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ALGEBRAS OF SUBNORMAL OPERATORS ON THE UNIT BALL

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Abstract. In this paper we show that each subnormal n-tuple $T \in L(H)^n$ with the property that the Taylor spectrum of T is contained in the closed Euclidean unit ball and is dominating in the open ball, is reflexive. The proof is based on the observation that the dual algebra generated by T possesses the factorization property $(\mathbb{A}_{1,\aleph_0})$. The same results are shown to hold for subnormal tuples that possess an isometric w*-continuous H^∞-functional calculus over the unit ball. Thus we extend a result of Olin and Thomson on the reflexivity of arbitrary single subnormal operators to the case of subnormal systems with rich spectrum in the Euclidean unit ball.

Keywords: *Subnormal systems, reflexive operators, dual operator algebras, Henkin measures, H^∞-functional calculus.*

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A result of Scott Brown ([4]) from 1978 shows that each subnormal operator $T \in \mathcal{L}(H)$ on a complex Hilbert space H has a non-trivial invariant subspace. Using Scott Brown's methods, Olin and Thomson ([15]) proved that, for each weak-$*$-continuous linear functional L on the weak-$*$-closed algebra $\mathfrak{A}_T$ generated by a single subnormal operator $T \in \mathcal{L}(H)$, there are vectors x and $y \in H$ such that $L(A) = \langle Ax, y\rangle$ for every $A \in \mathfrak{A}_T$. As a consequence of this result, Olin and Thomson were able to prove that each subnormal operator T on a Hilbert space H is reflexive, and that the weak-$*$-closed algebra generated by T coincides with the WOT-closed algebra generated by T (with the two topologies being identical on this algebra).

A result of K. Yan ([20]) shows that each subnormal n-tuple T, that is, each system $T = (T_1, \ldots, T_n) \in \mathcal{L}(H)^n$ of Hilbert-space operators that extends to a system $N = (N_1, \ldots, N_n) \in \mathcal{L}(K)^n$ of commuting normal operators on a larger Hilbert space K, possesses a non-trivial joint invariant subspace. It is an open question whether each subnormal n-tuple $T \in \mathcal{L}(H)^n$ is reflexive. A result of Bercovici ([2]) shows that each commuting system of isometries on a Hilbert space is reflexive. An extension of this result to jointly quasinormal systems was given by E.A. Azoff and M. Ptak ([1]). Apart from this, no general reflexivity results for subnormal systems seem to be known (see also [14]).

Sarason's decomposition theorem for compactly supported measures on the complex plane and corresponding decomposition theorems for subnormal operators (see [9]) allow the reduction of the reflexivity problem for single subnormal operators to the particular case of a subnormal operator $T \in \mathcal{L}(H)$ that possesses a minimal normal extension $N \in \mathcal{L}(K)$ with a scalar spectral measure μ for which $H^\infty(\mathbb{D}) = P^\infty(\overline{\mathbb{D}}, \mu)$. Here $H^\infty(\mathbb{D})$ is the Hardy space of all bounded analytic functions on the open unit disc $\mathbb{D}$ in $\mathbb{C}$, $P^\infty(\overline{\mathbb{D}}, \mu)$ denotes the weak-$*$-closure of the polynomials in $L^\infty(\overline{\mathbb{D}}, \mu)$, and we write $H^\infty(\mathbb{D}) = P^\infty(\overline{\mathbb{D}}, \mu)$ if $\mu|\partial\mathbb{D}$ is absolutely continuous with respect to the normalized Lebesgue measure m on the unit circle and if the identity map $\mathbb{C}[z] \to P^\infty(\overline{\mathbb{D}}, \mu)$, $p \mapsto p$, extends to a dual algebra isomorphism $H^\infty(\mathbb{D}) \to P^\infty(\overline{\mathbb{D}}, \mu)$.

In the present paper we show that the results of Olin and Thomson remain true for subnormal systems $T \in \mathcal{L}(H)^n$ with rich spectrum in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. More precisely, let $T \in \mathcal{L}(H)^n$ be a subnormal tuple such that the Taylor spectrum $\sigma(T)$ of T is contained in the closed ball $\overline{\mathbb{B}}$ and is dominating in the open ball $\mathbb{B}$. Then we show that, for each weak-$*$-continuous linear functional L on the weak-$*$-closed algebra $\mathfrak{A}_T$ generated by T, there are vectors x and y in H with

$$L(A) = \langle Ax, y \rangle \quad (A \in \mathfrak{A}_T).$$

As a corollary we obtain that the weak-$*$-closed algebra $\mathfrak{A}_T$ generated by T coincides with the WOT-closed algebra generated by T, and that both topologies agree on this algebra.

Under the same conditions we prove that, for each sequence $(L_k)_{k \geqslant 1}$ of weak-$*$-continuous linear functionals L_k on $\mathfrak{A}_T$, there are vectors x, y_k in H with

$$L_k(A) = \langle Ax, y_k \rangle \quad (A \in \mathfrak{A}_T,\ k \geqslant 1).$$

In the case that T is pure we deduce that the vectors x in H for which the induced cyclic invariant subspace

$$H_x = \bigvee_{k \in \mathbb{N}^n} T^k x \in \mathrm{Lat}(T)$$

is an analytic invariant subspace for T form a dense subset of H. As a consequence we obtain that each subnormal system $T \in \mathcal{L}(H)^n$ with rich spectrum in the unit ball $\mathbb{B}$ is reflexive.

The above results are also shown to be true for each subnormal system T in $\mathcal{L}(H)^n$ that possesses an isometric and weak-$*$-continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$ over the unit ball, or equivalently, for each subnormal system $T \in \mathcal{L}(H)^n$ that possesses a minimal normal extension $N \in \mathcal{L}(K)^n$ with a scalar spectral measure μ for which $\mu|\partial\mathbb{B}$ is a Henkin measure and $H^\infty(\mathbb{B}) = P^\infty(\overline{\mathbb{B}}, \mu)$. Subnormal tuples with an isometric H^∞-functional calculus over the unit ball are in particular absolutely continuous spherical contractions with a spherical dilation and isometric H^∞-functional calculus (see [10]). In the one-variable case, this latter class consists precisely of all absolutely continuous contractions of class $(\mathbb{A})$. By a result of Brown and Chevreau these contractions are reflexive. It is therefore natural to conjecture that each absolutely continuous spherical contraction of class $(\mathbb{A})$ is reflexive. But it seems that additional ideas are needed to decide this question in the multivariable case.

0. PRELIMINARIES

Let $T = (T_1, \ldots, T_n) \in \mathcal{L}(H)^n$ be a commuting system of continuous linear operators on a complex Hilbert space H. We denote by $\sigma(T)$ the Taylor spectrum of T (see [11]). The Banach space $\mathcal{L}(H)$ is the norm-dual of the space $C^1(H)$ of all trace-class operators on H via the duality

$$C^1(H) \times \mathcal{L}(H) \to \mathbb{C}, \quad (A, B) \mapsto \mathrm{Tr}(AB).$$

The smallest unital w*-closed subalgebra $\mathfrak{A}_T$ of $\mathcal{L}(H)$ containing $T_1, \ldots, T_n$ is the norm-dual of the Banach space $\mathcal{Q}_T = C^1(H)/{}^\perp\mathfrak{A}_T$. Thus $\mathfrak{A}_T$ becomes a dual algebra, that is, a Banach algebra $\mathcal{A}$ which is isometrically isomorphic to the norm-dual of a certain fixed Banach space $\mathcal{A}_*$ such that the multiplication in $\mathcal{A}$ is separately w*-continuous. Let $\mathcal{A}$ and $\mathcal{B}$ be dual algebras with preduals $\mathcal{A}_*$ and $\mathcal{B}_*$. By a dual algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ we mean an algebra homomorphism between $\mathcal{A}$ and $\mathcal{B}$ that is an isometric isomorphism and a w*-homeomorphism, or equivalently, an algebra homomorphism that is the adjoint of an isometric isomorphism $\varphi_* : \mathcal{B}_* \to \mathcal{A}_*$.

For $T \in \mathcal{L}(H)^n$ as above and $x, y \in H$, we denote by $[x \otimes y] \in \mathcal{Q}_T$ the equivalence class of the rank-one operator $H \to H$, $\xi \mapsto \langle \xi, y\rangle x$. Each w*-continuous linear functional $L : \mathfrak{A}_T \to \mathbb{C}$ is of the form

$$L = \sum_{k=1}^{\infty} [x_k \otimes y_k],$$

where (x_k) is a bounded sequence in H and $\sum_{k=1}^{\infty} \|y_k\| < \infty$. Let p, q be any cardinal numbers with $1 \leqslant p, q \leqslant \aleph_0$. The dual algebra $\mathfrak{A}_T$ possesses property $(\mathbb{A}_{p,q})$ if, for each matrix (L_{ij}) of functionals $L_{ij} \in \mathcal{Q}_T$ $(0 \leqslant i < p,\ 0 \leqslant j < q)$, there are vectors $(x_i)_{0 \leqslant i < p}$ and $(y_j)_{0 \leqslant j < q}$ in H solving the equations

$$L_{ij} = [x_i \otimes y_j] \quad (0 \leqslant i < p,\ 0 \leqslant j < q).$$

If $p = q$, then we write $\mathbb{A}_p$ instead of $\mathbb{A}_{p,p}$.

Let $G \subset \mathbb{C}^n$ be an open set. We denote by $\mathcal{O}(G)$ the Fréchet algebra of all analytic complex-valued functions on G, and we write $H^\infty(G)$ for the Banach algebra of all bounded analytic functions on G equipped with the norm $\|f\| = \sup_{z \in G} |f(z)|$. A set $\sigma \subset \mathbb{C}^n$ is dominating in G if $\|f\| = \sup\{|f(z)| : z \in \sigma \cap G\}$ for all $f \in H^\infty(G)$. The space $H^\infty(G)$ is a w*-closed subspace of $L^\infty(G)$ with respect to the duality $\langle L^1(G), L^\infty(G)\rangle$ (formed with respect to the $(2n)$-dimensional Lebesgue measure). A sequence (f_k) in $H^\infty(G)$ is a w*-zero sequence if and only if (f_k) is norm-bounded and converges to zero pointwise on G, or equivalently, uniformly on all compact subsets of G.

We write $P^\infty(G)$ for the w*-closure of the polynomials in $H^\infty(G)$. The space $P^\infty(G)$ is a dual algebra with predual $\mathcal{Q}_G = L^1(G)/{}^\perp P^\infty(G)$. For $\lambda \in G$ and $k \in \mathbb{N}^n$, the w*-continuous linear functionals

$$\mathcal{E}_\lambda : P^\infty(G) \to \mathbb{C}, \quad f \mapsto f(\lambda)$$
$$\mathcal{E}_\lambda^{(k)} : P^\infty(G) \to \mathbb{C}, \quad f \mapsto f^{(k)}(\lambda)$$

will be regarded as elements in $\mathcal{Q}_G$.

Let $\Phi : P^\infty(G) \to \mathcal{L}(H)$ be a unital w*-continuous algebra homomorphism with $\Phi(z_i) = T_i$ $(i = 1, \ldots, n)$. For $x, y \in H$,

$$x \otimes y : P^\infty(G) \to \mathbb{C}, \quad f \mapsto \langle \Phi(f)x, y\rangle$$

defines an element in $\mathcal{Q}_G$ with $x \otimes y = \Phi_*\big([x \otimes y]\big)$, where $\Phi_* : \mathcal{Q}_T \to \mathcal{Q}_G$ denotes the predual of $\Phi : P^\infty(G) \to \mathfrak{A}_T$. The map $\Phi : P^\infty(G) \to \mathfrak{A}_T$ is a dual algebra isomorphism if and only if Φ is isometric. In this case, the properties $(\mathbb{A}_{p,q})$ for $\mathfrak{A}_T$ admit an obvious reformulation in terms of the dual algebra $P^\infty(G)$.

Let X be a compact subset of $\mathbb{C}^n$. We write $M(X)$ for the Banach space of all regular complex Borel measures on X, and we set $M^+(X) = \{\mu \in M(X) : \mu \geqslant 0\}$. For $\mu \in M^+(X)$ and $1 \leqslant q < \infty$, we define $P^q(X,\mu)$ as the closure of the polynomials in $L^q(X,\mu)$, while $P^\infty(X,\mu)$ stands for the w*-closure of the set of all polynomials in $L^\infty(X,\mu)$. The space $P^\infty(X,\mu)$ is a dual algebra with predual $\mathcal{Q}(\mu) = L^1(X,\mu)/{}^\perp P^\infty(X,\mu)$. For a complex measure $\mu \in M(X)$, the space $P^q(X,\mu)$ is defined as $P^q(X,|\mu|)$, and the same convention is used for the other spaces defined above. When the context is clear, we omit the underlying space X in the above notations.

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open Euclidean unit ball in $\mathbb{C}^n$, and let $\mathbb{S} = \partial\mathbb{B}$ be the unit sphere. We write $A(\mathbb{B})$ (sometimes also $A(\mathbb{S})$) for the Banach algebra of all continuous complex functions on $\overline{\mathbb{B}}$ which are analytic on $\mathbb{B}$, equipped with the supremum-norm. A Montel sequence is a sequence (f_k) in $A(\mathbb{B})$ that is a w*-zero sequence in $H^\infty(\mathbb{B})$. A measure $\mu \in M(\mathbb{S})$ is a Henkin measure if

$$\lim_{k\to\infty} \int_{\mathbb{S}} f_k \, \mathrm{d}\mu = 0$$

for each Montel sequence (f_k). Examples of Henkin measures are the surface measure σ on $\mathbb{S}$, all measures that are absolutely continuous with respect to σ and all measures in $A(\mathbb{B})^\perp$. Here $A(\mathbb{B})$ is regarded as a closed linear subspace of $C(\mathbb{S})$. For details on Henkin measures, we refer the reader to Chapter 9 in [16]. We write $\mathrm{HM}(\mathbb{S})$ for the set of all Henkin measures $\mu \in M(\mathbb{S})$.

For each measure $\mu \in \mathrm{HM}(\mathbb{S})$, there is a unique w*-continuous algebra homomorphism $r_\mu : H^\infty(\mathbb{B}) \to P^\infty(\mathbb{S},\mu)$ extending the restriction map

$$A(\mathbb{B}) \to C(\mathbb{S}), \quad f \mapsto f|\mathbb{S}.$$

A measure $\mu \in M(\overline{\mathbb{B}})$ is a Henkin measure if $\mu|\mathbb{S} \in \mathrm{HM}(\mathbb{S})$. In this case, the map $r = r(\mu) : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}},\mu)$ with

$$r(f)|\mathbb{B} = f \quad \text{and} \quad r(f)|\mathbb{S} = r_{(\mu|\mathbb{S})}(f) \quad \big(f \in H^\infty(\mathbb{B})\big)$$

is a contractive w*-continuous algebra homomorphism. Furthermore, the map

$$\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}\big(L^2(\overline{\mathbb{B}},\mu)\big), \quad \Phi(f)g = r(f)g$$

and its restriction $\Phi|P^2(\overline{\mathbb{B}},\mu)$ to the invariant subspace $P^2(\overline{\mathbb{B}},\mu)$ are w*-continuous algebra homomorphisms. In particular, for $x,y \in L^2(\overline{\mathbb{B}},\mu)$,

$$x \otimes y : H^\infty(\mathbb{B}) \to \mathbb{C}, \quad f \mapsto \int_{\overline{\mathbb{B}}} r(f) x \overline{y} \, \mathrm{d}\mu$$

defines an element in the predual $\mathcal{Q} = L^1(\mathbb{B})/{}^\perp H^\infty(\mathbb{B})$ of $H^\infty(\mathbb{B})$. When the map $r : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}},\mu)$ is isometric, then r is the adjoint of an isometric isomorphism $r_* : \mathcal{Q}(\mu) = L^1(\overline{\mathbb{B}},\mu)/{}^\perp P^\infty(\overline{\mathbb{B}},\mu) \to \mathcal{Q}$, and we shall not distinguish between elements in $\mathcal{Q}$ and the corresponding functionals in $\mathcal{Q}(\mu)$.

