J. OPERATOR THEORY
41(1999), 261–289

COMPACT QUANTUM HYPERGROUPS

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Communicated by Norberto Salinas

ABSTRACT. A compact quantum hypergroup is a unital C^*-algebra equipped with a completely positive coassociative coproduct. The most important examples of such a structure are associated with double cosets of compact matrix pseudogroups in the sense of S.L. Woronowicz. We give a precise definition of a compact quantum hypergroup; prove existence and uniqueness of the Haar measure, establish orthogonality relations for matrix elements of irreducible corepresentations; construct a Peter-Weyl theory for irreducible corepresentations.

KEYWORDS: *C^*-algebra, coproduct, Haar measure, hypergroup, quantum group, quantum homogeneous space, representation.*

MSC (2000): Primary 81R50; Secondary 22A30.

INTRODUCTION

Let G be a compact group and K be a subgroup of it with Haar measures μ_G and μ_K, respectively. Then the algebra of all continuous functions on G that are bi-invariant with respect to translations by elements from K, $C_{KK}(G)$, can be equipped with the coproduct

$$(\delta f)(x,y) := \int_K f(xky)\,\mathrm{d}\mu_K, \quad f \in C_{KK}(G)$$

that plays an important role in the theory of zonal spherical functions (see, for example [9]) arising from irreducible representations of G. Such a structure is an example of what now is called a *hypergroup* (see [3], [10]). The situation becomes similar if one considers compact matrix pseudogroups in the sense of

S.L. Woronowicz ([19]). Here, for a pair of compact matrix pseudogroups (A_1, A_2) and a surjection $\pi : A_1 \to A_2$, we can also consider an algebra A of bi-invariant elements ([4], [12]) and define a coproduct on this algebra ([6], [7], [15], [16]). In this case, we say that A is endowed with a *hypergroup structure*. A number of examples of the mentioned type can be found in the papers cited above and in [18]. All of these examples are associated with various classes of special and q-special functions.

Analyzing these examples, one can come to the idea of considering a general structure consisting of a unital C^*-algebra equipped with a completely positive coassociative coproduct that preserves unit element and satisfies some additional natural axioms. Since compact matrix pseudogroups and usual hypergroups are included in this framework, we call such a structure a *compact quantum hypergroup*. The aim of the paper is to show that the theory of these objects is as rich as the theory of compact quantum groups ([19], [20]). In particular, we prove existence and uniqueness of the Haar measure and some basic results concerning harmonic analysis on a compact quantum hypergroup.

The paper is organized as follows. In Section 1, we define a hypergroup structure on a C^*-algebra and prove some elementary facts. In Section 2, we prove existence and uniqueness of the Haar measure. In Section 3, two examples of a hypergroup structure on a C^*-algebra are considered, those of a compact matrix pseudogroup itself and of a compact quantum homogeneous space arising from a pair of compact matrix pseudogroups (A_1, A_2) and an epimorphism $\pi : A_1 \to A_2$. In Section 4 we give a definition of a compact quantum hypergroup, prove that, in the case where the C^*-algebra is commutative, the compact quantum hypergroup is just the usual hypercomplex system ([3]), and, finally, we give some results that are used in Section 5. In Section 5, we summarize elements of the theory of corepresentations and prove theorems of the Peter-Weyl theory, namely, we prove that irreducible corepresentations are finite dimensional and their matrix elements are total both in the C^*-algebra and its L_2 completion with respect to the corresponding norms.

1. DEFINITION OF A HYPERGROUP STRUCTURE ON A C^*-ALGEBRA

Let $(A, \cdot, 1, *)$ be a separable unital C^*-algebra. We denote by $A \otimes A$ the injective or projective C^*-tensor square of A.

Definition 1.1. We will call $(A, \delta, \varepsilon, \star)$ a *hypergroup structure* on the C^*-algebra $(A, \cdot, 1, *)$ if:

(HS_1) $(A, \delta, \varepsilon, \star)$ is a $\star$-coalgebra with a counit ε, i.e. $\delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{C}$ are linear mappings, $\star : A \to A$ is an antilinear mapping such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta, \tag{1.1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \varepsilon) \circ \delta = \mathrm{id}, \tag{1.2}$$

$$\delta \circ \star = \Pi \circ (\star \otimes \star) \circ \delta, \tag{1.3}$$

$$\star \circ \star = \mathrm{id}, \tag{1.4}$$

where $\Pi : A \otimes A \to A \otimes A$ is the flip, $\Pi(a_1 \otimes a_2) = a_2 \otimes a_1$;

(HS_2) the mapping $\delta : A \to A \otimes A$ is positive, i.e. it maps the cone of positive elements of A into the cone of positive elements of $A \otimes A$;

(HS_3) the following identities hold

$$(a \cdot b)^\star = a^\star \cdot b^\star, \qquad \delta \circ * = (* \otimes *) \circ \delta, \tag{1.5}$$

$$\varepsilon(a \cdot b) = \varepsilon(a)\varepsilon(b), \qquad \delta(1) = 1 \otimes 1, \tag{1.6}$$

$$\star \circ * = * \circ \star. \tag{1.7}$$

Lemma 1.2. *Let $(A, \delta, \varepsilon, \star)$ be a hypergroup structure on a unital C^*-algebra $(A, \cdot, 1, *)$. Then:*

$$1^\star = 1, \tag{1.8}$$

$$\varepsilon(1) = 1, \tag{1.9}$$

$$\varepsilon(a^\star) = \overline{\varepsilon(a)}, \tag{1.10}$$

$$\varepsilon(a^*) = \overline{\varepsilon(a)}. \tag{1.11}$$

Proof. By using the first equality of (1.5), we have $1^\star = (1 \cdot 1)^\star = 1^\star \cdot 1^\star$. Hence, $1^\star \cdot (1 - 1^\star) = 0$. And so, $1 \cdot (1^\star - 1) = 1^\star - 1 = 0$. This shows (1.8).

By using relations (1.2) and the second relation of (1.6), we have $1 = (\varepsilon \otimes \mathrm{id}) \circ \delta(1) = \varepsilon(1)1$. Whence $\varepsilon(1) = 1$.

To show (1.10), let $a \in A$ and consider $(\varepsilon \otimes \overline{\varepsilon}) \circ (\mathrm{id} \otimes \star) \circ \delta(a^\star)$. First,

$$\begin{aligned}(\varepsilon \otimes \overline{\varepsilon}) \circ (\mathrm{id} \otimes \star) \circ \delta(a^\star) &= \overline{(\overline{\varepsilon} \otimes \varepsilon) \circ (\mathrm{id} \otimes \star) \circ \delta(a^\star)} \\ &= \overline{(\varepsilon \otimes \overline{\varepsilon}) \circ (\star \otimes \mathrm{id}) \circ (\star \otimes \star) \circ \delta(a)} \\ &= \overline{(\varepsilon \otimes \overline{\varepsilon}) \circ (\mathrm{id} \otimes \star) \circ \delta(a)} = \overline{\overline{\varepsilon}(a^\star)} = \varepsilon(a^\star).\end{aligned}$$

On the other hand,

$$(\varepsilon \otimes \overline{\varepsilon}) \circ (\mathrm{id} \otimes \star) \circ \delta(a^\star) = \overline{\varepsilon}((a^\star)^\star) = \overline{\varepsilon}(a).$$

Finally, (1.11) holds because of the following two identities:

$$\begin{aligned}(\varepsilon \otimes \overline{\varepsilon}) \circ (\mathrm{id} \otimes *) \circ \delta(a^*) &= \overline{(\overline{\varepsilon} \otimes \varepsilon) \circ (\mathrm{id} \otimes *) \circ \delta(a^*)} \\ &= \overline{(\overline{\varepsilon} \otimes \varepsilon) \circ (* \otimes \mathrm{id}) \circ \delta(a)} = \overline{\overline{\varepsilon}(a^*)} = \varepsilon(a^*)\end{aligned}$$

and

$$(\varepsilon \otimes \overline{\varepsilon}) \circ (\mathrm{id} \otimes *) \circ \delta(a^*) = \overline{\varepsilon}((a^*)^*) = \overline{\varepsilon}(a). \qquad \blacksquare$$

By A° we denote the set of all continuous linear functionals on the C^*-algebra A. For $\xi, \eta \in A^\circ$ we define a product $\cdot$ and an involution $^+$ by

$$\begin{aligned}(\xi \cdot \eta)(a) &= (\xi \otimes \eta)\delta(a) \\ \xi^+(a) &= \overline{\xi(a^\star)},\end{aligned} \tag{1.12}$$

$a \in A$, with the norm given by

$$\|\xi\| = \sup_{\|a\|=1} |\xi(a)|. \tag{1.13}$$

Lemma 1.3. *Let the product, involution and the norm on A° be given by* (1.12) *and* (1.13). *Then $(A^\circ, \cdot, \varepsilon, ^+)$ is a unital Banach $*$-algebra.*

Proof. It is clear that $(A^\circ, \cdot, \varepsilon, ^+)$ is a unital involutive algebra. Let us show that $\|\xi \cdot \eta\| \leqslant \|\xi\| \, \|\eta\|$. Indeed, because $\delta : A \to A \otimes A$ is a positive mapping, $\|\delta\| = 1$ ([5], Corollary 3.2.6). We also have for all $\xi, \eta \in A^\circ$ and $\widetilde{a} \in A \otimes A$ that $(\xi \otimes \eta)(\widetilde{a}) \leqslant \|\widetilde{a}\| \, \|\xi\| \, \|\eta\|$ ([14], Section IV.4). Thus

$$\begin{aligned}\|\xi \cdot \eta\| &= \sup_{a \in A,\, \|a\|=1} |(\xi \otimes \eta)\delta(a)| \leqslant \sup_{\widetilde{a} \in A \otimes A,\, \|\widetilde{a}\|=1} |(\xi \otimes \eta)(\widetilde{a})| \\ &\leqslant \sup_{\widetilde{a} \in A \otimes A,\, \|\widetilde{a}\|=1} \|\widetilde{a}\| \, \|\xi\| \, \|\eta\| = \|\xi\| \, \|\eta\|.\end{aligned}$$

It remains to show that $\|\xi^+\| = \|\xi\|$. First note that $\star$ is an antilinear mapping, which preserves the cone of positive elements of the C^*-algebra A. Indeed, it follows from (1.5) and (1.7) that

$$(a \cdot a^*)^\star = a^\star \cdot (a^*)^\star = a^\star \cdot (a^\star)^*.$$

Hence the mapping $\Phi : A \to A$ defined by $\Phi(a) = (a^*)^\star$ is a linear positive mapping. Thus $\|\Phi\| = 1$. So

$$\|\xi^+\| = \sup_{a \in A,\, \|a\|=1} |\overline{\xi(a^\star)}| = \sup_{a^* \in A,\, \|a^*\|=1} |\xi((a^*)^\star)| = \sup_{a \in A,\, \|a\|=1} |\xi(\Phi(a))| \leqslant \|\xi\|.$$

And, because $+ \circ + = \mathrm{id}$, we have that $\|\xi\| = \|\xi^{++}\| \leqslant \|\xi^+\|$, whence $\|\xi^+\| = \|\xi\|$.

