J. OPERATOR THEORY
41(1999), 291–319

K-THEORY FOR C^*-ALGEBRAS ASSOCIATED TO LATTICES IN HEISENBERG LIE GROUPS

SOO TECK LEE and JUDITH A. PACKER

Communicated by William B. Arveson

Abstract. We present methods for computing the K-groups of a variety of C^*-algebras associated to lattices in Heisenberg Lie groups, including twisted group C^*-algebras and Azumaya algebras over the corresponding nilmanifolds. A precise formula for the rank of the above K-groups is given, and it is shown that any twisted group C^*-algebra over such a lattice Γ is KK-equivalent to an ordinary group C^*-algebra corresponding to a possibly different lattice Γ_0. We also give applications of our methods to the calculation of K-groups for certain twisted transformation group C^*-algebras and certain continuous trace algebras whose spectra are tori.

Keywords: *Discrete Heisenberg groups, C^*-algebras, Brauer group,* K*-theory.*

MSC (2000): Primary 46L05, 46L80; Secondary 55R10, 55R20.

0. INTRODUCTION

In an earlier paper ([24]), an initial study was made of twisted group C^*-algebras corresponding to generalized discrete Heisenberg groups, that is, groups which can be expressed as cocompact discrete subgroups of simply connected Heisenberg Lie groups of dimension $2n+1$, $n \in \mathbb{Z}^+$. In particular, the second cohomology group with coefficients in the unit circle $\mathbb{T}$ was calculated, multipliers representing each element of this group were given, and a description was provided for the primitive ideal spaces of the twisted group C^*-algebras corresponding to specific multipliers. It was also shown that for such groups Γ of rank $2n+1$ with $n \geqslant 2$, any twisted group C^*-algebra $C^*(\Gamma, \sigma)$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle over the circle group $\mathbb{T}$ whose fibers are matrix algebras over non-commutative tori of rank $2n$. That such a decomposition exists leads one naturally to expect that exact sequences in topology, such as the Gysin sequence, could possibly play a role in the calculation of the K-groups for twisted discrete Heisenberg group C^*-algebras, and indeed this turns out to be the case, as has already been shown for certain untwisted discrete Heisenberg group C^*-algebras in [2].

It is our intention in this paper to study this problem in the slightly wider setting of twisted transformation group C^*-algebras $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$, of which twisted discrete Heisenberg group C^*-algebras are a special case. In most of the examples we will be considering, ρ will be an action on $C(\mathbb{T})$ corresponding to rotation by torsion elements of $\mathbb{T}$. If $\rho : \mathbb{Z}^k \to \mathbb{T}$ is the corresponding group homomorphism, then the kernel M of ρ is of finite index in $\mathbb{Z}^k$ and without loss of generality we can assume that the two-cocycle $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$ when restricted to $M \times M$ takes on its values in $[C(\mathbb{T}, \mathbb{T})]^{\mathbb{Z}^k}$, i.e. those elements of $C(\mathbb{T}, \mathbb{T})$ which are left invariant under the action of $\mathbb{Z}^k$. If $p : \mathbb{T} \to \mathbb{T}/\rho(\mathbb{Z}^k) = Z$ is the quotient map, then $[C(\mathbb{T}, \mathbb{T})]^{\mathbb{Z}^k}$ can be identified with $C(Z, \mathbb{T})$, and the corresponding twisted transformation group C^*-algebra $C(Z) \times_{\mathrm{id},\sigma_M} M$ can be identified with the subalgebra $p^*(C(Z)) \times_{\mathrm{id},\sigma_M} M$ of $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$. Results of M. Rieffel ([37]) show that there is a free and proper $\mathbb{Z}^k$ space X such that $(X/M, \mathbb{Z}^k)$ is equivalent to $(\mathbb{T}, \mathbb{Z}^k, \rho)$, and $(X/\mathbb{Z}^k, M)$ is equivalent to the trivial M space (Z, M). We thus can use the fact, due to A. Kumjian, I. Raeburn, and D. Williams ([23]), that the equivariant Brauer groups $\mathrm{Br}_{\mathbb{Z}^k}(X/M)$ and $\mathrm{Br}_M(X/\mathbb{Z}^k)$ are isomorphic, together with recent work of the second author, Raeburn and Williams on the equivariant Brauer groups of principal bundles ([36]), to prove the following result:

THEOREM 1.3. *Let $\rho : \mathbb{Z}^k \to \mathbb{T}$ be a homomorphism with finite range and kernel M, and denote the induced action of $\mathbb{Z}^k$ on $C(\mathbb{T})$ and $C(\mathbb{T}, \mathbb{T})$ by ρ also. Let $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$, and denote the restriction of σ to $M \times M$ by σ_M. Then, without loss of generality we may assume that σ_M takes on values in the trivial M module $p^*(C(Z, \mathbb{T}))$, and the twisted transformation group C^*-algebra $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ is strongly Morita equivalent to its C^*-subalgebra $p^*(C(Z)) \times_{\mathrm{id},\sigma_M} M \cong C(\mathbb{T}) \times_{\mathrm{id},\sigma_M} M$.*

The above theorem applies in particular to twisted discrete Heisenberg group C^*-algebras and allows us to deduce the following fact:

THEOREM 1.5. *Let* Γ *be a generalized discrete Heisenberg group of rank* $2n+1$*, and let* $[\sigma] \in \mathrm{H}^2(\Gamma, \mathbb{T})$*. Then there is a subgroup* Γ_0 *of* Γ *of finite index, and a multiplier* σ_0 *on* Γ_0 *which is homotopic to the trivial multiplier on* Γ_0*, such that* $C^*(\Gamma, \sigma)$ *is strongly Morita equivalent to* $C^*(\Gamma_0, \sigma_0)$

Since σ_0 is homotopic to the trivial multiplier, by work of the second author and I. Raeburn ([35]), the calculation of the K-groups of $C^*(\Gamma_0, \sigma_0)$, thus those of $C^*(\Gamma, \sigma)$, reduces to the calculation of the K-groups of the homogeneous space N/Γ_0, where here N represents the $2n+1$ dimensional simply connected Heisenberg Lie group. Since for such homogeneous spaces the Chern character $\mathrm{ch} : \mathrm{K}^*(N/\Gamma_0) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{H}^*(N/\Gamma_0, \mathbb{R})$ is an isomorphism, and since by Nomizu's Theorem $\mathrm{H}^*(N/\Gamma_0, \mathbb{R}) \cong \mathrm{H}^*(\mathfrak{n}, \mathbb{R})$, where $\mathfrak{n}$ is the Lie algebra of N, we can deduce that the ranks of all the K-groups involved depend on N alone, not Γ_0, and can be computed from knowledge about the cohomology groups $\mathrm{H}^k(\mathfrak{n}, \mathbb{R})$, $0 \leqslant k \leqslant 2n+1$, which have been recently computed by R. Howe ([18]). We thus are able to deduce the following result:

COROLLARY 2.7. *Let* Γ *be a generalized discrete Heisenberg group of rank* $2n+1$*, viewed as a subgroup of the* $2n+1$ *dimensional simply connected Heisenberg Lie group* N*, let* σ *be any multiplier on* Γ*, and let* $\mathcal{A}(N/\Gamma, \delta([\sigma]))$ *be the Azumaya algebra with spectrum* N/Γ *associated to* (Γ, σ) *as in* [35]*. Then*

$$\begin{aligned}\operatorname{rank}[\mathrm{K}_i(C^*(\Gamma,\sigma))] &= \operatorname{rank}[\mathrm{K}_i(\mathcal{A}(N/\Gamma,\delta([\sigma])))] \\ &= \operatorname{rank}[\mathrm{K}^i(N/\Gamma)] = \binom{2n+1}{n}, \quad i = 0, 1.\end{aligned}$$

This is the analogue for discrete generalized Heisenberg groups of a well-known work of G. Elliott ([13]), who studied the case $\Gamma = \mathbb{Z}^n$. We mention here that it has been conjectured that if $\mathcal{A}$ is any Azumaya algebra whose spectrum is a compact polyhedron X, then the K-groups of $\mathcal{A}$ have the same rank as the K-groups of $C(X)$ (see [38], Theorem 6.5 and [35], Remark 2.9); however, a detailed proof of this conjecture has never been published.

Section 2 of our paper also contains methods for computing the torsion subgroups of the K-groups of the homogeneous spaces N/Γ. Since N/Γ can be written as a principal $\mathbb{T}$-bundle over $\mathbb{T}^{2n}$, this problem reduces to the 6-term Gysin exact sequence in K-theory (c.f. [21]). We calculate the explicit form of certain of the connecting maps in this sequence, which together with knowledge about the ring $\mathrm{K}^*(\mathbb{T}^{2n})$ should allow one in principle to compute $\mathrm{K}^0(N/\Gamma)$ and $\mathrm{K}^1(N/\Gamma)$ via homological algebra. We carry through these computations for arbitrary Γ when $n = 1, 2$ and 3, (i.e., for N/Γ of dimension $3, 5$, and 7). We note that in the special case where Γ is the standard integer lattice in N, the K-groups for N/Γ have been computed for arbitrary n in [2].

We mention at this point that as far back as 1981, G.G. Kasparov ([22]) and J. Rosenberg ([39]) had independently proved results implying that for any cocompact discrete subgroup of a solvable simply connected Lie group G, $\mathrm{K}_*(C^*(\Gamma))$ was isomorphic, modulo a dimension shift, to $\mathrm{K}^*(G/\Gamma)$. However, explicit computation of the K-groups involved for specific classes of lattices do not appear to be common in the literature.

Although the computations carried out in Section 2 seem to deal with C^*-algebras of a very specialized nature, we show in Sections 1 and 3 that our methods have applications to the calculations of the K-groups of many more classes of C^*-algebras. In particular, our results in Section 1 show that any twisted transformation group C^*-algebra of the form $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ described in the statement of Theorem 1.3 is KK-equivalent to either $C(\mathbb{T}^{k+1})$ or to a C^*- algebra of the form $C^*(\Gamma) \otimes C(\mathbb{T}^{k-2n})$, where Γ is a discrete Heisenberg group of rank $2n+1$, for $n \leqslant [k/2]$. Also, as already indicated in the statement of Corollary 2.7, the problem of the calculation of the K-groups of twisted discrete Heisenberg C^*-algebras $C(\Gamma, \sigma)$ is by results of [35] equivalent to the problem of computing the K-groups for Azumaya algebras with spectrum N/Γ. In addition, in Section 3 of our paper, we indicate how C^*-algebras associated to discrete Heisenberg groups are KK-equivalent to certain twisted tori, by which we mean some continuous trace C^*-algebras whose spectra are tori having non-trivial Dixmier-Douady classes. Thus, the problem of calculating the K-groups for these twisted tori is also equivalent to the same problem for C^*-algebras associated to discrete Heisenberg groups, which adds one more subject to the already substantial list of “many apparently diverse topics where the Heisenberg group reveals itself as an important factor” ([17]).

We thank Professor Iain Raeburn for providing us with Proposition 3.1 and its proof, and for useful discussions.

1. KK-EQUIVALENCE FOR C^*-ALGEBRAS ASSOCIATED TO DISCRETE HEISENBERG GROUPS

In this section, we first give some background on generalized discrete Heisenberg groups, and then show how the study of the K-groups of a variety of twisted crossed products and continuous fields of non-commutative tori can be reduced to the problem of computing the K-groups of untwisted group C^*-algebras of discrete Heisenberg groups.

Fix $n \in \mathbb{Z}^+$, and suppose $d_1, d_2, \ldots, d_n \in \mathbb{Z}^+$ are chosen so that $d_1|d_2|\cdots|d_n$. As in [24], we define the generalized discrete Heisenberg group $H(d_1, \ldots, d_n)$ to be the set $\mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n$ with group operation

$$(1.1)\quad (r,s,t)(r',s',t') = \Big(r+r'+\sum_{i=1}^{n} d_i t_i s'_i,\, s+s',\, t+t'\Big) \quad r, r' \in \mathbb{Z},\ s, s', t, t' \in \mathbb{Z}^n.$$

The group $H(d_1, \ldots, d_n)$ can be given a matrix representation in $\mathrm{SL}\,(n+2, \mathbb{Z})$ as follows:

$$(r, s_1, \ldots, s_n, t_1, \ldots, t_n) \in \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n$$
$$\longrightarrow \begin{pmatrix} 1 & d_1t_1 & d_2t_2 & \cdot & \cdot & \cdot & d_nt_n & r \\ & 1 & & & & & & s_1 \\ & & 1 & & & & & s_2 \\ & & & \cdot & & 0 & & \cdot \\ & & & & \cdot & & & \cdot \\ & & 0 & & & \cdot & & \cdot \\ & & & & & & 1 & s_n \\ & & & & & & & 1 \end{pmatrix} \in \mathrm{SL}\,(n+2, \mathbb{Z}).$$

If one allows the parameters $r, s_1, \ldots, s_n, t_1, \ldots, t_n$ to take on real rather than integer values, one obtains the simply connected Heisenberg Lie groups of dimension $2n+1$, which we denote by N. It is clear that $H(d_1, \ldots, d_n)$ is a lattice in N, and conversely, it was shown in Section 1 of [24] that any lattice in N must be isomorphic to a group of the form $H(d_1, \ldots, d_n)$. More generally, we have the following result, from [4]:

PROPOSITION 1.1. *Let Γ be any 2-step nilpotent group which is a central extension of $\mathbb{Z}^k$ by $\mathbb{Z}$. Then there is a positive integer $n \leqslant [k/2]$ and positive integers $d_1|d_2|\cdots|d_n$ such that $\Gamma \cong H(d_1, \ldots, d_n) \times \mathbb{Z}^{k-2n}$.*

Proof. This is Corollary 3.4 of [4], with n and k replaced by r and m, respectively. ∎

We next relate discrete Heisenberg groups to the continuous fields of non-commutative tori first constructed by M. Rieffel. We recall that Rieffel showed in Section 2 of [37] that given X a locally compact Hausdorff space, G a locally compact second countable amenable group, and $\sigma \in \mathrm{Z}^2(G, C(X, \mathbb{T}))$ where $C(X, \mathbb{T})$ is viewed as a trivial G module, then the twisted transformation group C^*-algebra $C_0(X) \times_{\mathrm{id},\sigma} G$ can be viewed as the C^*-algebra of continuous sections vanishing at infinity for a C^*-bundle over X, whose fiber over $x \in X$ is the twisted group C^*-algebra $C^*(G, \sigma(\,\cdot\,,\cdot\,)(x))$, where $\sigma(\,\cdot\,,\cdot\,)(x) : G \times G \to \mathbb{T}$ is a 2-cocycle for G taking values in $\mathbb{T}$. If we specialize the above result to the case where $X = \mathbb{T}$ and $G = \mathbb{Z}^k$, it follows that $C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k$ can be viewed as the C^*-algebra of continuous sections for a C^*-bundle of non-commutative k-tori over the circle. The following result shows how discrete Heisenberg groups arise in the study of the K-theory for such C^*-algebras. It can also be viewed as a form of converse to [24], Theorem 3.4, which gave an isomorphism between certain twisted Heisenberg group C^*-algebras and C^*-algebras of continuous sections for C^*-bundles over the circle with fibers matrix algebras over non-commutative tori.

