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FINITELY STRONGLY REDUCTIVE OPERATORS

DOUGLAS R. FARENICK and DON HADWIN

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ABSTRACT. This paper initiates the study of those operators, acting on separable Hilbert spaces, that commute asymptotically with their sequences of almost-invariant finite-rank projections.

KEYWORDS: *Strongly reductive operator, almost-invariant projection.*

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In connection with the still open and nontractable question of whether every reductive operator acting on a separable infinite-dimensional Hilbert space is normal, Harrison ([9]) introduced the notion of a strongly reductive operator. We recall that a sequence $\{P_k\}_k$ of projections is said to be almost-invariant for an operator A if $\lim\limits_{k\to\infty} \|(1-P_k)AP_k\| = 0$, and that an operator T is called strongly reductive if $\|P_kT - TP_k\| \to 0$ for each of its sequences $\{P_k\}_k$ of almost invariant projections. In his paper Harrison asked whether strongly reductive operators must be normal and shortly thereafter Apostol, Foiaş, and Voiculescu ([2]) established that indeed they must be. The combined investigations of Harrison and Apostol, Foiaş, and Voiculescu show that an operator is strongly reductive if and only if it is normal and its spectrum is nowhere dense and has connected complement.

Our purpose here is to introduce and initiate the study of operators that are reduced asymptotically by sequences of almost-invariant finite-rank projections. Thus, an operator T will be called *finitely strongly reductive* if $\lim\limits_{k\to\infty} \|P_kT - TP_k\| = 0$ whenever $\{P_k\}_k$ is a sequence of finite-rank projections such that $\|(1-P_k)TP_k\| \to 0$. (As for the existence of almost-invariant projections of finite rank, Halmos proved in [8] that for every operator T there is a sequence of finite-rank projections P_k such that $\operatorname{rank} P_k = k$ and $\|(1-P_k)TP_k\| \to 0$.)

Given an arbitrary operator, it is in general difficult to decide whether or not it is finitely strongly reductive. Therefore most of the results presented in this article deal with necessary conditions for various types of operators. More precisely, we will show that all isometries are finitely strongly reductive and that there are normal operators that are not so. We will show that in many instances, however, an operator cannot be finitely strongly reductive without being normal. It remains an open problem to determine precisely which normal operators are finitely strongly reductive.

We first set some notation. If an operator S is unitarily equivalent to an operator T, meaning that $S = U^*TU$ for some unitary U, then this will be expressed by $S \sim T$. If S is approximately (unitarily) equivalent to T, which is to say that S is the norm-limit of a sequence of operators $U_k^*TU_k$ where each U_k is unitary, then we write $S \sim_{\mathrm{a}} T$. An operator A is called a direct summand (resp. an approximate direct summand) of T if there exists an operator B such that $A \oplus B \sim T$ (resp. $A \oplus B \sim_{\mathrm{a}} T$).

The following facts are for the most part straightforward to verify and are worth noting explicitly.

1. Suppose that T is finitely strongly reductive. If $S \sim_{\mathrm{a}} T$, then S is finitely strongly reductive; if $T \sim_{\mathrm{a}} A \oplus B$, then A and B are finitely strongly reductive.

2. An operator acting on a finite-dimensional space is finitely strongly reductive if and only if it is normal.

3. If T is finitely strongly reductive, then so is $T - \lambda 1$ for every $\lambda \in \mathbb{C}$.

4. If T is finitely strongly reductive, then $\ker T \subset \ker T^*$ and so $\operatorname{ind}(T-\lambda 1) \leqslant 0$ for all λ in the semi-Fredholm domain of T.

5. If a quasitriangular operator is finitely strongly reductive, then it is quasidiagonal and $\operatorname{ind}(T - \lambda 1) = 0$ for all λ in the semi-Fredholm domain of T. This is seen as follows. Because T is quasitriangular, there is a sequence of almost-invariant finite-rank projections P_k converging strongly to 1; as T is finitely strongly reductive, this same sequence of projections is almost-invariant for T^* as well and so T is quasidiagonal. As for the assertion concerning the index, if $T - \lambda 1$ is semi-Fredholm, then we have on the one hand that $\operatorname{ind}(T - \lambda 1) \geqslant 0$ (because T is quasitriangular) and on the other hand that $\operatorname{ind}(T - \lambda 1) \leqslant 0$ (because T is finitely strongly reductive).