1. FACTORIZATION RESULTS

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open Euclidean unit ball in $\mathbb{C}^n$, and let $\mathcal{Q} = L^1(\mathbb{B})/{}^\perp H^\infty(\mathbb{B})$ be the predual of $H^\infty(\mathbb{B})$ as explained in the preliminaries. Our factorization results for subnormal commuting systems will be based on corresponding factorization properties for the special subnormal system $M_z = (M_{z_1},\dots,M_{z_n})$ consisting of the multiplication operators by the coordinate functions on the space $P^2(\mu)$, where μ is a Henkin measure on $\overline{\mathbb{B}}$.

Lemma 1.1. *Let $L \in \mathcal{Q} = L^1(\mathbb{B})/{}^\perp H^\infty(\mathbb{B})$ be given with $\|L\| = 1 \neq |L(1)|$, and let $\nu \in M(\overline{\mathbb{B}})$ be a measure with $\|\nu\| \leqslant 1$ and*

$$L(f|\mathbb{B}) = \int_{\overline{\mathbb{B}}} f \, \mathrm{d}\nu \quad \big(f \in A(\mathbb{B})\big).$$

Then $|\nu|(\mathbb{B}) = 0$, $\|\nu\| = 1$, and $\nu|\mathbb{S}$ is a Henkin measure.

Proof. For each Montel sequence (f_i) in $A(\mathbb{B})$, the sequence

$$\int_{\mathbb{S}} f_i \, \mathrm{d}\nu = L(f_i|\mathbb{B}) - \int_{\mathbb{B}} f_i \, \mathrm{d}\nu$$

converges to zero. Hence $\nu|\mathbb{S}$ is a Henkin measure. By Hahn-Banach there is a function $f \in H^\infty(\mathbb{B})$ with $\|f\| = 1$ and $\langle L, f\rangle = 1$. Let (p_k) be a sequence of polynomials with $\|p_k\|_{\infty,B} \leqslant 1$ such that f is the w*-limit of (p_k) in $H^\infty(\mathbb{B})$. Then

$$\int_{\overline{\mathbb{B}}} p_k \, \mathrm{d}\nu = L(p_k|\mathbb{B}) \xrightarrow{k} 1.$$

On the other hand, we know that

$$\int_{\overline{\mathbb{B}}} p_k \,\mathrm{d}\nu = \int_{\mathbb{B}} p_k \,\mathrm{d}\nu + \int_{\mathbb{S}} p_k \,\mathrm{d}\nu \xrightarrow{k} \int_{\mathbb{B}} f \,\mathrm{d}\nu + \int_{\mathbb{S}} r_{\nu|\mathbb{S}}(f) \,\mathrm{d}\nu.$$

The estimate

$$1 = \Big| \int_{\mathbb{B}} f \,\mathrm{d}\nu + \int_{\mathbb{S}} r_{\nu|\mathbb{S}}(f) \,\mathrm{d}\nu \Big| \leqslant \|\nu\| \leqslant 1$$

implies that $\|\nu\| = 1$ and that

$$|\nu|(\mathbb{B}) = \Big| \int_{\mathbb{B}} f \,\mathrm{d}\nu \Big| \leqslant \int_{\mathbb{B}} |f| \,\mathrm{d}|\nu|.$$

We conclude that $|f| = 1$, $|\nu|$-almost everywhere on $\mathbb{B}$. Therefore the assumption that $|\nu|(\mathbb{B}) \neq 0$ would imply that f is a constant function of modulus 1, and hence that $1 = |\langle L, f\rangle| = |L(1)|$, which is not true by hypothesis. ∎

For each positive real number t, we set $B_t = \{z \in \mathbb{C}^n : |z| < t\}$. Define $A_t = \overline{\mathbb{B}} \setminus B_t$ for $0 < t < 1$. Let $\mu \in M(\overline{\mathbb{B}})$ be a Henkin measure, and let

$$r : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}}, \mu)$$

be the contractive w*-continuous algebra homomorphism defined in the preliminaries. We denote by $r_* : \mathcal{Q}(\mu) \to \mathcal{Q}$ the predual of the map r.

COROLLARY 1.2. *Let $\mu \in M^+(\overline{\mathbb{B}})$ be a Henkin measure such that the induced map*

$$r : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}}, \mu)$$

is a dual algebra isomorphism. Let $L \in \mathcal{Q}(\mu)$ be an element with $\|L\| = 1 \neq |L(1)|$, and let (u_k) be a sequence in $L^1(\overline{\mathbb{B}}, \mu)$ with

(i) $\overline{\lim\limits_{k\to\infty}} \|u_k\|_1 \leqslant 1$;

(ii) $\lim\limits_{k\to\infty} [u_k] = L$ *in* $\mathcal{Q}(\mu)$.

Then each measure $\nu \in M(\overline{\mathbb{B}})$ for which there is a subsequence (v_k) of (u_k) with

$$\nu = \mathrm{w}^*\text{-}\lim_{k\to\infty} v_k \mathrm{d}\mu$$

in $M(\overline{\mathbb{B}})$ is a Henkin measure with $|\nu|(\mathbb{B}) = 0$ and $\|\nu\| = 1$. For each t with $0 < t < 1$, we have

$$\lim_{k\to\infty} \int_{B_t} |u_k| \mathrm{d}\mu = 0.$$

Proof. For (v_k) and ν as above, we have

$$\|\nu\| \leqslant \overline{\lim_{k\to\infty}} \|v_k \mathrm{d}\mu\| = \overline{\lim_{k\to\infty}} \|v_k\|_1 \leqslant 1$$

and

$$(r_* L)(f|\mathbb{B}) = \lim_{k\to\infty} \int_{\overline{\mathbb{B}}} f\, u_k \mathrm{d}\mu = \int_{\overline{\mathbb{B}}} f\, \mathrm{d}\nu \quad \big(f \in A(\mathbb{B})\big).$$

Thus the first part of the assertion follows from Lemma 1.1.

Assume that there are real numbers $0 < t < 1$ and $\varepsilon > 0$ such that

$$\int_{B_t} |u_k| \mathrm{d}\mu \geqslant \varepsilon$$

for infinitely many k. By passing to a suitable subsequence, we may suppose that this estimate holds for all k and that the limits

$$\nu = \mathrm{w}^*\text{-}\lim_{k\to\infty} u_k \mathrm{d}\mu, \quad \eta = \mathrm{w}^*\text{-}\lim_{k\to\infty} u_k \chi_{B_t} \mathrm{d}\mu$$

exist in $M(\overline{\mathbb{B}})$. It follows that $|\eta|(\overline{\mathbb{B}} \setminus \overline{B}_t) = 0$, and by the first part, $|\nu|(\mathbb{B}) = 0$. Since $\nu - \eta = \mathrm{w}^*\text{-}\lim_{k\to\infty} u_k \chi_{A_t} \mathrm{d}\mu$ and

$$\|u_k \chi_{A_t} \mathrm{d}\mu\| = \int_{A_t} |u_k| \mathrm{d}\mu = \int_{\overline{\mathbb{B}}} |u_k| \mathrm{d}\mu - \int_{B_t} |u_k| \mathrm{d}\mu \leqslant \|u_k\|_1 - \varepsilon,$$

we obtain the contradiction

$$1 = \|\nu\| \leqslant \|\nu\| + \|\eta\| = \|\nu - \eta\| \leqslant \overline{\lim_{k\to\infty}} (\|u_k\|_1 - \varepsilon) \leqslant 1 - \varepsilon. \quad \blacksquare$$

Let σ be the surface measure on the unit sphere $\mathbb{S} \subset \mathbb{C}^n$, and let $H^\infty(\mathbb{S}) = P^\infty(\mathbb{S}, \sigma)$. Then the canonical map $r : H^\infty(\mathbb{B}) \to H^\infty(\mathbb{S})$ is a dual algebra isomorphism. We shall identify $H^\infty(\mathbb{B})$ with $H^\infty(\mathbb{S})$ and $\mathcal{Q} = L^1(\mathbb{B})/{}^\perp H^\infty(\mathbb{B})$ with $\mathcal{Q}(\sigma) = L^1(\mathbb{S}, \sigma)/{}^\perp H^\infty(\mathbb{S})$.

Lemma 1.3. *The set*

$$\overline{C}\{\mathcal{E}_\lambda : \lambda \in \mathbb{B}\} = \{L \in \mathcal{Q}(\sigma) : \|L\| = L(1) = 1\},$$

where on the left we mean the closed convex hull of the set of all point evaluations $\mathcal{E}_\lambda (\lambda \in \mathbb{B})$, *consists precisely of those elements* $L \in \mathcal{Q}(\sigma)$ *for which there is a Henkin measure* $\nu \in M^+(\mathbb{S})$ *with* $\|\nu\| = 1$ *and*

$$L(f) = \int_{\mathbb{S}} r_\nu(f)\, \mathrm{d}\nu \quad \big(f \in H^\infty(\mathbb{B})\big).$$

Furthermore, the set $\mathcal{M} = \{L \in \mathcal{Q}(\sigma) : \|L\| = |L(1)|\}$ *has no interior points.*

Proof. Define $\mathcal{M}_1 = \{L \in \mathcal{Q}(\sigma) : \|L\| = L(1) = 1\}$. Note that $\mathcal{M} = \mathbb{C}\mathcal{M}_1$. Fix an element $L \in \mathcal{M}_1$ and a sequence (u_k) in $L^1(\mathbb{S}, \sigma)$ with $L = [u_k]$ and $\|u_k\|_1 < 1 + \frac{1}{k}$ $(k \geqslant 1)$. After passing to a subsequence, we may suppose that

$$(u_k \mathrm{d}\sigma) \xrightarrow{\mathrm{w}^*} \nu$$

in $M(\mathbb{S})$. Then $\|\nu\| \leqslant 1$ and

$$L(f|\mathbb{S}) = \lim_{k\to\infty} \int_{\mathbb{S}} u_k f \,\mathrm{d}\sigma = \int_{\mathbb{S}} f \,\mathrm{d}\nu \quad \big(f \in A(\mathbb{B})\big).$$

It follows that $\|\nu\| = 1 = \nu(\mathbb{S})$ and that ν is a positive Henkin measure.

Next we show that each element $L \in \mathcal{Q}(\sigma)$ for which there is a Henkin measure $\nu \in M^+(\mathbb{S})$ with $\|\nu\| = 1$ and

$$L(f) = \int_{\mathbb{S}} r_\nu(f) \,\mathrm{d}\nu \quad \big(f \in H^\infty(\mathbb{B})\big)$$

belongs to $\overline{C}\{\mathcal{E}_\lambda : \lambda \in \mathbb{B}\}$. Otherwise, the separation theorem would yield a function $g \in H^\infty(\mathbb{B})$ and a real number α with

$$\sup_{\lambda\in\mathbb{B}} \operatorname{Re} g(\lambda) < \alpha < \operatorname{Re}\left(\int_{\mathbb{S}} r_\nu(g) \,\mathrm{d}\nu\right) = \int_{\mathbb{S}} \operatorname{Re} r_\nu(g) \,\mathrm{d}\nu.$$

By adding a sufficiently large positive constant to g if necessary, we may suppose that $\operatorname{Re} r_\nu(g) \geqslant 0$ ν-almost everywhere. Since $r_\nu : H^\infty(\mathbb{B}) \to L^\infty(\mathbb{S}, \nu)$ is a contractive unital homomorphism of Banach algebras, it follows that

$$\begin{aligned}\exp\big(\|\operatorname{Re} r_\nu(g)\|_{\infty,\nu}\big) &= \|\exp\big(r_\nu(g)\big)\|_{\infty,\nu} = \|r_\nu\big(\exp(g)\big)\|_{\infty,\nu} \\ &\leqslant \|\exp(g)\|_{\infty,\mathbb{B}} = \exp\Big(\sup_{\lambda\in\mathbb{B}} \operatorname{Re} g(\lambda)\Big).\end{aligned}$$

Thus we would obtain the contradiction that

$$\sup_{\lambda\in\mathbb{B}} \operatorname{Re} g(\lambda) < \alpha < \sup_{\lambda\in\mathbb{B}} \operatorname{Re} g(\lambda).$$

To conclude the proof, let us assume that the set $\mathcal{M} = \mathbb{C}\mathcal{M}_1$ has an interior point. Then there would be an element $L \in \operatorname{Int}(\mathcal{M}) \cap \mathcal{M}_1$. Let $g \in L^1(\mathbb{S}, \sigma)$ be a function with $\int_{\mathbb{S}} g \,\mathrm{d}\sigma = 0$. Then there is a real number $t_g \neq 0$ and an element $L_g \in \mathcal{M}_1$ such that

$$L + t_g[g] = \alpha_g L_g$$

for a suitable $\alpha_g \in \mathbb{C}$. By applying both sides to the function $1 \in H^\infty(\mathbb{S})$, one obtains that $\alpha_g = 1$. Hence

$$[g] = \frac{L_g - L}{t_g},$$

and there is a real Henkin measure $\nu \in M(\mathbb{S})$ with $\nu(\mathbb{S}) = 0$ and

$$\int_{\mathbb{S}} fg \,\mathrm{d}\sigma = \int_{\mathbb{S}} f \,\mathrm{d}\nu \quad \bigl(f \in A(\mathbb{B})\bigr).$$

Note that, for each $g \in L^1(\mathbb{S}, \sigma)$,

$$g \,\mathrm{d}\sigma = \Biggl(g - \int_{\mathbb{S}} g \,\mathrm{d}\sigma\Biggr) \mathrm{d}\sigma + \Biggl(\int_{\mathbb{S}} g \,\mathrm{d}\sigma\Biggr)\sigma.$$

Hence by Valskii's theorem (Theorem 9.2.1 in [16]), we could conclude that each Henkin measure $\mu \in M(\mathbb{S})$ is contained in

$$\{\nu : \nu \in M(\mathbb{S}) \text{ is a real Henkin measure with } \nu(\mathbb{S}) = 0\} + A(\mathbb{B})^\perp + \mathbb{C}\,\sigma.$$

Hence for each Henkin measure $\mu \in M(\mathbb{S})$,

$$\operatorname{Im}\mu \in \operatorname{Im}\bigl(A(\mathbb{S})^\perp\bigr) + \mathbb{R}\,\sigma,$$

$$\operatorname{Re}\mu = \operatorname{Im}(\mathrm{i}\mu) \in \operatorname{Im}\bigl(A(\mathbb{S})^\perp\bigr) + \mathbb{R}\,\sigma = \operatorname{Re}\bigl(A(\mathbb{S})^\perp\bigr) + \mathbb{R}\,\sigma.$$

But then each Henkin measure would be of the form

$$\mu = \operatorname{Re}\mu + \mathrm{i}\operatorname{Im}\mu \in \nu_1 + \overline{\nu}_2 + \mathbb{C}\,\sigma$$

with suitable measures $\nu_1, \nu_2 \in A(\mathbb{S})^\perp$. Here $\overline{\nu}_2$ is the complex conjugate of the measure ν_2. Since $\overline{A(\mathbb{S})^\perp} = \overline{A(\mathbb{S})}^\perp$ and since $A(\mathbb{S})^\perp + \mathbb{C}\,\sigma = A_0(\mathbb{S})^\perp$, where $A_0(\mathbb{S}) = \{f \in A(\mathbb{S}) : f(0) = 0\}$, we would obtain that

$$\mathrm{HM}(\mathbb{S}) = A_0(\mathbb{S})^\perp + \overline{A(\mathbb{S})}^\perp.$$

Note that $\overline{A(\mathbb{S})}^\perp = \overline{A(\mathbb{S})^\perp} \subset \mathrm{HM}(\mathbb{S})$ (for instance by Henkin's theorem). Since $\mathrm{HM}(\mathbb{S})$ is a closed subspace of $M(\mathbb{S})$, this implies that $A_0(\mathbb{S}) + \overline{A(\mathbb{S})}$ is a closed subspace of $C(\mathbb{S})$ and that

$$M(\mathbb{S}) = \bigl(A_0(\mathbb{S}) \cap \overline{A(\mathbb{S})}\bigr)^\perp = A_0(\mathbb{S})^\perp + \overline{A(\mathbb{S})}^\perp = \mathrm{HM}(\mathbb{S}).$$

This contradiction completes the proof. ∎

The following result is well known in the one-dimensional case (see Lemma V.4.3 in [8]). The extension to the multidimensional case is straightforward. We give the details for the convenience of the reader.