THEOREM 1.2. *Let* $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$ *where* $C(\mathbb{T}, \mathbb{T})$ *is a trivial* $\mathbb{Z}^k$*-module. If* $[\sigma]$ *is in the same path component as the identity* $\mathbb{1}$ *in* $\mathrm{H}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$*, then the central twisted crossed product* C^**-algebra* $C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k$ *has the same* K*-groups as* $C(\mathbb{T}^{k+1})$*. Otherwise, there is a positive integer* $n \leqslant [k/2]$ *and* $d_1, \dots, d_n \in \mathbb{Z}^+$ *with* $d_1|d_2|\cdots|d_n$ *such that* $C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k$ *has the same* K*-groups as* $C^*(H(d_1, \dots, d_n)) \otimes C(\mathbb{T}^{k-2n})$*.*

Proof. We recall from [34], Theorem 4.2, that the K-groups of $C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k$ depend only on the path component of $[\sigma]$ in $\mathrm{H}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$. Hence if $[\sigma]$ is in the same path component as $\mathbb{1}$, $\mathrm{K}_i(C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k) \cong \mathrm{K}_i(C(\mathbb{T}) \otimes C^*(\mathbb{Z}^k)) \cong \mathrm{K}_i(C(\mathbb{T}^{k+1}))$, $i = 0, 1$. Suppose now that $[\sigma]$ is in a different path component from the identity in $\mathrm{H}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$. By [31], Corollary 1.4, $\mathrm{H}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T})) \cong \prod_{i=1}^{k(k-1)/2} (C(\mathbb{T}, \mathbb{T}))_i$ where if we use the indexing $(f_{ij} :\ f_{ij} \in C(\mathbb{T}, \mathbb{T}),\ 1 \leqslant j < i \leqslant k)$ to represent an element in $\mathrm{H}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$, the corresponding representative in $\mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$ is given by

$$\omega\big((s_i)_{i=1}^k,\ (s_i')_{i=1}^k\big)(z) = \prod_{1 \leqslant j < i \leqslant k} (f_{ij}(z))^{s_i s_j'}.$$

The path component of ω will depend on the path component of each f_{ij} in $C(\mathbb{T}, \mathbb{T})$, and representatives for each of the path components in $C(\mathbb{T}, \mathbb{T})$ are known to be given by $\{z^c \mid c \in \mathbb{Z}\}$. So, given $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T},\mathbb{T}))$, there are integers $\{c_{ij} \mid 1 \leqslant j < i \leqslant k\}$ such that σ is homotopic to the 2-cocycle

$$\omega((c_{ij}))(s,s')(z) = \prod_{1\leqslant i<j\leqslant k} z^{c_{ij}s_i s'_j}.$$

Since σ is not homotopic to the identity, at least one of the c_{ij} is non-zero. Therefore $C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k$ has the same K-groups as $C(\mathbb{T}) \times_{\mathrm{id},\omega(c_{ij})} \mathbb{Z}^k$, and a simple argument involving Fourier transform in the $\mathbb{T}$ variable shows that $C(\mathbb{T}) \times_{\mathrm{id},\omega(c_{ij})} \mathbb{Z}^k$ is isomorphic to $C^*(\Gamma)$ where Γ is a central extension of $\mathbb{Z}^k$ by $\mathbb{Z}$ corresponding to the element of $\mathrm{Z}^2(\mathbb{Z}^k,\mathbb{Z})$ determined by (c_{ij}). By Proposition 1.1, there is a positive integer $n \leqslant [k/2]$ and $d_1, d_2, \ldots, d_n \in \mathbb{Z}^+$ with $d_1|d_2|\cdots|d_n$ such that $\Gamma \cong H(d_1,\ldots,d_n) \times \mathbb{Z}^{k-2n}$. Therefore $C(\mathbb{T}) \times_{\mathrm{id},\sigma} \mathbb{Z}^k$ has the same K-groups as $C^*(\Gamma) \cong C^*(H(d_1,\ldots,d_n) \times \mathbb{Z}^{k-2n}) = C^*(H(d_1,\ldots,d_n)) \otimes C^*(\mathbb{Z}^{k-2n}) \cong C^*(H(d_1,\ldots,d_n)) \otimes C(\mathbb{T}^{k-2n})$. ∎

Thus if we know the K-groups for C^*-algebras of discrete Heisenberg groups, the K-groups of the more general C^*-algebras of Theorem 1.2 can be computed via the Kunneth formula. Our ultimate aim of this section is to show how twisted group C^*-algebras for discrete Heisenberg groups are always KK-equivalent to untwisted discrete Heisenberg group C^*-algebras. In fact, we will be able to prove a generalization of this statement, again by considering twisted transformation group C^*-algebras of the form $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$, where now the action ρ is non-trivial, and by referring to the equivariant Brauer group for the system $(\mathbb{T},\rho,\mathbb{Z}^k)$.

For the statement of the following theorem, we first set up our notation. Let $k \in \mathbb{Z}^+$, and let $\rho : \mathbb{Z}^k \to \mathbb{T}$ be a homomorphism with finite range and kernel M. Then $\mathbb{Z}^k$ acts on $\mathbb{T}$ via

$$v \cdot z = \rho(v) \cdot z, \quad v \in \mathbb{Z}^k,\ z \in \mathbb{T}. \tag{1.2}$$

We also denote the corresponding action of $\mathbb{Z}^k$ on $C(\mathbb{T})$ by ρ. Note $\mathbb{Z}^k/M$ acts freely and properly on $\mathbb{T}$, and the orbit space $\mathbb{T}/(\mathbb{Z}^k/M) = \mathbb{T}/\mathbb{Z}^k$, which we denote by Z, is also homeomorphic to $\mathbb{T}$. Let $p : \mathbb{T} \to Z$ be the quotient map.

THEOREM 1.3. *Let $\rho : \mathbb{Z}^k \to \mathbb{T}$ be as described above, and let $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T},\mathbb{T}))$, where now the action of $\mathbb{Z}^k$ on $C(\mathbb{T},\mathbb{T})$ is non-trivial. Let $M = \ker\rho$. If we denote the restriction of σ to $M \times M$ by σ_M, then without loss of generality we may assume that σ_M takes on its values in $p^*(C(Z,\mathbb{T}))$ viewed as a trivial M module, and the twisted transformation group C^*-algebra $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ is strongly Morita equivalent to its C^*-subalgebra $p^*(C(Z)) \times_{\mathrm{id},\sigma_M} M$.*

Proof. Let l be the order of the range of ρ, so that $[\mathbb{Z}^k : M] = l$. Find $e_1 \in \mathbb{Z}^k$ such that e_1 is indivisible in $\mathbb{Z}^k$, and $\rho(e_1) = \mathrm{e}^{2\pi \mathrm{i} p/l}$ where $(p, l) = 1$. An easy argument then shows we can find $e_2, \ldots, e_k \in M$ such that $\{e_1, \ldots, e_k\}$ forms a basis for $\mathbb{Z}^k$. It follows that $\{le_1, \ldots, e_k\} \subseteq M$, and if we denote by M_1 the subgroup of M generated by $\{le_1, \ldots, e_k\}$, we have $[\mathbb{Z}^k : M_1] = l$ so that $M_1 = M$. Choose $a, b \in \mathbb{Z}$ such that $ap + bl = 1$, and let $X = \mathbb{R} \times \mathbb{Z}_a \times \mathbb{Z}^{k-1}$. Define a monomorphism $\alpha : \mathbb{Z}^k \to X$ by

$$\alpha\Big(\sum_{i=1}^{n} n_i e_i\Big) = \big(n_1, [bn_1], n_2, \ldots, n_k\big).$$

Since the image of α in X is a cocompact discrete subgroup, the corresponding action of $\mathbb{Z}^k$ on X induced by ρ is free and proper, and passing to the quotient space $X/\alpha(M) \cong \mathbb{T}$, the quotient action of $\mathbb{Z}^k$ on $X/\alpha(M)$ is equivalent to the original action $(\mathbb{T}, \rho, \mathbb{Z}^k)$, by the analysis of [37], Theorem 3. The same result of Rieffel shows that the action of M on the quotient space $X/\alpha(\mathbb{Z}^k) \cong \mathbb{T}$ is equivalent to the trivial action of M on $\mathbb{T}$. It follows that Z, the orbit space for the original action ρ of $\mathbb{Z}^k$ on $\mathbb{T}$, can be identified with $(X/\alpha(M))/(\alpha(\mathbb{Z}^k)/\alpha(M)) \cong X/\alpha(\mathbb{Z}^k)$. Now let $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$ be given, where $\mathbb{Z}^k$ acts on $C(\mathbb{T}, \mathbb{T})$ via ρ. It is a consequence of the Hochschild-Serre spectral sequence corresponding to the short exact sequence of groups

$$0 \to M \to \mathbb{Z}^k \to \mathbb{Z}_l \to 0 \tag{1.3}$$

that, multiplying σ by a coboundary if necessary, when we restrict σ to $M \times M$, the corresponding two-cocycle σ_M takes on values in the $\mathbb{Z}^k$-invariant subgroup $[C(\mathbb{T}, \mathbb{T})]^{\mathbb{Z}^k}$, which is exactly $p^*(C(Z, \mathbb{T}))$ and can be identified with $C(Z, \mathbb{T})$. By the stabilization trick ([33], Corollary 3.7), there are actions β and γ of $\mathbb{Z}^n$ and M on $C(\mathbb{T}) \otimes \mathcal{K}_1 \cong C(X/\alpha(M)) \otimes \mathcal{K}_1$ and $C(Z) \otimes \mathcal{K}_2 \cong C(X/\alpha(\mathbb{Z}^n)) \otimes \mathcal{K}_2$ such that $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ is stably isomorphic to $[C(X/\alpha(M)) \otimes \mathcal{K}_1] \times_\beta \mathbb{Z}^k$ and $C(Z) \times_{\mathrm{id},\sigma_M} M$ is stably isomorphic to $[C(X/\alpha(\mathbb{Z}^k)) \otimes \mathcal{K}_2] \times_\gamma M$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ denote the compact operators on the Hilbert spaces $\ell_2(\mathbb{Z}^k)$ and $\ell_2(M)$ respectively. To prove the theorem it will suffice to show that $[C(X/\alpha(M)) \otimes \mathcal{K}_1] \times_\beta \mathbb{Z}^k$ is strongly Morita equivalent to $[C(X/\alpha(\mathbb{Z}^k)) \otimes \mathcal{K}_2] \times_\gamma M$.

At this point we intend to use the equivariant Brauer groups $\mathrm{Br}_M(X/\alpha(\mathbb{Z}^k))$ and $\mathrm{Br}_{\mathbb{Z}^k}(X/\alpha(M))$ (c.f. [9], [23] and [36]) so we recall their definitions. If (Y, G) is a second countable locally compact transformation group, the elements of the equivariant Brauer group $\mathrm{Br}_G(Y)$ are equivalence classes of C^*-dynamical systems $[(A, G, \alpha)]$ in which A is a separable continuous trace C^*-algebra with spectrum Y, and $\alpha : G \to \mathrm{Aut}\,(A)$ is a strongly continuous action of G on $\widehat{A} = Y$ such that the induced action is equal to the original action (Y, G). We refer the reader to [9] for a more detailed description of the equivalence relation involved and group structure of $\mathrm{Br}_G(Y)$. By the main result of [23], if X is a second countable locally compact Hausdorff space admitting commuting free and proper actions of second countable groups G and H, the equivariant Brauer groups $\mathrm{Br}_H(G \setminus X)$ and $\mathrm{Br}_G(X/H)$ are mutually isomorphic via an isomorphism $\theta : \mathrm{Br}_H(G \setminus X) \to \mathrm{Br}_G(X/H)$ which in addition satisfies the condition that if $\theta[(B, \beta, H)] = [(A, \alpha, G)]$, then $B \times_\beta H$ is strongly Morita equivalent to $A \times_\alpha G$. The isomorphism θ is formed as the composite $\theta = \Phi_G^{-1} \circ \Phi_H$ where $\Phi_G : \mathrm{Br}_G(X/H) \to \mathrm{Br}_{G\times H}(X)$ and $\Phi_H : \mathrm{Br}_H(G \setminus X) \to \mathrm{Br}_{G\times H}(X)$.