6. Every quasitriangular, essentially normal, finitely strongly reductive operator T must be a compact perturbation of a normal operator. Proof: from (5) we see that T is an essentially normal operator for which $\operatorname{ind}(T - \lambda 1) = 0$ for all

λ in the semi-Fredholm domain of T (which is the complement of $\sigma_{\mathrm{e}}(T)$ in this case) and, therefore, the Brown-Douglas-Fillmore Theorem ([4]) states that T is a compact perturbation of a normal.

The conditions that define a finitely strongly reductive operator use almost-invariant projections P_k of finite rank, however the ranks of the projections can become arbitrarily large as k increases. If one were to impose the further requirement that the ranks of the projections be uniformly bounded, then the types of operators that would be reduced asymptotically by such sequences of finite-rank almost-invariant projections are easily characterised by a property of their approximate eigenvalues, which is described in the proposition below. Therefore, the use of almost-invariant finite-rank projections of arbitrarily large rank is a key feature in the study of finitely strongly reductive operators.

Proposition 1. *The following properties of an operator T are equivalent:*

(i) $\|P_jT - TP_j\| \to 0$ *whenever* $\{P_j\}_j$ *is a sequence of rank-*1 *almost-invariant projections for* T.

(ii) *For every positive integer* m, $\|P_jT - TP_j\| \to 0$ *whenever* $\{P_j\}_j$ *is a sequence of almost-invariant projections for* T *of rank at most* m.

(iii) *Every approximate eigenvalue of* T *is a normal approximate eigenvalue, meaning that* $\|(T - \lambda 1)x_j\| \to 0$ *for a sequence of unit vectors* x_j *implies that* $\|(T - \lambda 1)^* x_j\| \to 0$.

Proof. The proof that (i) implies (iii) is precisely the same as the proof given by Harrison in Theorem 3.3 of [9] — just take projections of rank-1 onto the subspace spanned by an individual approximate eigenvector.

That (ii) implies (i) is clear, and so we take up the proof that (iii) implies (ii). Suppose that $\{P_j\}_j$ is a sequence of projections and that $m < \infty$ is the least upper bound of the numbers $\mathrm{rank} P_j$. Suppose further that $\|(1 - P_j)TP_j\| \to 0$. Let n denote any one of the integers between 1 and m inclusive for which infinitely many of the projections P_j have rank n; denote the resulting subsequence of rank n projections by $\{P_j\}_j$ again. Let H_n denote a fixed n-dimensional Hilbert space. For each j we factor P_j as $P_j = V_jV_j^*$, where each V_j denotes an isometry $H_n \to H$ with range $P_j(H)$. The sequence $\{V_j^*TV_j\}$ in $B(H_n)$ has a convergent subsequence, call it $\{V_j^*TV_j\}_j$ once again, converging to an operator $\Lambda \in B(H_n)$. Thus,

$$\begin{aligned}\|TV_j - V_j\Lambda\| &= \|TV_j - V_j\Lambda + V_j(V_j^*TV_j) - V_j(V_j^*TV_j)\| \\ &\leqslant \|TV_j - V_jV_j^*TV_j\| + \|\Lambda - V_j^*TV_j\| \\ &= \|TP_j - P_jTP_j\| + \|\Lambda - V_j^*TV_j\|,\end{aligned}$$

and so $\|TV_j - V_j\Lambda\| \to 0$. Hence, Λ is an element of the (spatial) left $n \times n$ matricial spectrum of T ([7]). Because every approximate eigenvalue is a normal approximate eigenvalue, Example 1 of [7] shows that Λ is normal and moreover that $\lim_j \|T^*V_j - V_j\Lambda^*\| = 0$. The limit $\|P_jT - TP_j\| \to 0$ follows from the inequality

$$\begin{aligned}\|TP_j - P_jT\| &= \|(TV_j - V_j\Lambda)V_j^* + V_j\Lambda V_j^* - V_jV_j^*T\| \\ &\leqslant \|TV_j - V_j\Lambda\| + \|T^*V_j - V_j\Lambda^*\|.\end{aligned}$$

