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INTERPOLATION IN GENERALIZED NEVANLINNA AND STIELTJES CLASSES

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Abstract. A two-sided indefinite interpolation problem in the class of generalized Nevanlinna pairs is considered. In the case where the Pick matrix is nondegenerate a solvability criterion for the problem is given. All solutions of the problem are described as a fractional linear transformations of a parameter from a subclass of the Nevanlinna class. The nondegenerate interpolation problem in the generalized Stieltjes class has the same solution matrix and the parameter ranges over a subclass of the Stieltjes class. Sufficient conditions for the problems to have no excluded parameters are given in terms of the reproducing kernel matrices.

Keywords: *Interpolation, Nevanlinna pair, Pontryagin space, symmetric operator, selfadjoint extension, generalized resolvent, resolvent matrix.*

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0. INTRODUCTION

The generalized Nevanlinna class $N_\kappa(\mathbb{C}^n)$ consists of $n\times n$-matrix valued functions $F(\lambda)$ meromorphic on $\mathbb{C}_+$ such that the kernel $(F(z)-F(\overline{\zeta}))/(z-\overline{\zeta})$ has κ negative squares on $\mathbb{C}_+$. In 1981, A.A. Nudelman ([30]) has considered the following problem: there are given points $z_j \in \mathbb{C}_+$ ($j = 1,\dots,m$) and $n \times 1$-matrices V_{jp}, W_{jp} ($j = 1,\dots,m$; $p = 0,\dots,r_j - 1$; $n,m,r_j \in \mathbb{N}$). Find an $n \times n$-matrix function $F(z)$ which belongs to the generalized Nevanlinna class $N_\kappa(\mathbb{C}^n)$ and satisfies the equalities

$$\sum_{k=0}^{r_j-1} V_{jk}^*(\lambda - z_j)^k F(\lambda) = \sum_{k=0}^{r_j-1} W_{jk}^*(\lambda - z_j)^k + \mathrm{O}((\lambda - z_j)^{r_j}) \tag{0.1}$$

$\lambda \to \zeta_j$, $j = 1, \dots, m$.

For the problem to be solvable it is necessary and sufficient that the number $\mathrm{sq}_-(\mathbf{P})$ of negative eigenvalues of the Pick matrix $\mathbf{P}$ calculated by the data V, W of the problem does not exceed κ. In [31] a description of all solutions of the problem (0.1) was obtained under the assumptions that the matrix $\mathbf{P}$ is nondegenerate and $\mathrm{sq}_-(\mathbf{P}) = \kappa$. In the definite case ($\kappa = 0$) tangential and bitangential interpolation problems (0.1) have been studied in [18], [30], [5], [16], [4], [3]. The Nevanlinna–Pick problem for generalized Schur and Nevanlinna functions has been investigated in [1], [20], [19], [6], [14]. The solutions of this problem are parametrized via a fractional linear transformation over a subset of the extended class $\widetilde{N}_0(\mathbb{C}^n)$ of Nevanlinna pairs $\{\varphi, \psi\}$ (see Definition 1.1). A Nevanlinna pair which can not serve as a parameter of this transformation is said to be an excluded parameter. The cases where the Nevanlinna–Pick problem has no excluded parameters or has a unique excluded parameter were characterized in [14].

In the present paper the two-sided interpolation problem (0.1) in the class of generalized N_κ-pairs is considered. The operator approach we apply to the problem is similar to the one used in [3] for the case $\kappa = 0$. A symmetric operator acting in some Pontryagin reproducing kernel space is associated with the data of the problem. A description of all the solutions of the problem is shown to be reduced to the problem of description of generalized resolvents of this model operator. An application of the technique of boundary operators (see [21], [29], [13], [9]) enables us to prove some new results in extension theory of Pontryagin space symmetric operators (Propositions 1.9 and 2.11) to find a new formula (2.24) for the resolvent matrix and to simplify proofs of some statements from [3] (Theorem 2.4). A description of all the solutions of the problem (0.1) is given under the assumptions that the Pick matrix $\mathbf{P}$ is nondegenerate and $\mathrm{sq}_-(\mathbf{P}) \leqslant \kappa$. The cases where the interpolation problem (0.1) has no excluded parameters are characterized in terms of the reproducing kernel.

The Nevanlinna–Pick problem for Stieltjes functions has been considered in [26], [17] and its tangential and bitangential generalizations have been studied in [32], [4], [7]. The indefinite Nevanlinna–Pick problem for Stieltjes matrix valued functions was investigated in [2]. The set of excluded parameters for this problem was characterized in terms of two Pick matrices $\mathbf{P}$ and $\mathbf{P}_-$. In Section 5 the interpolation problem (0.1) in the classes of generalized Stieltjes pairs is considered. A solvability criterion is formulated in terms of the Pick matrices under the assumptions that $\mathbf{P}$ and $\mathbf{P}_-$ are nondegenerate. All solutions of the problem are described as a fractional linear transformations of a parameter which ranges over a subclass of the Stieltjes class. Sufficient conditions for the problem (0.1) to have no excluded parameters are given in terms of two reproducing kernels $K(t,\lambda)$ and $K_-(t,\lambda)$.

