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A GENERALIZATION OF BEURLING'S THEOREM AND A CLASS OF REFLEXIVE ALGEBRAS

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Abstract. We study the commutant $\left\{\rho(\sigma) \,\middle|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\right\}' =: \mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ of the right regular representation of the free product semigroup $\mathop{*}\limits_{i=1}^{n} P_i$, where P_i, $i = 1, 2, \ldots, n$, $n \geqslant 2$, are discrete semigroups with involution, no divisors of the identity, and the cancellation property. We obtain a description of the invariant subspace structure of the left regular representation $\left\{\lambda(\sigma) \,\middle|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\right\}$ extending Beurling's theorem, and show that the analytic Toeplitz algebra $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ is reflexive (resp. hyper-reflexive) and has property $\mathbb{A}_1$ if $n \geqslant 2$. This leads also to an inner-outer factorization and Szegö type theorem in this algebra when P_i $(i = 1, 2, \ldots, n)$ are certain totally ordered semigroups.

Keywords: *Reflexive algebra, free product semigroup, regular representation, inner-outer factorization.*

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1. INTRODUCTION AND PRELIMINARIES

Let P_i, $i = 1, 2, \ldots, n$, $n \geqslant 1$, be unital discrete semigroups with involution, no divisors of the identity, and the cancellation property. In this paper we study the analytic Toeplitz algebra $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ which is the commutant $\left\{\rho(\sigma) \,\middle|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\right\}'$ of the right regular representation of the free product semigroup $\mathop{*}\limits_{i=1}^{n} P_i$ on the Hilbert space $\ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$. Due to a canonical involution induced on $\mathop{*}\limits_{i=1}^{n} P_i$, this algebra is close related to

$$\mathcal{R}^\infty\Big(\mathop{*}_{i=1}^{n} P_i\Big) := \Big\{\lambda(\sigma)\,\Big|\,\sigma \in \mathop{*}_{i=1}^{n} P_i\Big\}'$$

where λ is the left regular representation of $\mathop{*}_{i=1}^{n} P_i$. Thus, all the properties of $\mathcal{L}^\infty\Big(\mathop{*}_{i=1}^{n} P_i\Big)$ have an analogue for $\mathcal{R}^\infty\Big(\mathop{*}_{i=1}^{n} P_i\Big)$.

These algebras have been already studied in some very important particular cases. When $n = 1$ and $P_1 = \mathbb{N}$ we obtain

$$\mathcal{L}^\infty(\mathbb{N}) = \mathcal{R}^\infty(\mathbb{N}) = \mathcal{T}(H^\infty),$$

the well-known algebra of all analytic Toeplitz operators on the Hardy space H^2. It was proved by Sarason ([28]) that $\mathcal{T}(H^\infty)$ is reflexive and by Davidson ([11]) that it is hyper-reflexive.

When $P_1 = P_2 = \cdots = P_n = \mathbb{N}$ and $n \geqslant 2$ we obtain

$$\mathcal{L}^\infty\Big(\mathop{*}_{i=1}^{n} \mathbb{N}\Big) = \mathcal{F}^\infty(H_n),$$

the noncommutative analytic algebra introduced in [22] as the WOT-closure of the noncommutative disc algebra $\mathcal{A}_n$ and used in [23] to obtain a WOT-continuous functional calculus for n-tuples of operators $(T_1, T_2, \ldots, T_n)$ satisfying the condition

$$T_1T_1^* + \cdots + T_nT_n^* \leqslant I,$$

extending the Sz.-Nagy–Foiaş H^∞-functional calculus for contractions ([29]).

A complete description of the invariant subspace structure of $\mathcal{F}^\infty(H_n)$ was obtained in [19] (even in a more general setting), using a noncommutative version of the Wold decomposition (see [20]), as well as an inner-outer factorization in this algebras (see [21]). This algebra can be seen as the noncommutative analytic Toeplitz algebra in n noncommuting variables. It has been studied later in [24], [25], and [1]. Let us mention that Arias and the author proved in [1] that $\mathcal{F}^\infty(H_n)$ is reflexive.

Recently, Davidson and Pitts ([13]) proved that this algebra is hyper-reflexive and has property $\mathbb{A}_1$. They also studied in [12] the algebraic structure of $\mathcal{F}^\infty(H_n)$ (in their notation $\mathcal{L}_n$).

In both the particular cases mentioned above ($P_1 = P_2 = \cdots = P_n = \mathbb{N}$, $n \geqslant 1$) a crucial role in proving reflexivity and hyper-reflexivity is played by the following facts:

(i) any invariant subspace of $\Big\{\lambda(\sigma)\,\Big|\,\sigma \in \mathop{*}_{i=1}^{n} \mathbb{N}\Big\}$ is determined by *inner* functions;

(ii) if $\mathcal{M}$ is any invariant subspace of λ, then $\lambda|\mathcal{M}$ is unitarily equivalent to a direct sum of copies of λ;

(iii) there are many invariant subspaces arising from the eigenvectors of the adjoint $\left\{\lambda(\sigma)^* \,\middle|\, \sigma \in \mathop{*}\limits_{i=1}^{n} \mathbb{N}\right\}$.

None of these facts is necessarily true in our setting (see Theorem 2.10 and the remarks preceding it). In Section 2, we give a characterization of the elements in $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ in terms of their *symbols*. Using the version of Wold's decomposition obtained in [26], we give a description of the invariant subspace structure of the left regular representation $\left\{\lambda(\sigma) \,\middle|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\right\}$, extending Beurling's theorem ([6]) to our setting. Some properties of inner and outer functions and many examples are also considered. We obtain an analogue of Szegö's theorem to our setting.

On the other hand, we characterize the elements in $\ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ which admit *inner-outer* factorization, when P_i $(i = 1, 2, \dots, n)$ are certain totally ordered semigroups.

Surprisingly, althought the properties (i), (ii), (iii) are not necessarily true in our setting, the analytic Toeplitz algebra $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ can be recovered from its invariant subspaces determined by *inner* functions. We prove that $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ is the set of all operators $T \in \ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ leaving each "inner"subspace invariant. In particular, we prove, in Section 3, that $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ is a reflexive algebra and has property $\mathbb{A}_1$.

In [14], Davidson and the author studied generalized Cuntz algebras ([10]) and noncommutative disc algebras $\mathcal{A}\left(\mathop{*}\limits_{i=1}^{n} G_i^+\right)$ associated to the free product $\mathop{*}\limits_{i=1}^{n} G_i^+$ of discrete subsemigroups G_i^+ of $\mathbb{R}^+$. Moreover, we established a dilation theorem for contractive representations of these semigroups which yielded a variant of the von Neumann inequality, extending some results from [7], [15], [18], [20], [22] and [25].

In Section 4, we prove that $\mathcal{L}^\infty\left(\mathop{*}\limits_{i=1}^{n} G_i^+\right)$ is hyper-reflexive with distance constant at most 56. In particular we show that the WOT-closure of the noncommutative disc algebra $\mathcal{A}\left(\mathop{*}\limits_{i=1}^{n} G_i^+\right)$ is hyper-reflexive with distance constant at most 113. Let us mention that these hyper-reflexivity results can be extended to a larger class of totally ordered semigroups P_i, $i = 1, 2, \dots, n$, for $n \geqslant 2$.

The case n=1 remains open. It would be interesting to know the structure of $\mathcal{L}^\infty(P)$ for P unital cancellative semigroup other than $\mathbb{N}$, for example, if P is the positive cone of an additive subgroup of $\mathbb{R}$.

After this paper was submitted for publication, we received a preprint from Bercovici ([5]), which has a different generalization of the Davidson-Pitts hyper-reflexivity result ([13]).

I am greatful to the referee for his helpful suggestions.

2. INVARIANT SUBSPACES AND INNER-OUTER FACTORIZATIONS

Let P be a unital discrete semigroup with the cancellation property, i.e., $xy = xz \Rightarrow y = z$ and $yx = zx \Rightarrow y = z$, and no divisors of the identity $e \in P$, i.e., $xy = e$ if and only if $x = y = e$. We say that $x \leqslant y$ if and only if there exists $z \in P$ such that $y = xz$. It is a routine to show that the relation "$\leqslant$" defines a partial order on P. Let us call it the left invariant order relation on P. Let P_i, $1 \leqslant i \leqslant n$, be n unital discrete semigroups with the cancellation property and no divisors of the identity. We also assume that P_i has an involution $x \mapsto \widetilde{x}$ such that, $\widetilde{\widetilde{x}} = x$ and $\widetilde{(xy)} = \widetilde{y}\widetilde{x}$ for $x, y \in P_i$. If P_i is commutative we may take the involution to be the identity on P_i, i.e., $\widetilde{x} = x$. Denote by $\mathop{*}\limits_{i=1}^{n} P_i$ the free product semigroup amalgamated over the identity $e \in P_i$. For every $f, g \in \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ we define their convolution $f \star g \in \ell^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ by

$$(f \star g)(\tau) = \sum_{\substack{\sigma, \omega \in \mathop{*}\limits_{i=1}^{n} P_i \\ \sigma\omega = \tau}} f(\sigma) g(\omega).$$

Let us denote by $\{\delta_\sigma\}_{\sigma \in \mathop{*}\limits_{i=1}^{n} P_i}$ the canonical basis of $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Let $\lambda : \mathop{*}\limits_{i=1}^{n} P_i \to B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ be the left regular representation of $\mathop{*}\limits_{i=1}^{n} P_i$ defined by

$$\lambda(\sigma)\delta_\tau = \delta_{\sigma\tau} \quad \text{for any } \sigma, \tau \in P.$$

It is clear that $\lambda(\sigma) f = \delta_\sigma \star f$ for any $f \in \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Similarly, we denote by $\rho : \mathop{*}\limits_{i=1}^{n} P_i \to B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ the right regular representation of $\mathop{*}\limits_{i=1}^{n} P_i$ defined by

$$\rho(\sigma)\delta_\tau = \delta_{\tau\sigma} \quad \text{for any } \sigma, \tau \in P.$$

Observe that λ and ρ commute, i.e.,

$$\rho(\sigma)\lambda(\omega) = \lambda(\omega)\rho(\sigma) \quad \text{for any } \sigma, \omega \in P.$$

Let us mention also that the left regular representation $\{\lambda(\sigma)\}_{\sigma\in \overset{n}{\underset{i=1}{*}} P_i}$ is irreducible (see [26] for a more general result). We shall denote by $\mathcal{P}$ the set of all polynomials $p \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ of the form

$$p = \sum_{\text{finite}} a_\sigma \delta_\sigma, \quad a_\sigma \in \mathbb{C}.$$