LEMMA 1.4. *Let* $K \subset \mathbb{C}^n$ *be a compact set, and let* $\mu \in M^+(K)$. *Fix a positive integer* $k \geqslant 1$ *and set* $p = 2k+1$ *and* $q = p/2k$. *For each* $h \in L^q(\mu)$ *with* $\|h\|_q = \sup\Big\{|\int\limits_K fh\,\mathrm{d}\mu| : f \in P^p(\mu)$ *with* $\|f\|_p \leqslant 1\Big\}$, *there is a function* $x \in P^2(\mu)$ *such that* $|h| = |x|^2$ μ*-almost everywhere.*

Proof. We may and shall suppose that $\|h\|_q = 1$. Since the closed unit ball of $P^p(\mu)$ is weakly compact, there is a function u of norm one in $P^p(\mu)$ with

$$1 = \int\limits_K uh\,\mathrm{d}\mu = \|u\|_p\|h\|_q = 1.$$

The above equality (in Hölder's inequality) can only occur if $|u|^p = |h|^q$ μ-almost everywhere (p. 190 in [12]). Using the concrete values of p and q, we obtain that $|u|^{2k} = |h|$ μ-almost everywhere.

Let us define $x = u^k \in L^2(\mu)$. To check that $x \in P^2(\mu)$, choose a sequence (p_j) of polynomials with $\lim\limits_{j\to\infty} \|p_j - u\|_{2k+1} = 0$. Using Hölder's inequality, one obtains

$$\begin{aligned}\int\limits_K |u^k - p_j^k|^2\,\mathrm{d}\mu &= \int\limits_K |u-p_j|^2 \Big|\sum_{i=0}^{k-1} u^i p_j^{k-1-i}\Big|^2 \mathrm{d}\mu \\ &\leqslant \Big(\int\limits_K |u-p_j|^{2k+1}\mathrm{d}\mu\Big)^{\frac{2}{2k+1}} \Big(\int\limits_K \Big|\sum_{i=0}^{k-1} u^i p_j^{k-1-i}\Big|^{2\frac{2k+1}{2k-1}} \mathrm{d}\mu\Big)^{\frac{2k-1}{2k+1}}.\end{aligned}$$

To see that the second factor is bounded in j, use Hölder's inequality and the observation that, for $i = 0, \dots, k-1$, with $s = \frac{2k+1}{2k-1}$

$$|u^i|^{2s} \in L^{\frac{2k-1}{2i}}(\mu) \quad (= L^\infty(\mu) \text{ for } i = 0),$$
$$|p_j^{k-1-i}|^{2s} \in L^{\frac{2k-1}{2(k-i)-1}}(\mu)$$

and that $2k+1 \geqslant \frac{2(k-1-i)(2k+1)}{2(k-i)-1}$. ∎

For the rest of this section, we shall suppose that $\mu \in M^+(\overline{\mathbb{B}})$ is a Henkin measure with $\|\mu\| = 1$ and such that the canonical map

$$r : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}}, \mu)$$

is a dual algebra isomorphism. As before we set $\mathcal{Q}(\mu) = L^1(\mu)/{}^\perp P^\infty(\mu)$. The map r is the adjoint of an isometric isomorphism $r_* : \mathcal{Q}(\mu) \to \mathcal{Q}$. We use the preceding result to show that each element in $\mathcal{Q}$ can almost be factorized.

LEMMA 1.5. *Let $\delta, t > 0$ be real numbers with $0 < t < 1$. Let $L \in \mathcal{Q}(\mu)$ be given with $\|L\| < \delta^2$. For any given $\varepsilon > 0$ and any given functions $g_1, \dots, g_l$, $h_1, \dots, h_l \in L^2(\mu)$ ($l \geqslant 1$ arbitrary), there are functions x in $P^2(\mu)$ and y in $L^2(\mu|A_t)$ with:*

(i) $\|x\| < \delta$, $\|y\| < \delta$;

(ii) $\|L - x \otimes y\| < \varepsilon$;

(iii) $\max\limits_{i=1,\dots,l} \|y\, g_i \chi_{\mathbb{B}}\|_1 < \varepsilon$, $\max\limits_{i=1,\dots,l} \|x\, h_i \chi_{\mathbb{B}}\|_1 < \varepsilon$.

Proof. Since by Lemma 1.3 the set

$$r_*\{L \in \mathcal{Q}(\mu) : \|L\| = |L(1)|\} = \{L \in \mathcal{Q} : \|L\| = |L(1)|\}$$

has no interior points, we may suppose that $\|L\| \neq |L(1)|$.

We choose a sequence (v_k) in $L^\infty(\mu)$ with $0 < \|v_k\|_1 < \min(\delta^2, \|L\| + \frac{1}{k})$ and $L = \lim\limits_{k\to\infty} [v_k]$ in $\mathcal{Q}(\mu)$. Let us fix an integer $k \geqslant 1$. Since

$$\lim_{q\downarrow 1} \|v_k\|_q = \|v_k\|_1,$$

we can choose an odd integer $p \geqslant 3$ (depending on k) such that the conjugate exponent q satisfies

$$\|v_k\|_q < \min\left(\delta^2, \|L\| + \frac{1}{k}\right).$$

By Hahn-Banach Theorem, there is a function $u_k \in L^q(\mu)$ with $u_k - v_k \in P^p(\mu)^\perp$ and

$$\|u_k\|_q = \sup\Big\{\Big|\int\limits_{\overline{\mathbb{B}}} f\, v_k \mathrm{d}\mu\Big| : f \in P^p(\mu) \text{ and } \|f\|_p \leqslant 1\Big\}.$$

By Lemma 1.4 there is a function $x_k \in P^2(\mu)$ with $|x_k|^2 = |u_k|$ μ-almost everywhere. Since $u_k \in L^1(\mu)$ with $[u_k] = [v_k]$ and since

$$\|u_k\|_1 \leqslant \|u_k\|_q \leqslant \|v_k\|_q,$$

we have $\overline{\lim\limits_{k\to\infty}} \|u_k\|_1 \leqslant \|L\|$, and Corollary 1.2 yields that

$$\lim_{k\to\infty} \int\limits_{B_s} |u_k| \mathrm{d}\mu = 0 \quad (0 < s < 1).$$

The measurable function $y_k : \overline{\mathbb{B}} \to \mathbb{C}$ defined by setting

$$y_k(z) = \chi_{A_t}(z)\big(\overline{u}_k(z)/\overline{x}_k(z)\big)$$

if $x_k(z) \neq 0$ and $y_k(z) = 0$ otherwise, satisfies $|y_k|^2 = |u_k|$ μ-almost everywhere on A_t. Hence $x_k \in P^2(\mu)$, $y_k \in L^2(\mu|A_t)$ with $\max(\|x_k\|, \|y_k\|) < \delta$. Because of

$$|\langle [u_k] - x_k \otimes y_k, f\rangle| = \left| \int\limits_{\overline{\mathbb{B}}} u_k f \,\mathrm{d}\mu - \int\limits_{\overline{\mathbb{B}}} x_k \overline{y}_k f \,\mathrm{d}\mu \right| \leqslant \int\limits_{B_t} |u_k| \mathrm{d}\mu \;\|f\|_{\infty,\mu}$$

for all $f \in P^\infty(\mu)$, it follows that $\|L - x_k \otimes y_k\| < \varepsilon$ for k sufficiently large.

Fix $\eta > 0$ with $\delta\eta < \frac{\varepsilon}{2}$ and choose a real number s with $0 < s < 1$ such that

$$\int\limits_{\mathbb{B}\setminus B_s} |h_i|^2 \mathrm{d}\mu < \eta^2, \qquad \int\limits_{\mathbb{B}\setminus B_s} |g_i|^2 \mathrm{d}\mu < \eta^2$$

for $i = 1, \ldots, l$. Then for k sufficiently large, the estimates

$$\begin{aligned} \|x_k h_i \chi_{\mathbb{B}}\|_1 &= \int\limits_{B_s} |x_k|\,|h_i| \mathrm{d}\mu + \int\limits_{\mathbb{B}\setminus B_s} |x_k|\,|h_i| \mathrm{d}\mu \\ &\leqslant \|h_i\|_2 \Bigg(\int\limits_{B_s} |u_k| \mathrm{d}\mu \Bigg)^{\frac{1}{2}} + \|x_s\|_2 \Bigg(\int\limits_{\mathbb{B}\setminus B_s} |h_i|^2 \mathrm{d}\mu \Bigg)^{\frac{1}{2}} < \varepsilon, \end{aligned}$$

$$\|y_k g_i \chi_{\mathbb{B}}\|_1 \leqslant \|g_i\|_2 \Bigg(\int\limits_{B_s} |u_k| \mathrm{d}\mu \Bigg)^{\frac{1}{2}} + \|y_k\|_2 \Bigg(\int\limits_{\mathbb{B}\setminus B_s} |g_i|^2 \mathrm{d}\mu \Bigg)^{\frac{1}{2}} < \varepsilon$$

hold for $i = 1, \ldots, l$. ∎

To improve the preceding almost factorization result, we need to know more about the possible boundary values of functions in $A(\mathbb{B})$.

Lemma 1.6. *Let $\kappa : \mathbb{S} \to \mathbb{R}$ be a Borel measurable function such that $c \leqslant \kappa \leqslant d$, where $c, d > 0$ are given real numbers. For any finite positive Borel measure ν on $\mathbb{S}$ and any real number $\varepsilon > 0$, there is a function $g \in A(\mathbb{B})$ with $|g| \leqslant d$ on $\overline{\mathbb{B}}$ and*

$$\nu\big(\{z \in \mathbb{S} : \kappa(z) \neq |g(z)|\}\big) < \varepsilon.$$

Proof. By Lusin's theorem (p. 227 in [7]) there is a real-valued continuous function $p : \mathbb{S} \to \mathbb{R}$ with $p \leqslant d$ and

$$\nu\big(\{z \in \mathbb{S} : \kappa(z) \neq p(z)\}\big) < \frac{\varepsilon}{2}.$$

Replacing p by $\max(p, c)$ if necessary, we may suppose that $c \leqslant p \leqslant d$. Choose a positive real number α with $\alpha\, p > 2$ on $\mathbb{S}$. Then there is a function $h \in A(\mathbb{B})$ (Theorem 15.2 in [17]) with $\operatorname{Re} h \leqslant \log(\alpha\, p)$ on $\mathbb{S}$ and

$$\nu\big(\{z \in \mathbb{S} : \log(\alpha\, p)(z) \neq \operatorname{Re} h(z)\}\big) < \frac{\varepsilon}{2}.$$

But then $g = \frac{\mathrm{e}^h}{\alpha}$ satisfies all the required conditions. ∎

The next result is the main tool to prove property $(\mathbb{A}_1)$ for subnormal tuples with rich spectrum in the unit ball. To prove it we modify corresponding ideas from [8] (Chapter VII) and [19].

Lemma 1.7. *Let t, δ and ε be positive real numbers with $t < 1$ and $\delta < \frac{1}{3}$. If $L \in \mathcal{Q}(\mu)$ and $a \in L^2(\mu)$, $b \in L^2(\mu|A_t)$ satisfy*

$$\|L - a \otimes b\| < \delta^4,$$

and if $h_1, \ldots, h_r \in L^2(\mu)$ are given functions, then there are functions $x \in P^2(\mu)$ and $y \in L^2(\mu|A_t)$, and a Borel set $Z \subset \mathbb{S}$ of measure $\mu(Z) < \varepsilon$ with:

(i) $\|L - (a + x) \otimes (b + y)\| < \varepsilon$;

(ii) $\|x\| < 3\delta$, $\|y\chi_{\mathbb{B}}\| < \delta^2$;

(iii) $\|(b + y)\chi_{\mathbb{S}}\| < \delta^2 + \frac{\|b\chi_{\mathbb{S}}\|}{1-2\delta}$, $\|b + y\| < \delta^2 + \frac{\|b\|}{1-2\delta}$;

(iv) $|a + x| \geqslant (1 - 2\delta)|a|$ *μ-almost everywhere on* $\mathbb{S} \setminus Z$;

(v) $\|x \otimes (h_j\chi_{\mathbb{B}})\| < \varepsilon$ *for* $j = 1, \ldots, r$.

Proof. By Lemma 1.5 there are functions $u \in P^2(\mu)$ and $v \in L^2(\mu|A_t)$ with $\|u\| < \delta^2$, $\|v\| < \delta^2$ and

$$\begin{aligned} &\|L - a \otimes b - u \otimes v\| < \frac{\varepsilon}{6}, \\ &\|u \otimes (b\chi_{\mathbb{B}})\| < \frac{\varepsilon}{6}, \quad \|a \otimes (v\chi_{\mathbb{B}})\| < \frac{\varepsilon}{6}, \\ &\|u \otimes (h_j\chi_{\mathbb{B}})\| < \frac{\varepsilon}{2} \quad (j = 1, \ldots, r). \end{aligned}$$

Choose a constant $\eta > 0$ with $\eta < \varepsilon$ such that

$$\int\limits_Z \Big(|uv| + \Big(1 + \frac{2}{\delta}\Big)|ub|\Big)\mathrm{d}\mu < \frac{\varepsilon}{6}$$

for each Borel set $Z \subset \mathbb{S}$ with $\mu(Z) < \eta$. Define $\kappa : \mathbb{S} \to \mathbb{R}$ by $\kappa(z) = \frac{2}{\delta}$ if $|a(z)| \leqslant |u(z)|/\delta$ and $\kappa(z) = 1$ otherwise. Here (as in all similar situations) $a(z)$ has to be understood as the value of a fixed representative of the equivalence class $a \in L^2(\mu)$. By Lemma 1.6 there is a function $g \in A(\mathbb{B})$ with $|g| \leqslant \frac{2}{\delta}$ on $\overline{\mathbb{B}}$ such that

$$Z_1 = \{z \in \mathbb{S} : |g(z)| \neq \kappa(z)\}$$

is a Borel set with $\mu(Z_1) < \eta/2$. By Theorem 3.5 in [17] there is a Montel sequence (p_i) in $A(\mathbb{B})$ with $|p_i| < 1$ on $\overline{\mathbb{B}}$ such that $\big(|p_i(z)|\big) \xrightarrow{i} 1$ μ-almost everywhere on $\mathbb{S}$. Egoroff's theorem (Proposition 3.1.3 in [7]) allows us to choose a Borel set $Z_2 \subset \mathbb{S}$ with $\mu(Z_2) < \eta/2$ such that $(|p_i|) \xrightarrow{i} 1$ uniformly on $\mathbb{S} \setminus Z_2$. Then $Z = Z_1 \cup Z_2$ is a measurable subset of $\mathbb{S}$ with $\mu(Z) < \eta$.

We fix a natural number j such that $1 - |p_j| < \delta/2$ on $\mathbb{S} \setminus Z_2$ and such that $f = p_j g \in A(\mathbb{B})$ satisfies

$$\|uf\chi_{\mathbb{B}}\| < \frac{\varepsilon}{6(\|h_j\| + \|b\| + \delta^2)} \quad (j = 1, \dots, r).$$

Then $x = (1 + f)u \in P^2(\mu)$ satisfies $\|x\| < 3\delta$. On $S_1 = \{z \in \mathbb{S} \setminus Z : |a(z)| \leqslant |u(z)|/\delta\}$

$$|a + u| \leqslant \left(1 + \frac{1}{\delta}\right)|u|, \quad |uf| \geqslant \left(1 - \frac{\delta}{2}\right)\frac{2}{\delta}|u| = \frac{2-\delta}{\delta}|u|,$$

and hence on the same set we have

$$|a + x| = |uf + a + u| \geqslant \left(\frac{1}{\delta} - 2\right)|u| \geqslant |u|.$$

On $S_1 \cap \{z \in \mathbb{S} : a(z) \neq 0\}$ this gives the estimate

$$\left|\frac{a+x}{a}\right| = \left|\frac{a+x}{u}\right|\left|\frac{u}{a}\right| \geqslant 1 - 2\delta.$$

Therefore on S_1 we obtain that $|a + x| \geqslant (1 - 2\delta)|a|$.