It is clear that A° is a Banach space with respect to the norm (1.13). $\blacksquare$

2. EXISTENCE OF A HAAR MEASURE

DEFINITION 2.1. Let $(A, \delta, \varepsilon, \star)$ be a hypergroup structure on a C^*-algebra A. A state $\nu \in A^\circ$ is called a *Haar measure* (with respect to the hypergroup structure) if

$$(\nu \otimes \mathrm{id}) \circ \delta(a) = (\mathrm{id} \otimes \nu) \circ \delta(a) = \nu(a)1 \tag{2.1}$$

for all $a \in A$.

DEFINITION 2.2. Let $(A, \delta, \varepsilon, \star)$ be a hypergroup structure on the C^*-algebra A. An element $a \in A$ is called *positive definite* if

$$\xi \cdot \xi^+(a) \geqslant 0 \tag{2.2}$$

for all $\xi \in A^\circ$.

To prove the following theorem, we combine the approaches of [11], [19], and [20].

THEOREM 2.3. *Let $(A, \delta, \varepsilon, \star)$ be a hypergroup structure on a C^*-algebra A. Suppose that the linear space spanned by the positive definite elements is dense in A. Then there exists a Haar measure ν, which is unique, and $\nu^+ = \nu$.*

Proof. Denote by Σ the set of all states on the C^*-algebra A. Because δ is positive and $\delta(1) = 1 \otimes 1$, Σ is closed with respect to the multiplication $\cdot$. Because $\star$ preserves the cone of positive elements, Σ is closed with respect to the involution $^+$. Clearly, it is convex and compact in the $*$-weak topology.

Let $\mathcal{L} = \{\Lambda : \Lambda \subset \Sigma\}$ denote the family of nonempty compact convex subsets of Σ such that $\Sigma \cdot \Lambda \subset \Lambda$. The family $\mathcal{L}$ is nonvoid since $\Sigma \in \mathcal{L}$. Consider $\mathcal{L}$ with the partial ordering induced by inclusion. A standard argument using Zorn's lemma shows that there is a minimal element $\Lambda_0 \in \mathcal{L}$.

For each $\lambda \in \Lambda_0$, $\Lambda_0 \cdot \lambda = \Lambda_0$. Indeed, $\Lambda_0 \cdot \lambda \subset \Sigma \cdot \Lambda_0 \subset \Lambda_0$. Moreover, $\Lambda_0 \cdot \lambda \in \mathcal{L}$ because $\Sigma \cdot (\Lambda_0 \cdot \lambda) = (\Sigma \cdot \Lambda_0) \cdot \lambda \subset \Lambda_0 \cdot \lambda$. And, because Λ_0 is minimal, $\Lambda_0 = \Lambda_0 \cdot \lambda$. This implies that, for each $\lambda, \mu \in \Lambda_0$ there exists an element $\chi \in \Lambda_0$ such that

$$\chi \cdot \lambda = \mu. \tag{2.3}$$

Denote by $\mathcal{R} = \{\Xi : \Xi \subset \Sigma\}$ the family of nonempty compact convex subsets of Σ such that $\Xi \cdot \Sigma \subset \Xi$. Denote $\Lambda_0^+ = \{\lambda^+ : \lambda \in \Lambda_0\}$. Then, because $\Lambda_0^+ \cdot \Sigma = (\Sigma \cdot \Lambda_0)^+ \subset \Lambda_0^+$, we see that $\Lambda_0^+ \in \mathcal{R}$. It is immediate that it is a minimal element of $\mathcal{R}$. Hence, in the same way as before, for each $\rho, \sigma \in \Lambda_0^+$, there is an element $\psi \in \Lambda_0^+$ such that

$$\rho \cdot \psi = \sigma. \tag{2.4}$$

Now, let $\Omega = \Lambda_0 \cap \Lambda_0^+$. The set Ω is nonempty because $\Lambda_0 \neq \emptyset$, $\Lambda_0^+ \neq \emptyset$ and $\Omega = \Lambda_0 \cap \Lambda_0^+ \supset \Lambda_0^+ \cdot \Lambda_0$. Let $\omega \in \Omega$. Then it follows from (2.3) that there is $\nu_\omega^{\mathrm{l}} \in \Lambda_0$ such that

$$\nu_\omega^{\mathrm{l}} \cdot \omega = \omega. \tag{2.5}$$

By using (2.4) we see that, for all $\rho \in \Lambda_0^+$, there is an element $\psi_\rho \in \Lambda_0^+$ such that $\omega \cdot \psi_\rho = \rho$. This, together with (2.5) implies that, for an arbitrary $\rho \in \Lambda_0^+$,

$$\nu_\omega^{\mathrm{l}} \cdot \rho = \rho. \tag{2.6}$$

Indeed,

$$\nu_\omega^{\mathrm{l}} \cdot \rho = \nu_\omega^{\mathrm{l}} \cdot (\omega \cdot \psi_\rho) = (\nu_\omega^{\mathrm{l}} \cdot \omega) \cdot \psi_\rho = \omega \cdot \psi_\rho = \rho.$$

By a similar argument we get $\nu_\omega^{\mathrm{r}} \in \Lambda_0^+$ such that

$$\lambda \cdot \nu_\omega^{\mathrm{r}} = \lambda \tag{2.7}$$

for all $\lambda \in \Lambda_0$. So (2.6) together with (2.7) mean that

$$\nu_\omega^{\mathrm{l}} = \nu_\omega^{\mathrm{l}} \cdot \nu_\omega^{\mathrm{r}} = \nu_\omega^{\mathrm{r}}.$$

Denote by $\nu_\omega = \nu_\omega^{\mathrm{l}} = \nu_\omega^{\mathrm{r}} \in \Omega$. If $\nu_{\omega'}$ is another element in Ω verifying (2.6) and (2.7), then clearly $\nu_{\omega'} = \nu_\omega$. Hence such an element is unique and we denote it by ν. Relations (2.7) and (2.6) now become

$$\lambda \cdot \nu = \lambda, \quad \nu \cdot \rho = \rho \tag{2.8}$$

for all $\lambda \in \Lambda_0$ and $\rho \in \Lambda_0^+$.

For each compact convex subset Υ of Σ and an arbitrary $\alpha \in \Upsilon$, $m \in \mathbb{Z}_+$, we denote

$$\alpha^{(m)} = \frac{1}{m} \sum_{i=1}^{m} \alpha^i.$$

Because Υ is compact, there is an accumulation point $\alpha^{(\infty)}$ of the sequence $\alpha^{(m)}$ in Υ and, if necessary, considering a subsequence we can assume that $\alpha^{(\infty)} = \lim\limits_{m\to\infty} \alpha^{(m)}$. Now

$$\alpha \cdot \alpha^{(m)} = \frac{1}{m} \sum_{i=2}^{m+1} \alpha^m = \frac{1}{m} \sum_{i=1}^{m} \alpha^i + \frac{1}{m}(\alpha^{m+1} - \alpha) = \alpha^{(m)} + \frac{1}{m}(\alpha^{m+1} - \alpha).$$

Since $\cdot$ is continuous, we get

$$\lim_{m\to\infty}(\alpha\cdot\alpha^{(m)}) = \alpha\cdot\lim_{m\to\infty}\alpha^{(m)} = \alpha\cdot\alpha^{(\infty)}.$$

On the other hand, because Υ is compact, it is bounded and hence we have that $\lim\limits_{m\to\infty}\frac{1}{m}(\alpha^{m+1}-\alpha) = 0$. This shows that

$$\lim_{m\to\infty}\left\{\alpha^{(m)} + \frac{1}{m}(\alpha^{(m+1)}-\alpha)\right\} = \alpha^{(\infty)}.$$

So

$$\alpha\cdot\alpha^{(\infty)} = \alpha^{(\infty)}. \tag{2.9}$$

In the same way we get

$$\alpha^{(\infty)}\cdot\alpha = \alpha^{(\infty)}. \tag{2.10}$$

Now let $\lambda\in\Lambda_0$. For $\lambda^{(\infty)}\in\Lambda_0$ and $\nu\in\Omega$ by using (2.3), we can find $\chi\in\Lambda_0$ such that

$$\chi\cdot\lambda^{(\infty)} = \nu.$$

But this means that

$$\nu = \chi\cdot\lambda^{(\infty)} = \chi\cdot\lambda^{(\infty)}\cdot\lambda = \nu\cdot\lambda.$$

In particular, for any $\omega\in\Omega$,

$$\nu = \nu\cdot\omega = \omega.$$

This shows that $\Omega = \Lambda_0^+\cdot\Lambda_0 = \{\nu\}$, i.e.

$$\rho\cdot\lambda = \nu \tag{2.11}$$

for all $\lambda\in\Lambda_0$, $\rho\in\Lambda_0^+$.

Now choose an arbitrary element $\lambda\in\Lambda_0$. We will prove that $\lambda=\nu$. Let $\widetilde{\Lambda}$ be the $*$-algebra generated by the elements $\lambda,\lambda^+,\nu,\varepsilon$. Denote $\zeta=\nu-\lambda$. It follows from (2.11) that

$$\zeta^+\cdot\zeta = (\nu-\lambda^+)\cdot(\nu-\lambda) = \nu\cdot\nu-\nu\cdot\lambda-\lambda^+\cdot\nu+\lambda^+\cdot\lambda = \nu-\nu-\nu+\nu = 0. \tag{2.12}$$

If $p\in A$ is a positive definite element, then, considered as a linear functional on $\widetilde{\Lambda}$, it is a positive linear functional, and, hence,

$$|(\nu-\lambda)(p)| \leqslant \varepsilon(p)((\nu-\lambda)^+\cdot(\nu-\lambda))(p) = 0.$$

This means that $\nu(p)=\lambda(p)$ for any positive definite $p\in A$ and, since each element in A can be approximated by a linear combination of positive definite elements, $\nu(a)=\lambda(a)$ for all $a\in A$, hence $\nu=\lambda$.

So, we get that $\Lambda=\Lambda^+=\{\nu\}$, which means that ν is a Haar measure, it is clearly unique, and by construction, $\nu^+=\nu$. ∎

REMARK 2.4. In defining a hypergroup structure on a C^*-algebra, it would be natural to follow the lines taken in [20] for defining a compact quantum group. In particular, if the following axiom, used in [20], is assumed to hold

(W): the linear subsets $\sum_{i=1}^{n}(b_i \otimes 1)\delta(a_i)$ and $\sum_{i=1}^{n}(1 \otimes b_i)\delta(a_i)$, $a_i, b_i \in A$, $n \in \mathbb{Z}_+$, are dense in $A \otimes A$,

then the claim in Theorem 2.3 would still be true and could be proved by using a slight modification of the proof found in [20]. Unfortunately, Condition (W) is too strong as an assumption, which can be seen from the following example, and thus we replaced it in the statement of Theorem 2.3, by the condition that positive definite elements are total in A.