Following [22], we define $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ as being the set of all $g \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ for which

$$\|g\|_\infty := \sup\{\|g \star p\|_2 \,\big|\, p \in \mathcal{P},\ \|p\|_2 \leqslant 1\} < \infty$$

where $\|\cdot\|_2 := \|\cdot\|_{\ell^2\left(\overset{n}{\underset{i=1}{*}} P_i\right)}$. If $f \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ and $g \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, then

$$f \star g = \lim_{n\to\infty} f \star p_n$$

(the convergence being in $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$), where $p_n \in \mathcal{P}$ and $\|p_n - g\|_2 \to 0$ as $n \to \infty$. Similarly to [22], Theorem 3.2, one can show that $\Big(F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big), \|\cdot\|_\infty\Big)$ is a noncommutative Banach algebra. In the particular case when $n = 1, P_1 = \mathbb{N}$ we can identify $F^\infty(\mathbb{N})$ with the Hardy space H^∞. Let us remark that the free semigroup $\overset{n}{\underset{i=1}{*}} P_i$ has the involution

$$\widetilde{(g_1 \cdots g_k)} = \widetilde{g}_k \cdots \widetilde{g}_1 \quad \text{for any } g_j \in P_{i_j}.$$

Let us define the operator

$$U : \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \to \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$$

by setting $U(\varphi) = \widetilde{\varphi}$, where for every $\varphi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ we denote by $\widetilde{\varphi}$ the element in $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ determined by $\widetilde{\varphi}(\sigma) = \varphi(\widetilde{\sigma})$, $\sigma \in \overset{n}{\underset{i=1}{*}} P_i$. It is clear that U is a unitary operator such that $U^2 = I$ and $U(\varphi \star \psi) = U(\psi) \star U(\varphi)$ for all $\varphi, \psi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Following [21], an operator $T \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ is called

(i) multi-analytic if $T\lambda(\sigma) = \lambda(\sigma)T$ for any $\sigma \in \overset{n}{\underset{i=1}{*}} P_i$;

(ii) inner if T is multi-analytic and isometric;

(iii) outer if T is multi-analytic and $T\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big)$ is dense in $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$.

On the other hand, we say that a function $\varphi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ is outer if and only if $\{\varphi \star p \mid p \in \mathcal{P}\}$ is dense in $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Similarly to [24], Proposition 1.1, one can prove the following.

THEOREM 2.1. *Let P_i, $1 \leqslant i \leqslant n$, be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. An operator $A \in B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ is multi-analytic if and only if there exists $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ such that*

$$Ah = h \star \widetilde{\varphi}, \quad h \in \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big).$$

We denote $A := R_{\widetilde{\varphi}}$, the right multiplication by $\widetilde{\varphi}$. To each $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ we associate an operator

$$L_\varphi : \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \to \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$$

uniquely defined by $L_\varphi g := \varphi \star g$ for $g \in \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Notice that if $\varphi \in \mathcal{P}$, and

$$\varphi = \sum a_\sigma \delta_\sigma$$

then $L_\varphi = \sum a_\sigma \lambda(\sigma)$. Observe that the mapping

$$\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \mapsto L_\varphi \in B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$$

is an isometric homomorphism. Denote the commutant of $\Big\{\rho(\sigma) \,\big|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\Big\}$ by $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ and, similarly, $\mathcal{R}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) := \Big\{\lambda(\sigma) \,\big|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\Big\}'$.

COROLLARY 2.2. *The double commutant of $\Big\{\lambda(\sigma) \,\big|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\Big\}$ is equal to*

$$\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) = \Big\{L_\varphi \,\big|\, \varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big\}. \tag{2.1}$$

Proof. According to Theorem 2.1, any element $X \in \Big\{\lambda(\sigma) \,\big|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\Big\}'$ has the form $X = R_{\widetilde{\varphi}}$ for some $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Let $A \in B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ such that $AR_{\widetilde{\varphi}} = R_{\widetilde{\varphi}}A$ for any $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Since $R_{\widetilde{\varphi}} = U^* L_\varphi U$, it follows that $UAU^* L_\varphi = L_\varphi UAU^*$ for any $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. In particular, we have $UAU^*\lambda(\sigma) = \lambda(\sigma) UAU^*$ for any $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$. Using again Theorem 2.1, there is $\psi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ such that $UAU^* = R_{\widetilde{\psi}}$. Hence, $A = U^* R_{\widetilde{\psi}} U = U^{*2} L_\varphi U^2 = L_\varphi$.

Conversely, if $\varphi, \psi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ and $\omega \in \mathop{*}\limits_{i=1}^{n} P_i$, then

$$L_\varphi R_{\widetilde{\psi}}(\delta_\omega) = \varphi \star (\delta_\omega \star \widetilde{\psi}) = (\varphi \star \delta_\omega) \star \widetilde{\psi} = R_{\widetilde{\psi}} L_\varphi(\delta_\omega).$$

Hence, $L_\varphi R_{\widetilde{\psi}} = R_{\widetilde{\psi}} L_\varphi$. Since $\lambda(\sigma) = U^* \rho(\widetilde{\sigma}) U$ and using again Theorem 2.1, we deduce the relation (2.1). This completes the proof. ∎

Is the algebra $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ equal to the WOT closure of the left regular representation algebra? This is the case when $P_1 = \cdots P_n = \mathbb{N}$.

Corollary 2.3. *$\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ coincides with its double commutant.*

Corollary 2.4. *Let $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Then L_φ is invertible in $B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ if and only if it is invertible in $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$.*

Let us remark that if $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ such L_φ is invertible, then φ is an outer function.

We say that $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ is inner if the multi-analytic operator $R_{\widetilde{\varphi}}$ is inner. The proof of the following characterization for inner functions is similar to [1], Proposition 1.6, so we omit it.

Proposition 2.5. *Let $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. The following statements are equivalent:*

(i) *φ is inner;*
(ii) *L_φ is an isometry;*
(iii) *$\big\{\varphi \star \delta_\sigma \,\big|\, \sigma \in \mathop{*}\limits_{i=1}^{n} P_i\big\}$ is an orthonormal set in $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$;*
(iv) *$\|\varphi\|_2 = \|\varphi\|_\infty = 1$.*

A closed subspace $\mathcal{M} \subset \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ is invariant for $\{\lambda(\sigma)\}_{\sigma \in \mathop{*}\limits_{i=1}^{n} P_i}$ if $\lambda(\sigma)\mathcal{M} \subset \mathcal{M}$ for any $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$. A subspace $\mathcal{L} \subset \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ is called wandering for $\{\lambda(\sigma)\}_{\sigma \in \mathop{*}\limits_{i=1}^{n} P_i}$ if

$$\lambda(\sigma)\mathcal{L} \perp \lambda(\omega)\mathcal{L}$$

for any $\sigma, \omega \in \mathop{*}\limits_{i=1}^{n} P_i$, $\sigma \neq \omega$. We say that two inner functions $\varphi, \psi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ are orthogonal if

$$\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \star \widetilde{\varphi} \perp \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \star \widetilde{\psi}.$$

Using the version of the Wold decomposition from [26], we can obtain a description of the invariant subspaces for $\lambda(\sigma)$, $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$. Our Beurling type theorem ([6]) is the following.

Theorem 2.6. *Let P_i, $1 \leqslant i \leqslant n$, be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. A closed subspace $\mathcal{M} \subset \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ is invariant for each $\lambda(\sigma)$, $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$, if and only if there is $\mathcal{N}_0$, $\mathcal{N}_1 \subset \mathcal{M}$ reducing subspaces for $\lambda(\sigma)|\mathcal{M}$, $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$, such that*

$$\mathcal{M} = \mathcal{N}_0 \oplus \mathcal{N}_1$$

where

$$\mathcal{N}_0 = \bigoplus_{j\in J} \left[\ell^2\Big(\mathop{*}_{i=1}^{n} P_i\Big) \star \widetilde{\varphi}_j\right], \tag{2.2}$$

with $\{\varphi_j\}_{j\in J}$ *orthogonal inner functions and*

$$\mathcal{N}_1 = P_{\mathcal{M}} \bigvee_{\substack{\sigma,\omega\in \mathop{*}_{i=1}^{n} P_i \\ \omega\not\leq\sigma}} \lambda(\sigma)^*\lambda(\omega)\mathcal{N}_1, \tag{2.3}$$

where $P_{\mathcal{M}}$ *is the orthogonal projection on* $\mathcal{M}$. *Moreover, this representation is essentially unique.*

Proof. Applying Theorem 2.4 from [26] to the semigroup of isometries $\{\lambda(\sigma)|\mathcal{M}\}_{\sigma\in \mathop{*}_{i=1}^{n} P_i}$, we obtain a unique orthogonal decomposition $\mathcal{M} = \mathcal{N}_0 \oplus \mathcal{N}_1$ with the property that $\mathcal{N}_0$ and $\mathcal{N}_1$ reduce each isometry $\lambda(\sigma)|\mathcal{M}, \sigma \in \mathop{*}_{i=1}^{n} P_i$, and

$$\mathcal{N}_0 = \bigoplus_{\sigma\in \mathop{*}_{i=1}^{n} P_i} \lambda(\sigma)(\mathcal{L}), \quad \mathcal{N}_1 = \mathcal{M} \ominus \mathcal{N}_0,$$

where $\mathcal{L}$ is the wandering subspace for $\{\lambda(\sigma)|\mathcal{M}\}_{\sigma\in \mathop{*}_{i=1}^{n} P_i}$ given by

$$\mathcal{L} = \mathcal{M} \ominus \left[\bigvee_{\substack{\sigma,\omega\in \mathop{*}_{i=1}^{n} P_i \\ \omega\not\leq\sigma}} P_{\mathcal{M}}\lambda(\sigma)^*\lambda(\omega)\mathcal{M} \right]. \tag{2.4}$$

Moreover, we have

$$\mathcal{N}_1 = P_{\mathcal{M}} \bigvee_{\substack{\sigma,\omega\in \mathop{*}_{i=1}^{n} P_i \\ \omega\not\leq\sigma}} \lambda(\sigma)^*\lambda(\omega)\mathcal{N}_1.$$

Let $\{\widetilde{\varphi}_j\}_{j\in J}$ be an orthonormal basis for the Hilbert space $\mathcal{L}$. Since $\mathcal{L}$ is a wandering subspace for $\lambda(\sigma)$, i.e.,

$$\lambda(\sigma)\mathcal{L} \perp \lambda(\omega)\mathcal{L} \tag{2.5}$$

for any $\sigma, \omega \in \mathop{*}_{i=1}^{n} P_i$, $\sigma \neq \omega$, it is clear that

$$\delta_\sigma \star \widetilde{\varphi}_j \perp \delta_\omega \star \widetilde{\varphi}_j \quad \text{for any } \sigma, \omega \in \mathop{*}_{i=1}^{n} P_i,\ \sigma \neq \omega.$$

According to Proposition 2.5, we deduce that φ_j is an inner function. The orthogonal decomposition

$$\mathcal{N}_0 = \bigoplus_{j\in J} \left[\ell^2\Big(\mathop{*}_{i=1}^{n} P_i\Big) \star \widetilde{\varphi}_j\right]$$

follows immediately using again the relation (2.5). The uniqueness part follows from [26], Theorem 2.4, so we omit it. ∎

An intrinsic description of the subspace $\mathcal{N}_1$ would be interesting.