1. PRELIMINARIES

GENERALIZED NEVANLINNA PAIRS. Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$. A linear subspace A in $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$ is said to be a linear relation in $\mathcal{H}$. Let us denote by $\widetilde{\mathcal{C}}(\mathcal{H})$ the set of closed linear relations in $\mathcal{H}$. Let $\mathcal{D}(A)$ and $\mathcal{R}(A)$ be the domain and the range of a relation $A \in \widetilde{\mathcal{C}}(\mathcal{H}); \ker A, \operatorname{mul} A := \ker A^{-1}$ be the kernel and the multivalued part of A respectively. Let $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ $(\mathcal{C}(\mathcal{H}_1,\mathcal{H}_2))$ be the set of bounded (closed) linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. The resolvent set $\rho(A)$ of a relation $A \in \widetilde{\mathcal{C}}(\mathcal{H})$ consists of the points $\lambda \in \mathbb{C}$ such that $(A-\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$. Let $\sigma(A) = \mathbb{C} \setminus \rho(A)$ be the spectrum and $\sigma_{\mathrm{p}}(A)$ be the set of eigenvalues of the relation A.

If $\rho(A) \neq \emptyset$, then the relation A admits a representation $A = \{\{\varphi h, \psi h\} \mid h \in \mathcal{H}\}$ where $\varphi, \psi \in \mathcal{B}(\mathcal{H})$. A family of linear relations $\tau(\lambda) \in \widetilde{\mathcal{C}}(\mathbb{C}^n)$ is said to be holomorphic on a domain $\mathcal{O}$ if there exist two holomorphic $n \times n$ matrix functions $\varphi(\lambda), \psi(\lambda)$ such that

$$\tau(\lambda) = \{\varphi(\lambda), \psi(\lambda)\} := \{\{\varphi(\lambda)h, \psi(\lambda)h\} \mid h \in \mathbb{C}^n\}. \tag{1.1}$$

A kernel $N(\lambda,\mu)$ with values in $\mathcal{B}(\mathbb{C}^n)$ is said to have κ negative squares on $\mathcal{O}$ if $N(\lambda,\mu) = N(\mu,\lambda)^*$ and for all $n \in \mathbb{Z}_+$, $\lambda_j \in \mathcal{O}$, $h_j \in \mathbb{C}^n$ $(j = 1,\dots,n)$ the matrix $((N(\lambda_i,\lambda_j)h_i,h_j)_{i,j=1}^n$ has at most κ negative eigenvalues and exactly κ negative eigenvalues for at least one collection of λ_j, h_j. To each pair of matrix valued functions $\{\varphi,\psi\}$ it is associated the kernel

$$N_{\varphi\psi}(\lambda,\mu) = \frac{\varphi(\mu)^*\psi(\lambda) - \psi(\mu)^*\varphi(\lambda)}{\lambda - \overline{\mu}}. \tag{1.2}$$

DEFINITION 1.1. A pair of $n \times n$-matrix functions $\{\varphi(\lambda), \psi(\lambda)\}$ holomorphic on a domain $\mathcal{O} = \overline{\mathcal{O}} \subset \mathbb{C}\backslash\mathbb{R}$ is said to be a *generalized Nevanlinna pair* (or N_κ-pair, $\kappa \in \mathbb{Z}_+$) if:

(i) the kernel $N_{\varphi\psi}(\lambda,\mu)$ has κ negative squares on $\mathcal{O}$;
(ii) $\psi(\overline{\lambda})^*\varphi(\lambda) - \varphi(\overline{\lambda})^*\psi(\lambda) = 0$, $\forall\, \lambda \in \mathcal{O}$;
(iii) $\operatorname{rank}\{\varphi(\lambda)^* : \psi(\lambda)^*\} = n$, $\forall\, \lambda \in \mathcal{O}$.

Each N_κ-pair $\{\varphi,\psi\}$ admits a holomorphic continuation on $\mathbb{C} \setminus \mathbb{R}$ (see Theorem 2.4 below). Let us save the same notation for the extended pair and denote by $\rho(\varphi,\psi)$ the set where φ,ψ are holomorphic and satisfy the assumptions (ii), (iii) of Definition 1.1. Two pairs $\{\varphi, \psi\}$ and $\{\varphi_1, \psi_1\}$ are said to be equivalent if $\varphi_1(\lambda) = \varphi(\lambda)H(\lambda)$, $\psi_1(\lambda) = \psi(\lambda)H(\lambda)$ for some holomorphic and invertible matrix function $H(\lambda)$ on $\mathcal{O}$. The set of classes of equivalent N_κ-pairs is denoted by $\widetilde{N}_\kappa(\mathbb{C}^n)$. If $\tau(\lambda) = \{\varphi(\lambda), \psi(\lambda)\} \in \widetilde{N}_\kappa(\mathbb{C}^n)$ and $\varphi(\lambda)$ is invertible, we shall write $\psi(\lambda)\varphi(\lambda)^{-1} \in N_\kappa(\mathbb{C}^n)$. Let us consider $N_\kappa(\mathbb{C}^n)$ as a subset of $\widetilde{N}_\kappa(\mathbb{C}^n)$ identifying a matrix $H(\lambda)$ with the linear relation $\{I, H(\lambda)\}$.