EXAMPLE 2.5. Let $I = [0, \pi]$, $A = C(I)$ be the commutative C^*-algebra of continuous complex-valued functions on I. Let $\delta : A \to A \otimes A$ be given by $\delta(f)(x, y) = \frac{1}{2}(f(\pi - |\pi - x - y|) + f(|x - y|))$, $\varepsilon(f) = f(0)$, $f^\star(x) = \overline{f(x)}$. For a cocommutative δ, $c \in A$ is called a *character* if $\delta(c) = c \otimes c$. In the case under consideration, the characters are $c_n = \cos nx$, $n \in \mathbb{Z}_+$. By the Weierstrass theorem, the linear span of the set $\mathcal{C} = \{c_n \mid n \in \mathbb{Z}_+\}$ is dense in A, but an easy argument shows that no elements from the linear span of the set $\{\delta(c_k)(c_l \otimes 1) \mid k, l \in \mathbb{Z}_+\}$ approximates, for example, the element $c_m \otimes c_n$ for $m < n$. Thus, condition (W) is violated. However, the characters are positive definite functions and, hence, the density condition of Theorem 2.3 holds and the Haar measure exists and is given by $\nu(f) = \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x$.

3. EXAMPLES

3.1. A HYPERGROUP STRUCTURE ASSOCIATED WITH A COMPACT QUANTUM GROUP.

Let $(A, \cdot, 1, *, \Delta, \varepsilon, S)$ be a compact matrix pseudogroup with A_0 being the involutive subalgebra generated by matrix elements of the fundamental corepresentation, A — the maximal C^*-closure of A_0 ([19]). We also use the following notations:

$$\xi \cdot a = (\mathrm{id} \otimes \xi) \circ \Delta(a), \quad a \cdot \xi = (\xi \otimes \mathrm{id}) \circ \Delta(a), \quad \xi \cdot \eta = (\xi \otimes \eta) \circ \Delta \tag{3.1}$$

for $\xi, \eta \in A^\circ$ and $a \in A$, and

$$\Delta^{(1)} = \Delta, \quad \Delta^{(2)} = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad \text{etc.}$$

It readily follows from (3.1) that

$$\xi \cdot (\eta \cdot a) = (\xi \cdot \eta) \cdot a, \quad (a \cdot \eta) \cdot \xi = a \cdot (\eta \cdot \xi). \tag{3.2}$$

Let $U^\alpha = (u^\alpha_{ij})^{d_\alpha}_{i,j=1}$ be an irreducible unitary corepresentation of A. Then there exists a unique, up to a positive constant, positive definite matrix $M^\alpha = (m^\alpha_{ij})^{d_\alpha}_{i,j=1}$ such that

$$M^\alpha \cdot U^\alpha = S^2(U^\alpha) \cdot M^\alpha, \tag{3.3}$$

where $\cdot$ here denotes the usual matrix multiplication ([19]).

For each $z \in \mathbb{C}$, we denote by $m^{\alpha(z)}_{ij}$ the matrix elements of the matrix $(M^\alpha)^z$. It is known that there exists a one-parameter family of homomorphisms $f_z : A_0 \to \mathbb{C}$, $z \in \mathbb{C}$, where, as before, A_0 denotes the $*$-subalgebra generated by matrix elements of the fundamental corepresentation. These homomorphisms are defined by

$$f_z(u^\alpha_{ij}) = m^{\alpha(z)}_{ij} \tag{3.4}$$

and possess the following properties ([19]):

(F_1) $f_z(1) = 1$ for all $z \in \mathbb{C}$;

(F_2) $f_z \cdot f_{z'} = f_{z+z'}$ and $f_0 = \varepsilon$;

(F_3) $f_z(S(a)) = f_{-z}(a)$;

(F_4) $f_z(a^*) = \overline{f_{-\overline{z}}(a)}$;

(F_5) $S^2(a) = f_{-1} \cdot a \cdot f_1$;

(F_6) $\nu(a \cdot b) = \nu(b \cdot (f_1 \cdot a \cdot f_1))$, where ν is the Haar measure on the compact matrix pseudogroup A.

As in [2], define now a mapping $\star : A_0 \to A_0$ by

$$a^\star = f_{-1/2} \cdot S(a)^* \cdot f_{1/2}. \tag{3.5}$$

Lemma 3.1.1. *Let the mapping $\star$ be defined by* (3.5)*. Then $(A_0, \Delta, \varepsilon, \star)$ is an involutive coalgebra. Moreover the first relation of* (1.5) *and relation* (1.7) *hold. The mapping $\star$ is continuous.*

Proof. It is clear from definition (3.5) that $\star$ is an antilinear and multiplicative mapping.

Let us show that $\Delta \circ \star = (\star \otimes \star) \circ \Delta$. Indeed, by using property (F_2) and definition (3.5), we have

$$\begin{aligned} \Delta \circ \star &= \Delta \circ (f_{-1/2} \otimes * \circ S \otimes f_{1/2}) \circ \Delta^{(2)} \\ &= \Pi \circ (f_{-1/2} \otimes * \circ S \otimes * \circ S \otimes f_{1/2}) \circ \Delta^{(3)} \\ &= \Pi \circ (f_{-1/2} \otimes * \circ S \otimes f_{1/2} \otimes f_{-1/2} \otimes * \circ S \otimes f_{1/2}) \circ \Delta^{(5)} \\ &= \Pi \circ (\star \otimes \star) \circ \Delta. \end{aligned}$$

To show that $\star \circ \star = \mathrm{id}$, first note that, by property (F_3), we have that, for all $z, z' \in \mathbb{C}$,

$$\begin{aligned} f_z \cdot S(a) \cdot f_{z'} &= (f_{z'} \otimes \mathrm{id} \otimes f_z) \circ \Delta^{(2)}(a) = (f_z \circ S \otimes S \otimes f_{z'} \circ S) \circ \Delta^{(2)}(a) \\ &= (f_{-z} \otimes S \otimes f_{-z'}) \circ \Delta^{(2)}(a) = S(f_{-z'} \cdot a \cdot f_{-z}), \end{aligned} \tag{3.6}$$

and, by property (F_4),

$$\begin{aligned} f_z \cdot a^* \cdot f_{z'} &= (f_{z'} \otimes \mathrm{id} \otimes f_z) \circ \Delta^{(2)}(a^*) = (f_{z'} \circ * \otimes * \otimes f_z \circ *) \circ \Delta^{(2)}(a) \\ &= (\overline{f}_{-\overline{z}'} \otimes * \otimes \overline{f}_{-\overline{z}}) \circ \Delta^{(2)}(a) = (f_{-\overline{z}} \cdot a \cdot f_{-\overline{z}'})^*. \end{aligned} \tag{3.7}$$

Now, by using (3.6) and (3.7), we obtain:

$$a^\star = f_{-1/2} \cdot S(a)^* \cdot f_{1/2} = (f_{1/2} \cdot S(a) \cdot f_{-1/2})^* = S(f_{1/2} \cdot a \cdot f_{-1/2})^*. \tag{3.8}$$

This implies that

$$\begin{aligned} (a^\star)^\star &= f_{-1/2} \cdot S(S(f_{1/2} \cdot a \cdot f_{-1/2})^*)^* \cdot f_{1/2} = f_{-1/2} \cdot (f_{1/2} \cdot a \cdot f_{-1/2}) \cdot f_{1/2} \\ &= (f_{-1/2} \cdot f_{1/2}) \cdot a \cdot (f_{-1/2} \cdot f_{1/2}) = a. \end{aligned}$$

We now show that $*$ and $\star$ commute. Indeed,

$$\begin{aligned} S((a^\star)^*) &= S((f_{-1/2} \cdot S(a)^* \cdot f_{1/2})^*) = S(f_{1/2} \cdot S(a) \cdot f_{-1/2}) \\ &= f_{1/2} \cdot S^2(a) \cdot f_{-1/2} = (f_{1/2} \cdot f_{-1}) \cdot a \cdot (f_1 \cdot f_{-1/2}) = f_{-1/2} \cdot a \cdot f_{1/2}. \end{aligned}$$

On the other hand,

$$S((a^*)^\star) = S(f_{-1/2} \cdot S(a^*)^* \cdot f_{1/2}) = f_{-1/2} \cdot S(S(a^*)^*) \cdot f_{1/2} = f_{-1/2} \cdot a \cdot f_{1/2}.$$

Because the antipode S is invertible, we see that $\star \circ * = * \circ \star$.

Continuity of $\star$ is a direct consequence of its commutativity with involution. ∎

REMARK 3.1.2. Because $\star$ is continuous and A_0 is dense in A, we can extend $\star$ by continuity to the whole C^*-algebra A and consider the coinvolution to be defined on A.

COROLLARY 3.1.3. *Let* $(A, \cdot\,, 1, *, \Delta, \varepsilon, S)$ *be a compact matrix pseudogroup. Then* $(A, \Delta, \varepsilon, \star)$ *is a hypergroup structure on the* C^**-algebra* $(A, \cdot\,, 1, *)$.

3.2. A HYPERGROUP STRUCTURE ASSOCIATED WITH A QUANTUM HOMOGENEOUS SPACE. Let now $(A_1, \cdot\,, 1, *, \Delta_1, \varepsilon_1, S_1)$ and $(A_2, \cdot\,, 1, *, \Delta_2, \varepsilon_2, S_2)$ be two compact matrix pseudogroups and let $\pi : A_1 \to A_2$ be a Hopf C^*-algebra epimorphism, i.e. π is a C^*-algebra epimorphism satisfying $(\pi \otimes \pi) \circ \Delta_1 = \Delta_2 \circ \pi$, $\varepsilon_2 \circ \pi = \varepsilon_1$ on A, and also $\pi(A_{10}) \subset A_{20}$ with $\pi \circ S_1 = S_2 \circ \pi$ on A_{10}, where A_{i0} is the $*$-subalgebra generated by matrix elements of the fundamental corepresentation of the corresponding compact matrix pseudogroup, $i = 1, 2$.

LEMMA 3.2.1. *If* $\pi : A_1 \to A_2$ *is a Hopf* $*$*-algebra epimorphism and the coinvolution* $\star$ *is defined by* (3.5) *on each* A_i, $i = 1, 2$, *then*

$$\star \circ \pi = \pi \circ \star. \tag{3.9}$$

Proof. Let H be a finite dimensional Hilbert space and $\iota : H \to A_1 \otimes H$ be an irreducible unitary corepresentation of A_1. Then $\pi_*(\iota) = (\pi \otimes \mathrm{id}) \circ \iota$ is a unitary corepresentation of A_2. Let $H = \bigoplus_{i=1}^{r} \widetilde{H}_i$, $\widetilde{H}_i = \bigoplus_{j=1}^{s_i} H_i$ be a decomposition of H such that the restriction of $\pi_*(\iota)$ onto each H_i is an irreducible unitary corepresentation of A_2 and, for every pair $i_1 \neq i_2$, the irreducible corepresentations $\pi_*(\iota)|H_{i_1}$ and $\pi_*(\iota)|H_{i_2}$ are inequivalent. For every $\widetilde{H}_i$, let us now choose an orthonormal basis in each copy of H_i from the decomposition of $\widetilde{H}_i$ such that the matrix elements $\widetilde{V}^i$ of the unitary corepresentation $\pi_*(\iota)|\widetilde{H}_i$ are written as

$$\widetilde{V}^i = \begin{pmatrix} V^i & 0 & \cdots & 0 \\ 0 & V^i & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & V^i \end{pmatrix} = V^i \otimes I_{s_i},$$

where V^i is the matrix consisting of matrix elements of the irreducible unitary corepresentaton $\pi_*(\iota)|H_i$ of A_2, I_{s_i} is the s_i-dimensional identity matrix.