In our situation, we let the space $X = \mathbb{R} \times \mathbb{Z}_a \times \mathbb{Z}^{k-1}$ be as described above, and recall that we have commuting free and proper actions of $G = \mathbb{Z}^n$ and $H = M$. By the theory of the equivariant Brauer group, it will suffice to show that $\Phi_{\mathbb{Z}^n}([(C(X/\alpha(M)) \otimes \mathcal{K}_1, \beta, \mathbb{Z}^n)]) = \Phi_M([(C(X/\alpha(\mathbb{Z}^n)) \otimes \mathcal{K}_2, \gamma, M)])$ in $\mathrm{Br}_{\mathbb{Z}^n\times M}(X)$. We define the action of $\mathbb{Z}^n \times M$ on X by

$$(v, m) \cdot \chi = \alpha(v) - \alpha(m) + \chi,$$

so that the stability subgroup for the action of $\mathbb{Z}^n \times M$ on X is the subgroup $\{(m, m) \mid m \in M\} \subseteq \mathbb{Z}^n \times M$, which we denote by N. We note that X is a principal $\mathbb{Z}^n \times M/N = \mathbb{Z}^n/M$ bundle over Z with respect to the action, so that by [36], Theorem 1.2, there are homomorphisms $\mathcal{M} : \mathrm{Br}_{\mathbb{Z}^n\times M}(X) \to C(Z, \mathrm{H}^2(N, \mathbb{T}))$ (the Mackey obstruction map), $S : \ker \mathcal{M} \to \mathrm{H}^1(Z, \widehat{\mathcal{N}})$ (the spectrum map) and $P : \mathrm{H}^2(Z, \mathbb{T}) \to \mathrm{Br}_{\mathbb{Z}^n\times M}(X)$ (the pullback map) with $\mathrm{im}\, P = \ker S$. Since $Z \cong \mathbb{T}$, we have $\check{\mathrm{H}}^2(Z, \mathcal{S}) \cong \check{\mathrm{H}}^3(Z, \mathbb{Z}) = \{0\}$ and $\check{\mathrm{H}}^1(Z, \widehat{\mathcal{N}}) = \check{\mathrm{H}}^1(Z, \widehat{\mathcal{M}}) \cong \check{\mathrm{H}}^1(Z, \widehat{\mathbb{Z}}^k) \cong \mathrm{H}^2(Z, \mathbb{Z}^k) = \{0\}$, so that $\mathrm{im}\, P = \ker S = \{0\}$, and consequently $\ker \mathcal{M} \cong \mathrm{im}\,(S) \subseteq \mathrm{H}^1(Z, \widehat{N}) = \{0\}$ so that $\mathcal{M}$ is one-to-one. On the other hand, since $N \cong \mathbb{Z}^k$ and N acts trivially on Z, by [31], Theorem 1.1, $C(Z, \mathrm{H}^2(N, \mathbb{T})) \cong \mathrm{H}^2(N, C(Z, \mathbb{T}))$. Thus to show that $[C(\mathbb{T}) \otimes \mathcal{K}_1] \times_\beta M$ is strongly Morita equivalent to $[C(Z) \otimes \mathcal{K}_2] \times_\gamma M$, it is enough to show that $\mathcal{M} \circ \Phi_{\mathbb{Z}^n}([(C(X/\alpha(M)) \otimes \mathcal{K}_1, \beta, \mathbb{Z}^n)]) = \mathcal{M} \circ \Phi_M([(C(X/\alpha(\mathbb{Z}^n)) \otimes \mathcal{K}_2, \gamma, M)])$. By [23], Proposition 7, $\Phi_{\mathbb{Z}^n}([(C(X/\alpha(M)) \otimes \mathcal{K}_1, \beta, \mathbb{Z}^n)]) = [(C_0(X) \otimes \mathcal{K}_3, \beta_1, \mathbb{Z}^n \times M)]$, where β_1 is the action of $\mathbb{Z}^n \times M$ on $C_0(X) \otimes \mathcal{K}_3$ formed via the stabilization trick from the twisted system $(X, (\alpha, -\alpha), \mathbb{Z}^n \times M, \omega)$, where $\mathcal{K}_3 = \mathcal{K}(\ell_2(\mathbb{Z}^n \times M))$ and $\omega \in \mathrm{Z}^2(\mathbb{Z}^n \times M, C(X, \mathbb{T}))$ is given by $\omega((v_1, m_1), (v_2, m_2))(x) = \sigma(v_1, v_2)(\pi_1(x))$, where $\pi_1 : X \to X/\alpha(M)$. Similarly, $\Phi_M([(C(X/\alpha(\mathbb{Z}^n)) \otimes \mathcal{K}_2, \gamma, M)]) = [(C_0(X) \otimes \mathcal{K}_3, \gamma_1, \mathbb{Z}^n \times M)]$ where γ_1 is the action of $\mathbb{Z}^n \times M$ on $C_0(X) \otimes \mathcal{K}_3$ formed via the stabilization trick from the twisted system $(X, (\alpha, -\alpha), \mathbb{Z}^n \times M, \eta)$ where $\eta \in \mathrm{Z}^2(\mathbb{Z}^n \times M, C(X, \mathbb{T}))$ is given by

$$\eta((v_1, m_1), (v_2, m_2))(x) = \sigma_M(m_1, m_2)(\pi_1(x)) = \sigma_M(m_1, m_2)(\pi_2(x)),$$

where $\pi_2 : X \to X/\alpha(\mathbb{Z}^n) \cong Z$. Now $\mathcal{M} \circ \Phi_{\mathbb{Z}^k}([(C(X/\alpha(M)) \otimes \mathcal{K}_1, \beta, \mathbb{Z}^k)]) = \omega|N \times N$ and $\mathcal{M} \circ \Phi_M([C(X/\alpha(\mathbb{Z}^n)) \otimes \mathcal{K}_2, \gamma, M]) = \eta_{N\times N}$ and since

$$\omega((m_1, m_1), (m_2, m_2))(x) = \sigma(m_1, m_2)(\pi_1 x) = \eta((m_1, m_1), (m_2, m_2))(x),$$

for all $((m_1, m_1), (m_2, m_2)) \in N \times N$, and $x \in X$, we see that $\Phi_{\mathbb{Z}^n}([(C(X/\alpha(M), \beta, \mathbb{Z}^k)]) = \Phi_M([(C(X/\alpha(\mathbb{Z}^k)), \gamma, M)])$ and consequently $C(X/\alpha(M)) \times_\beta \mathbb{Z}^k$ is strongly Morita equivalent to $C(X/\alpha(\mathbb{Z}^k)) \times_\gamma \mathbb{Z}^k$, so that $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ is strongly Morita equivalent to $C(Z) \times_{\mathrm{id},\sigma_M} M$, as desired. ∎

Theorems 1.2 and 1.3 give us the following result:

COROLLARY 1.4. *Let ρ be an action of $\mathbb{Z}^k$ on $C(\mathbb{T})$ corresponding to a homomorphism $\rho : \mathbb{Z}^k \to \mathbb{T}$ with finite range, and let $\sigma \in \mathrm{Z}^2(\mathbb{Z}^k, C(\mathbb{T}, \mathbb{T}))$. Then there is a non-negative integer $n \leqslant [k/2]$ and $d_1, \dots, d_n \in \mathbb{Z}^+$ with $d_1|d_2|\cdots|d_n$ such that the twisted transformation group C^*-algebra $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ has the same* K-*groups as*

$$\begin{array}{ll} C(\mathbb{T}^{k+1}), & \textit{if } n = 0, \\ C^*(H(d_1, \dots, d_n)) \otimes C(\mathbb{T}^{k-2n}), & \textit{if } n \geqslant 1. \end{array}$$

Proof. From the proof of Theorem 1.3 it is clear that $\ker \rho = M \cong \mathbb{Z}^k$ and $X/\rho(\mathbb{Z}^k) = Z \cong \mathbb{T}$. Thus $C(\mathbb{T}) \times_{\rho,\sigma} \mathbb{Z}^k$ has the same K-groups as $C(\mathbb{T}) \times_{\mathrm{id},\sigma_M} \mathbb{Z}^k$, and now we can apply Theorem 1.2 to obtain the desired result. ∎

We remark here that another amusing corollary of Theorem 1.3 is a proof of the known result that any rational rotation algebra $C^*(\mathbb{Z}^{k+1}, \sigma)$ is strongly Morita equivalent to $C(\mathbb{T}^{k+1})$ (see [12]). We omit details.

We now apply the results above to the study of twisted group C^*-algebras associated to discrete Heisenberg group. If Γ is a discrete group and $\sigma \in \mathrm{Z}^2(\Gamma, \mathbb{T})$, we recall that the twisted group C^*-algebra $C^*(\Gamma, \sigma)$ is by definition the C^*-enveloping algebra of the twisted L^1-algebra $\ell^1(\Gamma, \sigma)$, where multiplication and involution are defined on $\ell^1(\Gamma)$ by the formulas

$$\begin{aligned} f * g(\gamma) &= \sum_{\gamma_1 \in \Gamma} f(\gamma_1) g(\gamma_1^{-1}\gamma) \sigma(\gamma_1, \gamma_1^{-1}\gamma), \\ f^*(\gamma) &= \overline{\sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})}, \end{aligned} \tag{1.4}$$

for all $f, g \in \ell^1(\Gamma)$, $\gamma_1, \gamma \in \Gamma$. $C^*(\Gamma, \sigma)$ can also be described as the twisted crossed product $\mathbb{C} \times_{\mathrm{id},\sigma} \Gamma$.

We also recall from [24] that if $\Gamma = H(d_1, \ldots, d_n)$ for $n \geqslant 2$, then any multiplier on Γ is cohomologous to one of the form $c(\lambda_1, \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_n) \cdot \omega$, where ω is inflated from a multiplier on the quotient $\mathbb{Z}^{2n}$ of Γ modulo its center, and

$$
\begin{aligned}
c(\lambda_1, \mu_1, \lambda_2, \ldots, &\lambda_n, \mu_2, \ldots, \mu_n)((r, s, t), (r', s', t')) \\
&= \prod_{i=1}^{n} \lambda_i^{s_i' r} \prod_{i=1}^{n} \mu_i^{r' t_i} \lambda_1^{d_1 t_1 s_1' (s_1' - 1)/2} \mu_1^{d_1 s_1' t_1 (t_1 - 1)/2}
\end{aligned}
\tag{1.5}
$$

for $(\lambda_1, \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_n) \in (\mathbb{Z}_{d_2})^2 \times (\mathbb{Z}_{d_1})^{2(n-1)}$, where here $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ is the cyclic group of order k and is represented by the kth roots of unity. The multipliers

$$
\Big\{ c(\lambda_1, \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_n) \,\Big|\, (\lambda_1, \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_n) \\
\in (\mathbb{Z}_{d_2})^2 \times (\mathbb{Z}_{d_1})^{2(n-1)} \Big\}
$$

parametrize the distinct representatives of the path components of $\mathrm{H}^2(\Gamma, \mathbb{T})$. We now apply Theorem 1.3 to twisted group C^*-algebras corresponding to such multipliers.

THEOREM 1.5. *Let $\Gamma = H(d_1, \ldots, d_n)$ be a discrete Heisenberg group with $n \geqslant 2$, and let $\sigma = c(\lambda_1, \mu_1, \lambda_2, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) \cdot \omega$ be a multiplier on Γ. Let $l \in \mathbb{Z}^+$ be the order of the subgroup of $\mathbb{T}$ generated by $\{\lambda_1, \mu_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_n\}$. Then there is a subgroup Γ_0 of Γ of index l^2 such that $C^*(\Gamma, \sigma)$ is strongly Morita equivalent to $C^*(\Gamma_0, \sigma_0)$, where $[\sigma_0] = [\sigma | \Gamma_0 \times \Gamma_0]$ is homotopic to the identity multiplier in $\mathrm{H}^2(\Gamma_0, \mathbb{T})$.*

Proof. Denote by D the center of Γ; recall $D = \{(r, 0, 0) \mid r \in \mathbb{Z}\}$, and $\sigma | D \times D = 1$. By the decomposition theorem for twisted crossed product C^*-algebras, $C^*(\Gamma, \sigma) = \mathbb{C} \times_{\mathrm{id},\sigma} \Gamma \cong C^*(D) \times_{\rho,\gamma} \Gamma/D \cong C^*(\mathbb{Z}) \times_{\rho,\gamma} \mathbb{Z}^{2n} \cong C(\mathbb{T}) \times_{\rho,\gamma} \mathbb{Z}^{2n}$, where the action ρ of $\mathbb{Z}^{2n}$ on $C(\mathbb{T})$ is defined by

$$
\rho((s, t))(f)(z) = f\Big(\prod_{i=1}^{n} (\lambda_i^{-s_i} \mu_i^{t_i}) z \Big), \quad (s, t) \in \mathbb{Z}^{2n},\ z \in \mathbb{T}
$$

and $\gamma \in \mathrm{Z}^2(\mathbb{Z}^{2n}, C(\mathbb{T}, \mathbb{T}))$ is given by the formula stated in [33], Theorem 4.1. Thus, we are in exactly the situation covered by Theorem 1.3, and hence we can deduce that $C^*(\Gamma) = C(\mathbb{T}) \times_{\rho,\gamma} \mathbb{Z}^{2n}$ is strongly Morita equivalent to its C^*-subalgebra

$p^*(C(Z)) \times_{\mathrm{id},\gamma_M} M$, where $M = \ker\rho$ is by hypothesis a subgroup of $\mathbb{Z}^{2n}$ of index l, and $p : \mathbb{T} \to \mathbb{T}/\mathbb{Z}^{2n} = Z \cong \mathbb{T}$ is an l-to-one cover. Now $p^*(C(Z)) \times_{\mathrm{id},\gamma_M} M$ is clearly nothing more than the C^*-subalgebra $C^*(\Gamma_0, \sigma_0)$, where Γ_0 is the subgroup of Γ generated by $(l, 0, 0) \in D$ and $\{(0, s, t) \mid (s, t) \in M\}$, and $\sigma_0 = \sigma|\Gamma_0 \times \Gamma_0$. Since $[D : lD] = l$ and $[\mathbb{Z}^{2n} : M] = l$, it is clear that $[\Gamma : \Gamma_0] = l^2$. Furthermore, if we denote by D_0 the center of Γ_0, then $D_0 = lD$, and using [33], Theorem 4.1, again to decompose $C^*(\Gamma_0, \sigma_0) = \mathbb{C} \times_{\mathrm{id},\sigma_0} \Gamma_0$ as $C^*(D_0) \times_{\rho_0,\gamma_0} M$, then ρ_0 is the identity action, so that σ_0 is homotopic to the identity multiplier on Γ_0, by [24], Theorem 2.4. ∎

Corollary 1.6. *Let Γ be a generalized discrete Heisenberg group and let σ be a multiplier on Γ. Then there is a subgroup Γ_0 of Γ of finite index such that $C^*(\Gamma, \sigma)$ is* KK*-equivalent to $C^*(\Gamma_0)$.*