On the set $S_2 = \{z \in \mathbb{S} : |a(z)| > |u(z)|/\delta\} \cap (\mathbb{S} \setminus Z_1)$ we have $|x| \leqslant 2|u|$, and hence $|a + x| \geqslant (\frac{1}{\delta} - 2)|u| \geqslant |u|$. Furthermore, because of

$$|a| \leqslant |a + x| + |x| \leqslant |a + x| + 2|u| \leqslant |a + x| + 2\delta|a|$$

it follows that $|a + x| \geqslant (1 - 2\delta)|a|$ on S_2.

Combining these two estimates we obtain on $\mathbb{S} \setminus Z$

$$|a + x| \geqslant |u| \quad \text{and} \quad |a + x| \geqslant (1 - 2\delta)|a|.$$

Define a function $w \in L^2(\mu)$ by setting

$$w = \frac{\overline{u}}{\overline{a} + \overline{x}}\big(v - (1 + \overline{f})b\big)$$

on $W = (\mathbb{S} \setminus Z) \cap \{z \in \mathbb{S} : a(z) + x(z) \neq 0\}$, and $w = 0$ elsewhere. Set

$$y = v\chi_{\mathbb{B}} + w\chi_{\mathbb{S}} \in L^2(\mu|A_t).$$

The function $y+b \in L^2(\mu)$ satisfies the right estimate, since

$$\begin{aligned}
\|y+b\|^2 &= \int\limits_{\mathbb{B}} |v+b|^2 \,\mathrm{d}\mu + \int\limits_{W} \Big|\frac{\overline{u}}{\overline{a}+\overline{x}}v + \frac{\overline{a}}{\overline{a}+\overline{x}}b\Big|^2 \,\mathrm{d}\mu + \int\limits_{\mathbb{S}\setminus W} |b|^2 \,\mathrm{d}\mu \\
&= \int\limits_{\mathbb{B}} |v|^2 \,\mathrm{d}\mu + 2\mathrm{Re} \int\limits_{\mathbb{B}} v\bar{b} \,\mathrm{d}\mu + \int\limits_{\mathbb{B}} |b|^2 \,\mathrm{d}\mu + \int\limits_{\mathbb{S}\setminus W} |b|^2 \,\mathrm{d}\mu \\
&\quad + \int\limits_{W} |v|^2 \Big|\frac{u}{a+x}\Big|^2 \,\mathrm{d}\mu + 2\mathrm{Re} \int\limits_{W} v\bar{b}\frac{\overline{u}a}{|a+x|^2} \,\mathrm{d}\mu + \int\limits_{W} \Big|\frac{a}{a+x}\Big|^2 |b|^2 \,\mathrm{d}\mu \\
&\leqslant \Big(\|v\| + \frac{\|b\|}{1-2\delta}\Big)^2.
\end{aligned}$$

A similar estimate shows that $\|(y+b)\chi_{\mathbb{S}}\| < \delta^2 + \frac{\|b\chi_{\mathbb{S}}\|}{1-2\delta}$.

Our choices imply that

$$\begin{aligned}
&\|x \otimes h_j\chi_{\mathbb{B}}\| \leqslant \|u \otimes h_j\chi_{\mathbb{B}}\| + \|uf \otimes h_j\chi_{\mathbb{B}}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{6} < \varepsilon, \\
&\|x \otimes b\chi_{\mathbb{B}}\| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
\end{aligned}$$

We still have to estimate the norm of

$$L - (a+x)\otimes(b+y) = L - a\otimes b - x\otimes y - a\otimes y - x\otimes b.$$

For this purpose, write

$$x\otimes y = u\otimes(v\chi_{\mathbb{B}} + w\chi_{\mathbb{S}}) + (uf)\otimes(v\chi_{\mathbb{B}} + w\chi_{\mathbb{S}}) = u\otimes v + (uf)\otimes(v\chi_{\mathbb{B}}) + z,$$

where $z = u\otimes(-v\chi_{\mathbb{S}} + w\chi_{\mathbb{S}}) + (uf)\otimes w\chi_{\mathbb{S}}$. This gives

$$\begin{aligned}
L - (a+x)\otimes(b+y) &= (L - a\otimes b - u\otimes v) - (uf)\otimes(v\chi_{\mathbb{B}}) - a\otimes(v\chi_{\mathbb{B}}) \\
&\quad - x\otimes(b\chi_{\mathbb{B}}) - \big(z + a\otimes(w\chi_{\mathbb{S}}) + x\otimes(b\chi_{\mathbb{S}})\big).
\end{aligned}$$

Observe that, for $\varphi \in P^\infty(\mu)$,

$$\begin{aligned}
\big(z + a\otimes(w\chi_{\mathbb{S}}) + x\otimes(b\chi_{\mathbb{S}})\big)(\varphi) &= \int\limits_{\mathbb{S}} \varphi(-u\overline{v} + u\overline{w} + uf\overline{w} + a\overline{w} + x\overline{b}) \,\mathrm{d}\mu \\
&= \int\limits_{\mathbb{S}} \varphi\big((a+x)\overline{w} - u\overline{v} + x\overline{b}\big) \mathrm{d}\mu \\
&= \int\limits_{Z} \varphi(u\overline{b} + uf\overline{b} - u\overline{v}) \,\mathrm{d}\mu.
\end{aligned}$$

Hence

$$\|z + a\otimes(w\chi_{\mathbb{S}}) + x\otimes(b\chi_{\mathbb{S}})\| \leqslant \int\limits_{Z} |uv| + \Big(1 + \frac{2}{\delta}\Big)|ub| \mathrm{d}\mu < \frac{\varepsilon}{6},$$

and therefore $\|L - (a+x)\otimes(b+y)\| < \varepsilon$. ∎

The preceding result can be used to show that the dual algebra generated by the multiplication tuple $M_z \in \mathcal{L}\big(P^2(\mu)\big)^n$ satisfies property $(\mathbb{A}_1)$ whenever the measure $\mu \in M^+(\overline{\mathbb{B}})$ is a Henkin probability measure for which the canonical map $r = r(\mu) : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}}, \mu)$ is a dual algebra isomorphism.

THEOREM 1.8. *There is a universal constant $C > 0$ such that, for each element $L \in \mathcal{Q}(\mu)$ and any given functions $a, b \in P^2(\mu)$, there are $x, y \in P^2(\mu)$ with $L = x \otimes y$ and*

$$\|x - a\| \leqslant C\|L - a \otimes b\|^{\frac{1}{2}}, \quad \|y\| \leqslant C(\|L - a \otimes b\|^{\frac{1}{2}} + \|b\|).$$

Proof. Define $d = \|L - a \otimes b\|$. Without loss of generality we may suppose that $d > 0$. Set $\gamma = 1/16\sqrt{d}$, $L_1 = \gamma^2 L$, and $x_0 = \gamma a$, $y_0 = \gamma b$. ∎

Choose a sequence $(\varepsilon_k)_{k\geqslant 1}$ of positive real numbers with $1/4 < \varepsilon_1 < 1/3$, $\varepsilon_{k+1} < \varepsilon_k$ $(k \geqslant 1)$, and $\sum\limits_{k=1}^{\infty} \varepsilon_k < \infty$. Because of

$$\Big|\frac{1}{1 - 2\varepsilon_k} - 1\Big| = \frac{2\varepsilon_k}{1 - 2\varepsilon_k} < 6\varepsilon_k \quad (k \geqslant 1)$$

the product $\prod\limits_{k=1}^{\infty} \big(1/(1 - 2\varepsilon_k)\big)$ converges to a positive real number R. Define

$$\rho = \max\Big\{3\sum_{k=1}^{\infty} \varepsilon_k,\ R\Big(\sum_{k=1}^{\infty} \varepsilon_k^2\Big)\Big\}.$$

Since $\|L_1 - x_0 \otimes y_0\| < \varepsilon_1^4$, an inductive application of Lemma 1.7 yields sequences $(x_k)_{k\geqslant 1}$ in $P^2(\mu)$ and $(y_k)_{k\geqslant 1}$ in $L^2(\mu)$ with

$$\|L_1 - x_k \otimes y_k\| < \varepsilon_{k+1}^4,$$

$$\|x_{k+1} - x_k\| < 3\varepsilon_{k+1}, \quad \|y_{k+1}\| < \varepsilon_{k+1}^2 + \frac{\|y_k\|}{1 - 2\varepsilon_{k+1}}$$

for all $k \geqslant 0$. Then $x = \lim\limits_{k\to\infty} x_k \in P^2(\mu)$ exists and $\|x - x_0\| < \rho$. Because of the estimates

$$\begin{aligned}
\|y_k\| &< \varepsilon_k^2 + \frac{\|y_{k-1}\|}{1 - 2\varepsilon_k} \\
&< \varepsilon_k^2 + \frac{\varepsilon_{k-1}^2}{1 - 2\varepsilon_k} + \frac{\|y_{k-2}\|}{(1 - 2\varepsilon_{k-1})(1 - 2\varepsilon_k)} \\
&< \cdots \\
&< \rho + R\|y_0\|,
\end{aligned}$$

the sequence (y_k) is bounded. Hence there is a subsequence of (y_k) that converges weakly to some function $y \in L^2(\mu)$ with $\|y\| \leqslant \rho + R\|y_0\|$.

To conclude the proof it suffices to observe that

$$L = (x/\gamma) \otimes P(y/\gamma),$$

where P is the orthogonal projection from $L^2(\mu)$ onto $P^2(\mu)$ and

$$\|(x/\gamma) - a\| \leqslant 16\rho\sqrt{d}, \quad \|y/\gamma\| \leqslant 16\rho\sqrt{d} + R\|b\|. \quad \blacksquare$$

Let G be a bounded open subset of $\mathbb{C}^n$, and let $T = (T_1, \ldots, T_n) \in \mathcal{L}(H)^n$ be a subnormal tuple such that T possesses an isometric and w*-continuous functional calculus

$$\Phi : P^\infty(G) \to \mathcal{L}(H).$$

Denote by $N \in \mathcal{L}(K)^n$ the minimal normal extension of T defined on a larger Hilbert space $K \supset H$. We define $X = \sigma(N)$ and we fix a scalar-valued spectral measure $\mu_N \in M(X)$ of N. There is an isometric and w*-continuous isomorphism of von Neumann algebras

$$\Psi : L^\infty(X, \mu_N) \to W^*(N),$$

where $W^*(N)$ is the von Neumann algebra generated by $N_1, \ldots, N_n$ in $\mathcal{L}(H)$. The set

$$\mathcal{W} = \{f \in L^\infty(X, \mu_N) : \Psi(f)H \subset H\}$$

is a w*-closed subalgebra of $L^\infty(X, \mu_N)$ containing all polynomials. A standard argument (cf. Corollary II.2.17 in [8]) shows that the induced w*-continuous algebra homomorphism

$$\Psi_0 : \mathcal{W} \to \mathcal{L}(H), \quad f \mapsto \Psi(f)|H$$

is isometric again. Hence this map induces a dual algebra isomorphism

$$\Psi_0 : \mathcal{W} \to \mathcal{W}(T)$$

onto a w*-closed subalgebra $\mathcal{W}(T)$ of $\mathcal{L}(H)$ containing $\mathfrak{A}_T$. But then $P^\infty(X, \mu_N) = \Psi_0^{-1}(\mathfrak{A}_T)$, and Ψ_0 yields the dual algebra isomorphism

$$\Psi_T : P^\infty(X, \mu_N) \to \mathfrak{A}_T, \quad \Psi_T(f) = \Psi(f)|H.$$

The composition

$$\varphi : P^\infty(G) \xrightarrow{\Phi} \mathfrak{A}_T \xrightarrow{\Psi_T^{-1}} P^\infty(X, \mu_N)$$

is a dual algebra isomorphism, and the map

$$\Phi_N : P^\infty(G) \to \mathcal{L}(K), \quad f \mapsto \Psi\big(\varphi(f)\big)$$

is a w*-continuous algebra homomorphism with the property that $\Phi(f) = \Phi_N(f)|H$ for all $f \in P^\infty(G)$. Furthermore, for any two functions $f \in P^\infty(G)$ and $g \in P^\infty(X, \mu_N)$, the identity $\varphi(f) = g$ holds if and only if $\Phi(f) = \Psi(g)|H$.

Our next aim is to show that the dual algebra generated by a subnormal tuple $T \in \mathcal{L}(H)^n$ that possesses an isometric and w*-continuous functional calculus over the unit ball has property $(\mathbb{A}_1)$. To reduce this result to the particular case stated in Theorem 1.8, we use the multidimensional version of a result from [8].

PROPOSITION 1.9. *Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with minimal normal extension $N \in \mathcal{L}(K)^n$. For a given vector $h \in H$ and any given real number $\varepsilon > 0$, there is a separating vector $f \in H$ for N such that $\|f - h\| < \varepsilon$.* ∎

This result can be proved in exactly the same way as in the one-variable case (see Proposition V.17.4 in [8]). We omit the details.

THEOREM 1.10. *There is a constant $R > 0$ such that if $T \in \mathcal{L}(H)^n$ is a subnormal n-tuple with* w**-continuous isometric H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$, then for any functional $L \in \mathcal{Q}_T$ and any given vectors $a, b \in H$, there are vectors $x, y \in H$ with $L = [x \otimes y]$ and*

$$\|x - a\| \leqslant R\|L - [a \otimes b]\|^{\frac{1}{2}}, \quad \|y\| \leqslant R(\|L - [a \otimes b]\|^{\frac{1}{2}} + \|b\|).$$

Proof. Let $C > 0$ be the constant determined in Theorem 1.8. Let T in $\mathcal{L}(H)^n$ be a subnormal tuple as in Theorem 1.10, and let $N \in \mathcal{L}(K)^n$ be its minimal normal extension. Then $\sigma(N) \subset \sigma(T) \subset \overline{\mathbb{B}}$.

Fix elements $L \in \mathcal{Q}_T$ and $a, b \in H$ such that $d = \|L - [a \otimes b]\| > 0$. Choose a real number $\varepsilon > 0$ with

$$C(d + \varepsilon\|b\|)^{\frac{1}{2}} + \varepsilon < (C + 1)d^{\frac{1}{2}}.$$

By Proposition 1.9 there is a separating vector $h \in H$ for N with $\|h - a\| < \varepsilon$. Set $X = \sigma(N)$. Let E be the operator-valued spectral measure for N. Let $\mu \in M(\overline{\mathbb{B}})$ be the trivial extension of the scalar-valued spectral measure

$$\mu_N : \mathcal{B}(X) \to [0, \infty), \quad \mu_N(A) = \langle E(A)h, h\rangle$$

of N determined by h. Here $\mathcal{B}(X)$ denotes the σ-algebra of all Borel sets in X. The restriction map $P^\infty(\mu) \to P^\infty(X, \mu_N)$, $f \mapsto f|X$, is a dual algebra isomorphism which we use to identify the dual algebras $P^\infty(\mu)$ and $P^\infty(X, \mu_N)$.

We apply the remarks following Theorem 1.8 to the case $G = \mathbb{B}$. Using the same notations, we obtain a dual algebra isomorphism

$$\pi : H^\infty(\mathbb{B}) \xrightarrow{\varphi} P^\infty(X, \mu_N) \cong P^\infty(\mu).$$

For each $f \in H^\infty(\mathbb{B})$, the function $\pi(f)$ is the uniquely determined element in $P^\infty(\mu)$ with $\Psi\big(\pi(f)|X\big)|H = \Phi(f)$. In particular, π maps each function $f \in A(\mathbb{B})$ to its equivalence class in $P^\infty(\mu)$.

Since, for each Montel sequence (f_k) in $A(\mathbb{B})$,

$$\int_{\mathbb{S}} f_k \,\mathrm{d}(\mu|\mathbb{S}) = \int_{\overline{\mathbb{B}}} \chi_{\mathbb{S}} f_k \,\mathrm{d}\mu = \langle [\chi_{\mathbb{S}}], \pi(f_k|\mathbb{B})\rangle \xrightarrow{k} 0,$$

the measure μ is a Henkin measure, and π coincides with the canonical w*-continuous contractive algebra homomorphism $r(\mu) : H^\infty(\mathbb{B}) \to P^\infty(\mu)$ associated with the Henkin measure μ (see the preliminaries). Hence we can apply Theorem 1.8 to the measure μ.