COROLLARY 2.7. *If* $\varphi_1, \varphi_2 \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ *are inner functions such that*

$$\varphi_1 \star \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) = \varphi_2 \star \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big),$$

then there exists $\alpha \in \mathbb{C}$, $|\alpha| = 1$, *such that* $\varphi_1 = \alpha\varphi_2$.

COROLLARY 2.8. *If* $\psi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ *is an inner function, then* $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}$ *is an invariant subspace for* $\mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$.

Proof. Let $\psi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ be an inner function. Then $R_{\widetilde{\psi}}$ is an isometry and range $R_{\widetilde{\psi}} = \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}$ is closed. If $\varphi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, then $L_\varphi R_{\widetilde{\psi}} = R_{\widetilde{\psi}} L_\varphi$ and we have

$$\begin{aligned} L_\varphi\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}\Big) &= L_\varphi R_{\widetilde{\psi}}\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big) = R_{\widetilde{\psi}} L_\varphi\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big) \\ &\subset R_{\widetilde{\psi}}\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big) = \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}. \end{aligned}$$

This completes the proof. ∎

COROLLARY 2.9. *Let* P_i, $1 \leqslant i \leqslant n$, *be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. If* $P_1, \ldots, P_n$ *are totally ordered by the left invariant order relation "*$\leqslant$*", then the relations* (2.3) *and* (2.4) *are equivalent to*

$$\mathcal{N}_1 = \bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} P_i \\ \sigma \neq e}} \lambda(\sigma)\mathcal{N}_1 \quad \text{and} \quad \mathcal{L} = \mathcal{M} \ominus \Bigg[\bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} P_i \\ \sigma \neq e}} \lambda(\sigma)\mathcal{M}\Bigg].$$

In the particular case when $P_1 = \cdots = P_n = \mathbb{N}$, the subspace $\mathcal{N}_1 = \{0\}$, due to the Wold decomposition from [20]. Moreover, if $n = 1$ and $P_1 = \mathbb{N}$, then Theorem 2.6 coincides with Beurling's theorem ([6]). In our setting, according to Theorem 2.6, the invariant subspaces of $\{\lambda(\sigma)\}_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i}$ are not all generated by inner functions, i.e., of the form (2.2), so $\mathcal{N}_1 \neq \{0\}$.

Let us consider an example. Let $G_i^+ (1 \leqslant i \leqslant n)$ be n positive cones of discrete additive subgroups of $\mathbb{R}$, such that they are dense in $\mathbb{R}^+$. Define $\mathcal{M} \subset \ell^2\Big(\overset{n}{\underset{i=1}{*}} G_i^+\Big)$ by

$$\mathcal{M} = \bigoplus_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+ \\ \sigma \neq 0}} \lambda(\sigma)(\mathbb{C}\,\delta_0).$$

Now it is easy to see that

$$\mathcal{M} = \bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+ \\ \sigma \neq 0}} \lambda(\sigma)\mathcal{M}$$

and it cannot be of the form (2.2).

Notice that, according to the Wold decomposition ([26]), $\{\lambda(\sigma)|\mathcal{M}\}_{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+}$ is not unitarily equivalent to a direct sum of copies of $\{\lambda(\sigma)\}_{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+}$. Using the same idea, it is easy to construct many other invariant subspaces of $\{\lambda(\sigma)\}_{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+}$ which are not generated by inner functions.

The following result shows that there are very few invariant subspaces for $\{\lambda(\sigma)\}_{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+}$ arising from the eigenvectors of the adjoint $\{\lambda(\sigma)^*\}_{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+}$.

THEOREM 2.10. *Assume that all the semigroups G_i^+, $i = 1, 2, \ldots, n$, are dense in $\mathbb{R}^+$. Then there is only one* 1*-codimensional invariant subspace for $\lambda(\sigma)$, $\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+$, and this is $\mathcal{M} = \bigoplus_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+ \\ \sigma \neq 0}} \lambda(\sigma)(\mathbb{C})$.*

Proof. Assume that there is $\varphi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} G_i^+\Big)$, $\|\varphi\|_2 \leqslant 1$ such that $\{\varphi\}^\perp$ is invariant for each $\lambda(\sigma)$, $\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+$. This shows that, for each $\omega \in \overset{n}{\underset{i=1}{*}} G_i^+$, there is $\mu(\omega) \in \mathbb{C}$ such that $\lambda(\omega)^*\varphi = \mu(\omega)\varphi$ for any $\omega \in \overset{n}{\underset{i=1}{*}} G_i^+$. Since $\{\lambda(\sigma)\}_{\sigma \in \overset{n}{\underset{i=1}{*}} G_i^+}$ is a semigroup of operators, we infer that μ is a semicharacter of $\overset{n}{\underset{i=1}{*}} G_i^+$. Assume $\langle \varphi, \delta_0 \rangle = 1$. Then, for any $\omega \in \overset{n}{\underset{i=1}{*}} G_i^+$, we have

$$\langle \varphi, \delta_\omega \rangle = \langle \varphi, \lambda(\omega)\delta_0 \rangle = \langle \mu(\omega)\varphi, \delta_0 \rangle = \mu(\omega).$$

Therefore $\varphi = \sum_{\omega \in \overset{n}{\underset{i=1}{*}} G_i^+} \mu(\omega)\delta_\omega$. Since $\varphi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} G_i^+\Big)$, we must have

$$\sum_{\omega \in \overset{n}{\underset{i=1}{*}} G_i^+} |\mu(\omega)|^2 \leqslant 1.$$

According to [14], Theorem 3.2, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\mu(g) = 0$ for any $g \in G_i^+$ with $i \neq i_0$. On the other hand, as in the proof of [14], Theorem 1.4, there is $0 \leqslant r \leqslant 1$ and $\gamma \in \widehat{G}_{i_0}$ such that $\mu(g_{i_0}) = r^{g_{i_0}}\gamma(g_{i_0})$ for any $g_{i_0} \in G_{i_0}^+$. Therefore,

$$\varphi = \sum_{g_{i_0} \in G_{i_0}^+} r^{g_{i_0}}\gamma(g_{i_0})\delta_{g_{i_0}}$$

with

$$\sum_{g_{i_0} \in G_{i_0}^+} |r^{g_{i_0}}|^2 \leqslant 1.$$

Since $G_{i_0}^+$ is dense in $\mathbb{R}^+$, the later inequality is true if and only if $r = 0$. This shows that $\varphi = \delta_0$, which completes the proof. ∎

Let us also notice that if $\mathcal{M} \subset \ell^2\Big(\underset{i=1}{\overset{n}{*}} G_i^+\Big)$ is an invariant subspace for the left regular representation $\{\lambda(\sigma)\}_{\sigma \in \underset{i=1}{\overset{n}{*}} G_i^+}$ such that there is $g \in \mathcal{M}$ with $P_{\mathbb{C}\,\delta_0} g \neq 0$, then there is an inner function $\varphi \in F^\infty\Big(\underset{i=1}{\overset{n}{*}} G_i^+\Big)$ such that

$$\ell^2\Big(\underset{i=1}{\overset{n}{*}} G_i^+\Big) \star \widetilde{\varphi} \subset \mathcal{M}.$$

Indeed, since $g \in \mathcal{M}$ and $P_{\mathbb{C}\delta_0} g \neq 0$ it is clear that $\mathcal{M} \neq \bigvee_{\substack{\sigma \in \underset{i=1}{\overset{n}{*}} G_i^+ \\ \sigma \neq 0}} \lambda(\sigma)\mathcal{M}$ and therefore there exists a function $\widetilde{\varphi} \in \mathcal{M} \ominus \Bigg[\bigvee_{\substack{\sigma \in \underset{i=1}{\overset{n}{*}} G_i^+ \\ \sigma \neq 0}} \lambda(\sigma)\mathcal{M}\Bigg]$. This implies $\ell^2\Big(\underset{i=1}{\overset{n}{*}} G_i^+\Big) \star \widetilde{\varphi} \subset \mathcal{M}$. It would be nice to know if any invariant subspace of $\{\lambda(\sigma)\}_{\sigma \in \underset{i=1}{\overset{n}{*}} G_i^+}$ contains an "inner" invariant subspace. We expect a negative answer to this question.

Now let us prove some extremal properties of outer functions. The following theorem as well as its consequences were proved in [24] in the particular case when $P_1 = \cdots = P_n = \mathbb{N}$. Here, we extend those results to our setting, obtaining an analogue of Szegö's theorem.

THEOREM 2.11. *Let P_i, $1 \leqslant i \leqslant n$, be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. If $\varphi \in F^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$ is an outer function, then $|\varphi(e)| \geqslant |\psi(e)|$ for any $\psi \in F^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$ such that $L_\varphi^* L_\varphi = L_\psi^* L_\psi$.*

Conversely, if $\psi \in F^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$ is outer and $\varphi \in F^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$ such that $|\varphi(e)| \geqslant |\psi(e)|$ and $L_\varphi^* L_\varphi = L_\psi^* L_\psi$, then φ is outer.

Proof. Suppose that φ is an outer function in $F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. We have

$$|\varphi(e)|^2 = \inf_{k_\omega \in \ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)} \Bigg\| R_{\widetilde{\varphi}}(\delta_e) - \sum_{\substack{\omega \in \mathop{*}\limits_{i=1}^{n} P_i \\ \omega \neq e}} \lambda(\omega)(k_\omega) \Bigg\|^2.$$

Since this infimum is attained with $k_\omega = \lambda(\omega)^* R_{\widetilde{\varphi}}(\delta_e)$, $\omega \in \mathop{*}\limits_{i=1}^{n} P_i$, $\omega \neq e$, and $R_{\widetilde{\varphi}}\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ is dense in $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$, we deduce

$$\begin{aligned}
|\varphi(e)|^2 &= \inf_{h_\omega \in \ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)} \Bigg\| R_{\widetilde{\varphi}}(\delta_e) - \sum_{\substack{\omega \in \mathop{*}\limits_{i=1}^{n} P_i \\ \omega \neq e}} \lambda(\omega) R_{\widetilde{\varphi}}(h_\omega) \Bigg\|^2 \\
&= \inf_{h_\omega \in \ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)} \Bigg\langle R_{\widetilde{\varphi}}^* R_{\widetilde{\varphi}} \Bigg(\delta_e - \sum_{\substack{\omega \in \mathop{*}\limits_{i=1}^{n} P_i \\ \omega \neq e}} \lambda(\omega)(h_\omega)\Bigg), \Bigg(\delta_e - \sum_{\substack{\omega \in \mathop{*}\limits_{i=1}^{n} P_i \\ \omega \neq e}} \lambda(\omega)(h_\omega)\Bigg)\Bigg\rangle \\
&= \inf_{p \in \mathcal{P}_0} \langle R_{\widetilde{\varphi}}^* R_{\widetilde{\varphi}}(\delta_e - p), (\delta_e - p)\rangle.
\end{aligned}$$