REMARK 1.2. The assumptions (ii), (iii) of Definition 1.1 for a family of linear relations $\tau(\lambda) = \{\varphi(\lambda), \psi(\lambda)\}$ can be rewritten in the form $\tau(\overline{\lambda}) = \tau(\lambda)^*$ for all $\lambda \in \mathcal{O}$. Indeed, if $\tau(\overline{\lambda}) = \tau(\lambda)^*$ then, evidently, the condition (ii) is satisfied. The assumption $\operatorname{rank}\{\varphi(\lambda)^* : \psi(\lambda)^*\} < n$ yields $\ker\varphi(\lambda) \cap \ker\psi(\lambda) \neq \{0\}$ and, therefore, $\dim\tau(\lambda) < n$ for all $\lambda \in \mathcal{O}$. Hence $\dim\tau(\lambda)^* > n$ for all $\lambda \in \mathcal{O}$ which contradicts the assumption $\tau(\overline{\lambda}) = \tau(\lambda)^*$. Conversely, let the assumptions (ii), (iii) be fulfilled, then evidently $\tau(\overline{\lambda}) \subset \tau(\lambda)^*$ for all $\lambda \in \mathcal{O}$. It follows from (iii) that $\dim\tau(\lambda) = n$, hence $\tau(\overline{\lambda}) = \tau(\lambda)^*$.

DEFINITION 1.3. An N_κ-pair $\{\varphi(\lambda), \psi(\lambda)\}$ is said to be a *generalized Stieltjes pair* (or $N_\kappa^{\pm k}$-pair) if $\{\varphi(\lambda), \lambda^{\pm 1}\psi(\lambda)\} \in \widetilde{N}_k$. Moreover, $\widetilde{N}_\kappa^{\pm k}(\mathbb{C}^n)$ stands for the set of equivalence classes of $N_\kappa^{\pm k}$-pairs in $\mathbb{C}^n$.

Let $N_\kappa^{\pm k}(\mathbb{C}^n)$ be the set of matrix functions $F(\lambda)$ such that $\{I, F(\lambda)\}$ is an $N_\kappa^{\pm k}$-pair. The classes $N_\kappa^{\pm 0}(\mathbb{C}^n)$ and $N_0^{\pm k}(\mathbb{C}^n)$ were introduced in [23] and [13] respectively. In particular, the class $N_0^{+0}(\mathbb{C}^n)$ coincides with the Stieltjes class $\mathrm{S}(\mathbb{C}^n)$ of matrix valued functions $F \in N_0(\mathbb{C}^n)$ which admit holomorphic nonnegative continuations on the negative semiaxis ([25]).

BOUNDARY TRIPLES. Let S be a closed symmetric linear relation in a Pontryagin space $(\Pi, [\,\cdot\,,\cdot\,])$, $\widehat{\rho}(S)$ be the set of regular type points of S and let its defect subspaces $\mathcal{N}_\lambda = \ker(S^* - \lambda)$ $(\lambda \in \widehat{\rho}(S))$ be finite–dimensional and let the deficiency indices $n_\pm(S) = \dim\mathcal{N}_\lambda$ $(\lambda \in \mathbb{C}_\pm \cap \widehat{\rho}(S))$ coincide, $n_+(S) = n_-(S) = n < \infty$. Denote the Pontryagin index of the space Π by $\kappa^-(\Pi)$.

The operator S is identified with its graph $\operatorname{gr} S = \{\{f, Sf\} : f \in \mathcal{D}(S)\}$. An extension $\widetilde{A} \in \widetilde{\mathcal{C}}(\Pi)$ of the relation S is said to be proper if $S \subset \widetilde{A} \subset S^*$. Two proper extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of the relation S are said to be disjoint if $\widetilde{A}_1 \cap \widetilde{A}_2 = S$.

Let us remind (see [9], [12] and [21], [29] for the case $\kappa = 0$) the definitions of the boundary triple and the Weyl function of a symmetric linear relation S which are used for a description of generalized resolvents of S.