Thus, we have shown that there is a subsequence (of almost-invariant projections of rank n) of the original sequence of projections such that the elements of the subsequence commute asymptotically with T. Repeated applications of the preceding argument at those k between 1 and m for which infinitely many projections in the original sequence have rank k allows us to conclude finally that there is a subsequence of the original that commutes with T asymptotically. But if this is true for such subsequences, then it must be true for the original sequence as well, implying that T has property (ii). ∎

COROLLARY 2. *A compact operator T satisfies any one of the equivalent conditions in the proposition above if and only if* $\ker(T - \lambda 1)$ *reduces T for every $\lambda \in \mathbb{C}$.*

Proof. Assume T is compact. The necessity of the stated condition is clear; to prove that it is a sufficient condition, assume that $\lambda \in \sigma(T)$ and that x_j is a sequence of unit vectors for which $\|(T - \lambda 1)x_j\| \to 0$. Let x be a weak-limit of some subsequence of $\{x_j\}_j$. By the compactness of T,

$$0 = \lim_j (Tx_j - \lambda x_j) = Tx - \lambda x$$

and so $x \in \ker(T - \lambda 1) \subset \ker(T - \lambda 1)^*$. Hence, $\|(T - \lambda 1)^* x_j\| \to 0$, which completes the argument. ∎

Observe that the class of operators satisfying any one of the equivalent conditions of Proposition 1 is so large as to include, for example, all hyponormal operators and the classical Volterra integral operator.

We show next that in the study of finitely strongly reductive operators one may exclude from consideration those operators whose approximate point spectrum has interior.

THEOREM 3. *The approximate point spectrum of a finitely strongly reductive operator has no interior. In particular, in order for a normal operator to be finitely strongly reductive, or for an arbitary operator and its adjoint to be finitely strongly reductive, the operator must have nowhere dense spectrum.*

Proof. We partition the proof into three steps.

Step 1. (Berg) Suppose that m is an even integer such that $m \geqslant 16$. There exist operators N and P acting on an $4m^2$-dimensional space such that N is a normal contraction, P is a projection, $\|(1-P)NP\| < 100m^{-1}$, and $\|(1-P)N^*P\| \geqslant 1 - 100m^{-1}$.

To prove this assertion, suppose that $\{x_j\}_j$ is an orthornormal basis for a $4m^2$-dimensional Hilbert space. With respect to this basis, let A be the backward weighted shift operator with weights α_j as follows: when $1 \leqslant j \leqslant 2m^2$, $\alpha_j = j(2m^2)^{-1}$; when $2m^2 < j < 4m^2$, $\alpha_j = \alpha_{4m^2-j}$, and $\alpha_{4m^2} = 0$.

If P is the projection onto the span of $x_1, \ldots, x_{2m^2}$, then the range of P is A-invariant and $(1-P)AP = 0$. As for $(1-P)A^*P$, it is readily seen from the matrix representation of A^* that $\|(1-P)A^*P\| = \alpha_{2m^2} = 1$.

The operator A^* is a weighted unilateral shift operator of unit norm and with the properties that consecutive weights differ by less than m^{-2} and each of α_1 and α_{4m^2-1} are less than m^{-2}. Thus, the operator A^* satisfies the hypothesis of Theorem 1 of Berg ([3]). In [3], Theorem 1, Berg constructs a normal contraction, call it N^*, such that $\|A^* - N^*\| < 100m^{-1}$. Therefore, the normal operator N has the properties that

$$\|(1-P)NP\| = \|(1-P)AP + (1-P)(N-A)P\| \leqslant \|N - A\| \leqslant 100m^{-1}$$

and

$$1 = \|(1-P)A^*P\| = \|(1-P)N^*P + (1-P)(A^* - N^*)P\| \leqslant \|(1-P)N^*P\| + 100m^{-1}$$

and this completes the first step.

Step 2. There exist diagonal operators that are not finitely strongly reductive.