Write the positive definite matrix $M = M^\alpha$ from (3.3) which corresponds to the irreducible unitary corepresentation ι of A_1, according to the decompostion $H = \bigoplus_{i=1}^{r} \widetilde{H}_i$ as

$$M = \begin{pmatrix} \widetilde{M}_{11} & \cdots & \widetilde{M}_{1r} \\ \vdots & & \vdots \\ \widetilde{M}_{r1} & \cdots & \widetilde{M}_{rr} \end{pmatrix}.$$

Since π commutes with S, applying π to (3.3) we see that

$$\begin{pmatrix} \widetilde{M}_{11} & \cdots & \widetilde{M}_{1r} \\ \vdots & & \vdots \\ \widetilde{M}_{r1} & \cdots & \widetilde{M}^{\alpha}_{rr} \end{pmatrix} \begin{pmatrix} \widetilde{V}^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{V}^r \end{pmatrix} = \begin{pmatrix} S^2(\widetilde{V}^1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S^2(\widetilde{V}^r) \end{pmatrix} \begin{pmatrix} \widetilde{M}_{11} & \cdots & \widetilde{M}_{1r} \\ \vdots & & \vdots \\ \widetilde{M}_{r1} & \cdots & \widetilde{M}_{rr} \end{pmatrix}.$$

This implies that $\widetilde{M}_{kl}\widetilde{V}^l = S^2(\widetilde{V}^k)\widetilde{M}_{kl}$, $k,l = 1,\ldots,r$. But since the corepresentations $\pi_*(\iota)|\widetilde{H}_l$ and $\pi_*(\iota)|\widetilde{H}_k$ are inequivalent for $l \neq k$ by construction, it follows that $\widetilde{M}_{kl} = 0$ for $k \neq l$. Denote $\widetilde{M}_{kk} = \widetilde{M}^k$, $k = 1,\ldots,r$, and note that each $\widetilde{M}^k$ is positive definite and invertible.

For each $i = 1,\ldots,r$, writing $\widetilde{M}^i = (M^i_{kl})^{s_i}_{k,l=1}$ relatively to the decomposition of $\widetilde{H} = \bigoplus_{j=1}^{s_i} H_i$ and applying the same reasoning, we get that $M^i_{kl}V^l = S^2(V^l)M^i_{kl}$. Since the corepresentation $\pi_*(\iota)|H_i$ is irreducible, Theorem 5.4 from [19] implies that $M^i_{kl} = c^i_{kl}N^i$, where $c^i_{kl} \in \mathbb{C}$, N^i is a unique invertible positive definite matrix corresponding to the irreducible corepresentation V^i of A_2. Hence $\widetilde{M}^i = N^i \otimes C^i$, where $C^i = (c^i_{kl})^{s_i}_{k,l=1}$. Also note that the matrix C^i is positive definite and invertible since such is the matrix $\widetilde{M}^i$.

Let now U be the matrix consisting of matrix elements of the corepresentation ι of A_1 with respect to the basis in H obtained by taking the union of all the constructed bases in $\widetilde{H}_i$, $i = 1,\ldots,r$. Then

$$\pi(U) = \begin{pmatrix} V^1 \otimes I_{s_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V^r \otimes I_{s_r} \end{pmatrix}.$$

Now, if we write (3.5) as $U^\star = M^{1/2}S(U)^*M^{-1/2}$, and use that π commutes with

S and $*$, we get that

$$\pi((U)^\star) = \begin{pmatrix} N^1 \otimes C^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N^r \otimes C^r \end{pmatrix}^{1/2}$$
$$\times \begin{pmatrix} S(V^1)^* \otimes I_{s_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S(V^r)^* \otimes I_{s_r} \end{pmatrix}$$
$$\times \begin{pmatrix} N^1 \otimes C^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & N^r \otimes C^r \end{pmatrix}^{-1/2}$$
$$= \begin{pmatrix} (V^1)^\star & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (V^r)^\star \end{pmatrix} = \pi(U)^\star.$$

This shows that (3.9) holds for matrix elements of irreducible unitary corepresentations of A_1. Since such elements are dense with respect to the C^*-norm in A_1 and $\star$ is continuous, (3.9) holds on A_1. ∎

Consider now two compact matrix pseudogroups $(A_i, \cdot, 1, *, \Delta_i, \varepsilon_i, S_i)$, A_{i0} are the same as before, $i = 1, 2$, and let $\pi : A_1 \to A_2$ be a Hopf $*$-algebra epimorphism. Define

$$\begin{aligned} &A_{10}/A_{20} = \{a \in A_{10} : (\mathrm{id} \otimes \pi) \circ \Delta_1(a) = a \otimes 1\}, \\ &A_{20} \setminus A_{10} = \{a \in A_{10} : (\pi \otimes \mathrm{id}) \circ \Delta_1(a) = 1 \otimes a\}, \\ &A_{20} \setminus A_{10}/A_{20} = A_{20} \setminus A_{10} \cap A_{10}/A_{20}. \end{aligned} \tag{3.10}$$

It is immediate that A_{10}/A_{20}, $A_{20} \setminus A_{10}$, $A_{20} \setminus A_{10}/A_{20}$ are involutive algebras with the unit 1.

Following [6] and [7], we define $\delta : A_{10} \to A_{10} \otimes_{\mathrm{alg}} A_{10}$ by

$$\delta = (\mathrm{id} \otimes \nu_2 \circ \pi \otimes \mathrm{id}) \circ \Delta_1^{(2)}, \tag{3.11}$$

and let $\star : A_{10} \to A_{10}$ be given by

$$a^\star = f_{-1/2} \cdot S_1(a)^* \cdot f_{1/2}. \tag{3.12}$$

Here ν_2 is the Haar measure for A_2 and f_z, $z \in \mathbb{C}$, is the one-parameter family of modular homomorphisms on A_1.

It is easy to show that $\delta : A_{20} \setminus A_{10}/A_{20} \to A_{20} \setminus A_{10}/A_{20} \otimes_{\mathrm{alg}} A_{20} \setminus A_{10}/A_{20}$ and, since π and $\star$ commute, $\star : A_{20} \setminus A_{10}/A_{20} \to A_{20} \setminus A_{10}/A_{20}$. Denote by A^{inv} the C^*-algebra completion of $A_{20} \setminus A_{10}/A_{20}$ and extend the mappings δ and $\star$ by continuity to the corresponding mappings on A^{inv}.

THEOREM 3.2.2. *Let* $(A_i, \cdot, 1, *, \Delta_i, \varepsilon_i, S_i)$*,* $i = 1, 2$*, be two compact matrix pseudogroups and* $\pi : A_1 \to A_2$ *— a Hopf* $*$*-algebra epimorphism. Then, with the mappings* δ*,* $\star$ *defined by* (3.11) *and* (3.12)*, and* $\varepsilon = \varepsilon_1 | A^{\mathrm{inv}}$*,* $(A^{\mathrm{inv}}, \delta, \varepsilon, \star)$ *is a hypergroup structure on the* C^**-algebra* $(A^{\mathrm{inv}}, \cdot, 1, *)$*.*

Proof. (HS_1): Identity (1.4) has already been proved in Lemma 3.1. To see that (1.3) holds, first recall that $(A_2, \Delta_2, \varepsilon, \star)$ is a quantum hypergroup structure on the C^*-algebra $(A, \cdot, 1, *)$ and ν_2 is the Haar measure since it is unique. Moreover $\nu_2^+ = \overline{\nu}_2 \circ \star = \nu_2$, whence $\nu_2 \circ \star = \overline{\nu}_2$. Because $\Delta_1 \circ \star = \Pi \circ (\star \otimes \star) \circ \Delta_1$ (see the proof of Lemma 3.1) and $\star$ and π commute, we get (1.3). The proof of (1.1) and (1.2) is easy.

(HS_2): The mapping δ is a composition of the homomorphism $\Delta_1^{(2)}$, which is positive, and ν_2, which is a positive functional on a C^*-algebra. Hence δ is positive.

All properties in (HS_3) are immediate. ∎

4. COMPACT QUANTUM HYPERGROUP

DEFINITION 4.1. Suppose that $(A, \delta, \varepsilon, \star)$ is a hypergroup structure on a C^*-algebra $(A, \cdot, 1, *)$. We call $\mathcal{A} = (A, \cdot, 1, *, \delta, \varepsilon, \star, \sigma_t)$ a *compact quantum hypergroup* if

(QH_1) the mapping δ is completely positive ([14]) and the linear span of positive definite elements is dense in A;

(QH_2) σ_t, $t \in \mathbb{R}$, is a continuous one-parameter group of automorphisms of A such that:

(a) there exist dense subalgebras $A_0 \subset A$ and $\widetilde{A}_0 \subset A \otimes A$ such that the one-parameter groups σ_t and $\sigma_t \otimes \mathrm{id}$, $\mathrm{id} \otimes \sigma_t$ can be extended to complex one-parameter groups σ_z and $\sigma_z \otimes \mathrm{id}$, $\mathrm{id} \otimes \sigma_z$, $z \in \mathbb{C}$, of automorphisms of the algebras A_0 and $\widetilde{A}_0$ respectively;

(b) A_0 is invariant with respect to $*$ and $\star$, and $\delta(A_0) \subset \widetilde{A}_0$;

(c) the following relations hold on A_0 for all $z \in \mathbb{C}$:

$$\delta \circ \sigma_z = (\sigma_z \otimes \sigma_z) \circ \delta, \tag{4.1}$$

$$\nu(\sigma_z(a)) = \nu(a); \tag{4.2}$$

(d) there exists $z_0 \in \mathbb{C}$ such that the Haar measure ν satisfies the following strong invariance condition for all $a, b \in A_0$:

$$(\mathrm{id} \otimes \nu)\big[\big((* \circ \sigma_{z_0} \circ \star \otimes \mathrm{id}) \circ \delta(a)\big) \cdot (1 \otimes b)\big] = (\mathrm{id} \otimes \nu)\big((1 \otimes a) \cdot \delta(b)\big); \tag{4.3}$$

(QH_3) the Haar measure ν is faithful on A_0.

In the sequel it will be convenient to denote

$$\kappa = * \circ \sigma_{z_0} \circ \star \tag{4.4}$$

and call it an *antipode*. Note that κ is invertible with $\kappa^{-1} = \star \circ \sigma_{-z_0} \circ *$.