DEFINITION 1.4. A set $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ where Γ_0, Γ_1 are linear mappings from S^* into $\mathbb{C}^n$ is said to be a *boundary triple for the relation* S^* if the mapping $\Gamma : \widehat{f} \to \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$ from S^* into $\mathbb{C}^n \oplus \mathbb{C}^n$ is surjective and for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^*$ the following identity holds

$$[f', g] - [f, g'] = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathbb{C}^n} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathbb{C}^n}. \tag{1.3}$$

If $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ is a boundary triple for a linear relation S^*, then the mapping Γ establishes a one-to-one correspondence between the set of selfadjoint extensions $\widetilde{A}$ of the relation S and the set of selfadjoint relations τ in $\mathbb{C}^n$

$$\widetilde{A}_\tau \leftrightarrow \tau = -\Gamma \widetilde{A}_\tau = \{\{\Gamma_0 \widehat{f}, -\Gamma_1 \widehat{f}\} \mid \widehat{f} \in \widetilde{A}_\tau\}. \tag{1.4}$$

Conversely, for each $\tau = \tau^* \in \widetilde{\mathcal{C}}(\mathbb{C}^n)$ the corresponding extension $\widetilde{A}_\tau$ can be find by the equality $\widetilde{A}_\tau = \ker(\Gamma_1 + \tau\Gamma_0)$. Naturally associated with each boundary triple are two selfadjoint extensions of S, namely $A_j := \ker \Gamma_j$ $(j = 0, 1)$. Let π_j be the projection onto the j-th component of $\mathbb{C}^n \oplus \mathbb{C}^n$ $(j = 1, 2)$, $\widehat{\mathcal{N}}_\mu = \big\{\{f, \mu f\} \mid f \in \mathcal{N}_\mu\big\}$ $(\mu \in \widehat{\rho}(S))$.

PROPOSITION 1.5. ([8]) *Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triple for the linear relation S^* such that $\rho(A_0) \neq \emptyset$. Then the formula*

$$\gamma(\lambda) = \pi_1 (\Gamma_0 | \widehat{\mathcal{N}}_\lambda)^{-1}, \quad (\lambda \in \rho(A_0)) \tag{1.5}$$

correctly defines a holomorphic operator function on $\rho(A_0)$ with values in $\mathcal{B}(\mathbb{C}^n, \mathcal{N}_\lambda)$ which satisfies the equality

$$\gamma(\lambda) = \gamma(\mu) + (\lambda - \mu)(\widetilde{A} - \lambda)^{-1}\gamma(\mu) \quad (\lambda, \mu \in \rho(A_0)). \tag{1.6}$$

DEFINITION 1.6. A matrix function $M(\lambda)$ defined by the relation

$$M(\lambda)\Gamma_0 \widehat{f}_\lambda = \Gamma_1 \widehat{f}_\lambda \quad (\lambda \in \rho(\widetilde{A}),\ \widehat{f}_\lambda \in \widehat{\mathcal{N}}_\lambda) \tag{1.7}$$

is said to be the *Weyl function of the operator S corresponding to the boundary triple* $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$.

Let us note that the Weyl function $M(\lambda)$ is correctly defined and holomorphic on $\rho(\widetilde{A})$ and $M(\lambda)$ is a Q-function of the operator S corresponding to the selfadjoint extension A_0 in the sense of [23]. In the case where S is a simple operator (that is, c.l.s.$\{\mathcal{N}_\lambda : \lambda \in \widehat{\rho}(S)\} = \Pi$), the resolvent set $\rho(A_0)$ coincides with the holomorphy set of the Q-function $M(\lambda)$ ([25]). One can describe the spectrum of the extension $\widetilde{A}_\tau$ $(\tau = \tau^* \in \widetilde{\mathcal{C}}(\mathcal{H}))$ in terms of τ and $M(\lambda)$.

PROPOSITION 1.7. ([25], [12]) *Let S be a simple symmetric operator, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^*, $\tau = \tau^* \in \widetilde{\mathcal{C}}(\mathcal{H})$, $\lambda \in \rho(A_0)$ and let $M(\lambda)$ be the corresponding Weyl function. Then*

$$\lambda \in \rho(\widetilde{A}_\tau) \Leftrightarrow 0 \in \rho(M(\lambda) + \tau).$$

Moreover, $\rho(\widetilde{A}_\tau)$ coincides with the holomorphy set of the matrix function $(M(\lambda) + \tau)^{-1}$.

Let $\mathcal{S}_n$ be a set of selfadjoint $n \times n$-matrices.