Proof. Let $\Gamma = H(d_1, \dots, d_n)$ and $\sigma \in \mathrm{Z}^2(\Gamma, \mathbb{T})$. If $n = 1$, then by [24], Theorem 2.11, σ is homotopic to the identity multiplier on Γ, so that by [35], Theorem 2.3 and 2.6, $C^*(\Gamma, \sigma)$ is KK-equivalent to $C^*(\Gamma)$. If $n \geqslant 2$, by Theorem 1.5, $C^*(\Gamma, \sigma)$ is strongly Morita equivalent to $C^*(\Gamma_0, \sigma_0)$ where Γ_0 is a subgroup of Γ of finite index and $\sigma_0 = \sigma|\Gamma_0 \times \Gamma_0$ is homotopic to the identity multiplier on Γ_0. Using [35], Theorem 2.3 and 2.6, again, $C^*(\Gamma_0, \sigma_0)$ is KK-equivalent to $C^*(\Gamma_0)$. ∎

This immediately gives us our result about Azumaya algebras over N/Γ mentioned in our introduction. Recall ([10], [14], [35]) that an Azumaya algebra over a compact metric space X is a central separable C^*-algebra over the ring $C(X)$ all of whose irreducible representations are finite dimensional; such algebras are, up to stable isomorphism, exactly those continuous trace C^*-algebras with spectrum X whose Dixmier-Douady class is a torsion element of $\check{\mathrm{H}}^3(X, \mathbb{Z})$:

Corollary 1.7. *Let Γ be a generalized discrete Heisenberg group, and let $\mathcal{A}$ be an Azumaya algebra with spectrum N/Γ. Then there is a subgroup Γ_0 of Γ of finite index such that $\mathcal{A}$ is* KK*-equivalent to $C(N/\Gamma_0)$.*

Proof. By [35], Theorem 2.3, 2.5 and Corollary 2.8, there is a multiplier $\sigma \in \mathrm{Z}^2(\Gamma, \mathbb{T})$ such that $\mathcal{A} = \mathcal{A}(N/\Gamma, \delta([\sigma]))$, where δ is a homomorphism of $\mathrm{H}^2(\Gamma, \mathbb{T})$ onto the torsion subgroup of $\check{\mathrm{H}}(N/\Gamma, \mathbb{Z})$ whose kernel is the path component of $\mathbb{1}$ in $\mathrm{H}^2(\Gamma, \mathbb{T})$, so that $\mathrm{K}_i(\mathcal{A}) \cong \mathrm{K}_{i+1}(C^*(\Gamma, \sigma))$, $i = 0, 1$. Now for any discrete cocompact subgroup Γ_0 of N, we have $\mathrm{K}_i(C(N/\Gamma_0)) \cong \mathrm{K}_{i+1}(C^*(\Gamma_0))$, $i = 0, 1$, by [39], Theorem 3.6. Corollary 1.7 then gives us the desired result. ∎

The next example shows how one carries out the analysis of Theorem 1.5 for any multiplier on $H(d_1,\ldots,d_n)$ of the form $c(\mathrm{e}^{-2\pi\mathrm{i}/l_1},1,\mathrm{e}^{2\pi\mathrm{i}/l_2},\ldots,\mathrm{e}^{2\pi\mathrm{i}/l_n},1,\ldots,1)$, notation as in Equation (1.5), $n\geqslant 2$, where $l_n|l_{n-1}|\cdots|l_2|l_1|d_2$ and $l_2|d_1$. This choice of $(\lambda_1,\mu_1,\lambda_2,\ldots,\lambda_n,\mu_2,\ldots,\mu_n)\in(\mathbb{Z}_{d_2})^2\times(\mathbb{Z}_{d_1})^{2(n-1)}$ may seem somewhat contrived, but one can show through direct calculation that for any $(\lambda_1,\mu_1,\lambda_2,\ldots,\lambda_n,\mu_2,\ldots,\mu_n)\in(\mathbb{Z}_{d_2})^2\times(\mathbb{Z}_{d_1})^{2(n-1)}$, it is always possible to find an automorphism A of the group Γ and positive integers $l_1,\ldots,l_n$ as above such that $A^*(c(\lambda_1,\mu_1,\lambda_2,\ldots,\lambda_n,\mu_2,\ldots,\mu_n))$ is cohomologous to $c(\mathrm{e}^{-2\pi\mathrm{i}/l_1},1,\mathrm{e}^{2\pi\mathrm{i}/l_2},\ldots,\mathrm{e}^{2\pi\mathrm{i}/l_n},1,1,\ldots,1)$, where $A^*(\sigma)(\gamma_1,\gamma_2)=\sigma(A\gamma_1,A\gamma_2)$ for $\sigma\in\mathrm{Z}^2(\Gamma,\mathbb{T})$ and $\gamma_1,\gamma_2\in\Gamma$. Thus by [32], p. 189,

$$\begin{aligned}C^*(\Gamma,c(\lambda_1,\mu_1,\ldots,\mu_n))&\cong C^*(\Gamma,A^*(c(\lambda_1,\mu_1,\lambda_2,\ldots,\mu_n)))\\&\cong C^*(\Gamma,c(\mathrm{e}^{-2\pi\mathrm{i}/l_1},1,\mathrm{e}^{2\pi\mathrm{i}/l_2},\ldots,\mathrm{e}^{2\pi\mathrm{i}/l_n},1,1,\ldots,1)).\end{aligned}$$

EXAMPLE 1.8. Let $\Gamma=H(d_1,\ldots,d_n)$, $n\geqslant 2$, and let $\sigma=C(\mathrm{e}^{-2\pi\mathrm{i}/l_1},1,\mathrm{e}^{2\pi\mathrm{i}/l_2},\mathrm{e}^{2\pi\mathrm{i}/l_3},\ldots,\mathrm{e}^{2\pi\mathrm{i}/l_n},1,\ldots,1)$, where $l_n|l_{n-1}|\cdots|l_2|l_1$, $l_1|d_2$, and $l_2|\gcd(l_1,d_1)$. By Theorem 1.6, $C^*(\Gamma,\sigma)$ is KK-equivalent to $C^*(\Gamma_0)$, where Γ_0 has the generators $(l_1,0,0)$, $(0,s_j,0)$ and $(0,0,t_j)$, $1\leqslant j\leqslant n$ where $s_1=(l_1,0,\ldots,0)$, $s_j=(l_j,0,\ldots,1,\ldots,0)$, $2\leqslant j\leqslant n$, and $t_j=(0,\ldots,0,1,0,\ldots,0)$, $1\leqslant j\leqslant n$, are elements of $\mathbb{Z}^n$, and where $q_j=l_1/l_j$ so that $q_2|q_3|\cdots|q_n$. Evidently Γ_0 is a central extension of $\mathbb{Z}^{2n}$ by $\mathbb{Z}$ corresponding to the 2-cocycle $\sigma_B:\mathbb{Z}^{2n}\times\mathbb{Z}^{2n}\to\mathbb{Z}$, where $\sigma_B((s,t),(s',t'))=(s,t)B\begin{pmatrix}(s')^{\mathrm{T}}\\(t')^{\mathrm{T}}\end{pmatrix}$, $s,t,s',t'\in\mathbb{Z}^n$, and where B is the $2n\times 2n$ matrix $\begin{pmatrix}0_n&0_n\\A&0_n\end{pmatrix}$ where 0_n is the $n\times n$ zero matrix and

$$A=\begin{pmatrix}d_1&0&0&\cdot&\cdot&0\\\frac{d_1q_2}{l_1}&\frac{d_2}{l_1}&0&\cdot&\cdot&0\\\cdot&&\cdot&&&\\\cdot&&&\cdot&&\\\cdot&&&&\cdot&0\\\frac{d_1q_n}{l_1}&0&\cdot&\cdot&\cdot&\frac{d_n}{l_1}\end{pmatrix}.$$

Following the method of [4], Section 3 there exists $P,Q\in\mathrm{GL}(n,\mathbb{Z})$, and $c_1,\ldots,c_n\in\mathbb{Z}^+$, $c_1|c_2|\cdots|c_n$ with

$$P\begin{pmatrix}d_1&0&0&\cdot&\cdot&0\\\frac{d_1q_2}{l_1}&\frac{d_2}{l_1}&0&\cdot&\cdot&0\\\cdot&&\cdot&&&\\\cdot&&&\cdot&&\\\cdot&&&&\cdot&0\\\frac{d_1q_n}{l_1}&0&\cdot&\cdot&\cdot&\frac{d_n}{l_1}\end{pmatrix}\quad Q=\begin{pmatrix}c_1&&&&\\&c_2&&0&\\&&\cdot&&\\&&&\cdot&\\&0&&&\cdot\\&&&&c_n\end{pmatrix}$$

so that $\Gamma_0 \cong H(c_1, c_2, \ldots, c_n)$; [19], Theorem 3.9, p. 179 gives a precise formula for the $c_j, 1 \leqslant j \leqslant n$. In particular, if $l_2 = l_3 = \cdots = l_n = 1$, then we can take $q_2 = \cdots = q_n = 0$ and compute $c_1 = \gcd(d_1, d_2/l_1)$, and proceeding inductively, having defined $c_1, \ldots, c_{j-1}$ for $j < n$, from the formula in [19] we obtain

$$c_j = \gcd\left\{\prod_{i=1}^{j-1} \frac{d_i}{c_i} \cdot \frac{d_j}{l_1^{j-1}}, \prod_{i=1}^{j-1} \frac{d_{i+1}}{c_i} \cdot \frac{d_{j+1}}{l_1^j}\right\}, \quad (2 \leqslant j \leqslant n-1),$$

$$c_n = \frac{\prod\limits_{i=1}^{n} d_i}{\prod\limits_{i=1}^{n-1} c_i (l_1)^{n-1}}.$$

Specializing this example even further, if $l_1 = d_2/d_1$ and $l_j = 1$, $2 \leqslant j \leqslant n$, then $c_1 = d_1, c_2 = d_1, c_3 = d_3/l_1 = d_1 d_3/d_2, \ldots,$ and $c_n = d_n/l_1 = d_1 d_n/d_2$. If in this example d_1 is odd and l_1 is even, one can compute that the corresponding multiplier σ_0 on Γ_0 will be a non-trivial multiplier which is homotopic to the identity multiplier on Γ_0. ∎

EXAMPLE 1.9. Let $\Gamma = H(d_1, d_2)$ and consider $\sigma = c(\mathrm{e}^{-2\pi \mathrm{i}/l_1}, 1, \mathrm{e}^{2\pi \mathrm{i}/l_2}, 1)$ where $l_2 | \gcd(d_1, l_1)$ and $l_1 | d_2$. By Theorem 1.5, $C^*(\Gamma, \sigma)$ is strongly Morita equivalent to $C^*(\Gamma_0, \sigma_0)$ where Γ_0 is the subgroup of Γ generated by $(l_1, 0, 0, 0)$, $(0, l_1, 0, 0, 0)$, $(0, q, 1, 0, 0)$ and $(0, 0, 0, 0, 1)$, for $q = l_1/l_2$, and $\sigma_0 = \sigma/\Gamma_0 \times \Gamma_0$. Following the method of Example 1.8, we calculate that Γ_0 is isomorphic to $H(c_1, c_2)$ where $c_1 = \gcd(d_1 q/l_1, d_2/l_1) = \gcd(d_1/l_2, d_2/l_1)$ and $c_2 = d_1 d_2/l_1 c_1$. As a specific example, we let $d_1 = 4, d_2 = 72, l_1 = 6, l_2 = 2$ and we calculate that $C^*(H(4, 72), \sigma_f)$ is KK-equivalent to $C^*(H(2, 24))$. So using Example 2.4 from the next section,

$$\mathrm{K}_0(C^*(H(4,72), \sigma_f)) \cong \mathrm{K}_0(C^*(H(2,24))) \cong \mathbb{Z}^{10} \oplus (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_{24})^2,$$
$$\mathrm{K}_1(C^*(H(4,72), \sigma_f)) \cong \mathrm{K}_1(C^*(H(2,24))) \cong \mathbb{Z}^{10} \oplus (\mathbb{Z}_2)^2.$$

2. THE GYSIN SEQUENCE IN K-THEORY AND COHOMOLOGY FOR HEISENBERG NILMANIFOLDS

In Section 1, we showed that the problem of finding the K-groups for twisted discrete Heisenberg group C^*-algebras and the related Azumaya algebras is equivalent to finding the K-groups for homogeneous spaces of the form N/Γ, where N is a Heisenberg Lie group, and Γ is a cocompact discrete subgroup. In this section, we shall use the Gysin sequence for circle bundles over $\mathbb{T}^{2n}$ in K-theory and cohomology theory to set up an exact sequence for calculating these K-groups; note that both $\mathrm{K}^*(\mathbb{T}^{2n})$ and $\mathrm{H}^*(\mathbb{T}^{2n},\mathbb{Z})$ are torsion-free and their structure is well-understood. We first describe N/Γ as a circle bundle over $\mathbb{T}^{2n}$, and compute the corresponding characteristic classes in $\mathrm{H}^2(\mathbb{T}^{2n},\mathbb{Z})$. Note that singular and Čech cohomology are isomorphic for all the topological spaces we consider. Thus we denote these cohomologies with the notation H^*, except when referring to Čech cohomology with coefficients in a sheaf. The following discussion is similar to that in [2], Section 3, but we include it for the readers' convenience.

Fix $n \in \mathbb{Z}^+$, and let $\Gamma = H(d_1,\ldots,d_n)$ be as in the first section, where $d_1,\ldots,d_n \in \mathbb{Z}^+$ satisfy $d_1|d_2|\cdots|d_n$. Let N denote the $2n+1$ dimensional Heisenberg Lie group with multiplication defined by

$$(r,s,t)\cdot(r',s',t') = (r+r'+\langle t,s'\rangle, s+s', t+t'), \quad r\in\mathbb{R},\ s,t\in\mathbb{R}^n,$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the standard inner product on $\mathbb{R}^n$. We can identify Γ with the subgroup of N defined by $\{(r,s,t_1d_1,\ldots,t_nd_n)\ \mid r,t_1,\ldots,t_n\in\mathbb{Z},\ s\in\mathbb{Z}^n\}$. By Maltsev's theory, this embedding of Γ into N is unique up to automorphisms of N. Let L be the subgroup of N generated by Γ and $[N,N]$, i.e.,

$$L = \Gamma_1\cdot[N,N] = \{(r,s,t_1d_1,\ldots,t_nd_n)\ \mid r\in\mathbb{R},\ t_1,\ldots,t_n\in\mathbb{Z},\ s\in\mathbb{Z}^n\}.$$

Then $\Gamma\subseteq L$ so we have a fibration

$$\mathbb{T}\cong L/\Gamma \longrightarrow N/\Gamma \xrightarrow{\ p\ } N/L \cong (N/[N,N])/(\Gamma/(\Gamma\cap[N,N])) \cong \mathbb{T}^{2n}.$$

In particular, $(E = N/\Gamma, p, \mathbb{T}^{2n})$ is a principal circle bundle on N/Γ, since the action of $[N,N]=\mathbb{R}$ on N/Γ factors through $[N,N]/\Gamma\cap[N,N]=\mathbb{T}$.