With such a notation, relation (4.3) becomes

$$(\mathrm{id} \otimes \nu)\big((\kappa \otimes \mathrm{id}) \circ \delta(a) \cdot (1 \otimes b)\big) = (\mathrm{id} \otimes \nu)\big((1 \otimes a) \cdot \delta(b)\big). \tag{4.5}$$

REMARK 4.2. The definition of a compact quantum hypergroup involves the use of a continuous one-parameter group of automorphisms. At this point, we followed the approach used to define a *Woronowicz algebra* ([13]). If the group of automorphisms acts trivially, then this is a way a *Kac algebra* is defined (see [8], [17]).

EXAMPLE 4.3. The examples of hypergroup structures considered in Section 3 are, in fact, examples of compact quantum hypergroups. Consider, for instance, the example of $(A^{\mathrm{inv}}, \delta, \varepsilon, \star)$, a hypergroup structure associated with a quantum homogeneous space. Take here $A_0 = A_{20} \setminus A_{10}/A_{20}$ and $\widetilde{A}_0 = A_0 \otimes_{\mathrm{alg}} A_0$. The action of the group σ_t is defined by $\sigma_t(a) = f_{\mathrm{it}} \cdot a \cdot f_{-\mathrm{it}}$, $z_0 = -\frac{1}{2}\mathrm{i}$, and $\kappa = S$.

In this case δ is completely positive since it is a composition of completely positive maps. Axioms (QH_2) and (QH_3) summarize well-known properties of compact matrix pseudogroups ([19]).

LEMMA 4.4. *Let $\mathcal{A}$ be a compact quantum hypergroup and κ be defined by (4.4). Then, for all $a, b \in A_0$,*

$$\begin{gathered} \kappa(ab) = \kappa(b)\kappa(a), \quad \delta \circ \kappa(a) = \Pi \circ (\kappa \otimes \kappa) \circ \delta(a), \\ \nu \circ \kappa = \nu, \quad \kappa(1) = 1, \quad \varepsilon \circ \kappa = \varepsilon. \end{gathered} \tag{4.6}$$

Proof. The first four identities are obvious. To show the last identity, let us show that $\varepsilon(\sigma_z(a)) = \varepsilon(a)$, $a \in A_0$, $z \in \mathbb{C}$. By using (4.1), we have

$$\begin{aligned} \varepsilon(\sigma_z(a)) &= \varepsilon(\sigma_z \circ (\varepsilon \otimes \mathrm{id}) \circ \delta(a)) = (\varepsilon \otimes \varepsilon) \circ (\mathrm{id} \otimes \sigma_z) \circ \delta(a) \\ &= (\varepsilon \otimes \varepsilon) \circ (\sigma_{-z} \otimes \mathrm{id}) \circ (\sigma_z \otimes \sigma_z) \circ \delta(a) \\ &= (\varepsilon \otimes \varepsilon) \circ (\sigma_{-z} \otimes \mathrm{id}) \circ \delta(\sigma_z(a)) = \varepsilon(\sigma_{-z}(\sigma_z(a))) = \varepsilon(a). \quad \blacksquare \end{aligned}$$

If $\mathcal{A}$ is a compact quantum hypergroup, we use the GNS construction to complete A_0 or A to a Hilbert space H_ν with respect to the norm $\|\cdot\|_\nu$ induced by the inner product

$$\langle a, b \rangle = \nu(b^* a). \tag{4.7}$$

Proposition 4.5. *Let $\mathcal{A}$ be a compact quantum hypergroup and, for $a \in A_0$, let an operator $T_a : A_0 \to A_0$ be defined by*

$$T_a(x) = (\mathrm{id}\otimes\nu)\big((\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes x)\big) = (\mathrm{id}\otimes\nu)\big((1\otimes a)\cdot\delta(x)\big). \tag{4.8}$$

Then

(i) *the operator T_a, $a \in A_0$, is a Hilbert-Schmidt type operator if extended by continuity to the operator $T_a : H_\nu \to H_\nu$;*

(ii) *for $x \in H_\nu$, $a \in A_0$, the following relation holds*

$$\|T_a(x)\| \leqslant \|a\|\,\|x\|_\nu \tag{4.9}$$

and, hence, for $a \in A_0$, the range $\mathrm{Ran}\,(T_a) \subset A$;

(iii) *the adjoint operator is given by*

$$T_a^\dagger(x) = (\nu\otimes\mathrm{id})\big[\big((*\circ\kappa\circ*\otimes\mathrm{id})\circ\delta(a^*)\big)\cdot(x\otimes 1)\big]; \tag{4.10}$$

(iv) *denote by R the set $\{T_a(b) : a, b \in A_0\}$. Then R is total in H_ν with respect to the norm $\|\cdot\|_\nu$.*

Proof. (i) Let $e_i \in A_0$, $i \in \mathbb{Z}_+$, be an orthonormal basis in H_ν, and denote $\widetilde{a} = (\kappa\otimes\mathrm{id})\circ\delta(a) \in A\otimes A$. Then $T_a(x) = (\mathrm{id}\otimes\nu)(\widetilde{a}\cdot(1\otimes x))$. We have

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}_+}\langle T_a(e_i), e_j\rangle^2 &= \sum_{i,j\in\mathbb{Z}_+}\nu(e_j^*\cdot T_a(e_i))^2 = \sum_{i,j\in\mathbb{Z}_+}\nu\big(e_j^*\cdot(\mathrm{id}\otimes\nu)(\widetilde{a}\cdot(1\otimes e_i))\big)^2\\ &= \sum_{i,j\in\mathbb{Z}_+}(\overline{\nu}\otimes\nu)((*\otimes\mathrm{id})(\widetilde{a})\cdot(e_j\otimes e_i))^2.\end{aligned}$$

The last sum is finite since $(\overline{\nu}\otimes\nu)((*\otimes\mathrm{id})(\widetilde{a})\cdot(e_j\otimes e_i))$ is the Fourier coefficient of the element $(\mathrm{id}\otimes *)(\widetilde{a})$ considered in the Hilbert space $\overline{H}_\nu\otimes H_\nu$, where the Hilbert space $\overline{H}_\nu$ is obtained by completing A_0 with respect to the inner product $\langle a, b\rangle_{\overline{\nu}} = \overline{\nu}(b^*a)$.

(ii) Let φ be a state on the C^*-algebra A, $a \in A_0$, $x \in H$. Then we have that

$$\begin{aligned}\varphi(T_a(x)^*\cdot T_a(x)) &= \varphi\big((\mathrm{id}\otimes\nu)((1\otimes a)\cdot\delta(x))^*\cdot(\mathrm{id}\otimes\nu)((1\otimes a)\cdot\delta(x))\big)\\ &\leqslant \varphi\big((\mathrm{id}\otimes\nu)(\delta(x)^*\cdot(1\otimes a^*)\cdot(1\otimes a)\cdot\delta(x))\big)\\ &= (\varphi\otimes\nu)(\delta(x)^*\cdot(1\otimes a^*a)\cdot\delta(x))\\ &\leqslant (\varphi\otimes\nu)(\delta(x)^*\cdot\delta(x))\,\|1\otimes a\|^2 \leqslant (\varphi\otimes\nu)(\delta(x^*x))\,\|a\|^2\\ &= \nu(x^*x)\|a\|^2 = \|x\|_\nu^2\,\|a\|^2.\end{aligned}$$

Here we used the fact that the mappings $(\mathrm{id}\otimes\nu)$ and δ are completely positive and that ν is a Haar measure. This proves (4.9).

Finally, to show that $T_a(x)\in A$, let $x_n \xrightarrow{\nu} x$ with $x_n\in A_0$. All $T_a(x_n)\in A_0$ and relation (4.9) shows that $T_a(x_n)\to T_a(x)$ with respect to the C^*-norm. Since A is closed, the claim follows.

(iii) To prove formula (4.10), we have

$$\begin{aligned}\langle T_a(x),y\rangle &= \nu(y^*\cdot T_a(x)) = \nu\big(y^*\cdot(\mathrm{id}\otimes\nu)((\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes x))\big)\\ &= (\nu\otimes\nu)\big((y^*\otimes 1)\cdot(\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes x)\big)\\ &= \nu\big(((\nu\otimes\mathrm{id})((y^*\otimes 1)\cdot(\kappa\otimes\mathrm{id})\circ\delta(a))\cdot x\big).\end{aligned}$$

Hence

$$\begin{aligned}T_a^\dagger(y) &= (\nu\otimes\mathrm{id})\big((y^*\otimes 1)\cdot(\kappa\otimes\mathrm{id})\circ\delta(a)\big)^*\\ &= (\nu\otimes\mathrm{id})\big((*\circ\kappa\otimes *)\circ\delta(a)\cdot(y\otimes 1)\big)\\ &= (\nu\otimes\mathrm{id})\big((*\circ\kappa\circ *\otimes\mathrm{id})\circ\delta(a^*)\cdot(y\otimes 1)\big).\end{aligned}$$

(iv) Suppose that R is not total. Then there exists an element $y\in H_\nu$, $y\neq 0$, such that $y\perp T_a(b)$, $b\in A_0$, or, which is the same thing, that for all $a\in A_0$,

$$T_a^\dagger(y) = (\nu\otimes\mathrm{id})((*\circ\kappa\circ *\otimes\mathrm{id})\circ\delta(a^*)\cdot(y\otimes 1)) = 0.$$

By taking ε of both sides of this equality, we obtain that $\nu(\kappa(a)^*y)=0$. By taking $a=\kappa^{-1}(c^*)$, we see that $\nu(cy)=0$ for all $c\in A_0$. But this contradicts density of A_0 in H_ν. ∎

LEMMA 4.6. *Let $\mathcal{A}$ be a compact quantum hypergroup. Then*

$$*\circ\kappa\circ *\circ\kappa = \mathrm{id} \tag{4.11}$$

on A_0 and the operator $T_a^\dagger$, given by (4.10), *can be written in the form*

$$T_a^\dagger = T_{a^\dagger}, \tag{4.12}$$

where $a^\dagger$ on A_0 is defined by

$$a^\dagger = \kappa(a)^*. \tag{4.13}$$

Proof. Let $a,b\in A_0$. Rewrite the identity (4.5)

$$(\mathrm{id}\otimes\nu)\big((\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes b)\big) = (\mathrm{id}\otimes\nu)\big((1\otimes a)\cdot\delta(b)\big)$$

and take $*$ of both sides to get

$$(\mathrm{id}\otimes\overline{\nu})\big((*\circ\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes b)\big) = (\mathrm{id}\otimes\nu)(\delta(b^*)\cdot(1\otimes a^*)).$$

Now, by taking κ of both sides and using again (4.5), we obtain

$$\begin{aligned}(\mathrm{id}\otimes\overline{\nu})\big((\kappa\circ *\circ\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes b)\big) &= (\mathrm{id}\otimes\nu)\big((\kappa\otimes\mathrm{id})\circ\delta(b^*)\cdot(1\otimes a^*)\big)\\ &= (\mathrm{id}\otimes\nu)((1\otimes b^*)\cdot\delta(a^*)).\end{aligned}$$

Now again apply $*$ to both sides:

$$(\mathrm{id}\otimes\nu)\big((*\circ\kappa\circ *\circ\kappa\otimes\mathrm{id})\circ\delta(a)\cdot(1\otimes b)\big) = (\mathrm{id}\otimes\nu)((\delta(a)\cdot(1\otimes b)).$$

This means that on the element $T_a(b)\in R$, $*\circ\kappa * \circ\kappa = \mathrm{id}$. But R is dense in H_ν, and hence, in A_0. Thus (4.11) holds.