LEMMA 1.8. *Let $F(\lambda)$ be a polynomial $n \times n$-matrix function and let $\mathcal{Z}$ be the set of matrices $B = B^*$ such that the matrix function $(F(\lambda) + B)^{-1}$ has no holomorphic continuations at ∞. Then the set $\mathcal{S}_n \setminus \mathcal{Z}$ is nonempty, open, and $\mathcal{Z}$ is nowhere dense in $\mathcal{S}_n$.*

Proof. One can assume that $\det F(\lambda) \not\equiv 0$ without loss of generality. Let $F(\lambda) = (f_{ij}(\lambda))_{i,j=1}^n$ and let F_{ij} be the cofactors of the entries f_{ij} of the matrix function $F(\lambda)$. Assume that

$$\deg \det F(\lambda) < \max_{i,j} \deg F_{ij}(\lambda) = \deg F_{i_0 j_0}(\lambda).$$

Setting $B_0 = \{z\delta_{ii_0}\delta_{jj_0} + \overline{z}\delta_{ij_0}\delta_{ji_0}\}_{i,j=1}^n$, $F(\lambda, B) := F(\lambda) + B$, we have

$$\det F(\lambda, B_0) = \det F(\lambda) + (-1)^{i_0+j_0} \left(zF_{i_0j_0}(\lambda) + \overline{z}F_{j_0i_0}(\lambda)\right) - |z|^2 F_{i_0j_0}^{i_0j_0}(\lambda) \tag{1.8}$$

where $F_{i_0j_0}^{i_0j_0}(\lambda)$ is a minor of the order $n-2$ obtained by deleting the i_0-th and j_0-th rows and columns of the matrix $F(\lambda)$. By (1.8), there exists $z \in \mathbb{C}$ such that $\deg \det F(\lambda, B_0) > \deg \det F(\lambda)$. Iterating this procedure one obtains a matrix $B = B_0 + B_1 + \cdots + B_s$ such that the matrix function $F(\lambda, B)$ satisfies the inequality

$$\deg \det F(\lambda, B) \geqslant \max_{i,j} \deg F_{ij}(\lambda, B) \tag{1.9}$$

and, therefore, the matrix function $F(\lambda, B)^{-1}$ has a holomorphic continuation at ∞.

Let E be a small perturbation of the matrix B. The invertibility of the matrix $F(\lambda) + B + E$ and the holomorphy of the inverse matrix at ∞ are implied by the equality

$$F(\lambda) + B + E = (F(\lambda) + B)(I + (F(\lambda) + B)^{-1}E).$$

Let us consider the expansions of the polynomials $\det F(\lambda, B)$ and $F_{ij}(\lambda, B)$

$$\det F(\lambda, B) = \sum_{k=0}^{p} F_k(B)\lambda^k, \quad F_{ij}(\lambda, B) = \sum_{k=0}^{q} F_{ij}^k(B)\lambda^k,$$

where $F_k(B)$, $F_{ij}^k(B)$ are polynomials depending on $B \in \mathcal{S}_n$. For every i, j, there is an open set $\mathcal{O}_{ij} \subset \mathcal{S}_n$ such that $d \equiv \deg\det F(\lambda, B) \geqslant d_{ij} \equiv \deg F_{ij}(\lambda, B)$ for all $B \in \mathcal{O}_{ij}$.

Therefore, for all $B \in \mathcal{O}_{ij}$ the following equalities hold

$$F_k(B) = 0 \quad (k > d), \quad F_{ij}^k(B) = 0 \quad (k > d_{ij}). \tag{1.10}$$

Since $F_k(B) = 0$, $F_{ij}^k(B)$ are polynomials in n^2 real variables (entries of B) the equalities (1.10) are identities for all $B \in \mathcal{S}_n$. Therefore, the set $\mathcal{Z}$ is contained in the algebraic manifold of zeros $B \in \mathcal{S}_n$ of the equality $F_d(B) = 0$. The dimension of this manifold does not exceed $n^2 - 1$ and, therefore, $\mathcal{Z}$ is nowhere dense in $\mathcal{S}_n$. ∎

PROPOSITION 1.9. *Let S be a simple symmetric operator in a Pontryagin space Π. Then for every collection of points $z_j \in \mathbb{C}_+$ $(j = 1, \ldots, m)$ there is a selfadjoint extension $\widetilde{A}$ of the operator S such that $z_j \in \rho(\widetilde{A})$ $(j = 1, \ldots, m)$.*

Proof. Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* and let $M(\lambda)$ be the corresponding Weyl function. To prove the statement it sufficies (see Proposition 1.7) to construct an operator $B = B^* \in \mathcal{B}(\mathcal{H})$ such that the matrix function $(M(z) + B)^{-1}$ has a holomorphic continuation at the points z_j $(j = 1, \ldots, m)$. Assume that the matrix function $M(z)$ has a pole of order k_j at z_j that is

$$M(z) = \sum_{i=0}^{k} M_{ij}(z - z_j)^{-i} + M_j(z)(z - z_j), \quad (M_{ij} \in \mathcal{B}(\mathcal{H}),\ j = 1 \ldots, m), \tag{1.11}$$

where $M_j(z)$ is holomorphic at the point z_j. Let $F_j(\lambda, B) = \sum\limits_{i=0}^{k} M_{ij}\lambda^i + B$ and let $\mathcal{Z}_j \subset \mathcal{S}_n$ be the set of matrices $B \in \mathcal{S}_n$ such that the matrix function $F_j(\lambda, B)^{-1}$ has no holomorphic continuation at ∞ $(j = 1, \ldots, m)$. By virtue of Lemma 1.8, the set $\mathcal{Z} = \bigcup\limits_j \mathcal{Z}_j$ is nowhere dense in $\mathcal{S}_n$. Let $B \in \mathcal{S}_n \setminus \mathcal{Z}$. Then, from the equalities

$$M(z) + B = F_j(\lambda_j(z), B)\left(I + F_j(\lambda_j(z), B)^{-1} M_j(z)(z - z_j)\right)$$