Therefore,

$$|\varphi(e)| = \inf_{p \in \mathcal{P}_0} \|(\delta_e - p) \star \widetilde{\varphi}\|_2,$$

where $\mathcal{P}_0$ is the set of all polynomials p in $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ with $p(e) = 0$. Since $L_\varphi^* L_\varphi = L_\psi^* L_\psi$, we obtain

$$\begin{aligned}
|\varphi(e)|^2 &= \inf_{p \in \mathcal{P}_0} \langle R_{\widetilde{\varphi}}^* R_{\widetilde{\varphi}}(\delta_e - p), (\delta_e - p)\rangle = \inf_{p \in \mathcal{P}_0} \langle R_{\widetilde{\psi}}^* R_{\widetilde{\psi}}(\delta_e - p), (\delta_e - p)\rangle \\
&= \inf_{p \in \mathcal{P}_0} \|R_{\widetilde{\psi}}(\delta_e - p)\|^2 \geqslant \inf_{q \in \mathcal{P}_0} \|R_{\widetilde{\psi}}(\delta_e) - q\|^2 = |\psi(e)|^2.
\end{aligned}$$

Now, suppose $\varphi, \psi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ such that ψ is outer, $|\varphi(e)| \geqslant |\psi(e)|$, and $L_\varphi^* L_\varphi = L_\psi^* L_\psi$. Due to the later relation and since L_ψ has dense range, there is an isometry $X \in B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ such that $X L_\psi = L_\varphi$. Notice that

$$X \in \Big\{ R_{\widetilde{g}} \,\big|\, g \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big\}'. \tag{2.6}$$

Indeed, according to Corollary 2.2, we have

$$(R_{\widetilde{g}} X - X R_{\widetilde{g}}) L_\psi = (R_{\widetilde{g}} L_\varphi - L_\varphi R_{\widetilde{g}}) - X(L_\psi R_{\widetilde{g}} - R_{\widetilde{g}} L_\psi) = 0.$$

Since L_φ has dense range, it follows that (2.6) holds. According to Corollary 2.3 and Proposition 2.5, there exists an inner function $f \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that $X = L_f$. Since $L_f L_\psi = L_\varphi$, we have

$$\begin{aligned}|\psi(e)| &\leqslant |\varphi(e)| = |f(e)\psi(e)| = \|P_{\mathbb{C}\delta_e} L_f(\psi(e)\delta_e)\|_2 \\ &\leqslant \|L_f(\psi(e)\delta_e)\|_2 = \|\psi(e)f\|_2 \leqslant \|\psi(e)\delta_e\|_2 = |\psi(e)|.\end{aligned}$$

Hence, $\psi(e)f(e)\delta_e = \psi(e)f$. Therefore, $f(\omega) = 0$ for any $\omega \in \overset{n}{\underset{i=1}{*}} P_i$, $\omega \neq e$. Since f is inner, we deduce that $f = \alpha\delta_e$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Therefore $\alpha\psi = \varphi$. This completes the proof. ∎

COROLLARY 2.12. *If φ, ψ are outer functions in $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that $L_\varphi^* L_\varphi = L_\psi^* L_\psi$, then $\varphi = \alpha\psi$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.*

COROLLARY 2.13. *If $\varphi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ is an outer function, then*

$$|\varphi(e)| = \inf_{p \in \mathcal{P}_0} \|\varphi \star (\delta_e - p)\|_2 \quad (\textit{Szegö infimum})$$

where $\mathcal{P}_0$ is the set of all polynomials p in $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ with $p(e) = 0$.

In the particular case when $P_1 = \cdots = P_n = \mathbb{N}$ ($n \geqslant 2$), an inner-outer factorization for the elements in $\mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} \mathbb{N}\Big)$ (resp. $\ell^2\Big(\overset{n}{\underset{i=1}{*}} \mathbb{N}\Big)$) was obtained in [21] (resp. [1]). In what follows, we characterize the elements $\psi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, $\psi \neq 0$, which admit inner-outer factorization, when $P_1, \ldots, P_n$ are semigroups, as considered in this section, and totally ordered by the left invariant order relation "$\leqslant$". Denote

$$\mathcal{L}_0 = \Big[\bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} \lambda(\sigma)\widetilde{\psi}\Big] \ominus \Big[\bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} P_i \\ \sigma \neq e}} \lambda(\sigma)\widetilde{\psi}\Big].$$

We say that ψ has the property (L) if

$$\text{(L)} \qquad \bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} \lambda(\sigma)\widetilde{\psi} = \bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} \lambda(\sigma) P_{\mathcal{L}_0}\widetilde{\psi}$$

where $P_{\mathcal{L}_0}$ is the orthogonal projection onto $\mathcal{L}_0$, or equivalently, if $f \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ and $f \perp \lambda(\sigma)P_{\mathcal{L}_0}\widetilde{\psi}$ for any $\sigma \in \overset{n}{\underset{i=1}{*}} P_i$, then $f \perp \lambda(\sigma)\widetilde{\psi}$ for any $\sigma \in \overset{n}{\underset{i=1}{*}} P_i$.

THEOREM 2.14. *Let P_i, $1 \leqslant i \leqslant n$, be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. Assume that P_i, $1 \leqslant i \leqslant n$, are totally ordered by the left invariant order relation "$\leqslant$". Then $\psi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, $\psi \neq 0$ admits a factorization $\psi = \varphi \star g$ with φ inner and g outer functions if and only if ψ has property* (L).

Moreover, the factorization is essentially unique and $\psi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ if and only if $g \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ and $\|\psi\|_\infty = \|g\|_\infty$.

Proof. Suppose $\psi = \varphi \star g$ where φ is inner and g is outer function. Since g is outer there is $p_n \in \mathcal{P}$ such that $\|p_n \star \widetilde{g} - \delta_e\|_2 \to 0$ as $n \to \infty$. Hence,

$$\bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} \lambda(\sigma)\widetilde{\psi} = \bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} (\delta_\sigma \star \widetilde{g}) \star \widetilde{\varphi} = \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\varphi}. \tag{2.7}$$

Similarly, one can see that

$$\bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} P_i \\ \sigma \neq e}} \lambda(\sigma)\widetilde{\psi} = \bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} P_i \\ \sigma \neq e}} \lambda(\sigma)\widetilde{\varphi}.$$

Since φ is inner, we infer that

$$\mathcal{L}_0 = \Bigg[\bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} \lambda(\sigma)\widetilde{\psi}\Bigg] \ominus \Bigg[\bigvee_{\substack{\sigma \in \overset{n}{\underset{i=1}{*}} P_i \\ \sigma \neq e}} \lambda(\sigma)\widetilde{\psi}\Bigg] = \mathbb{C}\,\widetilde{\varphi}.$$

Hence, $P_{\mathcal{L}_0}\widetilde{\psi} = \alpha\widetilde{\varphi}$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Now, if $f \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ with $f \perp \lambda(\sigma)P_{\mathcal{L}_0}\widetilde{\psi}$ for any $\sigma \in \overset{n}{\underset{i=1}{*}} P_i$, then $f \perp \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\varphi}$. Taking into account (2.7), we deduce that $f \perp \lambda(\sigma)\widetilde{\psi}$ for any $\sigma \in \overset{n}{\underset{i=1}{*}} P_i$, which shows that ψ has property (L).

Conversely, suppose that ψ has property (L). Since $\mathcal{L}_0$ is a wandering subspace for $\{\lambda(\sigma)\}_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i}$, it follows that $\widetilde{\varphi} := P_{\mathcal{L}_0}\widetilde{\psi}$ is an inner function in $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Thus,

$$\mathcal{M} := \bigvee_{\sigma \in \overset{n}{\underset{i=1}{*}} P_i} \lambda(\sigma)\widetilde{\psi} = \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\varphi}.$$

Hence, there exists $\widetilde{g} \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that $\widetilde{\psi} = \widetilde{g} \star \widetilde{\varphi}$. Since $\widetilde{\varphi} \in \mathcal{M}$, there is $p_n \in \mathcal{P}$ such that $\|\widetilde{\varphi} - p_n \star \widetilde{\psi}\|_2 \to 0$ as $n \to \infty$. Therefore, we have

$$R_{\widetilde{\varphi}}(\delta_e) = \widetilde{\varphi} = \lim_{n\to\infty}(p_n \star \widetilde{\psi}) = \Big(\lim_{n\to\infty} p_n \star \widetilde{g}\Big) \star \widetilde{\varphi} = R_{\widetilde{\varphi}}\Big(\lim_{n\to\infty} p_n \star \widetilde{g}\Big).$$

Hence, $R_{\widetilde{\varphi}}\big(\delta_e - \lim_{n\to\infty} p_n \star \widetilde{g}\big) = 0$. Since $R_{\widetilde{\varphi}}$ is an isometry, it follows that $\lim_{n\to\infty} p_n \star \widetilde{g} = \delta_e$ which shows that g is an outer function.

Let us prove the uniqueness. Suppose that $\psi = \varphi_1 \star g_1 = \varphi_2 \star g_2$, where φ_1, φ_2 are inner and g_1, g_2 are outer functions. Then $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\varphi}_1 = \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\varphi}_2$ and according to Corollary 2.7, $\varphi_1 = \alpha\varphi_2$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$. On the other hand, we have

$$\widetilde{g}_1 \star \widetilde{\varphi}_1 - \widetilde{g}_2 \star \widetilde{\varphi}_2 = (\alpha\widetilde{g}_1 - \widetilde{g}_2) \star \widetilde{\varphi}_2 = R_{\widetilde{\varphi}_2}(\alpha\widetilde{g}_1 - \widetilde{g}_2) = 0.$$

Since φ_2 is inner, we infer that $\alpha\widetilde{g}_1 = \widetilde{g}_2$. Notice that, for any $p \in \mathcal{P}$, one has $\|(\varphi \star g) \star p\|_2 = \|\varphi \star (g \star p)\|_2 = \|g \star p\|_2$. Hence, we deduce that $g \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ if and only if $\psi = \varphi \star g \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. This completes the proof. ∎

Notice that the subspace $\mathcal{L}_0$ always has dimension 0 or 1. When do these two possibilities occur? What is the significance of the subspace when it is non-zero ? Perhaps an answer will show what property (L) really means.

3. Reflexivity and property $\mathbb{A}_1$ for some analytic Toeplitz algebras

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. If $A \in B(\mathcal{H})$ then the set of all invariant subspaces of A is denoted by $\operatorname{Lat} A$. For any $\mathcal{U} \subset B(\mathcal{H})$ define

$$\operatorname{Lat}\mathcal{U} = \bigcap_{A\in\mathcal{U}} \operatorname{Lat} A.$$

If $\mathcal{S}$ is any collection of subspaces of $\mathcal{H}$, then

$$\operatorname{Alg}\mathcal{S} := \{A \in B(\mathcal{H}) \mid \mathcal{S} \subset \operatorname{Lat} A\}.$$

An operator algebra $\mathcal{U} \subset B(\mathcal{H})$ is reflexive if $\operatorname{Alg}\operatorname{Lat}\mathcal{U} = \mathcal{U}$.

Throughout this section, P_i, $i = 1, 2, \ldots, n$, $n \geqslant 2$, are unital discrete cancellative semigroups with involution and no divisors of the identity. In what follows, we consider a few examples of inner functions in $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ which will be very useful to prove the main result of this section.