Let $H_h = \bigvee(T^k h; k \in \mathbb{N}^n) \in \mathrm{Lat}(T)$ be the cyclic invariant subspace of T generated by h. Because of

$$\|p(T)h\|^2 = \int_X |p|^2 \,\mathrm{d}\langle E(\cdot)h, h\rangle = \int_{\overline{\mathbb{B}}} |p|^2 \,\mathrm{d}\mu \quad \big(p \in \mathbb{C}[z]\big)$$

there is a (unique) unitary operator $U : P^2(\mu) \to H_h$ with $U(p) = p(T)h$ for all polynomials p. Let us denote by $\gamma_* : \mathcal{Q}_T \to \mathcal{Q}(\mu)$ the predual of the dual algebra isomorphism $\gamma = \Phi \circ \pi^{-1} = \Psi_T$.

An elementary exercise shows that

$$\gamma_*\big([U(f) \otimes U(g)]\big) = f \otimes g \quad (f, g \in P^2(\mu)).$$

Let P_h be the orthogonal projection from H onto H_h. Choose functions $\tilde{a}, \tilde{b}$ in $P^2(\mu)$ with $U(\tilde{a}) = P_h a$ and $U(\tilde{b}) = P_h b$. By Theorem 1.8 there are functions $f, g \in P^2(\mu)$ with $\gamma_*(L) = f \otimes g$ and

$$\begin{aligned} \|f - \tilde{a}\| &\leqslant C\|\gamma_*(L) - \tilde{a} \otimes \tilde{b}\|^{\frac{1}{2}}, \\ \|g\| &\leqslant C\big(\|\gamma_*(L) - \tilde{a} \otimes \tilde{b}\|^{\frac{1}{2}} + \|b\|\big). \end{aligned}$$

Define $x = U(f)$ and $y = U(g)$. Then $L = [x \otimes y]$ and

$$\begin{aligned} \|x - a\| &\leqslant \|f - \tilde{a}\| + \|P_h(a) - a\| \\ &\leqslant C\|L - P_h a \otimes P_h b\|^{\frac{1}{2}} + \|(P_h - I)(a - h)\| \\ &\leqslant C(d + \varepsilon\|b\|)^{\frac{1}{2}} + \varepsilon < (C+1)d^{\frac{1}{2}}. \end{aligned}$$

In the same way we obtain that

$$\|y\| \leqslant (C+1)(d^{\frac{1}{2}} + \|b\|).$$

Thus the assertion of Theorem 1.10 holds with $R = C + 1$. ∎

The preceding results can be used to show that each subnormal tuple T in $\mathcal{L}(H)^n$ such that the spectrum of T is contained in the closed unit ball and is dominating in the open unit ball generates a dual algebra of class $(\mathbb{A}_1)$.

Theorem 1.11. *There is a constant $\alpha > 0$ such that if $T \in \mathcal{L}(H)^n$ is a subnormal tuple with $\sigma(T) \subset \overline{\mathbb{B}}$ and such that $\sigma(T)$ is dominating in $\mathbb{B}$, then for any functional $L \in \mathcal{Q}_T$, there are vectors $x, y \in H$ with $\max(\|x\|, \|y\|) \leqslant \alpha\sqrt{\|L\|}$ and*

$$L = [x \otimes y].$$

Proof. Choose a minimal normal extension $N \in \mathcal{L}(K)^n$ of T. Let $\mu \in M(\overline{\mathbb{B}})$ be the trivial extension of a scalar spectral measure for N, as in the preceding proof. The canonical isomorphism of von Neumann algebras $\Psi : L^\infty(\overline{\mathbb{B}}, \mu) \to W^*(N)$ associated with N induces a dual algebra isomorphism (cf. the section following Theorem 1.8)

$$\Phi : P^\infty(\mu) \to \mathfrak{A}_T, \quad f \mapsto \Psi(f)|H.$$

Chaumat's lemma (Lemma V.17.10 in [8]) can be used to show (Proposition VI.1.11 in [8]) that the w*-closed subalgebra $P^\infty(\mu) \subset L^\infty(\overline{\mathbb{B}}, \mu)$ admits a decomposition

$$P^\infty(\mu) = L^\infty(\mu|\Delta_1) \oplus P^\infty(\mu)|\Delta_2,$$

where $\overline{\mathbb{B}} = \Delta_1 \cup \Delta_2$ is a Borel measurable partition of the closed unit ball and the two spaces on the right are regarded as subsets of the space on the left via trivial extension. Furthermore, one can choose Δ_1, Δ_2 in such a way that there is a complex measure $\eta \in M(\overline{\mathbb{B}})$ with

$$\int\limits_{\overline{\mathbb{B}}} f \,\mathrm{d}\eta = 0 \quad \bigl(f \in P^\infty(\mu)\bigr)$$

and such that $|\eta|$ is equivalent to the measure $\mu^{\Delta_2} \in M(\overline{\mathbb{B}})$ defined by

$$\mu^{\Delta_2}(A) = \mu(A \cap \Delta_2).$$

Since, for $f \in A(\mathbb{B})$,

$$\int\limits_{\mathbb{S}} f \,\mathrm{d}(\eta|\mathbb{S}) = -\int\limits_{\mathbb{B}} f \,\mathrm{d}(\eta|\mathbb{B}),$$

the measure $\eta|\mathbb{S}$ is a Henkin measure. By Henkin's theorem (Theorem 9.3.1 in [16]), the measure $|\eta|\mathbb{S}|$ is also a Henkin measure. Hence $\mu^{\Delta_2} \in M(\overline{\mathbb{B}})$ is a Henkin measure.

Since the characteristic functions $\chi_i : \overline{\mathbb{B}} \to \mathbb{C}$ of Δ_i $(i = 1, 2)$ belong to $P^\infty(\mu)$, the orthogonal projections $Q_i = \Psi(\chi_i)$ leave the space H invariant. The orthogonal projections $P_i = Q_i|H = \Phi(\chi_i)$ $(i = 1, 2)$ yield an orthogonal decomposition

$$H = H_1 \oplus H_2, \quad H_i = P_i H \quad (i = 1, 2)$$

which reduces the algebra $\mathfrak{A}_T$. Since

$$\Psi(f)H_1 = \Psi(f)Q_1 H = Q_1\Psi(f)Q_1 H = Q_1\Psi(f\chi_1)H \subset Q_1 H = H_1$$

for all $f \in L^\infty(\overline{\mathbb{B}}, \mu)$, it follows that the space H_1 is reducing for $W^*(N)$.

Since μ^{Δ_2} is a Henkin measure, the composition

$$\Psi_2 : H^\infty(\mathbb{B}) \longrightarrow P^\infty(\mu^{\Delta_2}) \cong \chi_2 P^\infty(\mu) \xrightarrow{\Phi} \mathfrak{A}_T \xrightarrow{\text{rest}} \mathcal{L}(H_2)$$

defines a w*-continuous contractive H^∞-functional calculus for $T|H_2$.

Write $\chi_{\mathbb{B}}, \chi_{\mathbb{S}} : \overline{\mathbb{B}} \to \mathbb{C}$ for the characteristic functions of the sets $\Delta_1 \cap \mathbb{B}$ and $\Delta_1 \cap \mathbb{S}$, respectively. The subspaces $H_{\mathbb{B}} \subset H$, $H_{\mathbb{S}} \subset H$ defined as the images of the projections $P_{\mathbb{B}} = \Psi(\chi_{\mathbb{B}})|H$ and $P_{\mathbb{S}} = \Psi(\chi_{\mathbb{S}})|H$ are reducing for $W^*(N)$. The normal tuple $T|H_{\mathbb{B}} = N|H_{\mathbb{B}}$ possesses the C_{00}-functional calculus

$$\Psi_{\mathbb{B}} : H^\infty(\mathbb{B}) \to \mathcal{L}(H_{\mathbb{B}}), \quad f \mapsto \Psi(\tilde{f}\chi_1)|H_{\mathbb{B}},$$

where $\tilde{f}$ is the trivial extension of f onto $\overline{\mathbb{B}}$.

We claim that

$$\mathfrak{A}_T = W^*(T|H_{\mathbb{S}}) \oplus W^*(T|H_{\mathbb{B}}) \oplus \mathfrak{A}_{(T|H_2)}.$$

Obviously, $\mathfrak{A}_T$ is contained in the direct sum on the right. To prove the opposite inclusion, we first show that each of the three compositions

$$L^\infty(\mu|\Delta_1 \cap \mathbb{S}) \hookrightarrow P^\infty(\mu) \xrightarrow{\Phi} \mathfrak{A}_T \xrightarrow{\text{rest}} W^*(T|H_{\mathbb{S}}),$$

$$L^\infty(\mu|\Delta_1 \cap \mathbb{B}) \hookrightarrow P^\infty(\mu) \xrightarrow{\Phi} \mathfrak{A}_T \xrightarrow{\text{rest}} W^*(T|H_{\mathbb{B}}),$$

$$P^\infty(\mu)|\Delta_2 \hookrightarrow P^\infty(\mu) \xrightarrow{\Phi} \mathfrak{A}_T \xrightarrow{\text{rest}} \mathfrak{A}_{(T|H_2)}$$

is a dual algebra isomorphism. Indeed, the first two maps are isometric, w*-continuous, unital $*$-homomorphisms mapping $(z_1, \ldots, z_n)$ to $T|H_{\mathbb{S}}$ and $T|H_{\mathbb{B}}$, respectively, while the third map is an isometric w*-continuous unital algebra homomorphism mapping the coordinate functions to the components of $T|H_2$. Since

$$P^\infty(\mu) = L^\infty(\mu|\Delta_1 \cap \mathbb{S}) \oplus L^\infty(\mu|\Delta_1 \cap \mathbb{B}) \oplus P^\infty(\mu)|\Delta_2,$$

the reverse inclusion is clear.

Since $\sigma(T|H_{\mathbb{S}})$ is contained in the unit sphere, it follows that $\sigma(T|H_{\mathbb{B}} \oplus H_2)$ is still dominating in $\mathbb{B}$. Hence $T|H_{\mathbb{B}} \oplus H_2$ is subnormal and possesses the w*-continuous isometric H^∞-functional calculus

$$\gamma : H^\infty(\mathbb{B}) \to \mathcal{L}(H_{\mathbb{B}} \oplus H_2), \quad f \mapsto \Psi_{\mathbb{B}}(f) \oplus \Psi_2(f).$$

Let $L \in \mathcal{Q}_T$ be arbitrary. Then the functionals defined by

$$L_0 : W^*(T|H_{\mathbb{S}}) \hookrightarrow \mathfrak{A}_T \xrightarrow{L} \mathbb{C},$$

$$L_1 : \mathfrak{A}_{(T|H_{\mathbb{B}} \oplus H_2)} \hookrightarrow \mathfrak{A}_T \xrightarrow{L} \mathbb{C},$$

are w*-continuous. Since $T|H_{\mathbb{S}}$ is normal, there are vectors $x_0, y_0 \in H_{\mathbb{S}}$ with $\max(\|x_0\|, \|y_0\|) \leqslant \sqrt{\|L_0\|} \leqslant \sqrt{\|L\|}$ such that

$$\langle L_0, A\rangle = \langle Ax_0, y_0\rangle \quad \big(A \in W^*(T|H_{\mathbb{S}})\big).$$

By Theorem 1.10 there are vectors $x_1, y_1 \in H_{\mathbb{B}} \oplus H_2$ with $\max(\|x_1\|, \|y_1\|) \leqslant R\sqrt{\|L_1\|} \leqslant R\sqrt{\|L\|}$ and

$$\langle L_1, A\rangle = \langle Ax_1, y_1\rangle \quad \big(A \in \mathfrak{A}_{(T|H_{\mathbb{B}} \oplus H_2)}\big).$$

Then $x = x_0 + x_1$ and $y = y_0 + y_1$ are vectors with $\max(\|x\|, \|y\|) \leqslant \sqrt{1+R^2}\sqrt{\|L\|}$ and such that

$$\langle L, A\rangle = \langle L_0, A|H_{\mathbb{S}}\rangle + \langle L_1, A|H_{\mathbb{B}} \oplus H_{\mathbb{S}}\rangle = \langle Ax_0, y_0\rangle + \langle Ax_1, y_1\rangle = \langle Ax, y\rangle$$

for all operators $A \in \mathfrak{A}_T$. ∎

In the setting of Theorem 1.10 or Theorem 1.11 the dual algebra $\mathfrak{A}_T$ generated by the subnormal tuple $T \in \mathcal{L}(H)^n$ satisfies the property $(\mathbb{A}_1(\rho))$ with $\rho = 1 + R^2$ in the sense of [3], Definition 2.01. As a well-known consequence (see Proposition 2.09 in [3]) we obtain the following result.

Corollary 1.12. *Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with $\sigma(T) \subset \overline{\mathbb{B}}$. Suppose that $\sigma(T)$ is dominating in $\mathbb{B}$ or that T possesses an isometric* w**-continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$. Then $\mathfrak{A}_T$ coincides with the* WOT*-closed algebra generated by T, and on $\mathfrak{A}_T$ the WOT and* w**-topology coincide.*

2. AN INFINITE FACTORIZATION THEOREM

In this section we show that the dual algebra generated by a subnormal system $T \in \mathcal{L}(H)^n$ with $\sigma(T) \subset \overline{\mathbb{B}}$ and the property that $\sigma(T)$ is dominating in $\mathbb{B}$ has property $(\mathbb{A}_{1,\chi_0})$. As before, let us fix a Henkin measure $\mu \in M^+(\overline{\mathbb{B}})$ with $\|\mu\| = 1$ such that the associated map

$$r : H^\infty(\mathbb{B}) \to P^\infty(\overline{\mathbb{B}}, \mu)$$

is a dual algebra isomorphism.

LEMMA 2.1. *Let $m \geqslant 1$ be an integer, let $L_1, \ldots, L_m \in \mathcal{Q}(\mu) = L^1(\mu)/{}^\perp P^\infty(\mu)$, and let $\varepsilon > 0$, $0 < \delta < \frac{1}{3}$, $0 < alpha_1, \ldots, \alpha_m < 1$, $\rho_1, \ldots, \rho_m > 0$ be given real numbers. Suppose that $a \in L^2(\overline{\mathbb{B}}, \mu)$ and $b_k \in L^2(\mu|A_{\alpha_k})$ $(k = 1, \ldots, m)$ are functions with*

$$\|L_k - a \otimes b_k\| < \rho_k \quad (k = 1, \ldots, m).$$

Then there are functions $x \in P^2(\mu)$ and $y_k \in L^2(\mu|A_{\alpha_k})$ $(k = 1, \ldots, m)$ with

$$\|L_k - (a + x) \otimes y_k\| < \varepsilon,$$
$$\|x\| < \frac{3}{\delta} \sum_{i=1}^m \sqrt{\rho_i},$$
$$\|(y_k - b_k)\chi_{\mathbb{B}}\| < \sqrt{\rho_k}, \quad \|y_k \chi_{\mathbb{S}}\| < \frac{\sqrt{\rho_k}}{(1 - 2\delta)^{m-k}} + \frac{\|b_k \chi_{\mathbb{S}}\|}{(1 - 2\delta)^m}$$

for $k = 1, \ldots, m$.

Proof. For $m = 1$, the result follows easily from Lemma 1.7. Therefore we may assume that $m > 1$.

Define $\mu_k = \mu|A_{\alpha_k}$ $(k = 1, \ldots, m)$ and set

$$\varepsilon_1 = \min_{1 \leqslant k \leqslant m} \frac{\rho_k - \|L_k - a \otimes b_k\|}{2}.$$

Choose $\delta_1 > 0$ such that

$$\int_Z |a\, b_k| \,\mathrm{d}\mu < \varepsilon_1 \quad (k = 1, \ldots, m)$$

for each Borel set $Z \subset \mathbb{S}$ with $\mu(Z) < \delta_1$.

