{T} \times \mathbb{Z}_n$ and $\mathbb{T} \times \mathbb{Z}_m$ are homeomorphic as spaces. Since $B(\mathbb{T} \times \mathbb{Z}_n) \cong C(\mathbb{T} \times \mathbb{Z}_n)$ and $B(\mathbb{T} \times \mathbb{Z}_m) \cong C(\mathbb{T} \times \mathbb{Z}_m)$ by Theorem 4.11, we have the desired result. ∎

Acknowledgements. The author wishes to thank Arlan Ramsay and Larry Baggett for their helpful guidance and constant encouragement.

REFERENCES

1. C. Berg, J.P.R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Springer-Verlag, New York 1984, pp. 66-92.
2. E.K. Blum, *Numerical Analysis and Computation Theory and Practice*, Addison-Wesley, Reading, MA 1972, pp. 230–232.
3. P. Eymard, L'algebre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **192**(1964), 181–236.
4. C.H. Fitzgerald, R.A. Horn, On fractional Hadamard powers of positive definite matrices, *J. Math. Anal. Appl.* **61**(1977), 633–642.
5. P. Hahn, Haar measure for measure groupoids, *Trans. Amer. Math. Soc.* **242**(1978), 1–34.
6. P. Hahn, The regular representations of measure groupoids, *Trans. Amer. Math. Soc.* **242**(1978), 35–72.
7. G.W. Mackey, Ergodic theory, group theory, and differential geometry, *Proc. Nat. Acad. Sci. U.S.A.* **50**(1963), 1184–1191.
8. G.W. Mackey, Ergodic theory and virtual groups, *Ann. of Math.* **166**(1966), 187–207.
9. P.S. Muhly, C. Qiu, J. Xia, Analyticity, uniform averaging, and K-theory, in *Algebraic Methods in Operator Theory*, Birkhäuser 1994, pp. 328–349.
10. A. Ramsay, Virtual groups and group actions, *Adv. Math.* **6**(1971), 253–322.
11. A. Ramsay, Nontransitive quasiorbits in Mackey's analysis of group extensions, *Acta Math.* **137**(1976), 17–48.
12. A. Ramsay, Topologies on measured groupoids, *J. Funct. Anal.* **47**(1982), 314–343.
13. A. Ramsay, The Mackey-Glimm dichotomy for foliations and other polish groupoids, *J. Funct. Anal.* **94**(1990), 358–374.
14. A. Ramsay, M.E. Walter, Fourier-Stieltjes algebras of locally compact groupoids, *J. Funct. Anal.*, to appear.
15. J. Renault, *A groupoid approach to* C^**-algebras*, Springer–Verlag, Berlin 1980, pp. 1–35, 47–74, 121–138.
16. J. Renault, The Fourier algebra of a measured groupoid and its multipliers, *J. Funct. Anal.* **145**(1997), 455–490.
17. M.E. Walter, W^*-algebras and nonabelian harmonic analysis, *J. Funct. Anal.* **11**(1972), 17–38.
18. B. Weiss, Measurable Dynamics, *Contemp. Math.* **26**(1984), 395–421.

KARLA J. OTY
Department of Mathematics
Southeaston Oklahoma State University
Durant, OK 74701
U.S.A.

E-mail: koty@sosu.edu

Received July 7, 1997; revised November 11, 1997.