LEMMA 3.1. *Let* $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$, $\sigma \neq e$, *and let* $\varphi(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ *be a function in the Hardy space* H^2. *Define* $\varphi_\sigma \in \ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ *by setting*

$$\varphi_\sigma := \sum_{k=0}^{\infty} a_k \delta_{\sigma^k} \qquad \textit{where } \sigma^k := \underbrace{\sigma \cdots \sigma}_{k\,\text{times}}. \tag{3.1}$$

Then φ *is inner (resp. outer) in* H^∞ *(resp.* H^2*) if and only if* φ_σ *is inner (resp. outer) in* $F^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ *(resp.* $\ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$*) .*

Proof. Let $\varphi \in H^2$ and let $\varphi_\sigma \in \ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ be defined as above. We show that $\{\varphi_\sigma \star \delta_\omega \mid \omega \in \mathop{*}\limits_{i=1}^{n} P_i\}$ is an orthonormal set in $\ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ if and only if φ is inner in H^∞.

Fix $\omega_1, \omega_2 \in \mathop{*}\limits_{i=1}^{n} P_i$ and denote

$$S = \langle \delta_{\widetilde{\omega}_1} \star \widetilde{\varphi}_\sigma, \delta_{\widetilde{\omega}_2} \star \widetilde{\varphi}_\sigma \rangle.$$

If ω_1, ω_2 are not comparable, i.e., $\omega_1 \not\leqslant \omega_2$ and $\omega_2 \not\leqslant \omega_1$ then $S = 0$. Suppose that $\omega_1 \leqslant \omega_2$, that is, $\omega_2 = \omega_1\omega_3$ for some unique $\omega_3 \in \mathop{*}\limits_{i=1}^{n} P_i$. Hence,

$$S = \langle \widetilde{\varphi}_\sigma, \delta_{\widetilde{\omega}_3} \star \widetilde{\varphi}_\sigma \rangle.$$

We have two subcases to consider. If ω_3 is infinitely divisible by $\sigma \neq e$, then it is easy to see that $S = 0$. Indeed, notice that

$$\langle \delta_{\sigma^n}, \delta_{\sigma^m} \star \delta_{\omega_3} \rangle = 0$$

for any $n, m \in \{0, 1, 2, \ldots\}$. It remains the case when $\omega_3 = \sigma^k \mu$ with $\sigma \not\leqslant \mu$ and $k = 0, 1, \ldots$. We have

$$S = \langle \varphi_\sigma, \varphi_\sigma \star \delta_{\sigma^k} \star \delta_\mu \rangle.$$

Now, if $\mu \neq e$, then it is clear that $S = 0$ because $\sigma \not\leqslant \mu$. On the other hand, if $\mu = e$, then

$$S = \langle \varphi_\sigma, \varphi_\sigma \star \delta_{\sigma^k} \rangle = \langle \varphi(z), \varphi(z) z^k \rangle_{H^2}.$$

Therefore, $\{\varphi_\sigma \star \delta_\omega \mid \omega \in \mathop{*}\limits_{i=1}^{n} P_i\}$ is an orthonormal set in $\ell^2\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$ if and only if $\{\varphi(z) z^k \mid k = 0, 1, \ldots\}$ is an orthonormal set in the Hardy space H^2. According to Proposition 2.5, we infer that φ is inner in H^∞ if and only if φ_σ is inner in $F^\infty\left(\mathop{*}\limits_{i=1}^{n} P_i\right)$.

Now suppose that $\varphi \in H^2$ is outer, that is, there exists a sequence $q_n \in H^2$ of analytic polynomials such that $\|\varphi q_n - 1\|_{H^2} \to 0$ as $n \to \infty$. This implies $\|\varphi_\sigma \star (q_n)_\sigma - \delta_e\|_2 \to 0$ as $n \to \infty$ and hence φ_σ is outer. Conversely, suppose that φ_σ is outer in $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Then there exist polynomials $p_n \in \mathcal{P}$ such that

$$\|\varphi_\sigma \star p_n - \delta_e\|_2 \to 0 \tag{3.2}$$

as $n \to \infty$. Let $H_\sigma^2 := \bigvee_{k\geqslant 0} \delta_{\sigma^k}$ and notice that

$$\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) = H_\sigma^2 \oplus \Big[\bigoplus_{\substack{\sigma \nleqslant \mu \\ \mu \neq e}} H_\sigma^2 \star \delta_\mu\Big] \oplus \Big[\bigvee_{\substack{\gamma \text{ is infinitely} \\ \text{divisible by } \sigma}} H_\sigma^2 \star \delta_\gamma\Big],$$

and H_σ^2 is a reducing subspace for L_{φ_σ}. Since $PL_{\varphi_\sigma} = L_{\varphi_\sigma}P$ where P is the orthogonal projection of $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ onto H_σ^2, the relation (3.2) implies

$$\|L_{\varphi_\sigma} P p_n - \delta_e\|_2 \to 0$$

as $n \to \infty$. It is clear that $Pp_n = (q_n)_\sigma$ for some analytic polynomial q_n in H^2. Therefore, we have

$$\|\varphi q_n - 1\|_{H^2} = \|\varphi_\sigma \star (q_n)_\sigma - \delta_e\|_2 \to 0$$

as $n \to \infty$. This completes the proof. ∎

Corollary 3.2. *The function $\lambda\delta_\sigma$ is inner in $F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ for every $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$ and $|\lambda| = 1$.*

Let us remark that one can also prove that the mapping

$$\varphi \in H^\infty \mapsto \varphi_\sigma \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$$

is an isometry.

The following result is an extension of [1], Lemma 3.2, which we need in what follows.

Lemma 3.3. *If $\omega \in \mathop{*}\limits_{i=1}^{n} P_i$, $\omega \neq e$, and $\lambda \in \mathbb{C}$, $|\lambda| < 1$, then*

$$f_{\omega,\lambda} = (\delta_\omega - \lambda\delta_e) \star (\delta_e - \overline{\lambda}\delta_\omega)^{-1}$$

is inner in $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ *and*

$$\bigcap_{\substack{0<|\lambda|<1 \\ \omega\in \overset{n}{\underset{i=1}{*}} P_i,\, \omega\neq e}} \Big[f_{\omega,\lambda}\star \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big] = \{0\}. \tag{3.3}$$

Proof. Let $b(z)=(z-\lambda)/(1-\overline{\lambda}z)$ be the Möbius map of the unit disc $\mathbb{D}=\big\{z\in\mathbb{C}\,\big|\,|z|<1\big\}$. Since $b(z)$ is inner in H^∞ and $b_\omega = f_{\omega,\lambda}$, according to Lemma 3.1, we deduce that $f_{\omega,\lambda}$ is inner in $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$.

Let us denote $\Phi_{\omega,\lambda} := \sum_{k=0}^\infty \lambda^k\delta_{\omega^k}$, where $\omega^0=e$. One can prove that $\Phi_{\omega,\lambda}\perp f_{\omega,\lambda}\star\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ for any $\lambda\in\mathbb{D}\setminus\{0\}$, and any $\omega\in \overset{n}{\underset{i=1}{*}} P_i$, $\omega\neq e$. Indeed, since $\delta_e-\overline{\lambda}\delta_\omega$ is invertible in $F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, we have

$$\langle(\delta_\omega-\lambda\delta_e)\star\delta_\sigma,\Phi_{\omega,\lambda}\rangle = \langle\delta_{\omega\sigma},\Phi_{\omega,\lambda}\rangle - \lambda\langle\delta_\sigma,\Phi_{\omega,\lambda}\rangle = 0$$

for any $\sigma\in \overset{n}{\underset{i=1}{*}} P_i$. If $\psi\in f_{\omega,\lambda}\star\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ for any $\lambda\in\mathbb{D}\setminus\{0\}$, $\omega\in \overset{n}{\underset{i=1}{*}} P_i$, $\omega\neq e$, then $\langle\psi,\Phi_{\omega,\lambda}\rangle=0$, i.e.,

$$\sum_{k=0}^\infty \lambda^k\langle\psi,\delta_{\omega^k}\rangle = 0 \quad \text{for any } \lambda\in\mathbb{D}\setminus\{0\},\ \omega\in \overset{n}{\underset{i=1}{*}} P_i,\ \omega\neq e.$$

This implies $\langle\psi,\delta_{\omega^k}\rangle=0$ for any $k=0,1,\dots$, $\omega\in \overset{n}{\underset{i=1}{*}} P_i$, $\omega\neq e$. Hence, $\langle\psi,\delta_\sigma\rangle=0$ for any $\sigma\in \overset{n}{\underset{i=1}{*}} P_i$, that is, $\psi=0$. Therefore, the relation (3.3) is satisfied. ∎

There is a canonical homomorphism of $\overset{n}{\underset{i=1}{*}} P_i$ onto $\prod_{i=1}^n P_i$ which is the identity on each P_i. Let the image of an element σ be denoted by $|\sigma|$, which we will call the lenght of σ.

EXAMPLE 3.4. For each $g=(g_1,\dots,g_n)\in\prod_{i=1}^n P_i$ denote

$$\Omega_g := \Big\{\omega\in \overset{n}{\underset{i=1}{*}} P_i\,\Big|\,|\omega|=g\Big\}.$$

Let $Y_g := \operatorname{span}\{\delta_\omega\mid\omega\in\Omega_g\}\subset\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. If $f\in Y_g$ with $\|f\|_2=1$, then f is inner. Indeed, if $\omega_1,\omega_2\in\Omega_g$ and $\sigma,\mu\in \overset{n}{\underset{i=1}{*}} P_i$, then $\omega_1\sigma=\omega_2\mu$ if and only if $\omega_1=\omega_2$ and $\sigma=\mu$. On the other hand, if $f=\sum a_\omega\delta_\omega\in Y_g$ and $\sigma,\mu\in \overset{n}{\underset{i=1}{*}} P_i$, $\sigma\neq\mu$, then $f \star \delta_\sigma \perp f \star \delta_\mu$. Since $\|f\|_2 = 1$, using Proposition 2.5, we infer that f is inner.

EXAMPLE 3.5. Let $\{\sigma_i\}_{i\in I} \subset \mathop{*}\limits_{i=1}^{n} P_i$ be with the property that for any $i, j \in I$, $i \neq j$, $\sigma_i \not\leqslant \sigma_j$ and $\sigma_j \not\leqslant \sigma_i$. If $\sum\limits_{i\in I} |a_i|^2 = 1$, then $\varphi := \sum\limits_{i\in I} a_i\delta_{\sigma_i}$ is inner. In particular, if $\{\sigma_1, \dots, \sigma_k\} \subset \mathop{*}\limits_{i=1}^{n} P_i$ has the property that any two monomials $\sigma_i, \sigma_j \in \{\sigma_1, \dots, \sigma_k\}$ start with elements belonging to different semigroups P_i $(i = 1, 2, \dots, n)$ and $\sum\limits_{i=1}^{k} |a_i|^2 = 1$, then $\varphi := \sum\limits_{i=1}^{k} a_i\delta_{\sigma_i}$ is inner.