$(\lambda_j(z) = 1/(z - z_j)$, $j = 1 \ldots, m)$ it follows that the matrix function $(M(z)+B)^{-1}$ is holomorphic at z_j. It follows now from Proposition 1.7 that $z_j \in \rho(\widetilde{A}_B)$ for all $j = 1, \ldots, m$. ∎

2. GENERALIZED RESOLVENTS

GENERALIZED RESOLVENTS OF SYMMETRIC OPERATORS. Let $\widetilde{A}$ be a selfadjoint extension of the relation S in a bigger Pontryagin space $\widetilde{\Pi}$, P_Π be the orthogonal projection from $\widetilde{\Pi}$ onto Π, and $\kappa = \kappa^-(\Pi)$. The operator function

$$\mathbf{R}_\lambda = P_\Pi(\widetilde{A} - \lambda)^{-1}|\Pi \quad (\lambda \in \rho(\widetilde{A})) \tag{2.1}$$

is said to be a generalized resolvent of the relation S. An extension $\widetilde{A} = \widetilde{A}^*$ of the relation S is said to be minimal if

$$\text{c.l.s.}\{\Pi + (\widetilde{A} - \lambda)^{-1}\Pi : \lambda \in \rho(\widetilde{A})\} = \widetilde{\Pi}. \tag{2.2}$$

DEFINITION 2.1. A generalized resolvent $\mathbf{R}_\lambda$ is said to be from *the class* $\Omega_\kappa(S)$ if it admits a representation (2.1) in which $\widetilde{A}$ is a minimal extension of S and $\kappa^-(\widetilde{\Pi}) = \widetilde{\kappa}$ ($\widetilde{\kappa} \in \mathbb{Z}_+$).

REMARK 2.2. If a generalized resolvent $\mathbf{R}_\lambda$ is holomorphic at the point $\lambda_0 \in \widehat{\rho}(S)$ and the extension $\widetilde{A}$ generating $\mathbf{R}_\lambda$ is minimal, then λ_0 is a regular point of $\widetilde{A}$. An explicit construction of such an extension for a densely defined symmetric operator S was given in [23]. This construction can be carried over to the case of a linear relation S.

THEOREM 2.3. ([8], [15]) *Let S be a simple symmetric operator in Π, $\kappa = \kappa^-(\Pi)$ and $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triple of the relation S^*, and let $M(\lambda)$ be the corresponding Weyl function, $\lambda_0 \in \rho(A_0) \cap \mathbb{C}_+$. Then:*

(i) *the formula*

$$\mathbf{R}_\lambda = (A_0-\lambda)^{-1} - \gamma(\lambda)\varphi(\lambda)(\psi(\lambda)+M(\lambda)\varphi(\lambda))^{-1}\gamma(\overline{\lambda})^* \quad (\lambda \in \rho(A_0)\cap\rho(\widetilde{A})) \tag{2.3}$$

establishes a one-to-one correspondence between the set of generalized resolvents $\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}(S)$ holomorphic at the point λ_0 and the set of classes of equivalent $N_{\widetilde{\kappa}-\kappa}$-pairs $\{\varphi, \psi\}$ holomorphic at λ_0 such that

$$\det(\psi(\lambda_0) + M(\lambda_0)\varphi(\lambda_0)) \neq 0. \tag{2.4}$$

(ii) *the formula* (2.3) *establishes also a one-to-one correspondence between the set of all generalized resolvents $\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}(S)$ and the set $\widetilde{N}_{\widetilde{\kappa}-\kappa}(\mathbb{C}^n)$ of classes of equivalent $N_{\widetilde{\kappa}-\kappa}$-pairs $\{\varphi, \psi\}$ such that*

$$\det(\psi(\lambda) + M(\lambda)\varphi(\lambda)) \not\equiv 0. \tag{2.5}$$

Proof. (i) In fact, the first statement was proved in [11] (see also [15] for the case of a standard operator), where the set of generalized resolvents was described by the Kreĭn's formula

$$\mathbf{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda) + M(\lambda))^{-1}\gamma(\bar{\lambda})^* \quad (\tau(\lambda) \in \widetilde{N}_{\widetilde{\kappa}-\kappa}). \tag{2.6}$$

It remains to note only that the condition $0 \in \rho(\tau(\lambda_0) + M(\lambda_0))$ for $\tau = \{\varphi, \psi\}$ is equivalent to the condition $0 \in \rho(\psi(\lambda_0) + M(\lambda_0)\varphi(\lambda_0))$ and in this case $(\tau(\lambda) + M(\lambda))^{-1} = \varphi(\lambda)(\psi(\lambda) + M(\lambda)\varphi(\lambda))^{-1}$.