The circle bundle $(E,p,\mathbb{T}^{2n})$ gives rise via transition functions to an element of the sheaf cohomology group $\check{\mathrm{H}}^1(\mathbb{T}^{2n},\mathcal{S}) = \check{\mathrm{H}}^1(\mathbb{T}^{2n},\mathcal{U}(1))$, and hence we can form the associated complex line bundle $(\widetilde{E},\widetilde{p},\mathbb{T}^{2n})$ to which one can associate an element $\lambda(\widetilde{E})\in\mathrm{K}^0(\mathbb{T}^{2n})$, via the canonical embedding of complex vector bundles over $\mathbb{T}^{2n}$ into $\mathrm{K}^0(\mathbb{T}^{2n})$.

Choosing the usual metric on $\mathbb{C}$, it follows from the construction of $(\widetilde{E}, \widetilde{p}, \mathbb{T}^{2n})$ that the sphere bundle $(S(\widetilde{E}), \widetilde{p}, \mathbb{T}^{2n})$ is exactly $(E, p, \mathbb{T}^{2n})$, and hence we can use the Gysin exact sequence in K-theory (see [21], IV. 1.13, p. 187) to compute the K-groups for $E = N/\Gamma$:

$$\begin{tikzcd}[column sep=small] & \mathrm{K}^0(\mathbb{T}^{2n}) \arrow[r, "\alpha_0^*"] & \mathrm{K}^0(\mathbb{T}^{2n}) \arrow[dr] & \\ \mathrm{K}^1(N/\Gamma) \arrow[ur] & & & \mathrm{K}^0(N/\Gamma) \arrow[dl] \\ & \mathrm{K}^1(\mathbb{T}^{2n}) \arrow[ul] & \mathrm{K}^1(\mathbb{T}^{2n}) \arrow[l, "\alpha_1^*"'] & \end{tikzcd} \tag{2.1}$$

where the maps α_j^*, $j = 0, 1$, are given by the product with $1 - \lambda(\widetilde{E}) \in \mathrm{K}^0(\mathbb{T}^{2n})$ (recall $\mathrm{K}^*(\mathbb{T}^{2n})$ has a graded ring structure). Since $\mathrm{K}^*(\mathbb{T}^{2n})$ is torsion free, our problem thus becomes a problem in multilinear algebra involving the ring structure of $\mathrm{K}^*(\mathbb{T}^{2n})$ and the computation of $1 - \lambda(\widetilde{E})$ as an endomorphism of $\mathrm{K}^*(\mathbb{T}^{2n})$.

It is well-known that the ring $\mathrm{K}^*(\mathbb{T}^m)$ is isomorphic to the exterior algebra over $\mathbb{Z}$ on m generators, $\bigwedge_{\mathbb{Z}}^*\{e_1, \ldots, e_m\}$ ([42], p. 185; [13]). Indeed, in this case the Chern character $\mathrm{ch} : \mathrm{K}^*(\mathbb{T}^m) \to \mathrm{H}^*(\mathbb{T}^m, \mathbb{Q})$ is integral and gives the isomorphisms $\mathrm{ch}_0 : \mathrm{K}^0(\mathbb{T}^m) \to \mathrm{H}^{\mathrm{even}}(\mathbb{T}^m, \mathbb{Z})$ and $\mathrm{ch}_1 : \mathrm{K}^1(\mathbb{T}^m) \to \mathrm{H}^{\mathrm{odd}}(\mathbb{T}^m, \mathbb{Z})$ (see [20]) for a proof) where $\mathrm{H}^*(\mathbb{T}^m, \mathbb{Z})$ under cup product, is well-known to be isomorphic to $\bigwedge_{\mathbb{Z}}^*\{e_1, \ldots, e_m\}$ with $\mathrm{H}^k(\mathbb{T}^m, \mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}^k\{e_1, \ldots, e_m\}$. Thus identifying $\mathrm{H}^*(\mathbb{T}^m, \mathbb{Z})$ with $\bigwedge_{\mathbb{Z}}^*\{e_1, \ldots, e_m\}$, in order to carry out the computations implicit in the sequence (2.1), we need to compute $\mathrm{ch}(1-\lambda(\widetilde{E})) = 1-\mathrm{ch}(\lambda(\widetilde{E})) \in \bigwedge_{\mathbb{Z}}^{\mathrm{even}}\{e_1, \ldots, e_m\}$. We will do this in Proposition 2.2.

We recall the following result of Massey, specialized to our context ([3]). Let Δ be a countable discrete group, let B be a connected CW-complex of type $K(\Delta, 1)$, and suppose that (E, p, B) is a principal $\mathbb{T}$-bundle over B such that the fundamental group $\pi_1(E)$ is a central extension of $\pi_1(B) = \Delta$ by $\pi_1(\mathbb{T}) = \mathbb{Z}$. Then the characteristic class c_1 of the bundle (E, p, B) as an element of $\check{\mathrm{H}}^2(B, \mathbb{Z}) \cong \check{\mathrm{H}}^1(B, \mathcal{S})$ (that is, the first obstruction to a cross-section), can be identified by the group cohomology class $\kappa \in \mathrm{H}^2(\Delta, \mathbb{Z})$ determined by the central group extension

$$0 \to \mathbb{Z} \to \pi_1(E) \to \pi_1(B) \cong \Delta \to 0$$

via the canonical isomorphism $\lambda_* : \mathrm{H}^2(\Delta, \mathbb{Z}) \to \check{\mathrm{H}}^2(B, \mathbb{Z})$ which is discussed in [35], Lemma 2.6, for example. In our case, $\pi_1(E) = \pi_1(N/\Gamma) = \Gamma = H(d_1, \ldots, d_n)$, $\pi_1(B) = \Delta \cong \mathbb{Z}^{2n}$, and, using the notation of [4], Section 3 $\kappa \in \mathrm{H}^2(\mathbb{Z}^{2n}, \mathbb{Z})$ is defined by $M = \sum\limits_{i=1}^{n} d_i E_{n+i,i}$ (here E_{jk} denotes the elementary matrix with 1 in the (j,k)th spot, 0's elsewhere). Thus we may write $\kappa = \sum\limits_{i=1}^{n} d_i[\sigma_{E_{n+i,i}}]$. From [4], Section 3 it is known that $\{[\sigma_{E_{jk}}] \mid 1 \leqslant k < j \leqslant n\}$ is a basis for $\mathrm{H}^2(\mathbb{Z}^{2n}, \mathbb{Z})$, and it is easily checked that $\lambda_*([\sigma_{E_{jk}}]) = e_k \wedge e_j \in \bigwedge_{\mathbb{Z}}^2\{e_1, \ldots, e_{2n}\} \cong \mathrm{H}^2(\mathbb{T}^{2n}, \mathbb{Z})$ (the line bundle corresponding to $e_k \wedge e_j$ is the pull-back of the standard Heisenberg non-trivial line bundle on $\mathbb{T}^2$ to $\mathbb{T}^{2n} = \prod\limits_{i=1}^{2n} \mathbb{T}$ via projection onto the kth and jth coordinates). It follows that the characteristic class of $(N/\Gamma, p, \mathbb{T}^{2n})$, or what is the same thing, the first Chern class of the complex line bundle $(\widetilde{E}, p, \mathbb{T}^{2n})$, is given by $c_1(\widetilde{E}) = \lambda_*([\sigma_M]) = \lambda_*\Big(\sum\limits_{i=1}^{n} d_i[\sigma_{E_{n+i,i}}]\Big) = \sum\limits_{i=1}^{n} d_i e_i \wedge e_{n+i} \in \mathrm{H}^2(\mathbb{T}^{2n}, \mathbb{Z})$. We have thus proved:

PROPOSITION 2.1. *Let N be the $2n+1$ dimensional Heisenberg Lie group, with cocompact discrete subgroup $\Gamma = H(d_1, \ldots, d_n)$, where $d_1|d_2|\cdots|d_n$. Then the characteristic class in $\mathrm{H}^2(\mathbb{T}^{2n}, \mathbb{Z}) = \bigwedge_{\mathbb{Z}}^2\{e_1, \ldots, e_{2n}\}$ and hence the first Chern class defined by the complex line bundle $(\widetilde{E}, \widetilde{p}, \mathbb{T}^{2n})$ associated to $(N/\Gamma, p, \mathbb{T}^{2n})$ is defined by $c_1(\widetilde{E}) = \sum\limits_{i=1}^{n} d_i e_i \wedge e_{n+i}$*

We now use Proposition 2.1 to deduce:

PROPOSITION 2.2. *Let N and Γ be as in Proposition 2.1, let $(\widetilde{E}, \widetilde{p}, \mathbb{T}^{2n})$ be the complex line bundle associated to the principal $\mathbb{T}$ bundle $(N/\Gamma, p, \mathbb{T}^{2n})$, and let $\lambda(\widetilde{E})$ be the corresponding representative in $\mathrm{K}^0(\mathbb{T}^{2n})$. Then the formula for $\mathrm{ch}(\lambda(\widetilde{E}))$ in $\mathrm{H}^{\mathrm{even}}(\mathbb{T}^{2n}, \mathbb{Z}) = \bigwedge_{\mathbb{Z}}^{\mathrm{even}}\{e_1, \ldots, e_{2n}\}$ is given by*

$$\begin{aligned}
\mathrm{ch}(\lambda(\widetilde{E})) &= \sum_{j=0}^{n} \frac{\Big(\sum\limits_{i=1}^{n} d_i e_i \wedge e_{n+i}\Big)^j}{j!} \\
&= 1 + \sum_{i=1}^{n} d_i e_i \wedge e_{n+i} + \sum_{1 \leqslant i_1 < i_2 \leqslant n} d_{i_1} d_{i_2} (e_{i_1} \wedge e_{n+i_1}) \wedge (e_{i_2} \wedge e_{n+i_2}) \\
&\quad + \cdots + \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n-1} \leqslant n} d_{i_1} \cdots d_{i_n} (e_{i_1} \wedge e_{n+i_1}) \wedge \cdots \wedge (e_{i_{n-1}} \wedge e_{n+i_{n-1}}) \\
&\quad + d_1 d_2 \cdots d_n (e_1 \wedge e_{n+1}) \wedge (e_2 \wedge e_{n+2}) \wedge \cdots \wedge (e_n \wedge e_{2n}).
\end{aligned}$$

Proof. The first equality follows from the formula for the Chern character of complex line bundles given in [27], p. 196, i.e. $\mathrm{ch}(\lambda(\widetilde{E})) = \exp(c_1(\widetilde{E}))$. The second formula follows by expanding the first expression. ∎

From the proposition we immediately obtain:

THEOREM 2.3. *Let N be the $2n+1$ dimensional Heisenberg Lie group and let $\Gamma = H(d_1, \ldots, d_n)$ be a cocompact discrete subgroup. The* K*-groups of N/Γ can in principal be computed form the exact sequence*

$$
(2.2)\qquad \begin{array}{ccccc}
 & \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1,\ldots,e_{2n}\} & \xrightarrow{\alpha_*^0} & \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1,\ldots,e_{2n}\} & \\
\nearrow & & & & \searrow \\
\mathrm{K}^1(N/\Gamma) & & & & \mathrm{K}^0(N/\Gamma) \\
\nwarrow & & & & \swarrow \\
 & \Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1,\ldots,e_{2n}\} & \xleftarrow{\alpha_*^1} & \Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1,\ldots,e_{2n}\} &
\end{array}
$$

where the maps $\alpha_*^i : \bigwedge_{\mathbb{Z}}^*\{e_1,\ldots,e_{2n}\} \to \bigwedge_{\mathbb{Z}}^*\{e_1,\ldots,e_{2n}\}$ are defined by

$$
\alpha_*^i(\gamma) = -\gamma\wedge\Bigg(\sum_{j=1}^{n}\Big(\sum_{1\leqslant i_1<i_2<\cdots<i_j\leqslant n} d_{i_1}d_{i_2}\cdots d_{i_j}(e_{i_1}\wedge e_{n+i_1})\wedge\cdots\wedge(e_{i_j}\wedge e_{n+i_j})\Big)\Bigg),
$$

for $\gamma \in \bigwedge_{\mathbb{Z}}^{\text{even}}\{e_1,\ldots,e_{2n}\}$ if $i=0$, and for $\gamma \in \bigwedge_{\mathbb{Z}}^{\text{odd}}\{e_1,\ldots,e_{2n}\}$ if $i=1$. In particular,

$$
\mathrm{K}^0(N/\Gamma) \cong \text{coker } \alpha_*^0 \oplus \ker\alpha_*^1,
$$
$$
\mathrm{K}^1(N/\Gamma) \cong \text{coker } \alpha_*^1 \oplus \ker\alpha_*^0.
$$

Proof. Using the Chern character ch : $\mathrm{K}^*(\mathbb{T}^{2n}) \to \bigwedge_{\mathbb{Z}}^*\{e_1,\ldots,e_{2n}\}$, the Gysin sequence for K-theory given in diagram (2.1) becomes exactly the diagram (2.2), and by Proposition 2.2, the maps α_*^i, $i=0,1$ of diagram (2.1) become the maps stated in the theorem, using the fact that $\mathrm{ch}(1_{\mathrm{K}^*}) = 1_{\bigwedge_{\mathbb{Z}}^*}$. We obtain the splitting for $\mathrm{K}^i(N/\Gamma)$, $i=0,1$, by using the fact the $\bigwedge_{\mathbb{Z}}^{\text{even}}\{e_1,\ldots e_{2n}\}$ and $\bigwedge_{\mathbb{Z}}^{\text{odd}}\{e_1,\ldots e_{2n}\}$ are finitely generated free abelian groups, so that the respective subgroups $\ker\alpha_*^i$, $i=0,1$, would have the same property. ∎