EXAMPLE 3.6. Let $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_p\}$ be disjoint subsets of $\{1, 2, \dots, n\}$. If $f \in \ell^2(P_{i_1} * \cdots * P_{i_k})$ with $\|f\|_2 = 1$ and $\sigma \in P_{j_1} * \cdots * P_{j_p}, \sigma \neq e$ then $f \star \delta_\sigma$ is an inner function in $F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Moreover, if φ is any inner function of type considered in Example 3.5, then $f \star \varphi$ is inner.

Let $\sigma_1, \sigma_2 \in P_{i_1} * \cdots * P_{i_k}$ and $\omega_1, \omega_2 \in \mathop{*}\limits_{i=1}^{n} P_i$. Notice that, for any $\sigma \in P_{j_1} * \cdots * P_{j_p}$, $\sigma_1\sigma\omega_1 = \sigma_2\sigma\omega_2$ if and only if $\sigma_1 = \sigma_2$ and $\omega_1 = \omega_2$. It is easy to show now that if $\omega_1, \omega_2 \in \mathop{*}\limits_{i=1}^{n} P_i, \omega_1 \neq \omega_2$, then $(f \star \delta_\sigma) \star \delta_{\omega_1} \perp (f \star \delta_\sigma) \star \delta_{\omega_2}$. Since $\|f \star \delta_\sigma\|_2 = 1$, it follows that $f \star \delta_\sigma$ is inner. The last part follows in a similar manner.

The following theorem extends the main result from [1] to our setting.

THEOREM 3.7. *Let P_i, $1 \leqslant i \leqslant n$, $n \geqslant 2$, be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. Then the algebra $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ is reflexive.*

Proof. For simplicity, denote $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) := \mathcal{L}^\infty$. We need to prove that $\operatorname{Alg}\operatorname{Lat}\mathcal{L}^\infty \subset \mathcal{L}^\infty$. Let us fix $A \in \operatorname{Alg}\operatorname{Lat}\mathcal{L}^\infty$. According to Corollary 2.8, for every φ inner in $F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$, $\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \star \widetilde{\varphi} \in \operatorname{Lat}\mathcal{L}^\infty$ and hence, $A\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \star \widetilde{\varphi}\Big) \subset \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) \star \widetilde{\varphi}$. Therefore, to each inner function φ corresponds a unique function $\psi \in \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ such that

$$A\widetilde{\varphi} = \psi \star \widetilde{\varphi}. \tag{3.4}$$

In particular, for each $\omega \in \mathop{*}\limits_{i=1}^{n} P_i$, there is $\psi_\omega \in \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ such that $A\delta_\omega = \psi_\omega \star \delta_\omega$. Let $g = (g_1, \dots, g_n) \in \prod\limits_{i=1}^{n} P_i$ with $g \neq (e, \dots, e)$ and

$$\Omega_g = \Big\{\omega \in \mathop{*}\limits_{i=1}^{n} P_i \,\Big|\, |\omega| = g\Big\}.$$

In general, Ω_g is not a singleton. Fix $\omega_0 \in \Omega_g$ and let $\omega \in \Omega_g, \omega \neq \omega_0$. According to Example 3.4, $\widetilde{f} = \frac{1}{\sqrt{2}}(\delta_{\widetilde{\omega_0}} + \delta_{\widetilde{\omega}})$ is inner (notice that $\sigma \in \Omega_g$ if and only if $\widetilde{\sigma} \in \Omega_{\widetilde{g}}$). According to (3.4), there is $\psi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that

$$Af = \psi \star f = \frac{1}{\sqrt{2}}(\psi \star \delta_{\omega_0} + \psi \star \delta_\omega). \tag{3.5}$$

On the other hand, we have

$$Af = \frac{1}{\sqrt{2}}(A\delta_{\omega_0} + A\delta_\omega) = \frac{1}{2}(\psi_{\omega_0} \star \delta_{\omega_0} + \psi_\omega \star \delta_\omega). \tag{3.6}$$

Since $\omega_0, \omega \in \Omega_g, \omega \neq \omega_0$ we have $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \delta_{\omega_0} \perp \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \delta_\omega$ (see Example 3.4). Hence, using the relations (3.5), (3.6), we infer that $\psi \star \delta_{\omega_0} = \psi_{\omega_0} \star \delta_{\omega_0}$ and $\psi \star \delta_\omega = \psi_\omega \star \delta_\omega$. Since δ_{ω_0} and δ_ω are inner, we have $\psi = \psi_{\omega_0} = \psi_\omega$. Therefore,

$$\psi_\omega = \psi_{\omega_0} \quad \text{ for any } \omega \in \Omega_g. \tag{3.7}$$

If Ω_g is a singleton, then (3.7) is trivial. Notice that

$$\overset{n}{\underset{i=1}{*}} P_i = \{e\} \cup \bigcup_{\substack{g \in \prod_{i=1}^n P_i \\ g \neq (e,\dots,e)}} \Omega_g.$$

Let us fix an element $g_0 = (g_1^0, \dots, g_n^0) \in \prod_{i=1}^n P_i$ with $g_i^0 \in P_i \setminus \{e\}$, $i = 1, \dots, n$, and fix

$$\omega_0 \in \Omega_0 = \big\{\omega \in \overset{n}{\underset{i=1}{*}} P_i \,\big|\, |\omega| = g_0\big\}.$$

Choose an arbitrary $g \in \prod_{i=1}^n P_i$ with $g \neq (e, \dots, e)$ and $g \neq g_0$. Since $n \geqslant 2$, there exist $\omega_1 \in \Omega_0$ and $\sigma_1 \in \Omega_g$ such that they start with elements belonging to different semigroups $P_i, i = 1, \dots, n$. The function $h = \frac{1}{\sqrt{2}}(\delta_{\omega_1} + \delta_{\sigma_1})$ is inner (see Example 3.5) and, according to (3.4), there exists $\psi_h \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that

$$A\widetilde{h} = \psi_h \star \widetilde{h} = \frac{1}{\sqrt{2}}(\psi_h \star \delta_{\widetilde{\omega_1}} + \psi_h \star \delta_{\widetilde{\sigma_1}}).$$

On the other hand, we have

$$A\widetilde{h} = \frac{1}{\sqrt{2}}(\psi_{\omega_1} \star \delta_{\widetilde{\omega_1}} + \psi_{\sigma_1} \star \delta_{\widetilde{\sigma_1}})$$

for some $\psi_{\omega_1}, \psi_{\sigma_1} \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Since $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \delta_{\widetilde{\omega_1}} \perp \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \delta_{\widetilde{\sigma_1}}$ we infer that $\psi_{\omega_1} = \psi_{\sigma_1} = \psi_h$.

Since $\omega_0, \omega_1 \in \Omega_0$, we already proved (see (3.7)) that $\psi_{\omega_0} = \psi_{\omega_1} = \psi_\omega$ for any $\omega \in \Omega_0$. On the other hand, due to similar reasons, $\psi_{\sigma_1} = \psi_\sigma$ for any $\sigma \in \Omega_g$. Therefore, $\psi_{\omega_0} = \psi_\omega = \psi_\sigma$ for any $\omega \in \Omega_0$ and $\sigma \in \Omega_g$. Hence, $\psi_{\omega_0} = \psi_\sigma$ for any $\sigma \in \overset{n}{\underset{i=1}{*}} P_i \setminus \{e\}$.

The above results show that there exists $h \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that

$$A\delta_\omega = h \star \delta_\omega$$

for any $\omega \in \overset{n}{\underset{i=1}{*}} P_i$, $\omega \neq e$. Since A is a bounded operator, it is clear that $h \in \mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Let $B := A - L_h$ and $h' := A\delta_e$. It is clear that $B \in \operatorname{Alg}\operatorname{Lat}\mathcal{L}^\infty$, $B\delta_\omega = 0$ if $\omega \neq e$, $\omega \in \overset{n}{\underset{i=1}{*}} P_i$, and $B\delta_e = h''$, where $h'' := h' - h$.

According to (3.4), for any inner function $\varphi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, $B\widetilde{\varphi} = \psi \star \widetilde{\varphi}$ for some $\psi \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. This shows that $\langle \widetilde{\varphi}, \delta_e \rangle h'' = \psi \star \widetilde{\varphi}$, hence, $h'' \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\varphi}$ for any inner function φ with $\langle \widetilde{\varphi}, \delta_e \rangle \neq 0$. According to Lemma 3.3, we infer that $h'' = 0$, which implies $A = L_h \in \mathcal{L}^\infty$. This completes the proof. ∎

Taking into account the results we have obtained so far, we can easily extend Theorem 2.10 from [13] to our setting, and show that $\mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ has property $\mathbb{A}_1$. The proof follows the same lines but we shall include it for completeness of exposition.

Theorem 3.8. *Let P_i, $1 \leqslant i \leqslant n$, $n \geqslant 2$, be unital discrete semigroups with involution, the cancellation property, and no divisors of the identity. If Φ is a weak-$*$ continuous linear functional on $\mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, and $\varepsilon > 0$, then there are elements $x, y \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ such that*

$$\Phi(A) = (Ax, y) \quad \text{for any } A \in \mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big),$$

and $\|x\|, \|y\| \leqslant \|\varphi\| + \varepsilon$.

Proof. Let Φ be a weak-$*$ continuous linear functional on $\mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ and fix $\varepsilon > 0$. According to Hahn-Banach theorem, there is a trace-class operator $K \in B\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big)$ with $\|K\|_1 := \operatorname{Tr}(K) \leqslant \|\varphi\| + \varepsilon$ and $\Phi(A) = \operatorname{Tr}(AK)$ for all $A \in \mathcal{L}^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. The singular decomposition of K yields

$$Kh = \sum_{k=1}^{\infty} s_k \langle h, f_k \rangle g_k$$

where $\{f_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$ are orthonormal sequences in $\ell^2\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$, $s_k \geqslant 0$, and $\sum_{k=1}^{\infty} s_k = \|K\|_1$. Define $x = \sum_{k=1}^{\infty} s_k^{1/2}(g_k \star \widetilde{\varphi}_k)$ and $y = \sum_{k=1}^{\infty} s_k^{1/2}(f_k \star \widetilde{\varphi}_k)$, where $\{\varphi_k\}_{k=1}^{\infty}$ are orthogonal inner functions, i.e., $\|\varphi_k\|_2 = \|\varphi_k\|_\infty = 1$ and

$$\ell^2\Big(\underset{i=1}{\overset{n}{*}} P_i\Big) \star \widetilde{\varphi}_j \perp \ell^2\Big(\underset{i=1}{\overset{n}{*}} P_i\Big) \star \widetilde{\varphi}_k$$