(ii) The second statement is implied by the first one in view of the equivalence given above. Thus, if $\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}(S)$ then the set $\rho(\widetilde{A}) \cap \rho(A_0)$ is nonempty and the corresponding $N_{\widetilde{\kappa}-\kappa}$-pair $\{\varphi, \psi\}$ satisfies the condition (2.5). Conversely, if an $N_{\widetilde{\kappa}-\kappa}$-pair $\{\varphi, \psi\}$ satisfies (2.5) then the operator function $\mathbf{R}_\lambda$ defined by (2.3) is a generalized resolvent of the relation S and $\rho(\widetilde{A}) \neq \emptyset$. ∎

It is interesting to note that Theorem 2.3 contains a description of generalized Nevanlinna pairs. In the case $\kappa = 0$ such description was obtained in another way in [3].

THEOREM 2.4. *Let a generalized N_κ-pair $\{\varphi, \psi\}$ be holomorphic on $\mathcal{O}$ and satisfy the condition $\psi(\lambda) + \lambda\varphi(\lambda) \equiv I$ $(\lambda \in \mathcal{O})$. Then it admits a representation*

$$\{\varphi(\lambda), \psi(\lambda)\} = \{-\mathbf{R}_\lambda, I + \lambda\mathbf{R}_\lambda\} \quad (\lambda \in \mathcal{O}), \tag{2.7}$$

where $\mathbf{R}_\lambda = G^*(\widetilde{A} - \lambda)^{-1}G$, $\widetilde{A}$ is a selfadjoint relation in a Pontryagin space $\widetilde{\Pi}$, $G \in \mathcal{B}(\mathbb{C}^n, \widetilde{\Pi})$, $G^*G = I$. The relation $\widetilde{A}$ can be chosen G-minimal, i.e. such that

$$\widetilde{\Pi} = \text{c.l.s.}\{Gf + (\widetilde{A} - \lambda)^{-1}Gh : \lambda \in \rho(\widetilde{A});\ f, h \in \mathbb{C}^n\}. \tag{2.8}$$

In this case $\kappa^-(\widetilde{\Pi}) = \kappa$.

Proof. Let us consider a trivial linear relation $S = \{0\}$ in $\mathbb{C}^n$ and a boundary triple $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ of the relation $S^* = \mathbb{C}^n \oplus \mathbb{C}^n$, setting $\Gamma_0\widehat{f} = f$, $\Gamma_1\widehat{f} = f'$ $(\widehat{f} = \{f, f'\} \in S^*)$. Then the extension $A_0 = \ker\Gamma_0$ coincides with the multivalued part of the relation S^* $(A_0 = \operatorname{mul} S^* = \{0, \mathbb{C}^n\})$, and the corresponding Weyl function takes the form $M(\lambda) = \lambda I$. In accordance with Theorem (2.3), there exist a Pontryagin space $\widetilde{\Pi}$ and a selfadjoint extension $\widetilde{A} \supset S$ in $\widetilde{\Pi}$ such that the minimal condition (2.8) is satisfied and the following equality holds

$$\mathbf{R}_\lambda = P_{\mathbf{C}^n}(\widetilde{A} - \lambda)^{-1}|\mathbb{C}^n = -\varphi(\lambda)(\psi(\lambda) + \lambda\varphi(\lambda))^{-1} = -\varphi(\lambda) \quad (\lambda \in \mathcal{O}). \tag{2.9}$$

It follows from (2.9) that

$$\varphi(\lambda) = -\mathbf{R}_\lambda, \quad \psi(\lambda) = I - \lambda\varphi(\lambda) = I + \lambda\mathbf{R}_\lambda \quad (\lambda \in \mathcal{O})$$

which coincides with (2.7). Here G is the embedding operator $\mathbb{C}^n \subset \widetilde{\Pi}$. ∎

It follows from Theorem 2.4 that each N_κ-pair $\{\varphi,\psi\}$ admits a meromorphic continuation at $\mathbb{C}_+\cup\mathbb{C}_-$ provided that $\Delta_{\varphi,\psi}\not\equiv 0$ and the determinant $\Delta_{\varphi,\psi}(\lambda)=\det(\psi(\lambda)+\lambda\varphi(\lambda))$ has at most κ zeros in $\mathbb{C}_+$

Remark 2.5. There exist N_κ-pairs $\{\varphi,\psi\}$ such that $\Delta_{\varphi,\psi}(\lambda)\equiv 0$. For example, $\tau=\{1,-\lambda\}$ is an N_1-pair and $\Delta_{\{1,-\lambda\}}(\lambda)\equiv 0$. The spectrum of the selfadjoint extension

$$\widetilde{A}=\{\{\operatorname{col}(a,0),\operatorname{col}(b,0)\}:a,b\in\mathbb{C}\}$$

corresponding to this pair $\{1,-\lambda\}$ fills out the whole plane $\mathbb{C}$.