EXAMPLE 2.4. Using the method of Theorem 2.3, we are able to compute the K-groups of all Heisenberg nilmanifold N/Γ of dimension $3, 5$, and 7, i.e. for $n = 1, 2$ and 3.

n	$\mathrm{K}^0(N/\Gamma)$	$\mathrm{K}^1(N/\Gamma)$
1	$\mathbb{Z}^3 \oplus \mathbb{Z}_{d_1}$	$\mathbb{Z}^3$
2	$\mathbb{Z}^{10} \oplus \mathbb{Z}_{d_1}^2$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_{d_1}^2 \oplus \mathbb{Z}_{d_2}^2$
3	$\mathbb{Z}^{35} \oplus (\mathbb{Z}_{d_1})^8 \oplus (\mathbb{Z}_{d_2})^4 \oplus \mathbb{Z}_{2d_2d_3/d_1}$	$\mathbb{Z}^{35} \oplus (\mathbb{Z}_{d_1})^8 \oplus (\mathbb{Z}_{d_2})^4$

EXAMPLE 2.5. Using Theorem 2.3 and by computing the diagonal forms for certain incidence matrices, we have computed the K-groups of N/Γ for $\Gamma =$

$H(1,1,\ldots,1)$ in [2]. Our results show

$$\mathrm{K}^0(N/\Gamma) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n \text{ odd},\\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{n/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n \text{ even}, \end{cases}$$

$$\mathrm{K}^1(N/\Gamma) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n \text{ odd},\\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-2)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n \text{ even}. \end{cases}$$

Theorem 2.3 shows us that $\mathrm{K}^0(N/\Gamma)$ and $\mathrm{K}^1(N/\Gamma)$ are finitely generated abelian groups, so that by the structure theorem for such groups, we can write $\mathrm{K}^0(N/\Gamma) \cong \mathbb{Z}^{m_0} \oplus T_0$, $\mathrm{K}^1(N/\Gamma) \cong \mathbb{Z}^{m_1} \oplus T_1$, where T_0 and T_1 are finite groups and m_0 and m_1 are non-negative integers giving the rank of $\mathrm{K}^0(N/\Gamma)$ and $\mathrm{K}^1(N/\Gamma)$ respectively. Now recall the Chern character defines an isomorphism $\mathrm{ch} : \mathrm{K}^*(N/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to \bigoplus_{i=0}^{2n+1} \mathrm{H}^i(N/\Gamma, \mathbb{Q})$ ([42], p. 174) so that by tensoring by $\mathbb{R}$ over $\mathbb{Q}$ we obtain the isomorphisms $\mathrm{ch} : \mathrm{K}^0(N/\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{m_0} \xrightarrow{\cong} \bigoplus \mathrm{H}^{\mathrm{even}}(N/\Gamma, \mathbb{R})$ and $\mathrm{ch} : \mathrm{K}^1(N/\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{m_1} \xrightarrow{\cong} \bigoplus \mathrm{H}^{\mathrm{odd}}(N/\Gamma, \mathbb{R})$. It follows that to calculate the rank of the abelian groups $\mathrm{K}^0(N/\Gamma)$ and $\mathrm{K}^1(N/\Gamma)$, it suffices to calculate the Betti numbers of the nilmanifold N/Γ. A celebrated theorem of Nomizu ([30], Theorem 1) tells us that if $\mathfrak{n}$ denotes the Lie algebra of N, then $\mathrm{H}^k(N/\Gamma, \mathbb{R}) \cong \mathrm{H}^k(\mathfrak{n}, \mathbb{R})$ so that these Betti numbers are independent of the cocompact discrete subgroup Γ chosen, and depend only on the Lie algebra structure of $\mathfrak{n}$. In particular, using these ideas, we obtain:

$$\begin{aligned} \operatorname{rank}(\mathrm{K}^1(N/\Gamma)) &= \sum_{k \,\mathrm{odd}} \dim_{\mathbb{R}}(\mathrm{H}^k(\mathfrak{n}, \mathbb{R})) = \sum_{k=0}^{n} \beta_{2k+1},\\ \operatorname{rank}(\mathrm{K}^0(N/\Gamma)) &= \sum_{k \,\mathrm{even}} \dim_{\mathbb{R}}(\mathrm{H}^k(\mathfrak{n}, \mathbb{R})) = \sum_{k=0}^{n} \beta_{2k}. \end{aligned} \tag{2.3}$$

Here β_k denote the kth Betti number of N/Γ, or equivalently, $\beta_k = \dim_{\mathbb{R}}(\mathrm{H}^k(N/\Gamma, \mathbb{R})) = \dim_{\mathbb{R}}(\mathrm{H}^k(\mathfrak{n}, \mathbb{R}))$.

The Betti numbers of N/Γ can be deduced from recent work of Dupré ([11]) and Howe ([18], A 4.3). Dupré calculated the Moore cohomology groups with coefficients in the reals for simply connected Heisenberg Lie groups. For Heisenberg Lie groups N the Moore cohomology can be identified with the continuous cohomology $\mathrm{H}^*_{\mathrm{conts}}(N, \mathbb{R})$, and it is a deep result of G. Mostow ([29]) that $\mathrm{H}^*_{\mathrm{conts}}(N, \mathbb{R}) \cong \check{\mathrm{H}}^*(N/\Gamma, \mathbb{R})$ for a large class of simply connected solvable Lie groups and cocompact discrete subgroups which includes the Heisenberg groups. Howe uses the representation theory of Sp_{2n} to calculate the Lie algebra cohomology of the Lie algebra of N, which by Nomizu's theory is exactly $\mathrm{H}^*(N/\Gamma, \mathbb{R})$.

THEOREM 2.6. (cf. [11], Theorem 4.6, [29], Theorem 8.1) *Let N be the $2n+1$ dimensional Heisenberg Lie group and let Γ be a cocompact discrete subgroup. Let β_k denote the kth Betti number of N/Γ, then*

$$\beta_k = \begin{cases} \binom{2n}{k} - \binom{2n}{k-2}, & 0 \leqslant k \leqslant n; \\ \binom{2n}{k-1} - \binom{2n}{k+1}, & n+1 \leqslant k \leqslant 2n+1. \end{cases}$$

(*Since* $\dim(N/\Gamma) = 2n+1$, $\beta_k = 0$ *for* $k \geqslant 2n+2$.) *Here we use the standard convention that* $\binom{m}{j} = 0$ *for* $j < 0$ *and* $j > m$.

COROLLARY 2.7. *Let N be the $2n+1$ dimensional Heisenberg Lie group, let Γ be any lattice in N, let σ be any multiplier on Γ, and let $\mathcal{A}(N/\Gamma, \delta([\sigma]))$ be the associated Azumaya algebra over N/Γ. Then for $i = 0, 1$,*

$$\operatorname{rank}[\mathrm{K}_i(C^*(\Gamma, \sigma))] = \operatorname{rank}[\mathrm{K}_i(\mathcal{A}(N/\Gamma, \delta([\sigma])))] = \operatorname{rank}[\mathrm{K}^i(N/\Gamma)] = \binom{2n+1}{n}.$$

Proof. We know that

$$\begin{aligned} \operatorname{rank}(\mathrm{K}^0(N/\Gamma)) = \sum_{k=0}^{n} \beta_{2k} &= 1 + \sum_{k=1}^{[n/2]} \left[\binom{2n}{2k} - \binom{2n}{2k-2} \right] \\ &\quad + \sum_{k=[n/2]+1}^{n-1} \left[\binom{2n}{2k-1} - \binom{2n}{2k+1} \right] + \binom{2n}{2n-1} \\ &= \begin{cases} \binom{2n}{n} - \binom{2n}{n-1}, & n \text{ even}, \\ \binom{2n}{n-1} - \binom{2n}{n}, & n \text{ odd}, \end{cases} \\ &= \binom{2n+1}{n} \end{aligned}$$

and similarly

$$\begin{aligned}\operatorname{rank}(\mathrm{K}^1(N/\Gamma)) = \sum_{k=0}^{n} \beta_{2k+1} &= 2n + \sum_{k=1}^{[n-1/2]} \left[\binom{2n}{2k+1} - \binom{2n}{2k-1} \right] \\ &\quad + \sum_{k=[n-1/2]+1}^{n-1} \left[\binom{2n}{2k} - \binom{2n}{2k+2} \right] + \binom{2n}{2n} \\ &= \begin{cases} \binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+1}{n}, & n \text{ odd} \\ \binom{2n}{n-1} + \binom{2n}{n}, & n \text{ even}, \end{cases} \\ &= \binom{2n+1}{n}.\end{aligned}$$

The lemma now follows from Corollary 1.7, Corollary 1.8, [39], Theorem 3.6 and [30], Theorem 1. ∎

REMARK 2.8. We note here how theoretically our methods can be used to calculate the K-groups and the Betti numbers for any principal $\mathbb{T}$-bundle $(E, p, \mathbb{T}^m)$ over an m-torus, for m even or odd. By the work of Massey mentioned in the proof of Proposition 2.1, the fundamental group of E will be a two-step nilpotent central extension of $\mathbb{Z}^m$ by $\mathbb{Z}$ so that in fact up to an homeomorphism E looks like $G/(\mathbb{Z}^m \times_{\sigma_M} \mathbb{Z})$ for $\sigma_M \in \mathrm{H}^2(\mathbb{Z}^m, \mathbb{Z})$, where G is the simply connected two-step nilpotent Lie group corresponding to $\mathbb{Z}^m \times_{\sigma_M} \mathbb{Z}$ by the work of Maltsev ([26]). This is accomplished as follows: as in our disussion prior to the proof of Proposition 2.1, any principal $\mathbb{T}$-bundle $(E, p, \mathbb{T}^m)$ is determined by its characteristic class $c_1(E) \in \mathrm{H}^2(\mathbb{T}^m, \mathbb{Z})$. Suppose that $c_1(E) = \sum_{1 \leqslant k < j \leqslant m} a_{jk} e_k \wedge e_j$, $a_{jk} \in \mathbb{Z}$ (recall $\{e_k \wedge e_j \mid 1 \leqslant k < j \leqslant m\}$ is a basis for $\mathrm{H}^2(\mathbb{T}^m, \mathbb{Z}) \cong \mathbb{Z}^{m(m-1)/2}$). Let M be the $m \times m$ matrix with integer entries defined by $M = \sum_{1 \leqslant k < j \leqslant m} a_{jk} E_{jk}$, and let $\Gamma = \mathbb{Z}^m \times_{\sigma_M} \mathbb{Z}$, σ_M as in [4], Section 3, and G the associated simply connected Lie group. As in our proof of Proposition 2.1, $(G/\Gamma, \pi, (G/[G,G])/(\Gamma/(\Gamma \cap [G,G])) \cong \mathbb{R}^m/\mathbb{Z}^m \cong \mathbb{T}^m$ is a principal $\mathbb{T}^m$-bundle with characteristic class $\sum_{1 \leqslant k < j \leqslant m} a_{jk} e_k \wedge e_j$. Thus, as a principal $\mathbb{T}$-bundle over $\mathbb{T}^m$, $(E, p, \mathbb{T}^m)$ is equivalent to $(G/\Gamma, \pi, \mathbb{T}^m)$. By [4], Corollary 3.4 there exists a positive integer n with $2n \leqslant m$ and positive integers $d_1, d_2, \ldots, d_n$ with $d_1 | d_2 | \cdots | d_n$ such that $\mathbb{Z}^m \times_{\sigma_M} \mathbb{Z}$ is isomorphic to $H(d_1, \ldots, d_n) \times \mathbb{Z}^{m-2n}$ and hence G is isomorphic to $N \times \mathbb{R}^{m-2n}$ (again by work of Maltsev) where N is the $2n+1$ dimensional Heisenberg Lie group. If $m = 2n$, we can immediately use Theorem 2.3 and 2.8. If $m > 2n$, E is homeomorphic to the homogeneous space

$$(N \times \mathbb{R}^{m-2n})/(H(d_1, \ldots, d_n) \times \mathbb{Z}^{m-2n}) = (N/H(d_1, \ldots, d_n)) \times \mathbb{T}^{m-2n},$$

and the K-groups and Betti numbers for E can be read off via the Kunneth formulas in K-theory and cohomology, respectively ([5], [41]). In particular, for letting $H = H(d_1, \ldots, d_n)$, since $\mathrm{K}^*(\mathbb{T}^{m-2n})$ is torsion free, we obtain from [5], Theorem 23.1.3,

$$\begin{aligned}\mathrm{K}^0(G/\Gamma) &\cong \mathrm{K}^0(N/H) \otimes \mathrm{K}^0(\mathbb{T}^{m-2n}) \oplus \mathrm{K}^1(N/H) \otimes \mathrm{K}^1(\mathbb{T}^{m-2n}) \\ &\cong [\mathrm{K}^0(N/H)]^{2^{m-2n-1}} \oplus [\mathrm{K}^1(N/H)]^{2^{m-2n-1}}.\end{aligned}$$

Similarly,

$$\mathrm{K}^1(G/\Gamma) = [\mathrm{K}^0(N/H)]^{2^{m-2n-1}} \oplus [\mathrm{K}^1(N/H)]^{2^{m-2n-1}}.$$

The Betti numbers for G/Γ would be computed similarly.

As a particular example we let $m = 3$. Any non-trivial $\mathbb{T}$-bundle over $\mathbb{T}^3$ will be homeomorphic as a topological space to $N/H(d) \times \mathbb{T}$, where N is the 3-dimensional simply connected Heisenberg Lie group and $d \in \mathbb{Z}^+$, and

$$\mathrm{K}^i(N/H(d) \times \mathbb{T}) \cong \mathrm{K}^0(N/H(d)) \oplus \mathrm{K}^1(N/H(d)) \cong \mathbb{Z}^6 \oplus \mathbb{Z}_d, \quad i = 0, 1,$$

by Example 2.4. Using the Kunneth formula in cohomology, one calculates that the Betti numbers for $N/H(d) \times \mathbb{T}$ are given by $\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = 4$, $\beta_3 = 3$, $\beta_4 = 1$.