Indeed by Step 1, there exist $k_n \times k_n$ matrices D_n and Q_n, with $k_n \to \infty$, and positive numbers $\varepsilon_n \to 0$ such that D_n is a diagonal contraction, Q_n is a projection, $\|(1-Q_n)D_nQ_n\| < \varepsilon_n$, and $\|(1-Q_n)D_n^*Q_n\| \geqslant 1 - \varepsilon_n$. Thus if D is the orthogonal direct sum of the diagonal matrices D_n and P_n is the finite-rank projection $P_n = 0 \oplus 0 \oplus \cdots \oplus Q_n \oplus 0 \cdots$, then

$$\lim_{n\to\infty} \|(1-P_n)DP_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(1-P_n)D^*P_n\| = 1.$$

Hence, the diagonal operator D fails to be finitely strongly reductive.

Step 3. The general case.

If T is finitely strongly reductive, then its approximate point spectrum consists of reducing approximate eigenvalues. Thus, from 6.1 of [11] it follows that if $\lambda \in \sigma_{\mathrm{ap}}(T)$, then either $\lambda \in \sigma_{\mathrm{le}}(T)$ or λ is an eigenvalue of finite multiplicity, isolated in $\sigma_{\mathrm{ap}}(T)$. Because every finite-dimensional eigenspace of T necessarily reduces T, there are at most countably many eigenvalues of the latter type. Therefore, if $\sigma_{\mathrm{ap}}(T)$ were to have interior, then so would $\sigma_{\mathrm{le}}(T)$, for $\sigma_{\mathrm{ap}}(T)$ is the union of the compact set $\sigma_{\mathrm{le}}(T)$ together with a countable or finite set of points isolated in $\sigma_{\mathrm{ap}}(T)$. But this would imply, after scaling T so that $\mathbb{D} \subset \sigma_{\mathrm{le}}(T)$, that every diagonal contraction is an approximate direct summand of T (see 4.6 of [11]); in particular, this would imply that T would have an approximate direct summand that is not finitely strongly reductive, which is impossible when T is finitely strongly reductive. Hence, the approximate point spectrum of T cannot have interior. ∎

Further work is necessary to better understand the notion of finitely strongly reductive operator within the class of normal operators. For example, we do not know whether a normal operator is finitely strongly reductive if and only if its spectrum is nowhere dense, nor do we know whether reductive normal operators are finitely strongly reductive.

THEOREM 4. *Every isometry is finitely strongly reductive.*

Proof. Let W be an isometry and let $\{P_n\}_n$ be a sequence of finite-rank projections satisfying $\|(1-P_n)WP_n\| < \varepsilon_n$, where $\varepsilon_n \to 0$. With respect to the decomposition $H = P_n(H) \oplus P_n(H)^\perp$, W is expressed as a 2×2 operator matrix W_n:

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}.$$

Thus, $\|B_n\| = \|P_nW(1-P_n)\|$ and $\|C_n\| = \|(1-P_n)WP_n\|$. The operator A_n acts on a finite-dimensional space and has, therefore, a polar decomposition of the form $A_n = U_nH_n$, where $H_n \geqslant 0$ and U_n is unitary. If we denote the identity operator on $P_n(H)$ by 1_n, then

$$\begin{pmatrix} 1_n & 0 \\ 0 & 1 \end{pmatrix} = W_n^*W_n = \begin{pmatrix} A_n^*A_n + C_n^*C_n & * \\ * & * \end{pmatrix}$$

implies that

$$\begin{aligned}(\varepsilon_n)^2 \geqslant \|C_n^*C_n\| = \|1_n - A_n^*A_n\| = \|U_n^*U_n - H_n^2\| = \|U_n^*(1_n - U_nH_n^2U_n^*)U_n\| \\ = \|U_n^*(1_n - A_nA_n^*)U_n\| = \|1_n - A_nA_n^*\|.\end{aligned}$$