Using again Lemma 1.7 one obtains functions $x_1 \in P^2(\mu)$, $y_1 \in L^2(\mu_1)$, and a Borel set $Z_1 \subset \mathbb{S}$ of measure $\mu(Z_1) < \delta_1$ with

$$\begin{aligned}
&\|L_1 - (a + x_1) \otimes y_1\| < \frac{\varepsilon}{m+1}, \\
&\|x_1\| < \frac{3}{\delta}\sqrt{\rho_1}, \quad \|(y_1 - b_1)\chi_{\mathbb{B}}\| < \sqrt{\rho_1}, \\
&\|y_1\chi_{\mathbb{S}}\| < \sqrt{\rho_1} + \frac{1}{1-2\delta}\,\|b_1\chi_{\mathbb{S}}\|, \\
&\|y_1\| < \sqrt{\rho_1} + \frac{1}{1-2\delta}\,\|b_1\|, \\
&|a + x_1| \geqslant (1-2\delta)|a| \quad \mu\text{-almost everywhere on } \mathbb{S} \setminus Z_1, \\
&\|x_1 \otimes b_k\chi_{\mathbb{B}}\| < \varepsilon_1 \quad (k = 1, \dots, m).
\end{aligned}$$

Define $w_1 \in L^\infty(\overline{\mathbb{B}}, \mu)$ by setting

$$w_1(z) = \frac{\overline{a}(z)}{\overline{a}(z) + \overline{x}_1(z)}$$

for $z \in V_1 = (\mathbb{S} \setminus Z_1) \cap \{z \in \mathbb{S} : a(z) + x_1(z) \neq 0\}$ and $w_1(z) = 0$ otherwise. The functions

$$b_k(1) = b_k\chi_{\mathbb{B}} + b_k w_1 \chi_{\mathbb{S}} \in L^2(\mu_k) \quad (k = 1, \dots, m)$$

satisfy the estimates $\|b_k(1)\| \leqslant \big(1/(1-2\delta)\big)\|b_k\|$ and

$$\|L_k - (a+x_1)\otimes b_k(1)\| \leqslant \|L_k - a\otimes b_k - x_1 \otimes b_k\chi_{\mathbb{B}}\| + \varepsilon_1 < \|L_k - a\otimes b_k\| + 2\varepsilon_1 \leqslant \rho_k$$

for $k = 1, \dots, m$.

Choose a positive real number $\varepsilon_2 \leqslant \varepsilon/(m+1)(m-1)$ with

$$\varepsilon_2 \leqslant \min_{1\leqslant k\leqslant m} \frac{\rho_k - \|L_k - (a + x_1) \otimes b_k(1)\|}{2}.$$

Fix a number $\delta_2 > 0$ such that

$$\int_Z |a + x_1|\,|y_1|\,\mathrm{d}\mu < \varepsilon_2, \quad \int_Z |a + x_1|\,|b_k(1)|\,\mathrm{d}\mu < \varepsilon_2 \quad (k = 1, \dots, m)$$

for each Borel set $Z \subset \mathbb{S}$ with $\mu(Z) < \delta_2$. Repeating the first step, but this time with L_1, a, b_k replaced by L_2, $a + x_1$, $b_k(1)$, we obtain functions $x_2 \in P^2(\mu)$, $y_2 \in L^2(\mu_2)$ and a Borel set $Z_2 \subset \mathbb{S}$ of measure $\mu(Z_2) < \delta_2$ satisfying all the corresponding estimates and such that in addition

$$\|x_2 \otimes y_1\chi_{\mathbb{B}}\| < \frac{\varepsilon}{m+1}.$$

Continuing in this way, one obtains functions $x_1, \dots, x_m \in P^2(\mu)$, and $y_k \in L^2(\mu_k)$ $(k = 1, \dots, m)$ together with Borel sets $Z_1, \dots, Z_m \subset \mathbb{S}$ such that

$$\int_{Z_{k+1}} \Big|a + \sum_{i=1}^{j} x_i\Big|\,|y_j|\,\mathrm{d}\mu < \frac{\varepsilon}{(m-1)(m+1)}$$

for $k = 1, \dots, m-1$ and $j = 1, \dots, k$, and such that

$$\Big\|L_k - \Big(a + \sum_{i=1}^{k} x_i\Big) \otimes y_k\Big\| < \frac{\varepsilon}{m+1},$$

$$\|x_k\| < \frac{3}{\delta}\sqrt{\rho_k}, \quad \|(y_k - b_k)\chi_{\mathbb{B}}\| < \sqrt{\rho_k},$$

$$\|y_k\chi_{\mathbb{S}}\| < \sqrt{\rho_k} + \frac{1}{1-2\delta}\|b_k\, w_1 \cdots w_{k-1}\chi_{\mathbb{S}}\| \leqslant \sqrt{\rho_k} + \Big(\frac{1}{1-2\delta}\Big)^k \|b_k\chi_{\mathbb{S}}\|,$$

$$\|y_k\| < \sqrt{\rho_k} + \frac{1}{1-2\delta}\|b_k(\chi_{\mathbb{B}} + w_1 \cdots w_{k-1}\chi_{\mathbb{S}})\| \leqslant \sqrt{\rho_k} + \Big(\frac{1}{1-2\delta}\Big)^k \|b_k\|,$$

$$\Big|a + \sum_{i=1}^{k} x_i\Big| \geqslant (1-2\delta)\Big|a + \sum_{i=1}^{k-1} x_i\Big| \quad \mu\text{-almost everywhere on } \mathbb{S} \setminus Z_k,$$

$$\|x_k \otimes y_i\chi_{\mathbb{B}}\| < \frac{\varepsilon}{m+1} \quad (i = 1, \dots, k-1)$$

for all $k = 1, \dots, m$. Here the functions $w_k \in L^\infty(\overline{\mathbb{B}}, \mu)$ $(k = 1, \dots, m)$ are given by

$$w_k(z) = \Big(\overline{a}(z) + \sum_{i=1}^{k-1} \overline{x}_i(z)\Big) \Big/ \Big(\overline{a}(z) + \sum_{i=1}^{k} \overline{x}_i(z)\Big)$$

for $z \in V_k = (\mathbb{S} \setminus Z_k) \cap \Big\{z \in \mathbb{S} : \Big(a + \sum_{i=1}^{k} x_i\Big)(z) \neq 0\Big\}$ and $w_k(z) = 0$ for $z \in \overline{\mathbb{B}} \setminus V_k$.

Define $x = \sum_{i=1}^{m} x_i \in P^2(\mu)$, $h_m = y_m \in L^2(\mu_m)$, and

$$h_k = y_k(\chi_{\mathbb{B}} + w_{k+1} \cdots w_m\chi_{\mathbb{S}}) \in L^2(\mu_k) \quad (1 \leqslant k < m).$$

With these definitions we obtain, for $k = 1, \dots, m$,

$$\|h_k\chi_{\mathbb{S}}\| \leqslant \Big(\frac{1}{1-2\delta}\Big)^{m-k} \|y_k\chi_{\mathbb{S}}\| \leqslant \Big(\frac{1}{1-2\delta}\Big)^{m-k} \Big(\sqrt{\rho_k} + \Big(\frac{1}{1-2\delta}\Big)^k \|b_k\chi_{\mathbb{S}}\|\Big),$$

$$\|(h_k - b_k)\chi_{\mathbb{B}}\| = \|(y_k - b_k)\chi_{\mathbb{B}}\| < \sqrt{\rho_k},$$

$$\|x\| < \frac{3}{\delta}\sum_{i=1}^{m} \sqrt{\rho_i}.$$

For $1 \leqslant k < m$ and $Y_k = V_{k+1} \cap \cdots \cap V_m$, it follows that

$$
\begin{aligned}
&\|L_k - (a+x) \otimes h_k\| \\
&\quad = \Big\| L_k - \Big(a + \sum_{i=1}^{k} x_i\Big) \otimes y_k \chi_{\mathbb{B}} - \Big(\sum_{i=k+1}^{m} x_i\Big) \otimes y_k \chi_{\mathbb{B}} \\
&\qquad - \Big(a + \sum_{i=1}^{m} x_i\Big) \otimes y_k w_{k+1} \cdots w_m \chi_{Y_k} \Big\| \\
&\quad = \Big\| L_k - \Big(a + \sum_{i=1}^{k} x_i\Big) \otimes y_k \chi_{\mathbb{B}} - \sum_{i=k+1}^{m} x_i \otimes (y_k \chi_{\mathbb{B}}) \\
&\qquad - \Big(a + \sum_{i=1}^{m} x_i\Big) \otimes \Big(y_k \Big[\Big(\overline{a} + \sum_{i=1}^{k} \overline{x}_i\Big) / \Big(\overline{a} + \sum_{i=1}^{m} \overline{x}_i\Big)\Big] \chi_{Y_k}\Big) \Big\| \\
&\quad \leqslant \Big\| L_k - \Big(a + \sum_{i=1}^{k} x_i\Big) \otimes y_k \Big\| + \sum_{i=k+1}^{m} \|x_i \otimes (y_k \chi_{\mathbb{B}})\| \\
&\qquad + \sup \Big\{ \Big| \int_{\mathbb{S} \setminus Y_k} f \Big(a + \sum_{i=1}^{k} x_i\Big) \overline{y}_k \, \mathrm{d}\mu \Big| : f \in P^\infty(\mu) \text{ with } \|f\| \leqslant 1 \Big\}.
\end{aligned}
$$

Note that $a + \sum\limits_{i=1}^{k} x_i = 0$ μ-almost everywhere on $\big((\mathbb{S} \setminus Z_{k+1}) \cap \cdots \cap (\mathbb{S} \setminus Z_m)\big) \setminus Y_k$. Hence the above supremum can be estimated from above against

$$
\sum_{j=k+1}^{m} \int_{Z_j} \Big| a + \sum_{i=1}^{k} x_i \Big| \, |y_k| \, \mathrm{d}\mu < \frac{\varepsilon}{m+1}.
$$

Thus, for $1 \leqslant k < m$, the estimate

$$
\|L_k - (a+x) \otimes h_k\| < \frac{\varepsilon}{m+1} + (m-k) \frac{\varepsilon}{m+1} + \frac{\varepsilon}{m+1} \leqslant \varepsilon
$$

results, while for $k = m$,

$$
\|L_m - (a+x) \otimes h_m\| < \frac{\varepsilon}{m+1} < \varepsilon.
$$

The corresponding estimates for the functions y_k yield

$$
\begin{aligned}
\|h_k\| &= \|y_k (\chi_{\mathbb{B}} + w_{k+1} \cdots w_m \chi_{\mathbb{S}})\| \\
&\leqslant \|y_k\| \Big(\frac{1}{1-2\delta}\Big)^{m-k} \leqslant \sqrt{\rho_k} \Big(\frac{1}{1-2\delta}\Big)^{m-k} + \Big(\frac{1}{1-2\delta}\Big)^{m} \|b_k\|
\end{aligned}
$$

for $k = 1, \ldots, m$. ∎

Let $\mu \in M^+(\overline{\mathbb{B}})$ be a Henkin measure as in Lemma 2.1. The next result implies that the dual algebra generated by the multiplication tuple $M_z \in \mathcal{L}\big(P^2(\mu)\big)^n$ satisfies property $(\mathbb{A}_{1,\aleph_0})$.

Proposition 2.2. *Let $(L_k)_{k\geqslant 1}$ be a sequence in $\mathcal{Q}(\mu) = L^1(\mu)/{}^\perp P^\infty(\mu)$. Then, for any given $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that, for each $a \in P^2(\mu)$, there are functions $x, y_k \in P^2(\mu)$ $(k \geqslant 1)$ with $\|x - a\| < \varepsilon$ and*

$$L_k = x \otimes y_k, \quad \|y_k\| \leqslant C(\varepsilon) k^{12} \|L_k\| \quad (k \geqslant 1).$$

Proof. Without loss of generality we may suppose that $L_k \neq 0$ for all $k \geqslant 1$.

Choose a real number $\eta > 0$ so small that $12\sqrt{\eta} \sum\limits_{k=1}^{\infty} k^{-2} < \varepsilon$ and define $\delta_k = 1/4k^3$ for $k \geqslant 1$. Since

$$0 \leqslant \frac{1}{1 - 2\delta_k} - 1 \leqslant 4\delta_k \quad (k \geqslant 1)$$

and since $(1+x_1)\cdots(1+x_m) \leqslant \exp(x_1+\cdots+x_m)$ for any finite set of real numbers $x_i \geqslant -1$, it follows that

$$\left(\frac{1}{1-2\delta_k}\right)^k - 1 \leqslant \exp(4k\delta_k) - 1 \quad (k \geqslant 1).$$

Since $4k\delta_k \leqslant 1$ $(k \geqslant 1)$ and since $\mathrm{e}^x - 1 < 3x$ for $0 < x \leqslant 1$, we obtain

$$\left(\frac{1}{1-2\delta_k}\right)^k - 1 \leqslant 12k\delta_k = \frac{3}{k^2} \quad (k \geqslant 1).$$

In particular, the product

$$\tau = \prod_{k=1}^{\infty} \left(\frac{1}{1-2\delta_k}\right)^k$$

converges.

Define $\rho_k = \eta/k^{12}$ $(k \geqslant 1)$. Then

$$\sum_{k=1}^{\infty} \frac{k}{\delta_k}\sqrt{\rho_k} = 4\sqrt{\eta} \sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{\varepsilon}{3}.$$

Define $M_k = \rho_k L_k / 2\|L_k\|$ $(k \geqslant 1)$. Then $\|M_k\| < \rho_k$ for $k \geqslant 1$. Set $x^0 = a \in P^2(\mu)$ and $y^0 = 0 \in L^2(\mu)$. Using Lemma 2.1 we construct inductively sequences $(x^m)_{m\geqslant 1}$ and $(y^m)_{m \geqslant 1}$ with

(i) $x^m \in P^2(\mu)$, $y^m = (y^m_1, \ldots, y^m_m) \in L^2(\mu)^m$;
(ii) $\|M_k - x^m \otimes y^m_k\| < \rho_{m+1}$;

(iii) $\|x^m - x^{m-1}\| < \frac{3}{\delta_m} m\sqrt{\rho_m}$;
(iv) $\|y_k^m \chi_{\mathbb{S}}\| < \left(\frac{1}{1-2\delta_m}\right)^m \left[\|y_k^{m-1}\chi_{\mathbb{S}}\| + (1-2\delta_m)^k\sqrt{\rho_m}\right]$;
(v) $\|(y_k^m - y_k^{m-1})\chi_{\mathbb{B}}\| < \sqrt{\rho_m}$

for all $m \geqslant 1$ and $k = 1, \ldots, m$. Here $y_m^{m-1} = 0$ by definition.