3. K-theory for some twisted tori

In this section we shall briefly discuss how we can calculate the K-groups for certain continuous trace C^*-algebras whose spectra are tori of odd dimension and whose Dixmier-Douady classes are of a specific form from the knowledge obtained in previous sections about the K-groups for generalized discrete Heisenberg group C^*-algebras. Since the K-groups of continuous trace C^*-algebras have been termed "twisted K-groups" for the spectrum (this terminology is due to Rosenberg [38]), by abuse of terminology we shall refer to the continuous trace C^*-algebras whose spectra are tori as "twisted" tori (not to be confused with the "non-commutative tori" of Rieffel).

We start first with a proposition whose statement and proof was first shown to us by I. Raeburn; we thank him for giving us permission to reproduce it here.

PROPOSITION 3.1. *Let Γ be a generalized discrete Heisenberg group of rank $2n+1$. Then $C^*(\Gamma)$ is strongly Morita equivalent to a C^*-crossed product $\mathcal{A}_\sigma \times_\tau \mathbb{R}^{2n}$, where $\mathcal{A}_\sigma$ is a continuous trace C^*-algebra whose spectrum is homeomorphic to $\mathbb{T}^{2n+1}$. Thus $C^*(\Gamma)$ is* KK*-equivalent to a twisted $(2n+1)$ torus.*

Proof. Let $\Gamma = H(d_1, \dots, d_n)$ where $d_1|d_2|\cdots|d_n$. We write Γ as a central extension of $\mathbb{Z}^{2n}$ by $\mathbb{Z}$ corresponding to $\sigma \in \mathrm{Z}^2(\mathbb{Z}^{2n}, \mathbb{Z})$ defined by

$$\sigma((s,t),(s',t')) = \sum_{i=1}^{n} d_i t_i s'_i. \tag{3.1}$$

Taking $N = \mathbb{Z},\ G = \Gamma$, and $G/N = \mathbb{Z}^{2n}$ in [33], Theorem 4.1 we have, using the theory of decomposition of twisted crossed products,

$$C^*(\Gamma) = \mathbb{C} \times_{\mathrm{id}} \Gamma = (\mathbb{C} \times_{\mathrm{id}} \mathbb{Z}) \times_{\mathrm{id}, v_\sigma} \mathbb{Z}^{2n} \cong (C^*(\mathbb{Z})) \times_{\mathrm{id}, v_\sigma} \mathbb{Z}^{2n},$$

where

$$v_\sigma((s,t),(s',t')) = i_{\mathbb{Z}}(\sigma((s,t),(s',t'))) = \delta_{\sigma((s,t),(s',t'))}, \quad (s,t),(s',t') \in \mathbb{Z}^{2n}. \tag{3.2}$$

Using the isomorphism $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ given by Fourier transform we obtain an isomorphism $C^*(\Gamma) \cong C(\mathbb{T}) \times_{\mathrm{id}, \omega} \mathbb{Z}^{2n}$ where $\omega : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to C(\mathbb{T})$ is defined by

$$\omega((s,t),(s',t'))(z) = z^{\sigma((s,t),(s',t'))}, \quad (s,t),(s',t') \in \mathbb{Z}^{2n},\ z \in \mathbb{T}. \tag{3.3}$$

Using the stabilization trick of [33] and inducing from $\mathbb{Z}^{2n}$ to $\mathbb{R}^{2n}$, we have, as summarized in [32], Equation (2.8)

$$C^*(\Gamma) \otimes \mathcal{K} \cong (C(\mathbb{T}) \times_{\mathrm{id},\omega} \mathbb{Z}^{2n}) \otimes \mathcal{K} \cong \mathrm{Ind}_{\mathbb{Z}^{2n}}^{\mathbb{R}^{2n}}(\beta)[C(\mathbb{T}) \otimes \mathcal{K}] \times_\tau \mathbb{R}^{2n},$$

where the action β of $\mathbb{Z}^{2n}$ on $C(\mathbb{T}) \otimes \mathcal{K}$ is obtained from [33], Theorem 3.4 and is explicitly defined in [32], Equations (2.1) and (2.3) of [32] (replacing "σ" in [32] by our "ω"). Since the original action of $\mathbb{Z}^{2n}$ on $C(\mathbb{T})$ was trivial, by applying [34], Lemma 3.5 we see that $\mathrm{Ind}_{\mathbb{Z}^{2n}}^{\mathbb{R}^{2n}}(\beta)[C(\mathbb{T}) \otimes \mathcal{K}]$ is a continuous trace C^*-algebra with spectrum $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}) \times \mathbb{T} = \mathbb{T}^{2n+1}$. Setting $\mathcal{A}_\sigma = \mathrm{Ind}_{\mathbb{Z}^{2n}}^{\mathbb{R}^{2n}}(\beta)[C(\mathbb{T}) \otimes \mathcal{K}]$ and applying the Thom isomorphism Theorem of Connes ([8]), we have

$$\mathrm{K}_i(C^*(\Gamma)) \cong \mathrm{K}_i(\mathcal{A}_\sigma \times_\tau \mathbb{R}^{2n}) = \mathrm{K}_i(\mathcal{A}_\sigma), \quad i = 0, 1, \tag{3.4}$$

completing the proof of the proposition. ∎

We now discuss the problem of identifying the Dixmier-Douady class of the twisted torus $\mathcal{A}_\sigma$ described in Proposition 3.1. In fact, results in [35] give an explicit formula for this Dixmier-Douady class in the sheaf cohomology group $\check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathcal{S})$ (here $\mathcal{S} = \underline{\mathbb{T}}$ represents the sheaf of germs of $\mathbb{T}$-valued functions on $\mathbb{T}^{2n+1}$):

LEMMA 3.2. *Let* Γ, σ *and* $\mathcal{A}_\sigma$ *be as in Proposition* 3.1. *Viewing* $\mathbb{R}^{2n}$ *as a principal* $\mathbb{Z}^{2n}$ *bundle over* $\mathbb{T}^{2n}$, *let* $\{N_i\}_{i\in I}$ *be an open cover of* $\mathbb{T}^{2n}$ *with continuous sections* $c_i : N_i \to \mathbb{R}^{2n}$, *and define* $\lambda_{ij} : N_{ij} \to \mathbb{Z}^{2n}$ *by* $\lambda_{ij}(x) = c_i(x)c_j(x)^{-1}$, $x \in N_{ij}$. *Let* $\{U_i = \mathbb{T} \times N_i\}$ *be the corresponding open cover of* $\mathbb{T}^{2n+1} = \mathbb{T}^1 \times \mathbb{T}^{2n}$. *Then the Dixmier-Douady class* $\delta(\mathcal{A}_\sigma) \in \check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathcal{S})$ *is represented with respect to this cover by the Čech two-cocycle* $\{\eta_{ijk} : U_{ijk} \to \mathbb{T}\}$ *where*

$$\eta_{ijk}((z,x)) = z^{\sigma(\lambda_{ij}(x),\lambda_{jk}(x))}, \quad z \in \mathbb{T},\ x \in N_{ijk} \subseteq \mathbb{T}^{2n}, \tag{3.5}$$

σ *as in Equation* (3.1).

Proof. This is a direct application of Equation (3.3) together with [34], Theorem 3.6 and Corollary 3.4 (i). ∎

Since our aim is to recognize which twisted tori are KK-equivalent to generalized discrete Heisenberg group C^*-algebras, whose K-groups are known in principle, the formula in Equation (3.5) will not be very helpful as it stands. Instead, we want to calculate the element of $\mathrm{H}^3(\mathbb{T}^{2n+1}, \mathbb{Z})$ corresponding to the Čech two cocycle defined in Equation (3.5). Recall from Section 2 that $\mathrm{H}^3(\mathbb{T}^{2n+1}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\binom{2n+1}{3}}$, with standard generators given by $\{e_i \wedge e_j \wedge e_k \mid 1 \leqslant i < j < k \leqslant 2n+1\}$.

LEMMA 3.3. *Let* $\sigma \in \mathrm{Z}^2(\mathbb{Z}^{2n}, \mathbb{Z})$ *be as defined in Equation* (3.1), *and let* $\mathcal{A}_\sigma$ *be the corresponding twisted* $(2n+1)$*-torus constructed in Proposition* 3.1. *Then the Dixmier-Douady class of* $\mathcal{A}_\sigma$ *in* $\mathrm{H}^3(\mathbb{T}^{2n+1}, \mathbb{Z})$ *is given by*

$$\delta(A_\sigma) = \sum_{i=1}^{n} d_i e_1 \wedge e_{i+1} \wedge e_{n+i+1}. \tag{3.6}$$

Proof. We have already computed the Dixmier-Douady class of $\mathcal{A}_\sigma$ as an element of $\check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathcal{S})$ in Equation (3.5). To obtain the result as stated in Equation (3.6), we need to compute the image of the sheaf two-cocycle given in Equation (3.5) under the Bockstein map $d_2 : \check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathcal{S}) \to \check{\mathrm{H}}^3(\mathbb{T}^{2n+1}, \mathbb{Z}) = \mathrm{H}^3(\mathbb{T}^{2n+1}, \mathbb{Z})$ in the long exact sequence in sheaf cohomology corresponding to the short exact sequence of groups $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{T} \longrightarrow 0$.

We first write $\mathbb{T}^{2n+1}$ as a cartesian product $\mathbb{T} \times \mathbb{T}^{2n}$. Let e be the generator for $\check{\mathrm{H}}^1(\mathbb{T}, \mathbb{Z})$ and $f_1, \dots, f_{2n}$ be the standard generators for $\check{\mathrm{H}}^1(\mathbb{T}^{2n}, \mathbb{Z})$. The projections $\pi_1 : \mathbb{T} \times \mathbb{T}^{2n} \to \mathbb{T}$ and $\pi_2 : \mathbb{T} \times \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ give pullbacks $\pi_1^* : \check{\mathrm{H}}^1(\mathbb{T}, \mathbb{Z}) \to \check{\mathrm{H}}^1(\mathbb{T}^{2n+1}, \mathbb{Z})$ and $\pi_2^* : \check{\mathrm{H}}^1(\mathbb{T}^{2n}, \mathbb{Z}) \to \check{\mathrm{H}}^1(\mathbb{T}^{2n+1}, \mathbb{Z})$ such that $\pi_1^*(e) = e_1$ and $\pi_2^*(f_i) = e_{i+1}$, $1 \leqslant i \leqslant 2n$, where now $\{e_i \mid 1 \leqslant i \leqslant 2n+1\}$ represent the standard generators of $\check{\mathrm{H}}^1(\mathbb{T}^{2n+1}, \mathbb{Z}) = \check{\mathrm{H}}^1(\mathbb{T}^{2n+1}, \mathbb{Z})$. We now resort to an axiomatic argument similar to that used by J. Brylinski in [7], Theorem 4.14 and Corollary 4.15. Note that the sheaf two-cocycle $\{\eta_{ijk}\}$ defined in Equation (3.5) is by definition the cup product of the Čech 0-cocycle $g_z \in \check{Z}^0(\mathbb{T}^{2n+1}, \mathbb{T})$ defined by $g_z((z, x)) = z$, $(z, x) \in \mathbb{T} \times \mathbb{T}^{2n}$, and the Čech two-cocycle $g_\sigma \in \mathbb{Z}^2(\mathbb{T}^{2n+1}, \mathbb{Z})$, where $\{(g_\sigma)_{ijk} \mid \ U_{ijk} = \mathbb{T} \times N_{ijk} \to \mathbb{Z}\}$ is defined by

$$(g_\sigma)_{ijk}(z, \chi) = \sigma(\lambda_{ij}(x), \lambda_{jk}(x)) \quad (z, \chi) \in \mathbb{T} \times N_{ijk}, \tag{3.7}$$

i.e.

$$\{\eta_{ijk}\} = \{(g_z)_i\} \cup \{(g_\sigma)_{ijk}\}, \tag{3.8}$$

where here our cup product $\cup$ maps $\check{\mathrm{H}}^0(\mathbb{T}^{2n+1}, \mathcal{S}) \otimes \check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathbb{Z})$ to $\check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathcal{S} \otimes \mathbb{Z}) \cong \check{\mathrm{H}}^2(\mathbb{T}^{2n+1}, \mathcal{S})$. Using our proof of Proposition 2.1 and the explicit structure of the isomorphism $\lambda_* : \mathrm{H}^2(\mathbb{Z}^{2n}, \mathbb{Z}) \to \check{\mathrm{H}}^2(\mathbb{T}^{2n}, \mathbb{Z})$ discussed there as defined in [35], Lemma 2.6 it is easy to see that

$$\begin{aligned}
\{(g_\sigma)_{ijk}\} &= \pi_2^* \circ \lambda_*([\sigma]) = \pi_2^* \circ \lambda_* \left(\sum_{i=1}^n d_i [\sigma_{E_{n+i,i}}] \right) \\
&= \pi_2^* \left(\sum_{i=1}^n d_i f_i \wedge f_{n+i} \right) = \sum_{i=1}^n d_i e_{i+1} \wedge e_{n+i+1}.
\end{aligned}$$

Therefore

$$\{(\eta_{ijk})\} = (g_z) \cup \left(\sum_{i=1}^n d_i e_{i+1} \wedge e_{n+i+1} \right).$$

By using axiomatic results from sheaf theory ([6], Theorem 7.1(a)), we get

$$d_2(\{\eta_{ijk}\}) = d_2 \left((g_z) \cup \left(\sum_{i=1}^n d_i e_{i+1} \wedge e_{n+i+1} \right) \right) = (d_0(g_z)) \cup \left(\sum_{i=1}^n d_i e_{i+1} \wedge e_{n+i+1} \right)$$

where $d_0 : \check{\mathrm{H}}^0(\mathbb{T}^{2n+1}, \mathcal{S}) \to \check{\mathrm{H}}^1(\mathbb{T}^{2n+1}, \mathbb{Z})$ is the Bockstein map in dimension 0. Now it is well known that if $g_j : \mathbb{T}^{2n+1} \to \mathbb{T}$ is the continuous function given by projection onto the jth coordinate, $1 \leqslant j \leqslant 2n+1$, then $\{g_j\} \in \check{\mathrm{H}}^0(\mathbb{T}^{2n+1}, \mathcal{S})$ and $d_0(g_j) = e_j$. Since $g_z = g_1$ we have

$$d_2(\{\eta_{ijk}\}) = (e_1) \cup \left(\sum_{i=1}^{n} d_i e_{i+1} \wedge e_{n+i+1} \right).$$

In Section 2, we have already identified $\mathrm{H}^*(\mathbb{T}^{2n+1}, \mathbb{Z})$ with the exterior algebra on $2n+1$ generators $\{e_1, \ldots, e_{2n}\}$ and under this identification the cup product is identified with the exterior product. Hence

$$d_2(\{\eta_{ijk}\}) = e_1 \wedge \sum_{i=1}^{n} d_i e_{i+1} \wedge e_{n+i+1} = \sum_{i=1}^{n} d_i e_1 \wedge e_{i+1} \wedge e_{n+i+1},$$

as desired. ∎

Using Lemma 3.3 we can identify certain twisted tori as being KK-equivalent to generalized discrete Heisenberg group C^*-algebras:

THEOREM 3.4. *Let $\mathcal{A}$ denote a twisted $2n+1$ torus, that is, a continuous trace C^*-algebra with spectrum $\mathbb{T}^{2n+1}$, and suppose that there exist positive integers $\{d_i \mid 1 \leqslant i \leqslant n\}$ with $d_1 | d_2 | \cdots | d_n$ such that $\delta(\mathcal{A}) = \sum_{i=1}^{n} d_i e_1 \wedge e_{i+1} \wedge e_{n+i+1}$. Then $\mathcal{A}$ is* KK*-equivalent to $C^*(\Gamma)$, where $\Gamma = H(d_1, \ldots, d_n)$, and thus the* K*-groups of $\mathcal{A}$ can be computed using Theorem* 2.3.