for $j \neq k$. For example, fix $g \in P_n \setminus \{e\}$, $\omega \in P_1 \setminus \{e\}$, and define $\psi_j := \delta_{g^j\omega}$, $j = 1, 2, \ldots$, where $g^j = \underbrace{g \cdots g}_{j \text{ times}}$. It is easy to see now that $\|x\|_2 = \|y\|_2 = \sum_{k=1}^{\infty} s_k = \|K\|_1$. Notice that $\ell^2\Big(\underset{i=1}{\overset{n}{*}} P_i\Big) \star \widetilde{\varphi}_j$ are invariant subspaces (see Theorem 2.6) for $\{\lambda(\sigma)\}_{\sigma \in \underset{i=1}{\overset{n}{*}} G_i^+}$, and hence, according to Corollary 2.8, they are invariant to any $A \in \mathcal{L}^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$. On the other hand,

$$R^*_{\widetilde{\varphi}_j} A R_{\widetilde{\varphi}_j} = A \tag{3.8}$$

for any $A \in \mathcal{L}^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$. Indeed, since $A = L_\psi$ for some $\psi \in F^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$, we have

$$A R_{\widetilde{\varphi}_j}(\delta_\sigma) = \psi \star (\delta_\sigma \star \widetilde{\varphi}_j) = (\psi \star \delta_\sigma) \star \widetilde{\varphi}_j = R_{\widetilde{\varphi}_j} A(\delta_\sigma)$$

for any $\sigma \in \underset{i=1}{\overset{n}{*}} P_i$. Since φ_j is inner, $R_{\widetilde{\varphi}_j}$ is an isometry. Therefore, the relation (3.8) holds. Using all these facts, we deduce

$$\begin{aligned}
\varphi(A) &= \operatorname{Tr}(AK) = \sum_{k=1}^{\infty} s_k \langle A g_k, f_k \rangle = \sum_{k=1}^{\infty} s_k \langle R^*_{\varphi_k} A R_{\varphi_k} g_k, f_k \rangle \\
&= \sum_{k=1}^{\infty} s_k \langle A(g_k \star \widetilde{\varphi}_k), f_k \star \widetilde{\varphi}_k \rangle = \Big\langle \sum_{k=1}^{\infty} s_k^{1/2} A(g_k \star \widetilde{\varphi}_k), \sum_{k=1}^{\infty} s_k^{1/2} f_k \star \widetilde{\varphi}_k \Big\rangle \\
&= \langle Ax, y \rangle.
\end{aligned}$$

This completes the proof. ∎

COROLLARY 3.9. *The weak-$*$ and* WOT *topologies on* $\mathcal{L}^\infty\Big(\underset{i=1}{\overset{n}{*}} P_i\Big)$ *coincide.*

4. HYPER-REFLEXIVITY FOR SOME ANALYTIC TOEPLITZ ALGEBRAS

The algebra $\mathcal{U} \in B(\mathcal{H})$ is said to be hyper-reflexive ([4]) if there is a constant M such that

$$\text{dist}(T,\mathcal{U}) \leqslant M \sup_{\mathcal{L}\in \operatorname{Lat}\mathcal{U}} \|P_{\mathcal{L}}^{\perp} T P_{\mathcal{L}}\| \tag{4.1}$$

for any $T \in B(\mathcal{H})$, where $P_{\mathcal{L}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{L}$ and $P_{\mathcal{L}}^{\perp} = I_{\mathcal{H}} - P_{\mathcal{L}}$. The constant of hyper-refexivity is the smallest number M with property (4.1). The list of algebras known to be hyper-reflexive is rather short. It includes, for example, nest algebras ([3], [17]), injective von Neumann algebras ([8]), the analytic Toeplitz algebra $\mathcal{L}^{\infty}\Big(\mathop{*}\limits_{i=1}^{n} \mathbb{N}\Big)$ (see [11] for $n = 1$ and [13] for $n \geqslant 2$), and very few others (see [9], [27]).

In this section, we provide a new class of hyper-reflexive algebras, including $\mathcal{L}^{\infty}\Big(\mathop{*}\limits_{i=1}^{n} G_i^{+}\Big)$, where G_i^{+}, $i = 1, 2, \ldots, n$, $n \geqslant 2$, are positive cones of discrete additive subgroups of real numbers. In particular, we show that the WOT-closure of the noncommutative disc algebra $\mathcal{A}\Big(\mathop{*}\limits_{i=1}^{n} G_i^{+}\Big)$ (see [14]) is hyper-reflexive.

We need first a few preliminary results. As in Section 3, let P_i, $i = 1, 2, \ldots, n$, $n \geqslant 2$, be unital discrete cancellative semigroups with involution and no divisors of the identity.

A WOT-closed algebra $\mathcal{U}$ is said to have infinite multiplicity if it is unitarily equivalent to an algebra of the form $\mathcal{B} \otimes I$ where I is the identity operator on an infinite dimensional space. We need to recall a well-known result about algebras of infinite multiplicity.

Theorem 4.1. *Every* WOT-*closed algebra of infinite multiplicity is hyper-reflexive with distance constant at most* 9.

This theorem is a consequence of some results from [2], [8], [11], and [16] (see [13], Theorem 2.7). The following result will be constantly used in this section. To simplify our notation, denote $\mathcal{L}^{\infty}\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big) := \mathcal{L}^{\infty}$.

Lemma 4.2. *Let $\mathcal{W} \subset \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ be an infinite dimensional wandering subspace for $\lambda(\sigma)$, $\sigma \in \mathop{*}\limits_{i=1}^{n} P_i$, and denote $\mathcal{X} := \bigoplus\limits_{\sigma\in \mathop{*}\limits_{i=1}^{n} P_i} \lambda(\sigma)\mathcal{W}$. Then $\mathcal{X}$ is invariant for $\mathcal{L}^{\infty}$ and for any $T \in B\Big(\ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)\Big)$ we have*

$$\text{dist}\,(T|\mathcal{X}, \mathcal{L}^{\infty}|\mathcal{X}) \leqslant 10 \sup_{\mathcal{L}\in \operatorname{Lat}\,(\mathcal{L}^{\infty}|\mathcal{X})} \|P_{\mathcal{L}}^{\perp} (T|\mathcal{X}) P_{\mathcal{L}}\| \tag{4.2}$$

where $P_{\mathcal{L}}$ *is the orthogonal projection of* $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$ *onto* $\mathcal{L}$.

Proof. As in the proof of Theorem 2.6, we infer that there exist orthogonal inner functions $\psi_j \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, $j \in J$, with card $J = \dim \mathcal{W}$ such that

$$\mathcal{X} = \bigoplus_{j\in J} \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}_j.$$

According to Corollary 2.8, $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}_j$ is invariant for $\mathcal{L}^\infty$ and hence $\mathcal{X}$ is invariant for $\mathcal{L}^\infty$. Since

$$\mathcal{L}^\infty = \Big\{ R_{\widetilde{\psi}} \,\Big|\, \psi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big\}',$$

it is clear that $\mathcal{L}^\infty$ is WOT-closed in $B\Big(\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)\Big)$ and hence, $\mathcal{L}^\infty|\mathcal{X}$ is WOT-closed. For each $j \in J$, define $W_j : \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \to \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}_j$ by $W_j f = R_{\widetilde{\psi}_j} f$ for all $f \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Since ψ_j is an inner function, it follows that W_j is a unitary operator. On the other hand, if $\psi \in F^\infty\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$, then

$$L_\varphi W_j f = L_\varphi R_{\widetilde{\psi}_j} f = R_{\widetilde{\psi}_j} L_\varphi f = W_j L_\varphi f$$

for any $f \in \ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big)$. Therefore,

$$W_j^*\Big(L_\varphi|\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}_j\Big) W_j = L_\varphi$$

for any $j \in J$. Since $\mathcal{L}^\infty|\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}_j$ is unitarily equivalent to $\mathcal{L}^\infty$, and the subspaces $\ell^2\Big(\overset{n}{\underset{i=1}{*}} P_i\Big) \star \widetilde{\psi}_j$, $j \in J$, are orthogonal and invariant to $\mathcal{L}^\infty$, it is easy to see that $\mathcal{L}^\infty|\mathcal{X}$ is unitarily equivalent to $\mathcal{L}^\infty \otimes I$, where I is the identity operator on a Hilbert space of dimension equal to $\dim \mathcal{W}$.

Since $\mathcal{L}^\infty|\mathcal{X}$ is a WOT-closed algebra of infinite multiplicity, using Theorem 4.1, we infer that it is hyper-reflexive with distance constant at most 9. Therefore, we can deduce that

$$\operatorname{dist}(P_{\mathcal{X}}(T|\mathcal{X}), \mathcal{L}^\infty|\mathcal{X}) \leqslant 9 \sup_{\mathcal{L}\in \operatorname{Lat}(\mathcal{L}^\infty|\mathcal{X})} \|P_{\mathcal{L}}^\perp (T|\mathcal{X}) P_{\mathcal{L}}\|. \tag{4.3}$$

On the other hand,

$$\operatorname{dist}(P_{\mathcal{X}}^\perp(T|\mathcal{X}), \mathcal{L}^\infty|\mathcal{X}) \leqslant \|P_{\mathcal{X}}^\perp(T|\mathcal{X})\| \leqslant \sup_{\mathcal{L}\in \operatorname{Lat}(\mathcal{L}^\infty|\mathcal{X})} \|P_{\mathcal{L}}^\perp (T|\mathcal{X}) P_{\mathcal{L}}\|.$$

Combining this inequality with (4.3), we obtain (4.2). This completes the proof. ∎

LEMMA 4.3. *Let* $\mathcal{M} \subset \ell^2\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$ *be an invariant subspace for* $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. *If there is a wandering subspace* $\mathcal{W} \subset \mathcal{M}$ *for* $\{\lambda(\sigma)\}_{\sigma \in \mathop{*}\limits_{i=1}^{n} P_i}$, *then*

$$\|L_\varphi\| = \|L_\varphi|\mathcal{M}\|$$

for any $\varphi \in F^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$.

Proof. According to Corollary 2.8, the subspace $\mathcal{N} := \bigoplus\limits_{\sigma \in \mathop{*}\limits_{i=1}^{n} P_i} \lambda(\sigma)\mathcal{W}$ is invariant for $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$. Notice that

$$\|L_\varphi\| \geqslant \|L_\varphi|\mathcal{M}\| \geqslant \|L_\varphi|\mathcal{N}\| = \|L_\varphi\|.$$

The last equality holds because $L_\varphi|\mathcal{N}$ is unitarily equivalent to a direct sum of $\alpha := \dim \mathcal{W}$ copies of L_φ. This ends the proof. ∎

We provide now a new class of hyper-reflexive algebras, including $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} G_i^+\Big)$, where G_i^+, $i = 1, 2, \ldots, n$, $n \geqslant 2$, are positive cones of discrete additive subgroups of real numbers. For the sake of simplicity, we prove the hyper-reflexivity for $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} G_i^+\Big)$.

The proof uses some ideas from [13], Theorem 2.9, but is quite different at some points because of the new obstructions which occur in our more general setting (see Section 1).