For $m \geqslant k \geqslant 1$, we obtain

$$\begin{aligned}
\|y_k^m \chi_{\mathbb{S}}\| &< \frac{\sqrt{\rho_m}}{(1-2\delta_m)^{m-k}} + \frac{\|y_k^{m-1}\chi_{\mathbb{S}}\|}{(1-2\delta_m)^m} \\
&< \frac{\sqrt{\rho_m}}{(1-2\delta_m)^{m-k}} + \left(\frac{1}{1-2\delta_m}\right)^m \left[\frac{\sqrt{\rho_{m-1}}}{(1-2\delta_{m-1})^{m-1-k}} + \frac{\|y_k^{m-2}\chi_{\mathbb{S}}\|}{(1-2\delta_{m-1})^{m-1}}\right] \\
&< \cdots \\
&\leqslant \sum_{i=k}^{m} \left(\prod_{j=i}^{m} \left(\frac{1}{1-2\delta_j}\right)^j\right)\sqrt{\rho_i} \leqslant \left[\prod_{j=k}^{\infty} \left(\frac{1}{1-2\delta_j}\right)^j\right] \sum_{i=k}^{\infty} \sqrt{\rho_i}.
\end{aligned}$$

From (v) it follows that $y_{k,\mathbb{B}} = \lim_{m\to\infty} y_k^m \chi_{\mathbb{B}} \in L^2(\mu)$ exists for $k \geqslant 1$ and that

$$\|y_{k,\mathbb{B}}\| = \lim_{m\to\infty} \left\| \sum_{i=k}^{m} (y_k^i - y_k^{i-1})\chi_{\mathbb{B}} \right\| \leqslant \sum_{i=k}^{\infty} \sqrt{\rho_i}.$$

For $k \geqslant 1$, let us fix a weak limit $y_{k,\mathbb{S}} \in L^2(\mu)$ of a subsequence of $(y_k^m \chi_{\mathbb{S}})_{m \geqslant k}$. The above estimates imply that

$$\|y_{k,\mathbb{S}}\| \leqslant \tau \sum_{i=k}^{\infty} \sqrt{\rho_i} \quad (k \geqslant 1).$$

Then $y_k = y_{k,\mathbb{B}} + y_{k,\mathbb{S}}$ $(k \geqslant 1)$ is a weak limit of a subsequence of $(y_k^m)_m$ and

$$\|y_k\| \leqslant (1+\tau) \sum_{i=k}^{\infty} \sqrt{\rho_i} \quad (k \geqslant 1).$$

It follows from (iii) that $x = \lim_{m\to\infty} x^m \in P^2(\mu)$ exists and that

$$\|x - a\| \leqslant \sum_{m=1}^{\infty} \|x^m - x^{m-1}\| \leqslant 3 \sum_{m=1}^{\infty} \frac{m}{\delta_m} \sqrt{\rho_m} < \varepsilon.$$

Condition (ii) ensures that

$$M_k = x \otimes y_k \quad (k \geqslant 1).$$

To conclude the proof it suffices to replace the sequence (y_k) by the sequence $P(2\|L_k\| y_k/\rho_k)$, where P is the orthogonal projection of $L^2(\mu)$ onto $P^2(\mu)$. Our definitions show that the constant $C(\varepsilon)$ can be chosen as $C(\varepsilon) = C/\varepsilon$ with a suitable universal constant C. ∎

As in Section 1 we use the above measure theoretic results to obtain corresponding factorizations for subnormal tuples.

Theorem 2.3. *Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with an isometric* w*-*continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$. Then, for any given $\varepsilon > 0$, there is a constant $C(\varepsilon)$ (only depending on ε) such that, for each sequence $(L_k)_{k\geqslant 1}$ in $\mathcal{Q}_T$ and each given vector $a \in H$, there are vectors x, y_k $(k \geqslant 1)$ in H with $\|x - a\| < \varepsilon$ and*

$$L_k = [x \otimes y_k], \quad \|y_k\| \leqslant C(\varepsilon) k^{12} \|L_k\| \quad (k \geqslant 1).$$

Proof. In exactly the same way as in the proof of Theorem 1.10 the assertion can be reduced to the corresponding measure theoretic result contained in Proposition 2.2. To make sure that the chosen vector $x \in H$ satisfies the condition $\|x-a\| < \varepsilon$, one should choose the separating vector $h \in H$ for the minimal normal extension N of T close enough to the given vector $a \in H$. ∎

As a corollary we obtain that the dual algebra generated by a subnormal tuple $T \in \mathcal{L}(H)^n$ with a w*-continuous isometric H^∞-functional calculus over the unit ball in $\mathbb{C}^n$ possesses property $(\mathbb{A}_{1,\aleph_0})$. We obtain the same result for subnormal tuples $T \in \mathcal{L}(H)^n$ with rich spectrum in the unit ball.

Theorem 2.4. *Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with $\sigma(T) \subset \overline{\mathbb{B}}$ and $\sigma(T)$ dominating in $\mathbb{B}$. Then, for each sequence $(L_k)_{k\geqslant 1}$ in $\mathcal{Q}_T$, there are vectors $x, y_k \in H$ $(k \geqslant 1)$ with $L_k = [x \otimes y_k]$ $(k \geqslant 1)$.*

Proof. The Chaumat decomposition used to prove Theorem 1.11 allows us to reduce the assertion to the case considered in Theorem 2.3 and to the fact that the von Neumann algebra generated by a commuting tuple of normal operators satisfies property $(\mathbb{A}_{1,\chi_0})$.

For the convenience of the reader, we briefly discuss the case of normal tuples. Let $N \in \mathcal{L}(K)^n$ be a normal tuple on a Hilbert space K. Choose a separating vector $f \in K$ for N and denote by E the operator-valued spectral measure for N. Then $\mu = \langle E(\cdot)f, f\rangle \in M(X)$, where $X = \sigma(N)$, is a scalar-valued spectral measure for N, and we have the usual isomorphism of von Neumann algebras

$$L^\infty(X, \mu) \xrightarrow{\phi} W^*(N),$$

which is the adjoint of a corresponding isometric isomorphism

$$L^1(X, \mu) \xleftarrow{\phi_*} C^1(K)/{}^\perp W^*(N).$$

Since

$$\|p(N)f\|^2 = \langle |p|^2(N)f, f\rangle = \int\limits_X |p|^2 \,\mathrm{d}\langle E(\cdot)f, f\rangle = \|p\|_{2,\mu}^2$$

for all polynomials p in z and $\overline{z}$, there is a unitary operator

$$U : L^2(X,\mu) \longrightarrow \bigvee_{k,\ell\geqslant 0} N^k N^{*\ell} f$$

intertwining M_z on $L^2(X,\mu)$ and the restriction of N to the space on the right.

Let $(L_k)_{k\geqslant 1}$ be a sequence of w*-continuous linear forms $L_k : W^*(N) \to \mathbb{C}$. Then $(h_k) = \big(\phi_*(L_k)\big)$ is a sequence in $L^1(X,\mu)$. Since

$$\langle [U(f)\otimes U(g)],\, \phi(\theta)\rangle = \int\limits_X \theta f\overline{g}\,\mathrm{d}\mu$$

for $f, g \in L^2(X,\mu)$ and $\theta \in L^\infty(X,\mu)$, it suffices to check that there are functions $f, g_k \in L^2(X,\mu)$ with $h_k = fg_k$ for $k \geqslant 1$. To prove this, we are of course allowed to assume that $h_k \neq 0$ for all k. Define an L^1-function by

$$h = \sum_{k=1}^{\infty} 2^{-k} \frac{|h_k|}{\|h_k\|_1} \in L^1(X,\mu)$$

and choose functions $f, g \in L^2(X,\mu)$ with $h = fg$ and f non-zero almost everywhere. Then it suffices to define $g_k = h_k/f$ and to observe that $g_k \in L^2(X,\mu)$ because of the estimates

$$|g_k| = |h_k|/|f| \leqslant 2^k \|h_k\|_1 h/|f| = 2^k\|h_k\|_1 |g|. \quad \blacksquare$$

3. REFLEXIVITY

Let $T \in \mathcal{L}(H)^n$ be a commuting subnormal tuple on a Hilbert space H. We denote by $\operatorname{Alg}\operatorname{Lat}(T)$ the subalgebra of $\mathcal{L}(H)$ consisting of all operators $C \in \mathcal{L}(H)$ with $\operatorname{Lat}(C) \supset \operatorname{Lat}(T)$. We show that T is reflexive, that is, $\operatorname{Alg}\operatorname{Lat}(T)$ coincides with the WOT-closed unital subalgebra of $\mathcal{L}(H)$ generated by T, whenever $\sigma(T) \subset \overline{\mathbb{B}}$ and $\sigma(T)$ is dominating in the open ball $\mathbb{B}$. We obtain the same result under the condition that T possesses an isometric w*-continuous $H^\infty(\mathbb{B})$-functional calculus.

Our reflexivity proof will be based on the following consequence of Theorem 2.3.

Theorem 3.1. *Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with an isometric* w*-*continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$. Let $(\mu_j)_{j\geqslant 0}$ be a dense sequence in $\mathbb{B}$. Then, for any $\varepsilon > 0$ and any vector $a \in H$, there are vectors $x, y_j^{(k)} \in H$ $(j \geqslant 0,\ k \in \mathbb{N}^n)$ with $\|x - a\| < \varepsilon$ and*

$$x \otimes y_j^{(k)} = \mathcal{E}_{\mu_j}^{(k)}/k! \quad (j \geqslant 0,\ k \in \mathbb{N}^n)$$

and such that, for each $j \geqslant 0$, the power series

$$f_j(\lambda) = \sum_{k\in\mathbb{N}^n} y_j^{(k)}(\lambda - \mu_j)^k$$

converges on the polydisc with centre μ_j and multiradius $\rho_j = (1 - |\mu_j|)/\sqrt{n}$, i.e. on

$$D_j = \{\lambda \in \mathbb{C}^n : |\lambda_i - \mu_{j,i}| < \rho_j \text{ for } i = 1, \ldots, n\}.$$

Proof. Let us choose an enumeration $(L_k)_{k\geqslant 1}$ of the set

$$\{\mathcal{E}_{\mu_j}^{(k)}/k! : j \geqslant 0 \text{ and } k \in \mathbb{N}^n\}$$

in the following way. Set $L_1 = \mathcal{E}_{\mu_0}^{(0)}$. Then enumerate all functionals $\mathcal{E}_{\mu_0}^{(k)}/k!$ $(|k| = 1)$, then all functionals $\mathcal{E}_{\mu_1}^{(k)}/k!$ $(|k| \leqslant 1)$, then all functionals

$$\mathcal{E}_{\mu_0}^{(k)}/k! \quad (|k| = 2), \quad \mathcal{E}_{\mu_1}^{(k)}/k! \quad (|k| = 2), \quad \mathcal{E}_{\mu_2}^{(k)}/k! \quad (|k| \leqslant 2),$$

and continue in this way.

Fix a natural number $m \geqslant 0$. For $i \geqslant 0$, each functional

$$\mathcal{E}_{\mu_m}^{(j)}/j! \quad (|j| = m + i)$$

occurs in the sequence $(L_k)_{k\geqslant 1}$ with an index $k \leqslant (m + i + 1)^{n+1}$. The norm of the functionals $\mathcal{E}_{\mu_m}^{(j)}/j!$ $(|j| = m + i)$ can be estimated by (Theorem 2.2.7 in [13])

$$\|\mathcal{E}_{\mu_m}^{(j)}/j!\| \leqslant \left(\frac{1}{\rho_m}\right)^{m+i}.$$

According to Theorem 2.3 one can choose vectors $x, y_i^{(j)} \in H$ $(i \geqslant 0,\ j \in \mathbb{N}^n)$ with $\|x - a\| < \varepsilon$ and

$$x \otimes y_i^{(j)} = \mathcal{E}_{\mu_i}^{(j)}/j! \quad (i \geqslant 0,\ j \in \mathbb{N}^n)$$

and such that with a suitable constant $C > 0$ (independent of i and j) the vector $y_m^{(j)}$ corresponding to $\mathcal{E}_{\mu_m}^{(j)}$ $(|j| = m+i,\ i \geqslant 0)$ satisfies

$$\|y_m^{(j)}\| \leqslant C(m+i+1)^{12(n+1)} \left(\frac{1}{\rho_m}\right)^{m+i}.$$

Let $\lambda \in D_m$. Then, for $|j| = m+i$ $(i \geqslant 0)$,

$$\|y_m^{(j)}(\lambda - \mu_m)^j\| \leqslant \|y_m^{(j)}\|\rho^{|j|} \leqslant C(m+i+1)^{12(n+1)}(\rho/\rho_m)^{m+i},$$

where $\rho = \max\limits_{\nu=1,\ldots,n} |\lambda_i - \mu_{m,\nu}| < \rho_m$. The terms on the right of the last inequality remain bounded for $i \geqslant 0$, because

$$\lim_{k\to\infty} \left((k+1)^{12(n+1)}\right)^{\frac{1}{k}} = 1.$$

Therefore the power series

$$f_m(\lambda) = \sum_{j\in\mathbb{N}^n} y_m^{(j)}(\lambda - \mu_m)^j$$

converges on the polydisc D_m. ∎

Let $T \in \mathcal{L}(H)^n$ be as in Theorem 3.1, and let us choose vectors $x, y_j^{(k)}$ $(j \geqslant 0,\ k \in \mathbb{N}^n)$ as explained there. Define

$$Y = Y_x = \bigvee_{k\in\mathbb{N}^n} T^k x \in \mathrm{Lat}(T)$$

and set $\tilde{y}_j^{(k)} = P_x y_j^{(k)}$, where P_x is the orthogonal projection from H onto Y_x. With the notations from Theorem 3.1,

$$\mathbb{B} = \bigcup_{j\in\mathbb{N}} D_j,$$

and, for each $j \in \mathbb{N}$, the series

$$e_j(\lambda) = \sum_{k\in\mathbb{N}^n} \tilde{y}_j^{(k)}(\overline{\lambda} - \overline{\mu}_j)^k \quad (\lambda \in D_j)$$

defines a conjugate analytic function $e_j : D_j \to Y$.

LEMMA 3.2. *In the situation explained above, $e_i = e_j$ on $D_i \cap D_j$ for $i, j \in \mathbb{N}$, and the induced conjugate analytic function $e : \mathbb{B} \to Y$ satisfies*

$$x \otimes e(\lambda) = \mathcal{E}_\lambda \quad (\lambda \in \mathbb{B}).$$

Proof. For $\lambda = D_j$ and $f \in H^\infty(\mathbb{B})$,

$$\langle x \otimes e_j(\lambda), f\rangle = \sum_{k \in \mathbb{N}^n} \langle x \otimes y_j^{(k)}, f\rangle (\lambda - \mu_j)^k = f(\lambda).$$

Therefore we obtain that $x \otimes e_j(\lambda) = \mathcal{E}_\lambda$ for $j \in \mathbb{N}$ and $\lambda \in D_j$. Since, for $i, j \in \mathbb{N}$ and $\lambda \in D_i \cap D_j$,

$$\langle T^k x, e_j(\lambda)\rangle = \lambda^k = \langle T^k x, e_i(\lambda)\rangle \quad (k \in \mathbb{N}^n),$$

there is a conjugate analytic function $e : \mathbb{B} \to Y$ with $e_j = e|D_j$ for $j \geqslant 0$. ∎

Note that in the setting of Lemma 3.2 we have

$$(\lambda_i - T_i|Y)^* e(\lambda) = 0 \quad (\lambda \in B,\ i = 1, \ldots, n).$$

To prove our reflexivity results we use the concept of an analytic invariant subspace.

DEFINITION 3.3. Let $A \in \mathcal{L}(H)^n$ be a commuting tuple of Hilbert-space operators. Let $G \subset \mathbb{C}^n$ be an open connected set. A space $Y \in \operatorname{Lat}(A)$ is a G-*analytic invariant subspace* for A if there is a non-zero conjugate analytic function $e : G \to Y$ such that $(\lambda_i - A_i|Y)^* e(\lambda) = 0$ for $\lambda \in G$ and $i = 1, \ldots, n$.

For a commuting tuple $A \in \mathcal{L}(H)^n$ and $x \in H$, we denote by

$$Y_x = \bigvee_{k \in \mathbb{N}^n} A^k x$$

the smallest space in $\operatorname{Lat}(A)$ containing the vector x, and we write P_x for the orthogonal projection of H onto Y_x. Suppose that Y_x is a G-analytic invariant subspace via the conjugate analytic function $e : G \to Y_x$. Then the zero set of the function e coincides with the set $\{\lambda \in G : \langle x, e(\lambda)\rangle = 0\}$. Indeed, if $\langle x, e(\lambda)\rangle = 0$, then

$$\langle A^k x, e(\lambda)\rangle = \langle x, (A|Y_x)^{*k} e(\lambda)\rangle = \lambda^k \langle x, e(\lambda)\rangle = 0$$

for all $k \in \mathbb{N}^n$, and hence $e(\lambda) = 0$.

For e as above, the map

$$\iota = \iota_e : H \to \mathcal{O}(G), \quad h \mapsto \langle h, e\rangle$$

becomes continuous linear if $\mathcal{O}(G)$ is equipped with its natural Fréchet-space topology. For $u \in Y_x$ and $p \in \mathbb{C}[z]$, we have

$$\iota\big(p(A)u\big) = \langle p(A)u, e\rangle = \langle u, \tilde{p}\big((A|Y_x)^*\big)e\rangle = p\iota(u) \quad \big(\tilde{p}(\lambda) = \overline{p(\bar{\lambda})}\big).$$

For $h \in H$, we denote by $Z(h)$ the zero set of the analytic function $\iota(h)$.