Formula (4.12) follows immediately from (4.10) by noticing that (4.11) implies the relation $*\circ\kappa = \kappa^{-1}\circ *$. ∎

Formulas (4.11) and (4.13) show that, for all $a\in A_0$, $a^{\dagger\dagger}=a$. This means that any element in A_0 can be written as $a = a_{\mathrm{r}} + \mathrm{i}\, a_{\mathrm{im}}$ with $a_{\mathrm{r}}^\dagger = a_{\mathrm{r}}$, $a_{\mathrm{im}}^\dagger = a_{\mathrm{im}}$ by setting $a_{\mathrm{r}} = \frac{1}{2}(a+a^\dagger)$, $a_{\mathrm{im}} = \frac{1}{2\mathrm{i}}(a-a^\dagger)$. It also follows from (4.12) and part (i) of Proposition 4.5 that T_a is a compact self adjoint operator if $a^\dagger = a$.

PROPOSITION 4.7. *The set $R' = \{T_a(b) : a, b\in A_0, a^\dagger = a\}$ is total in A with respect to the C^*-norm.*

Proof. Suppose that the closure $\overline{R}' \neq A$. Then there exists $a\in A_0\setminus\overline{R}'$ and a continuous linear functional φ such that $\varphi(R')=0$ and $\varphi(a)=\|a\|$. Let us consider the element $b=(\varphi\otimes\mathrm{id})\circ\delta(a)$. Because $\varepsilon(b)=(\varphi\otimes\varepsilon)\circ\delta(a)=\varphi(a)\neq 0$, we see that $b\neq 0$. This means that $\nu(b^*b)\neq 0$ since ν is faithful. On the other hand,

$$\begin{aligned}\nu(b^*b) &= \nu(b^*\cdot(\varphi\otimes\mathrm{id})\circ\delta(a)) = (\varphi\otimes\nu)((1\otimes b^*)\cdot\delta(a))\\ &= \varphi(T_{b^*}(a)) = \varphi(T_{b^*_{\mathrm{r}}}(a) + \mathrm{i}\, T_{b^*_{\mathrm{im}}}(a)) = 0.\end{aligned}$$

This contradiction proves the claim. ∎

Suppose now that the C^*-algebra A is commutative and denote its spectrum by P. Each element $\xi\in P$ defines a linear operator on A defined by

$$R_\xi = (\mathrm{id}\otimes\xi)\circ\delta, \tag{4.14}$$

which will be called a *generalized translation operator*. For $\xi\in P$ and $a\in A_0$, we define

$$\xi^\dagger(a) = \overline{\xi(a^\dagger)}. \tag{4.15}$$

From the definition of $a^\dagger$ it immediately follows that $\xi^\dagger$ is a homomorphism $A_0\to\mathbb{C}$ and, hence, continuous. Being extended by continuity to A, it becomes a point in P.

LEMMA 4.8. *Let $\mathcal{A}$ be a commutative compact quantum hypergroup. Let R_ξ, $\xi \in P$, be given by* (4.14)*. Then:*

(i) *R_ξ is a bounded operator and the mapping $\xi \mapsto R_\xi$ is strongly continuous;*

(ii) *for all $\xi, \eta \in P$ and $a \in A_0$, $\overline{\xi^\dagger(R_{\eta^\dagger}(a^\dagger))} = \eta(R_\xi(a))$;*

(iii) *the counit ε belongs to P and $R_\varepsilon = \mathrm{id}$;*

(iv) *if $a \in A$ is positive, then $R_\xi(a)$ is also positive for all $\xi \in P$;*

(v) *for all $\xi, \eta \in P$, $\eta(R_\xi(1)) = 1$;*

(vi) *if $R_\xi^\dagger$ denotes the operator adjoint to R_ξ with respect to inner product* (4.7)*, then $R_\xi^\dagger = R_{\xi^\dagger}$.*

Proof. (i) Because ξ is a homomorphism of a C^*-algebra, it is completely positive, and, hence, R_ξ is completely positive since δ is. This implies that $R_\xi(a)^* \cdot R_\xi(a) \leqslant R_\xi(a^* \cdot a)$ (see [14]). So, we have

$$\begin{aligned}\|R_\xi(a)\|_\nu^2 &= \nu(R_\xi(a)^* \cdot R_\xi(a)) \leqslant \nu(R_\xi(a^*a)) = (\xi \otimes \nu) \circ \delta(a^*a) \\ &= (\xi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nu) \circ \delta(a^* \cdot a) \\ &= \xi(\nu(a^*a)1) = \nu(a^*a)\xi(1) = \|a\|_\nu^2\, \xi(1).\end{aligned}$$

This shows that the operator R_ξ is bounded. It also follows that, if $\xi \to \eta$, then $\|(R_\xi - R_\eta)(a)\|_\nu \to 0$, which means that R_ξ is strongly continuous.

(ii) Let $\xi, \eta \in P$. By using (4.14) and (1.12), we have

$$\begin{aligned}\overline{\xi^\dagger(R_{\eta^\dagger}(a^\dagger))} &= \overline{(\xi^\dagger \otimes \eta^\dagger)} \circ \delta(a^\dagger) = (\xi \otimes \eta) \circ (\dagger \otimes \dagger) \circ \delta(a^\dagger) \\ &= (\eta \otimes \xi) \circ \delta(a) = \eta(R_\xi(a)).\end{aligned}$$

(iii) The counit ε is a homomorphism and hence $\varepsilon \in P$. Also $R_\varepsilon = \mathrm{id}$.

(iv) Because δ is positive by definition of a quantum hypergroup and ξ is positive as a homomorphism, R_ξ is positive.

(v) Because ξ and η are homomorphisms, this immediately follows from (1.6).

(vi) Let $a, b \in A_0$ and $\xi \in P$. Then

$$\langle R_\xi(a), b\rangle = \nu(b^* \cdot R_\xi(a)) = \nu(b^* \cdot (\mathrm{id} \otimes \xi) \circ \delta(a)) = (\nu \otimes \xi)((b^* \otimes 1) \cdot \delta(a)).$$

On the other hand, by using the definition of $\dagger$, the fact that $\delta \circ \kappa^{-1} = \Pi \circ (\kappa^{-1} \otimes \kappa^{-1}) \circ \delta$ and that κ is an antihomomorphism, as well as that $\nu \circ \kappa = \nu$ and (4.5), we obtain

$$\begin{aligned}\langle a, R_{\xi^\dagger}(b)\rangle &= \nu(R_{\xi^\dagger}(b)^* \cdot a) = \nu\big((\mathrm{id}\otimes\xi^\dagger)\circ\delta(b)^*\cdot a\big)\\ &= \nu\big((*\otimes\xi\circ\dagger)\circ\delta(b)\cdot a\big)\\ &= (\nu\otimes\xi)\big((*\otimes *\circ\kappa)\circ\delta(b)\cdot(a\otimes 1)\big)\\ &= (\nu\otimes\xi)\big((*\otimes\kappa^{-1}\circ *)\circ\delta(a)\cdot(a\otimes 1)\big)\\ &= (\nu\otimes\xi)\big((\kappa\circ\kappa^{-1}\otimes\kappa^{-1})\circ\delta(b^*)\cdot(a\otimes 1))\\ &= (\xi\otimes\nu)\big((\mathrm{id}\otimes\kappa)\circ\delta(\kappa^{-1}(b^*))\cdot(1\otimes a)\big)\\ &= (\xi\otimes\nu)\big((1\otimes\kappa^{-1}(a))\cdot\delta(\kappa^{-1}(b^*))\big)\\ &= (\xi\otimes\nu)\big((\kappa\otimes\mathrm{id})\circ\delta(\kappa^{-1}(a))\cdot(1\otimes\kappa^{-1}(b^*))\big)\\ &= (\nu\otimes\xi)\big((\kappa^{-1}\otimes\mathrm{id})\circ\delta(a)\cdot(\kappa^{-1}(b^*)\otimes 1)\big)\\ &= (\nu\otimes\xi)((b^*\otimes 1)\cdot\delta(a)).\end{aligned}$$

By comparing the two identities, we see that $R_{\xi^\dagger} = R_\xi^\dagger$. ∎

By using Lemma 4.8 and applying Theorem 2.1 from [3], we immediately get the following theorem.

THEOREM 4.9. *Let $\mathcal{A}$ be a commutative compact quantum hypergroup. Let P denote the spectrum of the commutative C^*-algebra. Then P is the basis of a normal hypercomplex system $L_1(P,\nu)$ with a basis unit ε.*

5. COREPRESENTATIONS OF COMPACT QUANTUM HYPERGROUPS AND A PETER-WEYL THEOREM

Let A be a Banach space, and $\mathcal{A} = (A,\delta,\varepsilon)$ be a coalgebra ([1]). Let V be a Banach space and $\iota : V \to A\otimes V$ be a continuous linear map such that

$$\begin{aligned}(\delta\otimes\mathrm{id})\circ\iota &= (\mathrm{id}\otimes\iota)\circ\delta,\\ (\varepsilon\otimes\mathrm{id})\circ\delta &= \mathrm{id},\end{aligned} \tag{5.1}$$

where $A\otimes V$ denotes the Banach space obtained by completion of the algebraic tensor product with respect to the injective or projective cross-norm ([14]). The Banach space V will be called a *left comodule* over the coalgebra $\mathcal{A}$ and (V,ι) — a *corepresentation* of the coalgebra. If V is finite dimensional, then the corepresentation (V,ι) is called *finite dimensional*. If (V,ι) is a finite dimensional corepresentation of a coalgebra $\mathcal{A}$ and $\mathcal{E} = \{e_i \mid i = 1,\ldots,d\}$ is a basis in V, $d = \dim V$, then $\iota(e_i) = \sum_{j=1}^{d} t_{ij}\otimes e_j$ for some elements $t_{ij}\in A$ which are called *matrix elements* of the corepresentation (V, ι) with respect to the basis $\mathcal{E}$. For matrix elements t_{ij}, we have the following identities:

$$\begin{aligned} \delta(t_{ij}) &= \sum_k t_{ik} \otimes t_{kj}, \\ \varepsilon(t_{ij}) &= \delta_{ij}, \end{aligned} \tag{5.2}$$

where δ_{ij} denotes the Kronecker symbol.

Two finite dimensional corepresentations of a coalgebra $\mathcal{A}$, (V_1, ι_1) and (V_2, ι_2) are called *equivalent* if there is an invertible operator $F : V_1 \to V_2$ such that

$$\iota_2 \circ F = (\mathrm{id} \otimes F) \circ \iota_1. \tag{5.3}$$

Let (V_1, ι_1) and (V_2, ι_2) be two equivalent finite dimensional corepresentations of a coalgebra $\mathcal{A}$ with matrix elements $(t^1) = (t^1_{ij})^{d_1}_{i,j=1}$, $(t^2) = (t^2_{ij})^{d_2}_{i,j=1}$ with respect to bases in V_1 and V_2. It follows from elementary linear algebra and definition (5.3) that $d^1 = d^2$ and

$$(t^1) = (f_{ij})' \cdot (t^2) \cdot (f_{ij})'^{-1}, \tag{5.4}$$

where (f_{ij}) denotes the matrix of F with respect to the chosen bases in V_1 and V_2, $'$ denotes the matrix transpose, and $\cdot$ is the usual matrix multiplication.