Proof. The proof is a direct application of Proposition 3.1, Lemma 3.2 and Lemma 3.3. ∎

REMARK 3.5. By applying elements of $\mathrm{Aut}(\mathbb{T}^{2n+1}) = \mathrm{GL}(2n+1, \mathbb{Z})$ to the elements of $\mathrm{H}^3(\mathbb{T}^{2n+1}, \mathbb{Z})$ given in Theorem 3.4, we can arrive at a much wider class of elements of $\mathrm{H}^3(\mathbb{T}^{2n+1}, \mathbb{Z})$ whose corresponding continuous trace C^*-algebras can have their K-groups computed via the methods of Section 2. This idea used together with the Kunneth formula as in Remark 2.8 should allow the K-groups of a wide range of twisted tori of both odd and even dimension to be computed. We plan to take up this problem in detail in a future paper; for now we content ourselves with the following corollary:

COROLLARY 3.6. *Let $\mathcal{A}$ denote a nontrivial twisted m-torus for $m \geqslant 3$, and suppose that there exist integers $\{a_{jk} \mid 2 \leqslant k < j \leqslant m\}$ such that $\delta(\mathcal{A}) = \sum_{2 \leqslant k < j \leqslant m} a_{jk} e_1 \wedge e_k \wedge e_j \in \mathrm{H}^3(\mathbb{T}^3, \mathbb{Z})$. Then, modulo a dimension shift equal to the parity of $(m-1)$, $\mathcal{A}$ is* KK*-equivalent to $C^*(\Gamma \times \mathbb{Z}^{m-(2n+1)})$, where Γ is a generalized discrete Heisenberg group of rank $2n+1 \leqslant m$, and thus the* K*-groups of $\mathcal{A}$ can be computed by the method of Remark* 2.8.

Proof. Let M be the $(m-1)\times(m-1)$ matrix defined by $M = \sum_{2\leqslant k<j\leqslant m} a_{jk}\cdot E_{j-1,k-1}$, and let $\Gamma_1 = \mathbb{Z}^{m-1}\times_{\sigma_M}\mathbb{Z}$, that is, the central extension of $\mathbb{Z}^{m-1}$ by $\mathbb{Z}$ corresponding to $\sigma_M \in \mathrm{Z}^2(\mathbb{Z}^{m-1},\mathbb{Z})$ as defined in [4], Section 3. By [4], Corollary 3.4, Γ_1 is isomorphic to $\Gamma\times\mathbb{Z}^{m-(2n+1)}$, where Γ is a generalized discrete Heisenberg group of rank $2n+1\leqslant m$. By the same method as in Proposition 3.1, $C^*(\Gamma_1)$ is strongly Morita equivalent to $\mathrm{Ind}_{\mathbb{Z}^{m-1}}^{\mathbb{R}^{m-1}}(\beta)[C(\mathbb{T})\otimes\mathcal{K}]\times_\tau\mathbb{R}^{m-1}$. Let $\mathcal{A} = \mathrm{Ind}_{\mathbb{Z}^{m-1}}^{\mathbb{R}^{m-1}}(\beta)[C(\mathbb{T})\otimes\mathcal{K}]$. Then the method of Lemma 3.2 shows that $\mathcal{A}$ is a continuous trace C^*-algebra with spectrum $\mathbb{T}^m$ and the desired Dixmier-Douady class, and we have, by Connes' Thom isomorphism theorem ([8]),

$$\mathrm{K}_i(\mathcal{A}) = \mathrm{K}_{i+(m-1)}\big(C^*(\Gamma_1)\big) = \mathrm{K}_{i+(m-1)}\big(C^*(\Gamma\times\mathbb{Z}^{m-(2n+1)})\big),\quad i=0,1.$$

The K-groups of the last C^*-algebra can be computed using the formulas in Remark 2.8.

Example 3.7. Let $n=1$, and consider continuous trace C^*-algebras with spectrum $\mathbb{T}^{2n+1}=\mathbb{T}^3$. Any such C^*-algebra is either strongly Morita equivalent to $C(\mathbb{T}^3)$, hence has both K_0 and K_1 groups isomorphic to $\mathbb{Z}^4$, or is stably isomorphic to the stable continuous trace C^*-algebra $\mathcal{A}_d$ having Dixmier-Douady class $de_1\wedge e_2\wedge e_3 \in \check{\mathrm{H}}^3(\mathbb{T}^3,\mathbb{Z}) \cong \mathbb{Z}(e_1\wedge e_2\wedge e_3)$, for some $d\in\mathbb{Z}\setminus\{0\}$. If $d\in\mathbb{Z}^+$, by Theorem 3.4, $\mathcal{A}_d$ is KK-equivalent to $C^*(H(d))$ so that by Example 2.4,

$$\mathrm{K}_i(\mathcal{A}_d) = \begin{cases}\mathbb{Z}^3 & i=0,\\ \mathbb{Z}^3\oplus\mathbb{Z}_d & i=1.\end{cases}$$

If $d<0$, easy arguments using the automorphism $(z,x,y)\to(z^{-1},x,y)$ show that $\mathcal{A}_d\cong\mathcal{A}_{|d|}$ so that

$$\mathrm{K}_i(\mathcal{A}_d) = \begin{cases}\mathbb{Z}^3 & i=0,\\ \mathbb{Z}^3\oplus\mathbb{Z}_{|d|} & i=1;\end{cases}$$

for all $d\in\mathbb{Z}\setminus\{0\}$.

Example 3.8. Let $n=2$ so that $2n+1=5$. If $\mathcal{A}_\delta$ is a twisted 5-torus with Dixmier-Douady class given by $\delta = d_1e_1\wedge e_2\wedge e_4 + d_2e_1\wedge e_3\wedge e_5$ for positive integers d_1,d_2 with $d_1|d_2$, then Example 2.4 and Theorem 3.4 show that

$$\mathrm{K}_i(\mathcal{A}_\delta) = \begin{cases}\mathbb{Z}^{10}\oplus(\mathbb{Z}_{d_1})^2\oplus(\mathbb{Z}_{d_2})^2 & i=0,\\ \mathbb{Z}^{10}\oplus(\mathbb{Z}_{d_1})^2 & i=1.\end{cases}$$

REFERENCES

1. J. Anderson, W. Paschke, The rotation algebra, *Houston J. Math.* **15**(1989), 1–26.
2. H. Aslaksen, S.T. Lee, J.A. Packer, K-theory for the integer Heisenberg groups, *K-theory*, to appear.
3. L. Auslander, L. Green, F. Hahn, *Flows on Homogeneous Spaces*, Ann. of Math. Stud., vol. 53, Princeton Univ. Press, Princeton 1963.
4. L. Baggett, J.A. Packer, The primitive ideal space of two-step nilpotent group C^*-algebras, *J. Funct. Anal.* **124**(1994), 389–426.
5. B. Blackadar, *K-theory for Operator Algebras*, Springer-Verlag, New York 1986.
6. G. Bredon, *Sheaf Theory*, McGraw Hill Book Co., New York 1967.
7. J.L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser, Boston 1993.
8. A. Connes, An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbb{R}$, *Adv. Math.* **39**(1981), 31–55.
9. D. Crocker, A. Kumjian, I. Raeburn, D. Williams, An equivariant Brauer group and actions of groups on C^*-algebras, *J. Func. Anal.*, to appear.
10. P. Donovan, M. Karoubi, Graded Brauer groups and K-theory with local coefficients, *Inst. Hautes Étude Sci. Publ. Math.* **38**(1970), 5–25.
11. A. Dupré, Combinatorial Extension Cohomology I. Groups, *Adv. Math.* **106**(1994), 96–117.
12. S. Echterhoff, J. Rosenberg, Fine structure of the Mackey machine for actions of abelian groups with constant Mackey obstruction, *Pacific J. Math.* **170**(1995), 17–52.
13. G. Elliott, The K-Theory of the C^*-algebra generated by a projective representation of a torsion-free discrete abelian group, in *Operator Algebras and Group Representations*, vol. 1, Pitman, London 1984, pp. 157–184.
14. A. Grothendieck, Le groupe de Brauer, Séminaire Bourbaki, vol. 290, W.A. Benjamin Inc., New York–Amsterdam 1965.
15. G. Hochschild, J.-P. Serre, Cohomology of group extensions, *Trans. Amer. Math. Soc.* **74**(1953), 110–134.
16. R. Howe, The Brauer group of a compact Hausdorff space and n-homogeneous C^*-algebras, *Proc. Amer. Math. Soc.* **34**(1972), 209–214.
17. R. Howe, On the role of the Heisenberg group in harmonic analysis, *Bull. Amer. Math. Soc.* **3**(1980), 821–843.
18. R. Howe, Prospectives on invariant theory: Schur duality, multiplicity free actions, and beyond, in *The Schur Lectures* 1992, Israel Mathematical Conference Proceedings, vol. 8, Amer. Math. Soc., Providence 1995, pp. 1–182.
19. N. Jacobson, *Basic Algebra I*, W.H. Freeman and Co., San Francisco 1974.
20. R. Ji, On the crossed product C^*-algebras associated with Furstenberg transformations on tori, Ph.D. Dissertation, SUNY at Stony Brook, 1986.
21. M. Karoubi, *K-Theory: An introduction*, Grundlehren Math. Wiss., vol. 226, Springer-Verlag, Heidelberg–New York 1978.
22. G.G. Kasparov, K-theory, group C^*-algebras, and higher signatures (Conspectus, first distributed 1981), in *Novikov conjectures, index theorems and rigidity*, London Math. Soc. Lecture Note Ser. vol. 226, Cambridge University Press, Cambridge 1995, pp. 101–146.
23. A. Kumjian, I. Raeburn, D. Williams, The equivariant Brauer groups of free and proper actions are isomorphic, *Proc. Amer. Math. Soc.* **124**(1996), 809–817.

24. S.T. Lee, J.A. Packer, Twisted group C^*-algebras for two-step nilpotent and generalized discrete Heisenberg groups, *J. Operator Theory* **34**(1995), 91–124.
25. G. Mackey, Unitary representations of group extensions. I, *Acta Math.* **99**(1958), 265–311.
26. A. Maltsev, On a class of homogeneous spaces, *Amer. Math. Soc. Transl.*, vol. 39, Amer. Math. Soc., Providence 1951, pp. 276–307.
27. J. Milnor, J. Stasheff, *Characteristic Classes*, Ann. of Math. Stud., vol. 76, Princeton Univ. Press, Princeton 1963.
28. C. Moore, Extensions and low-dimensional cohomology theory of locally compact group. II, *Trans. Amer. Math. Soc.* **113**(1964), 64–86.
29. G.D. Mostow, Cohomology of topological groups and solvimanifolds, *Ann. Math.* **73**(1961), 20–48.
30. K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* **59**(1954), 531–538.
31. J.A. Packer, C^*-algebras generated by projective representations of the discrete Heisenberg group, *J. Operator Theory* **18**(1987), 42–66.
32. J.A. Packer, Transformation group C^*-algebras: A selective survey, Contemp. Math., vol. 167, Amer. Math. Soc., Providence 1994, pp. 183–217.
33. J.A. Packer, I. Raeburn, Twisted crossed products of C^*-algebras, *Math. Proc. Cambridge Philos. Soc.* **106**(1989), 293–311.
34. J.A. Packer, I. Raeburn, Twisted crossed products of C^*-algebras. II, *Math. Ann.* **287**(1990), 595–612.
35. J.A. Packer, I. Raeburn, On the structure of twisted group C^*-algebras, *Trans. Amer. Math. Soc.* **334**(1992), 685–718.
36. J.A. Packer, I. Raeburn, D. Williams, The equivariant Brauer groups of principal bundles, *J. Operator Theory* **36**(1996), 73–105.
37. M.A. Rieffel, The cancellation theorem for projective modules over irrational rotation algebras, *Proc. London Math. Soc.* **47**(1983), 285–302.
38. J. Rosenberg, Homological invariants of extensions of C^*-algebras, *Proc. Sympos. Pure. Math.*, vol. 38, Amer. Math. Soc., Providence 1982, pp. 35–75.
39. J. Rosenberg, Group C^*-algebras and topological invariants, in *Operator Algebras and Group Representations*, Pitman, New York 1984, pp. 95–115.
40. H. Rouhani, Quasi-rotation C^*-algebras, *Pacific J. Math.* **148**(1991), 131–151.
41. E. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York 1966.
42. J.L. Taylor, Banach algebras and topology, in *Algebras in Analysis*, J.H. Williamson, ed., Academic Press, London–New York 1995, pp. 118–186.

SOO TECK LEE
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260
REPUBLIC OF SINGAPORE
E-mail: matleest@nus.edu.sg

JUDITH A. PACKER
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260
REPUBLIC OF SINGAPORE
E-mail: matjpj@nus.edu.sg

Received February 10, 1997; revised August 18, 1998.