Lemma 3.4. *Let $A \in \mathcal{L}(H)^n$ be a commuting tuple, and let $x \in H$. Suppose that Y_x is a G-analytic invariant subspace for A via the conjugate analytic function $e : G \to Y_x$. The function*

$$G \setminus Z(x) \to Y_x, \quad \lambda \mapsto e(\lambda)/\langle e(\lambda), x\rangle$$

extends to a conjugate analytic function $\psi : G \to Y_x$. Furthermore, ψ is the unique conjugate analytic function on G with

$$\psi(\lambda) \in \operatorname{Ker}(\lambda_i - A_i|Y_x)^* \quad (\lambda \in G,\ i = 1, \ldots, n)$$

and $\langle x, \psi(\lambda)\rangle = 1$ for all $\lambda \in G$.

Proof. Let $\iota : H \to \mathcal{O}(G)$, $h \mapsto \langle h, e\rangle$, be the continuous linear map considered before. Since

$$\iota\big(p(A)x\big) \in \iota(x)\mathcal{O}(G) \quad \big(p \in \mathbb{C}[z]\big)$$

and since the principal ideal $\iota(x)\mathcal{O}(G) \subset \mathcal{O}(G)$ is closed, it follows that $\iota(Y_x) \subset \iota(x)\mathcal{O}(G)$. In particular, for each $u \in Y_x$, the function

$$G \setminus Z(x) \to \mathbb{C}, \quad \lambda \mapsto \langle u, e(\lambda)\rangle/\langle x, e(\lambda)\rangle$$

has a unique extension to an analytic function $f_u \in \mathcal{O}(G)$. As an application of the uniform boundedness principle, it follows that the formula

$$\langle u, \psi(\lambda)\rangle = f_u(\lambda) \quad (u \in Y_x,\ \lambda \in G)$$

defines a conjugate analytic function $\psi : G \to Y_x$ with

$$\psi(\lambda) = e(\lambda)/\langle e(\lambda), x\rangle \quad \big(\lambda \in G \setminus Z(x)\big).$$

Clearly, $\langle x, \psi(\lambda)\rangle = 1$ for all $\lambda \in G$, and the uniqueness part of the assertion is obviously true. ∎

We need a few more elementary properties of analytic invariant subspaces.

Lemma 3.5. *Let Y be a G-analytic invariant subspace for a commuting tuple A in $\mathcal{L}(H)^n$ via the conjugate analytic function $e : G \to Y$. Define*

$$X = \{v \in Y : \langle v, e(\lambda)\rangle = 0 \textit{ for all } \lambda \in G\}.$$

(i) *For $u \in Y \setminus X$, the space Y_u is a G-analytic invariant subspace via the function $e_u : G \to Y_u$, $\lambda \mapsto P_u e(\lambda)$.*

(ii) *Suppose that $Y = Y_x$ for some $x \in H$ and that A possesses a* w*-*continuous functional calculus $\Phi : P^\infty(G) \to \mathcal{L}(H)$. For $h \in P^\infty(G)$ and $\lambda \in G$,*

$$\langle \Phi(h)x, e(\lambda)\rangle = h(\lambda)\langle x, e(\lambda)\rangle.$$

In particular $\Phi(h)x \in Y \setminus X$ for each non-zero $h \in P^\infty(G)$.

Proof. (i) For $u \in Y \setminus X$, the function $e_u : G \to Y_u$, $\lambda \mapsto P_u\big(e(\lambda)\big)$, is non-zero and satisfies, for all $\lambda \in G$,

$$(\lambda_i - A_i|Y_u)^* e_u(\lambda) = P_u(\lambda_i - A_i|Y)^* P_u\big(e(\lambda)\big) = P_u(\lambda_i - A_i|Y)^* e(\lambda) = 0.$$

(ii) For $h \in P^\infty(G)$, choose a net (p_j) of polynomials with w*-limit h. Then

$$\langle \Phi(h)x, e(\lambda)\rangle = \lim_{j\to\infty} \langle x, \tilde{p}_j\big((A|Y)^*\big)e(\lambda)\rangle = h(\lambda)\langle x, e(\lambda)\rangle$$

for $\lambda \in G$. ∎

Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with an isometric w*-continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$. By Theorem 3.1 and the remark preceding Definition 3.3, the set

$$\mathcal{C} = \{x \in H : Y_x \text{ is a } \mathbb{B}\text{-analytic invariant subspace for } T\}$$

is a dense subset of H. Let $x \in \mathcal{C}$ and let $\psi : \mathbb{B} \to Y_x$ be the unique conjugate analytic function with

$$x \otimes \psi(\lambda) = \mathcal{E}_\lambda \quad (\lambda \in \mathbb{B}).$$

Fix an operator $C \in \operatorname{Alg}\operatorname{Lat}(T)$. Since $(C|Y_x)^* \in \operatorname{Alg}\operatorname{Lat}\big((T|Y_x)^*\big)$, there are (unique) complex numbers $g(\lambda)$ ($\lambda \in \mathbb{B}$) such that

$$(C|Y_x)^*\psi(\lambda) = \overline{g(\lambda)}\psi(\lambda) \quad (\lambda \in \mathbb{B}).$$

For $u \in Y_x$ and $\lambda \in \mathbb{B}$,

$$\langle Cu, \psi(\lambda)\rangle = \langle u, (C|Y_x)^*\psi(\lambda)\rangle = g(\lambda)\langle u, \psi(\lambda)\rangle.$$

The induced map

$$\Psi = \Psi_x : \operatorname{Alg}\operatorname{Lat}(T) \to H^\infty(\mathbb{B}), \quad C \mapsto g \text{ (defined as above)}$$

is a contractive unital algebra homomorphism with $\Psi(T_i) = z_i$ for $i = 1, \dots, n$.

Now a standard procedure following for instance [6] can be used to prove that T is reflexive.

PROPOSITION 3.6. *Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with an isometric* w**-continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$. Let $x \in \mathcal{C}$ and let $C \in \operatorname{Alg}\operatorname{Lat}(T)$. Then $g = \Psi_x(C)$ is the unique function in $H^\infty(\mathbb{B})$ with*

$$\Phi(g)|Y_x = C|Y_x.$$

Proof. To simplify the notation we write Y instead of Y_x. Since $\sigma(T|Y) = \overline{\mathbb{B}}$, the restriction of Φ to Y gives an isometric w*-continuous H^∞-functional calculus for $T|Y$. Thus the uniqueness part of the assertion is obvious.

To prove that the function g has the claimed property, denote by $\psi : \mathbb{B} \to Y$ the conjugate analytic function with $x \otimes \psi(\lambda) = \mathcal{E}_\lambda$ for $\lambda \in \mathbb{B}$ and define $X = \{u \in Y : \langle u, \psi(\lambda)\rangle = 0$ for all $\lambda \in \mathbb{B}\}$. It suffices to show that

$$\langle \Phi(g)u, v\rangle = \langle C\, u, v\rangle \quad (u \in Y \setminus X,\, v \in Y).$$

We first show that this equality holds if, in addition, $u \otimes v = \mathcal{E}_{\lambda_0}$ for some λ_0 in $\mathbb{B}$. Let $\psi_u : \mathbb{B} \to Y_u$ be the unique conjugate analytic function with $u \otimes \psi_u(\lambda) = \mathcal{E}_\lambda$ for $\lambda \in \mathbb{B}$. By Lemma 3.4 and Lemma 3.5 we know that

$$\psi_u(\lambda) = P_u\big(\psi(\lambda)\big)/\langle \psi(\lambda), u\rangle$$

for all $\lambda \in \mathbb{B}\setminus Z(u)$, where $Z(u) = \{\lambda \in \mathbb{B} : \langle u, \psi(\lambda)\rangle = 0\}$. Since $u \otimes v = u \otimes \psi_u(\lambda_0)$, it follows that

$$\langle Cu, v\rangle = \langle Cu, \psi_u(\lambda_0)\rangle = \lim_{\substack{\lambda \to \lambda_0 \\ \lambda \notin Z(u)}} \langle Cu, \psi(\lambda)\rangle/\langle u, \psi(\lambda)\rangle = g(\lambda_0) = \langle \Phi(g)u, v\rangle.$$

In a second step we prove that $\langle \Phi(g)u, v\rangle = \langle Cu, v\rangle$ for all $u \in Y \setminus X$, $v \in Y$ with

$$u \otimes v \in \mathcal{E} = \mathrm{LH}\{\mathcal{E}_\lambda : \lambda \in \mathbb{B}\},$$

where on the right we mean the linear hull of the set of all $\mathcal{E}_\lambda$ $(\lambda \in \mathbb{B})$. Let $u \otimes v = \sum\limits_{i=1}^{m} t_i \mathcal{E}_{\lambda_i}$ with pairwise distinct $\lambda_1, \dots, \lambda_m \in \mathbb{B}$ and $t_1, \dots, t_m \in \mathbb{C} \setminus \{0\}$. Choose polynomials $p_i \in \mathbb{C}[z]$ with $p_i(\lambda_j) = \delta_{ij}/t_i$ for $i, j = 1, \dots, m$ and define $v_i = p_i(T|Y)^* v$. Then

$$u \otimes v_i(f) = \langle \Phi(fp_i)u, v\rangle = f(\lambda_i) \quad \big(f \in H^\infty(\mathbb{B})\big),$$

and the first step yields that

$$\langle \Phi(g)u, v\rangle = \sum_{i=1}^{m} t_i g(\lambda_i) = \sum_{i=1}^{m} t_i \langle Cu, v_i\rangle = \Big\langle Cu,\ \sum_{i=1}^{m} \bar{t}_i v_i \Big\rangle = \langle Cu, v\rangle.$$

To check the last equality note that $u \otimes v = u \otimes \Big(\sum_{i=1}^{m} \bar{t}_i v_i \Big)$.

To complete the proof we show that $\langle \Phi(g)u, v\rangle = \langle Cu, v\rangle$ for all $u \in Y \setminus X$ and $v \in Y$. Fix $u \in Y \setminus X$ and $v \in Y$ such that $L = u \otimes v \notin \mathcal{E}$. Choose a sequence $(L_k)_{k \geqslant 1}$ in $\mathcal{E}$ with $(L_k) \xrightarrow{k} L$. Theorem 1.10 applied to $T|Y$ allows us to choose sequences $(u_k)_{k\geqslant 1}$ and $(v_k)_{k\geqslant 1}$ in Y with $L_k = u_k \otimes v_k$ and

$$\|u_k - u\| \leqslant R\, d_k^{1/2}, \quad \|v_k\| \leqslant R\, d_k^{1/2} + R\|v\|$$

for $k \geqslant 1$. Here $d_k = \|L_k - u \otimes v\|$. After passing to suitable subsequences we may suppose that $(v_k)_{k\geqslant 1}$ converges weakly to a vector $w \in Y$ and that $u_k \notin X$ for all $k \geqslant 1$. By the second step of the proof

$$\langle Cu, w\rangle = \lim_{k\to\infty} \langle Cu_k, v_k\rangle = \lim_{k\to\infty} \langle \Phi(g)u_k, v_k\rangle = \langle \Phi(g)u, v\rangle.$$

Since, for all $f \in H^\infty(\mathbb{B})$,

$$u \otimes v(f) = \lim_{k\to\infty} \langle \Phi(f)u_k, v_k\rangle = \langle \Phi(f)u, w\rangle = u \otimes w(f),$$

it follows that $\langle Cu, v\rangle = \langle Cu, w\rangle$. This observation completes the proof. ∎

Let $T \in \mathcal{L}(H)^n$ be a subnormal tuple with an isometric w*-continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$. Let $C \in \operatorname{Alg}\operatorname{Lat}(T)$. By Proposition 3.6, for each vector $x \in \mathcal{C}$, there is a unique function g_x in $H^\infty(\mathbb{B})$ with $C|Y_x = \Phi(g_x)|Y_x$. Since $\mathcal{C}$ is dense in H, the reflexivity of T is proved if we can show that $g_x = g_y$ for all $x, y \in \mathcal{C}$.

Theorem 3.7. *Each subnormal tuple $T \in \mathcal{L}(H)^n$ with an isometric* w*-*continuous H^∞-functional calculus $\Phi : H^\infty(\mathbb{B}) \to \mathcal{L}(H)$ is reflexive.*

Proof. Fix an operator $C \in \operatorname{Alg}\operatorname{Lat}(T)$. For each $x \in \mathcal{C}$, we denote by g_x the unique function in $H^\infty(\mathbb{B})$ with $C|Y_x = \Phi(g_x)|Y_x$. We know that $\|g_x\| \leqslant \|C\|$ for all $x \in \mathcal{C}$.

We claim that, for each $v \in H$, there is a function $g \in H^\infty(\mathbb{B})$ with $Cv = \Phi(g)v$. To check this, choose a sequence $(x_k)_{k\geqslant 1}$ in $\mathcal{C}$ with $\lim_{k\to\infty} x_k = v$. By passing to a subsequence we can achieve that the associated sequence $(g_{x_k})_{k\geqslant 1}$ possesses a w*-limit g in $H^\infty(\mathbb{B})$. But then we obtain, for all $y \in H$,

$$\langle \Phi(g)v, y\rangle = \lim_{k\to\infty} \langle \Phi(g_{x_k})v, y\rangle = \lim_{k\to\infty} \langle \Phi(g_{x_k})x_k, y\rangle = \lim_{k\to\infty} \langle Cx_k, y\rangle = \langle Cv, y\rangle.$$

Let $x, y \in \mathcal{C}$. To show that $g_x = g_y$ we choose a function $h \in H^\infty(\mathbb{B})$ with $C(x + y) = \Phi(h)(x + y)$, and we observe that $\Phi(g_x - h)x = \Phi(h - g_y)y$. Since the restrictions of Φ to Y_x and Y_y are isometric, we conclude that $\Phi(g_x - h)x = 0$ if and only if $g_x = h$, and that $\Phi(h - g_y)y = 0$ if and only if $h = g_y$. To prove that $g_x = g_y$ we may therefore assume that $g_x \neq h$ and $h \neq g_y$.

By Lemma 3.5 the vector $u = \Phi(g_x - h)x$ belongs to $\mathcal{C}$, and since $Y_u \subset Y_x$, the uniqueness part of Proposition 3.6 implies that $g_u = g_x$. But in exactly the same way we obtain that $u = \Phi(h - g_y)y$ has the associated function $g_u = g_y$. Therefore $g_x = g_y$, and the reflexivity proof is complete. ∎

The Chaumat decomposition carried out in the proof of Theorem 1.11 can be used to prove the reflexivity for subnormal tuples with rich spectrum in the unit ball.

THEOREM 3.8. *Each subnormal tuple $T \in \mathcal{L}(H)^n$ with $\sigma(T) \subset \overline{\mathbb{B}}$ and $\sigma(T)$ dominating in $\mathbb{B}$ is reflexive.*

Proof. As shown in the proof of Theorem 1.11, there is an orthogonal decomposition $H = H_{\mathbb{S}} \oplus H_0$ ($H_0 = H_{\mathbb{B}} \oplus H_2$ in the notation of Theorem 1.11) that is reducing for T such that $T|H_{\mathbb{S}}$ is normal, $T|H_0$ possesses an isometric w*-continuous $H^\infty(\mathbb{B})$-functional calculus and

$$\mathfrak{A}_T = W^*(T|H_{\mathbb{S}}) \oplus \mathfrak{A}_{(T|H_0)}.$$

Let $C \in \mathrm{Alg}\,\mathrm{Lat}(T)$. Then $C|H_{\mathbb{S}} \in \mathrm{Alg}\,\mathrm{Lat}(T|H_{\mathbb{S}})$ and $C|H_0 \in \mathrm{Alg}\,\mathrm{Lat}(T|H_0)$. By [18] we know that $C|H_{\mathbb{S}} \in W^*(T|H_{\mathbb{S}})$ and from Theorem 3.7 we deduce that $C|H_0 \in \mathfrak{A}_{(T|H_0)}$. Therefore

$$C = (C|H_{\mathbb{S}}) \oplus (C|H_0) \in \mathfrak{A}_T$$

as was to be shown. ∎

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