THEOREM 4.4. *Let* G_i^+, $i = 1, 2, \ldots, n$, $n \geqslant 2$, *be positive cones of discrete additive subgroups of real numbers. Then the algebra* $\mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ *is hyper-reflexive and for any* $T \in B\Big(\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)\Big)$ *we have*

$$\operatorname{dist}\Big(T, \mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)\Big) \leqslant 56 \sup_{\mathcal{L} \in \operatorname{Lat} \mathcal{L}^\infty\left(\mathop{*}\limits_{k=1}^{n} G_k^+\right)} \|P_\mathcal{L}^\perp T P_\mathcal{L}\|,$$

where $P_\mathcal{L}$ *is the orthogonal projection from* $\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ *onto* $\mathcal{L}$.

Proof. Let $T \in B\Big(\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)\Big)$ be a fixed operator. Setting

$$C = \sup_{\mathcal{L} \in \operatorname{Lat} \mathcal{L}^\infty\left(\mathop{*}\limits_{k=1}^{n} G_k^+\right)} \|P_\mathcal{L}^\perp T P_\mathcal{L}\|,$$

we need to prove that

$$\operatorname{dist}\left(T, \mathcal{L}^\infty\left(\mathop{*}_{k=1}^{n} G_k^+\right)\right) \leqslant 56C. \tag{4.4}$$

For each $i = 1, 2, \ldots, n$, choose a decreasing sequence $\{g_{im}\}_{m=1}^\infty \subset G_i^+ \setminus \{0\}$ such that $g_{im} \to \inf(G_i^+ \setminus \{0\})$ as $m \to \infty$. For each $j = 1, 2$, and $m = 1, 2, \ldots$, define the subspace

$$X_{jm} = \left[\bigoplus_{\substack{i \in \{1,2,\ldots,n\} \\ i \neq j \\ g_j \in G_j^+ \cap [g_{jm}, \infty)}} \ell^2\left(\mathop{*}_{k=1}^{n} G_k^+\right) \star \delta_{g_{im}} \star \delta_{g_j} \right] \oplus \left[\bigoplus_{\substack{i \in \{1,2,\ldots,n\} \\ i \neq j}} \ell^2\left(\mathop{*}_{k=1}^{n} G_k^+\right) \star \delta_{g_{im}} \right].$$

According to Corollary 2.8, the subspaces $\mathcal{X}_{jm}$, $j = 1, 2$, are invariant to $\mathcal{L}^\infty\left(\mathop{*}_{k=1}^{n} G_k^+\right)$. Notice that $\mathcal{X}_m := \mathcal{X}_{1m} + \mathcal{X}_{2m} = \bigoplus_{i=1}^{n} \ell^2\left(\mathop{*}_{k=1}^{n} G_k^+\right) \star \delta_{g_{im}}$. According to Lemma 4.2, there are elements $A_{im} \in \mathcal{L}^\infty\left(\mathop{*}_{k=1}^{n} G_k^+\right)$, $i = 1, 2, \ldots$, such that

$$\|(T - A_{im})|\mathcal{X}_{im}\| \leqslant 10C. \tag{4.5}$$

Notice that $\mathcal{X}_{1m} \cap \mathcal{X}_{2m}$ is an invariant subspace for $\mathcal{L}^\infty\left(\mathop{*}_{k=1}^{n} G_k^+\right)$ containing a wandering subspace, for example $\delta_{g_{1m}} \star \delta_{g_{2m}}$. According to Lemma 4.3, we have

$$\begin{aligned} \|A_{1m} - A_{2m}\| &= \|(A_{1m} - A_{2m})|\mathcal{X}_{1m} \cap \mathcal{X}_{2m}\| \\ &\leqslant \|(A_{1m} - T)|\mathcal{X}_{1m}\| + \|(T - A_{2m})|\mathcal{X}_{2m}\| \leqslant 20C, \end{aligned}$$

and hence

$$\|A_m - A_{im}\| \leqslant 10C, \quad i = 1, 2, \tag{4.6}$$

where $A_m := (A_{1m} + A_{2m})/2$. Combining (4.5) with (4.6), we obtain

$$\|(T - A_m)|\mathcal{X}_{im}\| \leqslant 20C \quad \text{for each } i = 1, 2. \tag{4.7}$$

For any $h \in \mathcal{X}_m$, there exist $f_i \in \mathcal{X}_{im}$, $i = 1, 2$ such that $f_1 \perp f_2$ and $h = f_1 + f_2$ (notice that $P_{\mathcal{X}_{1m}} P_{\mathcal{X}_{2m}} = P_{\mathcal{X}_{2m}} P_{\mathcal{X}_{1m}}$). Therefore, we have

$$\begin{aligned} \|(T - A_m)|\mathcal{X}_m\| &\leqslant \sup_{\substack{h \in \mathcal{X}_m \\ \|h\|=1}} \{\|(T - A_m)f_1\| + \|(T - A_m)f_2\|\} \\ &\leqslant \left(\|(T - A_m)|\mathcal{X}_{1m}\|^2 + \|(T - A_m)|\mathcal{X}_{2m}\|^2\right)^{1/2} \leqslant 20\sqrt{2}C. \end{aligned}$$

Since $\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \delta_{g_{im}} \subset \ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \delta_{g_{i(m+1)}}$, it is clear that $\mathcal{X}_m \subset \mathcal{X}_{m+1}$ and $\{\delta_0\}^\perp$ is an increasing union of these subspaces. On the other hand, applying again Lemma 4.3 and using (4.5), we have

$$\|A_{im}\| = \|A_{im}|\mathcal{X}_{im}\| \leqslant 10C + \|T\|.$$

Therefore, $\|A_m\| \leqslant 10C + \|T\|$. Since the unit ball of $B\Big(\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)\Big)$ is WOT-compact, let A be a WOT-limit of a subsequence of $\{A_m\}$. We deduce that $A \in \mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ and

$$\|(T-A)|\{\delta_0\}^\perp\| \leqslant 20\sqrt{2}C. \tag{4.8}$$

Let us fix $\omega \in \mathop{*}\limits_{k=1}^{n} G_k^+$, $\omega \neq 0$ and $\lambda \in \mathbb{C}$, $|\lambda| < 1$. According to Lemma 3.3, the Möbius function

$$\varphi_{\omega,\lambda} := (\delta_\omega - \lambda\delta_0) \star (\delta_0 - \overline{\lambda}\delta_\omega)^{-1}$$

is inner in $F^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$. Therefore, $\mathcal{N} := \ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \varphi_{\omega,\lambda}$ is an invariant subspace for $\mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$. Let us fix $g \in G_n^+ \setminus \{0\}$ and define $\psi_j := \delta_{\omega g_j}$, $j = 1, 2, \ldots$, where $g_j = \underbrace{g + \cdots + g}_{j \text{ times}}$. Since $n \geqslant 2$, it is easy to see that we can choose ω and g such that

$$\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \psi_j \perp \ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \psi_k.$$

for $j \neq k$, and

$$\ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \psi_j \perp \ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \varphi_{\omega,\lambda}$$

for any $j = 1, 2, \ldots$. For example, take $\omega \in G_1^+ \setminus \{0\}$ and $g \in G_n^+ \setminus \{0\}$. Notice that

$$\mathcal{M} := \Big[\bigoplus_{j=1}^{\infty} \ell^2\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big) \star \psi_j\Big] \oplus \mathcal{N}$$

is an invariant subspace for $\mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ with infinite dimensional wandering subspace containing $\varphi_{\omega,\lambda}$. Using again Lemma 4.2, we deduce that there is an operator $B \in \mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ such that

$$\|(T-B)|\mathcal{N}\| \leqslant \|(T-B)|\mathcal{M}\| \leqslant 10C. \tag{4.9}$$

Notice that $\mathcal{N} \cap \{\delta_0\}^\perp$ is an invariant subspace for $\mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ containing an wandering subspace, namely the one generated by $\delta_g * \varphi_{\lambda,\omega}$, where $g \in G_n^+ \setminus \{0\}$ is fixed as above. According to Lemma 4.3, the relations (4.8) and (4.9), we deduce that

$$\begin{aligned}\|B-A\| &= \|(B-A)|\mathcal{N}\cap\{\delta_0\}^\perp\| \\ &\leqslant \|(B-T)|\mathcal{N}\| + \|(T-A)|\{\delta_0\}^\perp\| \leqslant (10+20\sqrt{2})C.\end{aligned} \tag{4.10}$$

Since $\varphi_{\lambda,\omega} = -\lambda\delta_0 + g$ for some $g \in \{\delta_0\}^\perp$ and $\|\varphi_{\lambda,\omega}\|_2 = 1$, we have

$$\|(T-A)(\lambda\delta_0)\| \leqslant \|(T-A)\varphi_{\lambda,\omega}\| + \|(T-A)g\| \leqslant \|(T-A)|\mathcal{N}\| + \|(T-A)|\{\delta_0\}^\perp\|\,\|g\|_2.$$

On the other hand, using again Lemma 4.3, we infer that

$$\|(T-A)|\mathcal{N}\| \leqslant \|(T-B)|\mathcal{N}\| + \|(B-A)|\mathcal{N}\| \leqslant \|(T-B)|\mathcal{N}\| + \|B-A\|.$$

Combining these inequalities, we obtain

$$\|(T-A)(\lambda\delta_0)\| \leqslant \|(T-B)|\mathcal{N}\| + \|B-A\| + \|(T-A)|\{\delta_0\}^\perp\|\,\|h\|_2.$$

Since $\|h\|_2 = \sqrt{1-|\lambda|^2}$, using (4.8), (4.9), and (4.10), we obtain

$$|\lambda|\,\|(T-A)\delta_0\| \leqslant 20(1+\sqrt{2})C + 20\sqrt{2}C\sqrt{1-|\lambda|^2}.$$

Since this inequality holds for any $0<\lambda<1$, setting $\lambda\to 1$, we deduce

$$\|(T-A)\delta_0\| \leqslant 20(1+\sqrt{2})C.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|T-A\| \leqslant \big(\|(T-A)\delta_0\|^2 + \|(T-A)|\{\delta_0\}^\perp\|^2\big)^{1/2} = 20\sqrt{5+2\sqrt{2}}C < 56C.$$

Hence, the relation (4.4) follows, so $\mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ is hyper-reflexive. This completes the proof. ∎

A consequence of Theorem 3.8, Theorem 4.4, and [11] or [16] is the following.

COROLLARY 4.5. *Every* WOT-*closed unital subalgebra of* $\mathcal{L}^\infty\Big(\mathop{*}\limits_{k=1}^{n} G_k^+\Big)$ *is hyper-reflexive with constant at most* 113.

In particular, the WOT-closure of the noncommutative disc algebra $\mathcal{A}\Big(\mathop{*}\limits_{i=1}^{n} G_i^+\Big)$ is hyper-reflexive. Let us remark that Theorem 4.4 holds true for the Toeplitz algebra $\mathcal{L}^\infty\Big(\mathop{*}\limits_{i=1}^{n} P_i\Big)$, when P_i ($i=1,\dots,n$; $n\geqslant 2$) are unital discrete cancellative semigroups with involution, and totally ordered by the left invariant order "$\leqslant$". Notice that the proof is similar to that of Theorem 4.4.

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