The $(1,1)$-entries of the inequality $0 \leqslant 1 - WW^* = 1 - W_n W_n^*$ yield $0 \leqslant 1 - (A_n A_n^* + B_n B_n^*)$, whence $B_n B_n^* \leqslant 1 - A_n A_n^*$ and

$$\|(1-P_n)W^*P_n\|^2 = \|B_n^*\|^2 = \|B_n B_n^*\| \leqslant \|1_n - A_n A_n^*\| \leqslant (\varepsilon_n)^2.$$

From $\|(1-P_n)W^*P_n\| \to 0$ it follows that W is finitely strongly reductive. ∎

Corollary 5. *If W is an isometry such that W^* is finitely strongly reductive, then W is a unitary operator.*

Proof. Although the unilateral shift S is finitely strongly reductive, its adjoint S^* is not (since $\sigma_{\mathrm{ap}}(S^*)$ has interior). Thus, if any copies of S appear in the Wold-von Neumann decomposition of W, then S^* will appear as a direct summand of W^*, which is impossible if W^* is finitely strongly reductive. Hence, all that can appear in the Wold-von Neumann decomposition of W is a unitary, which is to say that W is unitary. ∎

Theorem 4 indicates that the unilateral shift and the bilateral shift are finitely strongly reductive. In fact these are actually the only periodic weighted shifts with this property. Suppose that T is a weighted (unilateral or bilateral) shift operator with (nonnegative) periodic weights $\alpha_1, \ldots, \alpha_n$. Let W denote the unweighted (unilateral if T is unilateral or bilateral otherwise) shift. Following 2.2 in [5], there is a unitary $U : H \to H \otimes \mathbb{C}^n$ such that

$$UTU^* = \begin{pmatrix} 0 & & & & \alpha_n W \\ \alpha_1 1 & 0 & & & \\ & \alpha_2 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{n-1} 1 & 0 \end{pmatrix}.$$

Let $\widetilde{W}$ denote the image of W in the Calkin algebra. Choose n^2 characters ϕ_{ij} on the commutative C^*-algebra $C^*(\widetilde{W})$. The map $C^*(\widetilde{W}) \otimes M_n \to M_n$ given by $[X_{ij}] \mapsto [\phi_{ij}(X_{ij})]$ induces a $*$-homomorphism $\rho : C^*(T) \to M_n$ that factors through the compact operators. Hence $T \sim_{\mathrm{a}} T \oplus \rho(T)$ by [12], where

$$\rho(T) = \begin{pmatrix} 0 & & & & \alpha_n e^{\mathrm{i}\theta} \\ \alpha_1 & 0 & & & \\ & \alpha_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{n-1} & 0 \end{pmatrix} \in M_n,$$

for some $\theta \in \mathbb{R}$. As $\rho(T)$ must be finitely strongly reductive and hence normal, the weights $\alpha_1, \ldots, \alpha_n$ must coincide.

The discussion above concerning periodic shifts is an example of a more general phenomenon, which is summarised by the following proposition.

PROPOSITION 6. *Finitely strongly reductive essentially n-normal operators are essentially normal.*

Proof. The key fact in showing this is again Voiculescu's Theorem ([12]). Assume T is essentially n-normal and consider $\widetilde{T}$ in the Calkin algebra. Let ϱ be a faithful representation of $C^*(\widetilde{T})$ on a separable Hilbert space H_ϱ; therefore $\varrho(\widetilde{T})$ is n-normal. If π is an arbitrary irreducible representation of $C^*(\varrho(\widetilde{T}))$, then this representation takes place on a space of dimension n or less. So, $\pi(\varrho(\widetilde{T}))$, which by Voiculescu's Theorem is a finite-rank approximate direct summand of T, is irreducible and normal and therefore is a scalar. This is to say that every irreducible representation of $C^*(\varrho(\widetilde{T}))$ takes place on a 1-dimensional space. Voiculescu's Theorem states that the identity representation of $C^*(\varrho(\widetilde{T}))$ is approximately equivalent to a direct sum of irreducible representations and so $\varrho(\widetilde{T})$ is the norm-limit of a sequence of diagonal operators. Thus $\varrho(\widetilde{T})$ is normal, and because ϱ is faithful, $\widetilde{T}$ is normal as well. Hence T is essentially normal and this completes the proof. ∎

Theorem 8 below is a technical result which we will put to use in Proposition 10 to show that for several types of operators it is impossible to be finitely strongly reductive without being normal.