A corepresentation (V, ι) of a coalgebra $\mathcal{A}$ is called *irreducible* if there is no proper linear closed subspace $V' \subset V$ such that (V', ι) is a corepresentation of $\mathcal{A}$. A finite dimensional corepresentation (V, ι) is irreducible if and only if an operator $F : V \to V$ such that $(\mathrm{id} \otimes F) \circ \iota = \iota \circ F$ is necessarily a multiple of the identity operator ([19]).

Definition 5.1. Let $\mathcal{A} = (A, \cdot, 1, *, \delta, \varepsilon, \star, \sigma_t)$ be a compact quantum hypergroup. We call (V, ι) a *corepresentation* of $\mathcal{A}$ if (V, ι) is a corepresentation of the coalgebra (A, δ, ε). All notions concerning corepresentations of a compact quantum hypergroups are understood in the sense of the corresponding notions for its coalgebra structure.

Let (V, ι) be a corepresentation of a compact quantum hypergroup $\mathcal{A}$ and let A° denote the set of all continuous linear functionals on the C^*-algebra A. Then the first formula in (1.12) defines the structure of an algebra on A° and the corepresentation (V, ι) gives rise to a representation (V, ι°) of the algebra A°, $A^\circ \ni \xi \mapsto \iota^\circ(\xi)$, defined by the formula

$$\iota^\circ(\xi) = (\xi \otimes \mathrm{id}) \circ \iota. \tag{5.5}$$

A corepresentation of a compact quantum hypergroup $\mathcal{A}$, (V, ι), is irreducible if and only if such is the representation (V, ι°) of the algebra A°. Two finite dimensional corepresentations of $\mathcal{A}$, (V_1, ι_1) and (V_2, ι_2), are equivalent if and only if the corresponding representations of A°, (V_1, ι_1°) and (V_2, ι_2°) are equivalent.

By using the proof of Lemma 4.8 in [19], we get a similar result.

Lemma 5.2. *Let Q_0 be a finite set of irreducible finite dimensional corepresentations (V^q, ι^q) of a compact quantum hypergroup $\mathcal{A}$ and let λ_{ij}^q, $q \in Q_0$, $i, j = 1, \ldots, d^q$ (d^q is the dimension of V^q) be an arbitrary set of complex numbers. If t_{ij}^q are matrix elements with respect to some bases in V^q, then there exists a continuous linear functional $\alpha \in A^\circ$ such that $\alpha(t_{ij}^q) = \lambda_{ij}^q$. Hence the matrix elements are linearly independent.*

Proposition 5.3. *Let $\mathcal{A}$ be a compact quantum hypergroup and (V, ι) be a finite dimensional corepresentation. Then any element that belongs to the linear span of the matrix elements t_{ij}, in any basis of V, is entire analytic relatively to the one-parameter group σ_t.*

Proof. Consider the algebra $\iota^\circ(A^\circ)$. This is a finite dimensional subalgebra of $M_d(\mathbb{C})$, the algebra of $d \times d$-matrices over $\mathbb{C}$, $d = \dim(V)$. Because, for $t \in \mathbb{R}$, σ_t is a C^*-algebra automorphism, it is continuous and, hence, for $\xi \in A^\circ$, we have that $\sigma_t^\circ(\xi) = \xi \circ \sigma_t \in A^\circ$. Identity (4.1) implies that σ_t° is an automorphism of the algebra A° for each $t \in \mathbb{R}$. This means that $\iota^\circ \circ \sigma_t^\circ : \iota^\circ(A^\circ) \to \iota^\circ(A^\circ)$ is an automorphism. If A_ι will denote the Hilbert space of matrix elements of the corepresentation (V, ι) endowed with the inner product defined by (4.7), then $\sigma_t : A_\iota \to A_\iota$ and it is easy to check that σ_t is a one-parameter group of unitary operators, and so $\sigma_t = \mathrm{e}^{\mathrm{i}Nt}$, where $N : A_\iota \to A_\iota$ is a self-adjoint operator, $\mathrm{i} = \sqrt{-1}$, $t \in \mathbb{R}$. Let us define $\sigma_z : A_\iota \to A_\iota$ by $\sigma_z = \mathrm{e}^{\mathrm{i}Nz}$, $z \in \mathbb{C}$. Clearly this is an analytic extension of σ_t. ∎

Let (V, ι) be a finite dimensional corepresentation of a compact quantum hypergroup $\mathcal{A}$ and let A_ι be the linear span of the matrix elements. Then A_ι is a coalgebra and the restriction mapping $\rho_\iota : A^\circ \to A_\iota^\circ$ defined by $\xi \mapsto \xi_\iota = \rho_\iota(\xi) = \xi|A_\iota$ is an epimorphism of the algebra A° onto the algebra A_ι° and ι° is a representation of A_ι° on the linear space V.

Let now H be a finite dimensional Hilbert space with an inner product $(\cdot\,, \cdot)$ and let (H, ι) be a corepresentation of a compact quantum hypergroup $\mathcal{A}$. Since, by Proposition 5.3, matrix elements of a finite dimensional corepresentation are analytic, we can make the following definition.

DEFINITION 5.4. A finite dimensional corepresentation (H,ι) of a compact quantum hypergroup $\mathcal{A}$ is called a †-*corepresentation* if, for all $u,v\in H$, we have

$$\sum_{i=1}^{d}(u,v_i)b_i=\sum_{i=1}^{d}(u_i,v)a_i^\dagger, \tag{5.6}$$

where $\iota(u)=\sum_{i=1}^{d}a_i\otimes u_i$, $\iota(v)=\sum_{i=1}^{d}b_i\otimes v_i$, $a_i,b_i\in A$, $u_i,v_i\in H$, and $d=\dim H$.

LEMMA 5.5. *Let t_{ij}, $i,j=1,\dots,d$, be matrix elements of a finite dimensional †-corepresentation (H,ι) with respect to an orthonormal basis in H. Then*

$$t_{ij}^\dagger=t_{ji}. \tag{5.7}$$

Proof. The claim immediately follows from (5.6) by setting $u=e_i$, $v=e_j$. ∎

LEMMA 5.6. *Let (H,ι) be a finite dimensional corepresentation of a compact quantum hypergroup $\mathcal{A}$. Define an involution $\xi_\iota\mapsto\xi_\iota^\dagger$ on A_ι° by*

$$\xi_\iota^\dagger(a)=\overline{\xi}_\iota(a^\dagger),\quad a\in A_\iota. \tag{5.8}$$

Then $\xi_\iota^\dagger$ is a continuous functional on A_ι and the corepresentation (H,ι) is a †-corepresentation if and only if ι° is a †-representation of the involutive algebra A_ι°.

Proof. Because A_ι is finite dimensional and $\xi_\iota^\dagger$ is a linear mapping, it is continuous. Now, for $\xi_\iota\in A_\iota^\circ$, $u,v\in A_\iota^\circ$ with $\iota(u)=\sum_{i=1}^{d}a_i\otimes u_i$, $\iota(v)=\sum_{i=1}^{d}b_i\otimes v_i$, we have

$$(\iota^\circ(\xi_\iota)(u),v)=\Big(\sum_{i=1}^{d}(\xi_\iota(a_i)u_i,v\Big)=\xi\Big(\sum_{i=1}^{d}(u_i,v)a_i\Big).$$

On the other hand,

$$(u,\iota^\circ(\xi_\iota^\dagger)(v))=\Big(u,\sum_{i=1}^{d}\xi_\iota^\dagger(b_i)v_i\Big)=\sum_{i=1}^{d}(u,v_i)\overline{\xi_\iota^\dagger}(b_i)=\xi_\iota\Big(\sum_{i=1}^{d}(u,v_i)b_i^\dagger\Big).$$

The proof follows if we compare these two identities. ∎

LEMMA 5.7. *Let (H,ι) be a finite dimensional †-corepresentation of a compact quantum hypergroup $\mathcal{A}$. Then (H,ι) is a finite direct sum of irreducible finite dimensional †-corepresentations, i.e. $H=\bigoplus_{i=1}^{k}H_i$ and (H_i,ι_i) is an irreducible †-corepresentation with $\iota_i=\iota|H_i$.*

Proof. Clearly it will be sufficient to show that, for an invariant subspace H_1, the subspace $H_1^\perp = H \ominus H_1$ will be invariant, i.e. $\iota : H_1^\perp \to A \otimes H_1^\perp$. Choose an orthonormal basis in H_1, $e_1, \dots, e_m$, and let $e_{m+1}, \dots, e_{m+n}$ be an orthonormal basis in $H_1^\perp$. Set $u = e_i$, $v = e_{m+j}$, where $i \leqslant m$, $j \leqslant n$. Then we have that

$$\iota(u) = \iota(e_i) = \sum_{k=1}^{m} t_{ik} \otimes e_k$$

since H_1 is an invariant subspace, and

$$\iota(v) = \iota(e_{m+j}) = \sum_{k=1}^{m+n} t_{m+j\,k} \otimes e_k.$$

By using (5.6), we find that

$$t_{m+j\,i} = \sum_{k=1}^{m+n} (e_i, e_k) t_{m+j\,k} = \sum_{k=1}^{m} (e_k, e_{m+j}) t_{ik}^\dagger = 0$$

for all $i = 1, \dots, m$, $j = 1, \dots, n$. This shows that $H_1^\perp$ is invariant. ∎

We now prove some orthogonality relations.

THEOREM 5.8. *Let (V^p, ι^p) and (V^q, ι^q) be finite dimensional irreducible corepresentations of a compact quantum hypergroup $\mathcal{A}$. Let t_{ij}^p and t_{kl}^q denote matrix elements of the corresponding corepresentations. Then*

$$\nu(t_{ij}^p \kappa(t_{kl}^q)) = 0 \tag{5.9}$$

if either the corepresentations are not equivalent or $i \neq l$.