LEMMA 7. (see 5.2 of [10]) *If T is quasitriangular, and if $e, f \in H$ are orthonormal, then there exists a sequence of finite-rank projections P_n and a positive contraction A such that:*

(i) $\|(1-P_n)TP_n\| \to 0$;

(ii) $P_n \to A$ *in the weak-operator-topology;*

(iii) A *is not* 0 *or* 1; *and*

(iv) $1/4 \leqslant \varphi(P_n) \leqslant 3/4$ *for all n, where φ is the state on $B(H)$ given by* $\varphi(R) = 1/2[(Re,e)+(Rf,f)]$.

THEOREM 8. *Suppose that $T = N \oplus S$ is the decomposition of an operator T as the direct sum of a normal operator N and a completely nonnormal operator S. If T is a finitely strongly reductive quasitriangular operator, then the intersection of $C^*(S)$ with the compact operators on H is zero.*

Proof. From the decomposition $T = N \oplus S$ both N and S must be finitely strongly reductive. Suppose that Ω is a connected component of the semi-Fredholm domain of S. Because the interior of $\sigma(N)$ is empty, we can always find a $\lambda \in \Omega$ that is not in $\sigma(N)$; such λ are in the semi-Fedholm domain of T. So

$$(\dagger) \qquad 0 = \operatorname{ind}(T-\lambda 1) = \operatorname{ind}(N-\lambda 1) + \operatorname{ind}(S-\lambda 1) = \operatorname{ind}(S-\lambda 1) \leqslant 0$$

and therefore $\operatorname{ind}(S-\lambda 1)=0$. As the index is constant over Ω it follows from [1] that S is quasitriangular. Therefore we may assume that T itself is completely nonnormal.

Assume, contrary to what we wish to prove, that $C^*(T)\cap K(H)\neq\{0\}$. Then H has a decomposition as $H=H_0\oplus\sum_i^{\oplus}H_i^{(n(i))}$ such that $C^*(T)\cap K(H)=\{0\}\oplus\sum_i^{\oplus}K(H_i)^{(n(i))}$, where $1\leqslant n(i)<\infty$ for all i. With respect to this decomposition, write $T=T_0\oplus\sum_i^{\oplus}T_i^{(n(i))}$. At least one summand in the compact portion must be present; say that it is $T_1^{(n(1))}$.

Claim. We can restrict ourselves to the case $n(1)=1$. If we write $T=C\oplus T_1^{(n(1))}$ and $B=C\oplus T_1$, then B is a direct summand of T and thus is completely nonnormal and finitely strongly reductive. Moreover, $C^*(B)$ contains a nonzero compact operator. A completely nonnormal finitely strongly reductive operator A must satisfy $\ker(A-\lambda 1)=\{0\}$ for all $\lambda\in\mathbb{C}$; thus λ is in the semi-Fredholm domain if and only if $A-\lambda 1$ has closed range. Because B is a completely nonnormal finitely strongly reductive direct summand of T, the semi-Fredholm domains of B and T coincide. The index argument used in (†) shows that B is quasitriangular. Thus, we can reduce to the case $n(1)=1$.

Thus far we have decomposed T as $T=C\oplus T_1$, where $C^*(T)\cap K(H)=\{0\}\oplus K(H_1)$ and where the dimension of H_1 is at least 2 (as T_1 is completely nonormal). Therefore H_1 contains two orthonormal vectors e and f. Define the state φ on $B(H)$ by $\varphi(R)=1/2[(Re,e)+(Rf,f)]$ for $R\in B(H)$. By Lemma 7, there exist finite-rank projections P_n that converge weakly to a positive contraction $A\neq 0,1$ and that satisfy $\|(1-P_n)TP_n\|\to 0$ and $\varphi(P_n)\in[1/4,3/4]$ for all n. As T is finitely strongly reductive, $\|P_nT-TP_n\|\to 0$ and so $AT=TA$. Hence A is in the commutant of $C^*(T)$. Now because $\{0\}\oplus K(H_1)\subset C^*(T)$, A must be of the form $A=A_0\oplus\alpha 1$ for some real number $\alpha\geqslant 0$. Moreover, $\alpha=\varphi(A)$ and so $\alpha\in[1/4,3/4]$. Let P be the rank-1 projection $e\otimes e$. Then $P\in\{0\}\oplus K(H_1)\subset C^*(T)$ and so P commutes asymptotically with the projection sequence $\{P_n\}_n$. In particular at the point e,