Proof. Let us apply the strong invariance condition for ν given by (4.5) to elements $a = t_{ij}^p$ and $b = \kappa(t_{kl}^q)$. For these elements, the left-hand side of (4.5) becomes:

$$(\mathrm{id} \otimes \nu)((\kappa \otimes \mathrm{id}) \circ \delta(t_{ij}^p) \cdot (1 \otimes \kappa(t_{kl}^q))) = \sum_r \nu(t_{rj}^p \kappa(t_{kl}^q)) \kappa(t_{ir}^p),$$

whereas the right-hand side will be

$$\begin{aligned}(\mathrm{id} \otimes \nu)((1 \otimes t_{ij}^p) \cdot \delta(\kappa(t_{kl}^q))) &= (\mathrm{id} \otimes \nu)((1 \otimes t_{ij}^p) \cdot \Pi\Big(\sum_s \kappa(t_{ks}^q) \otimes \kappa(t_{sl}^q)\Big)\Big) \\ &= (\mathrm{id} \otimes \nu)((1 \otimes t_{ij}^p) \cdot \sum_s \kappa(t_{sl}^q) \otimes \kappa(t_{ks}^q)) \\ &= \sum_s \nu(t_{ij}^p \kappa(t_{ks}^q)) \kappa(t_{sl}^q).\end{aligned}$$

Now, if we recall that the matrix elements of corepresentations are linearly independent and κ is invertible, comparing the last two expressions we see that relation (5.9) holds. ∎

REMARK 5.9. If the corepresentations (V^p, ι^p) and (V^q, ι^q) are †-corepresentations then, by using (5.7), we can rewrite (5.9) in the following form

$$\nu(t^p_{ij} {t^q_{lk}}^*) = 0. \tag{5.10}$$

The following proposition is a modification of the well-known Hilbert-Schmidt theorem.

PROPOSITION 5.10. *Let the operator* T_a, $a \in A_0$, $a^\dagger = a$, *be defined by* (4.8). *Let* $y = T_a(x)$ *for some* $x \in H_\nu$ *and*

$$y = \sum_{i=1}^{\infty} \langle y, v^{\lambda_i}\rangle v^{\lambda_i} \tag{5.11}$$

be the Fourier expansion of y *with respect to an orthonormal set of the eigenvectors* v^{λ_i} *of the self-adjoint compact operator* T_a, *where* λ_i *is a corresponding eigenvalue,* $\lambda_i \neq 0$. *Then* $v^\lambda \in A$ *and the series* (5.11) *converges in the* C^**-norm.*

Proof. First of all, if v^{λ_i} is the eigenvector corresponding to an eigenvalue $\lambda_i \neq 0$, then by Proposition 4.5 (ii), $v^\lambda = \frac{1}{\lambda} T_a(v^\lambda) \in \text{Ran } T_a \subset A$. Hence $v^{\lambda_i} \in A$.

Now, to prove convergence of (5.11), we use the Cauchy criterion. So let $m, n \in \mathbb{Z}_+$. We have

$$\begin{aligned}
\Big\| \sum_{i=m}^{m+n} \langle y, v^{\lambda_i}\rangle v^{\lambda_i} \Big\| &= \Big\| \sum_{i=m}^{m+n} \langle T_a(x), v^{\lambda_i}\rangle v^{\lambda_i} \Big\| = \Big\| \sum_{i=m}^{m+n} \langle x, T_a(v^{\lambda_i})\rangle v^{\lambda_i} \Big\| \\
&= \Big\| \sum_{i=m}^{m+n} \langle x, v^{\lambda_i}\rangle \lambda_i v^{\lambda_i} \Big\| = \Big\| \sum_{i=m}^{m+n} \langle x, v^{\lambda_i}\rangle T_a(v^{\lambda_i}) \Big\| \\
&= \Big\| T_a\Big(\sum_{i=m}^{m+n} \langle x, v^{\lambda_i}\rangle v^{\lambda_i} \Big) \Big\| \leqslant \|a\| \, \Big\| \sum_{i=m}^{m+n} \langle x, v^{\lambda_i}\rangle v^{\lambda_i} \Big\|_\nu .
\end{aligned}$$

In order to obtain the last inequality, we used estimate (4.9). Finally, since $x \in H_\nu$, $\Big\| \sum_{i=m}^{m+n} \langle x, v^{\lambda_i}\rangle v^{\lambda_i} \Big\|_\nu \to 0$ as $m \to \infty$ and hence series (5.11) converges in the C^*-norm. It clearly converges to y. ∎

THEOREM 5.11. *Let* Q *be the set of all finite dimensional irreducible non-equivalent* †*-corepresentations* (V^q, ι^q), $q \in Q$, *of a compact quantum hypergroup* $\mathcal{A}$ *and* $\mathcal{B} = \{t^q_{ij} : q \in Q,\ i, j = 1, \ldots, d_q = \dim V^q\}$ *be the set of all matrix elements of these corepresentations with respect to some bases. Then the linear span of the set* $\mathcal{B}$ *is dense in* A *with respect to the* C^**-norm.*

Proof. Let us again consider the operator $T_a : H_\nu \to H_\nu$. Because this operator is compact and self-adjoint,

$$\text{Ran } T_a = \bigoplus_{\lambda \neq 0} V^\lambda, \tag{5.12}$$

where V^λ is the finite dimensional eigenspace corresponding to the eigenvalue λ.

Let us show that

$$\delta(V^\lambda) \subset A \otimes V^\lambda. \tag{5.13}$$

It follows from Proposition 5.10 that $V^\lambda \subset A$ and hence $\delta : V^\lambda \to A \otimes A$. Now, to prove (5.13), let $v^\lambda \in V^\lambda$ and let φ be an arbitrary continuous functional on A. It will suffice to prove that $u = (\varphi \otimes \text{id}) \circ \delta(v^\lambda) \in V^\lambda$. To see that, by using coassociativity of δ, we find

$$\begin{aligned}
T_a(u) &= (\text{id} \otimes \nu)((1 \otimes a) \cdot \delta(u)) \\
&= (\text{id} \otimes \nu)\big((1 \otimes a) \cdot \delta((\varphi \otimes \text{id}) \circ \delta(v^\lambda))\big) \\
&= (\text{id} \otimes \nu)((1 \otimes a) \cdot (\varphi \otimes \delta) \circ \delta(v^\lambda)) \\
&= (\varphi \otimes \text{id} \otimes \nu)((1 \otimes 1 \otimes a) \cdot (\text{id} \otimes \delta) \circ \delta(v^\lambda)) \\
&= (\varphi \otimes \text{id} \otimes \nu)((1 \otimes 1 \otimes a) \cdot (\delta \otimes \text{id}) \circ \delta(v^\lambda)) \\
&= (\varphi \otimes \text{id}) \circ \delta((\text{id} \otimes \nu)((1 \otimes a) \cdot \delta(v^\lambda))) \\
&= (\varphi \otimes \text{id}) \circ \delta(T_a(v^\lambda)) = \lambda\, u.
\end{aligned}$$

This proves (5.13), which means that $\mathcal{V}^\lambda = (V^\lambda, \delta|V^\lambda)$ is a finite dimensional corepresentation.

Let us now show that $(V^\lambda, \delta|V^\lambda)$ is a †-corepresentation. Equip V^λ with the inner product defined by

$$(u, v) = \nu(uv^*).$$

Then condition (5.6) can be written as

$$(\text{id} \otimes \nu)((1 \otimes u) \cdot (\text{id} \otimes *) \circ \delta(v)) = (\text{id} \otimes \nu)((\dagger \otimes \text{id}) \circ \delta(u) \cdot (1 \otimes v^*)).$$

By applying the map $(* \otimes \text{id})$ to both sides of this equality, we obtain

$$(\text{id} \otimes \nu)((1 \otimes u) \cdot \delta(v^*)) = (\text{id} \otimes \nu)((\kappa \otimes \text{id}) \circ \delta(u) \cdot (1 \otimes v^*)),$$

which is just the strong invariance condition (4.5).

Next, if we apply the map $(\text{id} \otimes \varepsilon)$ to the right-hand side of (5.13), we see that V^λ is contained in the linear span of the matrix elements of the corepresentation $\mathcal{V}^\lambda$, which because of (5.12) and Proposition 5.10, implies that the linear span of matrix elements of $\mathcal{V}^\lambda$ for all eigenvalues $\lambda \neq 0$ is dense in Ran T_a. So now application of Proposition 4.7 yields the claim. ∎

The following corollary is immediate.

COROLLARY 5.12. *Let $\mathcal{B}$ be defined as in Theorem* 5.11. *Then the linear span of $\mathcal{B}$ is total in H_ν with respect to the L_2-norm.*

THEOREM 5.13. *Let V be a Banach space, (V,ι) be an irreducible corepresentation of a compact quantum hypergroup $\mathcal{A}$. Then V is finite dimensional.*

Proof. Consider the finite dimensional linear spaces A_q formed by the linear span of matrix elements of the finite dimensional †-corepresentations (V^q,ι^q), and let $P_q : A \to A_q$ denote the orthogonal projection in H_ν onto the subspace A_q. Then clearly P_q is continuous in the C^*-norm and

$$\delta \circ P_q = (P_q \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes P_q) \circ \delta = (P_q \otimes P_q) \circ \delta. \tag{5.14}$$

Choose an arbitrary vector $v \in V$ and a finite dimensional corepresentation (V^q,ι^q) such that $(P_q \otimes \mathrm{id}) \circ \iota(v) \neq \{0\}$. This is always possible since, for an arbitrary $v \in V$ and $v \neq 0$, $\iota(v) \neq 0$, and hence, if $\iota(v) = \sum a_i \otimes v_i$ with $a_i \in A$ and linearly independent $v_i \in V$, there is an index i_0 such that $a_{i_0} \neq 0$. By using Corollary 5.12, we can expand a_{i_0} in H_ν with respect to matrix elements $t^q_{ij} \in A_{\iota^q}$, and thus find a projection P_q with $P_q(a_{i_0}) \neq 0$.

Now, let $V_v = \{(\xi \circ P_q \otimes \mathrm{id}) \circ \iota(v) \mid \xi \in A^\circ\}$, where A° denotes the linear space of all continuous functionals on A. It is clear that V_v is finite dimensional since the linear space $\{\xi \circ P_q \mid \xi \in A^\circ\}$ is. We will prove that V_v is an invariant subspace of the corepresentation (V,ι), i.e. $\iota(V_v) \subseteq A \otimes V_v$.

Let $u_\xi = (\xi \circ P_q \otimes \mathrm{id}) \circ \iota(v)$ and $\varphi \in A^\circ$ be a continuous functional on the C^*-algebra A. Then

$$\begin{aligned}
(\varphi \otimes \mathrm{id}) \circ \iota(u_\xi) &= (\varphi \otimes \mathrm{id}) \circ \iota \circ (\xi \circ P_q \otimes \mathrm{id}) \circ \iota(v) \\
&= (\xi \otimes \varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes P_q \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \iota) \circ \iota(v) \\
&= (\xi \otimes \varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes P_q \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id}) \circ \iota(v) \\
&= (\xi \otimes \varphi \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id}) \circ (P_q \otimes \mathrm{id}) \circ \iota(v) \\
&= (\xi \cdot \varphi \circ P_q \otimes \mathrm{id}) \circ \iota(v) = u_{\xi\cdot\varphi}
\end{aligned}$$

is in V_v and, hence, it is invariant. Clearly, it is closed and, because the corepresentation was assumed to be irreducible, $V_v = V$. ∎

Acknowledgements. We are grateful to the referree for his (her) useful remarks, especially concerning the proof of Lemma 3.2.1 and the formulation of Definition 4.1.

This work has been supported in part by the Ukrainian Committee for Fundamental studies and by the International Science Foundation.

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Received January 15, 1997; revised June 15, 1998.