$$\begin{aligned}\|(P_n-(P_ne,e)1)e\|&=\|P_ne-(P_ne,e)e\|=\|P_ne-PP_ne\|\\&=\|(P_nP-PP_n)e\|\to 0.\end{aligned}$$

As $(P_ne,e)\to(Ae,e)=\alpha$, it follows that $\|P_ne-\alpha e\|\to 0$ and so $\alpha\in\{0,1\}$, which is in contradiction to $\alpha\in[1/4,3/4]$. ∎

COROLLARY 9. *If the unital C^*-algebra generated by a quasitriangular finitely strongly reductive operator T contains a nonzero compact operator, then T has a nontrivial reducing invariant subspace.*

A short list of those finitely strongly reductive operators that must be normal is given by the following result.

PROPOSITION 10. *If T is a finitely strongly reductive operator and is any one of the following types, then T is normal:*

(i) *Algebraic.*

(ii) *Compact or polynomially compact.*

(iii) *Essentially unitary of index* 0.

(iv) *Quasitriangular and essentially normal.*

Proof. It is not difficult to show that algebraic finitely strongly reductive operators must be normal — the argument is essentially the same as the one used in Example 3 of [7] — so let us detail the proof of (ii). Suppose that T is polynomially compact and take any faithful representation ϱ of $C^*(\widetilde{T})$, where $\widetilde{T}$ is the image of T in the Calkin algebra on a separable Hilbert space H_ϱ. By Voiculescu's Theorem, T is approximately equivalent to $T \oplus \varrho(\widetilde{T})$. Therefore, $\varrho(\widetilde{T})$ is finitely strongly reductive and algebraic, and so by (i), $\rho(\widetilde{T})$ is a normal operator. Because ϱ is faithful, $\widetilde{T}$ must be normal as well, which means that T is essentially normal. The essential spectrum of T is the set of roots of the minimal monic annihilating polynomial of $\widetilde{T}$ and so by the Brown-Douglas-Fillmore Theorem, T is a compact perturbation of a normal operator and is, more generally, quasitriangular. The conclusion that T is normal will follow from (iv).

Statement (iii) is a special case of (iv). To prove (iv) we use Theorem 8. Suppose that T is quasitriangular, essentially normal, and finitely strongly reductive. Assume, on the contrary, that T is nonnormal. Then there exists a normal operator N and a completely nonnormal operator S (acting on a space of dimension at least 2) such that $T = N \oplus S$. Because S is essentially normal, $S^*S - SS^* \in C^*(S) \cap K(H_0)$, where H_0 is the Hilbert space on which S acts. However S is also quasitriangular and finitely strongly reductive and therefore, by Theorem 8, $C^*(S) \cap K(H_0) = \{0\}$, which implies that S is normal (in contradiction to the fact that S is completely nonnormal). Thus, it must be that T is normal, completing the proof of the proposition. ∎

As a concrete application of Proposition 10, observe that we can now deduce that if T is a weighted bilateral shift operator with nonnegative weight sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ satisfying $\alpha_n \to 1$ as $|n| \to \infty$, then T is finitely strongly reductive if and only if $\alpha_n = 1$ for every $n \in \mathbb{Z}$. The sufficiency of the claim is true by Theorem 4, so suppose that T is such a weighted bilateral shift and that T is finitely strongly reductive. The hypothesis on the weights α_n implies that T is a compact perturbation of a unitary operator; more precisely, T is essentially unitary of index zero. Hence, by Proposition 10, T is normal. The desired conclusion is reached by noting that a bilateral weighted shift operator is normal if and only if the weight sequence is constant.

Future research in this area may begin, perhaps, with the following two open questions, which are of some interest.

(Q1) Is it sufficient that a normal operator have nowhere dense spectrum in order for it to be finitely strongly reductive, and are reductive normal operators finitely strongly reductive? (It is to be noted that reductive normal operators need not be strongly reductive.)

(Q2) The adjoint of an isometry is finitely strongly reductive only if the isometry is unitary. More generally, is it true that if both T and T^* are finitely strongly reductive, then T is normal?

We conclude with some observations concerning the second open question. First of all, to resolve this question in the affirmative, it would be enough to prove that if T and T^* are finitely strongly reductive, then T is essentially normal. The proof that T is normal whenever T is essentially normal and T and T^* are finitely strongly reductive runs as follows. As every essentially normal operator is a direct sum of a normal operator and at most countably many irreducible essentially normal operators, we may assume that if T is nonnormal, then $T = A \oplus B$, where A is an irreducible essentially normal operator. Now A and A^* are finitely strongly reductive and so $\operatorname{ind}(A - \lambda 1) = 0$ for all λ in the semi-Fredholm domain of A. Thus, A is quasitriangular and essentially normal; by Proposition 10, A is, therefore, normal. This contradiction implies that T must have been normal to begin with.

Another approach to solving the problem posed in Q2 is to consider the following problem concerning finite matrices.

(Q3) Suppose that for each n, $A_n \in M_n(\mathbb{C})$ is a contraction, and suppose that the sequence $\{A_n\}_n$ has the property that if $\|(1 - P_n)A_nP_n\| \to 0$, where each P_n is an $n \times n$ projection, then $\|(1 - P_n)A_n^*P_n\| \to 0$ as well. Does it then follow that the distance between each A_n and the set of $n \times n$ normal matrices converges to 0 as $n \to \infty$?

An affirmative answer to Question 3 above is equivalent to the assertion that every reductive element is normal in the C^*-direct product of the algebras M_n modulo the C^*-direct sum the algebras M_n (because projections in the quotient lift to projections in the direct product). Question 3 is of interest in the present paper in that an affirmative answer to Q3 implies an affirmative answer to Q2. The reasons for this are as follows. Assume that Q3 has been answered in the affirmative and suppose that T^* and T are both finitely strongly reductive. The hypothesis on T implies that the index of $T - \lambda 1$ is zero for all λ in the semi-Fredholm domain of T and so T is a quasitriangular finitely strongly reductive operator; hence T is quasidiagonal. Therefore, T is a compact perturbation of a block diagonal operator. To the blocks in the block-diagonal part of T we apply the affirmative answer to Q3 to deduce that T is an essentially normal operator. As has been explained above, every essentially normal operator T for which T and T^* are finitely strongly reductive must be normal.

Yet another observation on Q2 stems from the work of Davidson, Herrero, and Salinas ([6]): if both T and T^* are finitely strongly reductive and if the complement of $\sigma_{\rm e}(T)$ is connected, then T is a norm-limit of algebraic quasidiagonal operators. To prove this under the assumption that the essential spectrum of T has connected complement, one need only show, by Corollary 2.5 of [6], that if T and T^* are finitely strongly reductive, then $\varrho(\widetilde{T})$ quasidiagonal for some faithful representation of $C^*(\widetilde{T})$ on a separable Hilbert space. Take, therefore, any faithful representation ϱ of $C^*(\widetilde{T})$ on a separable Hilbert space. Voiculescu's Theorem asserts that $\varrho(\widetilde{T})$ is an approximate direct summand of T; thus, $\varrho(\widetilde{T})$ and $\varrho(\widetilde{T}^*)$ are finitely strongly reductive. As has been argued before, this shows that the index of $\varrho(\widetilde{T}) - \lambda 1$ is zero for all λ in the semi-Fredholm domain of $\varrho(\widetilde{T})$; hence $\varrho(\widetilde{T})$ is a quasitriangular finitely strongly reductive operator and is, therefore, quasidiagonal.

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DOUGLAS R. FARENICK
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S 0A2
CANADA

DON HADWIN
Mathematics Department
University of New Hampshire
Durham, New Hampshire 03824
U.S.A.

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