Lemma 2.6. *Let $\{\varphi,\psi\}$ be an N_κ-pair which satisfies the assumptions of Definition* 1.1 *on a domain $\mathcal{O}$ and $z_j\in\mathcal{O}$ $(j=1,\dots,m)$. Then there is a nonnegative invertible matrix X such that*

$$\det(\psi(z_j)+z_jX\varphi(z_j))\neq 0\quad (j=1,\dots,m). \tag{2.10}$$

Proof. In view of the condition (iii) of Definition 1.1, the ranks of the matrices $\{\psi(z_j)^*:\overline{z}_j\varphi(z_j)^*\}$ $(j=1,\dots,m)$ are equal to n. Let us choose a nondegenerate minor $\mathcal{M}$ of this matrix of the order n and denote by $i_1,i_2,\dots,i_k$ the indices of the columns of the matrix $\varphi(z_j)^*$ which are contained in $\mathcal{M}$. Setting $x_l=t$ for $l=i_1,i_2,\dots,i_k$ and $x_l=0$ otherwise, we obtain the matrix $X_j=\operatorname{diag}\{x_1,\dots,x_n\}$ such that

$$\Delta_j:=\det(\psi(z_j)^*+\overline{z}_j\varphi(z_j)^*X_j)\neq 0\quad (j=1,\dots,m) \tag{2.11}$$

for t large enough.

Let us consider the functions $\Delta_j(X)$ on the set of real diagonal matrices $X=\operatorname{diag}\{x_1,\dots,x_n\}$. Since the function $\Delta_j(X)$ does not vanish identically on $\mathbb{R}^n$, its set $\mathcal{X}_j$ of zeros is at most an $(n-1)$-dimensional real variety in $\mathbb{R}^n$. On the other hand, the set of positive diagonal $n\times n$-matrices is an n-dimensional variety. Therefore, there exists a positive diagonal matrix X such that (2.10) holds for all $j(=1,\dots,m)$. ∎

Symmetric operators with N^k-property

Definition 2.7. A closed linear Pontryagin space symmetric operator S is said to have N^k-*property* $(k\in\mathbb{Z}_+)$ if the form $[A\,\cdot,\cdot]$ has k negative squares and $\widehat{\rho}(S)\neq\emptyset$.

In particular, a linear relation S with the N^0-property is nonnegative in Π. Denote by $\operatorname{Ext}_S^{\widetilde{k}}(-\infty,0)$ $(k\leqslant\widetilde{k})$ the set of selfadjoint extensions $\widetilde{A}$ of S which have $N^{\widetilde{k}}$-property.

Let J be a fundamental symmetry in Π and let $H = JS$ be a Hilbert space symmetric operator in $(\Pi, [J\,\cdot\,,\cdot\,])$ with N^k-property. There are two extremal extensions $H_{\rm F}$ and $H_{\rm K}$ of the operator H in the class $\mathrm{Ext}_H^k(-\infty, 0)$ defined by:

$$H_{\rm F} := \text{s-R-}\lim_{x\to-\infty} H_x, \quad H_{\rm K} := \text{s-R-}\lim_{x\to 0} H_x,$$

where $H_x = H \dot{+} \widehat{\mathcal{N}}_x(H)$, $\widehat{\mathcal{N}}_x(H) = \{\{f_x, xf_x\} \mid f_x \in \mathcal{N}_x(H)\}$. The extensions $S_{\rm F} := JH_{\rm F}$, $S_{\rm K} := JH_{\rm K}$ are said to be the Friedrichs and the Kreĭn–von Neumann extensions of S. One can characterize them in terms of the Weyl function $M(\lambda)$.

Proposition 2.8. ([9], [12]) *Let S be a closed linear Pontryagin space symmetric operator with N^k-property, $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triple for the relation S^* such that $\rho(A_0) \neq \emptyset$, $0 \notin \sigma_p(S)$, $M(\lambda)$ be the corresponding Weyl function. Then the following equivalences hold:*

(i) $A_0 = S_{\rm F} \Leftrightarrow \lim\limits_{x\downarrow-\infty} M(x) = \infty$;

(ii) $A_0 = S_{\rm K} \Leftrightarrow \lim\limits_{x\uparrow 0} M(x) = \infty$;

(iii) $A_0 \cap S_{\rm F} = S \Leftrightarrow M(-\infty) := \lim\limits_{x\to-\infty} M(x) \in [\mathbb{C}^n]$*; in this case* $S_{\rm F} = \ker(\Gamma_1 - M(-\infty)\Gamma_0)$;

(iv) $A_0 \cap S_{\rm K} = S \Leftrightarrow M(0) := \lim\limits_{x\to 0} M(x) \in [\mathbb{C}^n]$*; in this case* $S_{\rm K} = \ker(\Gamma_1 - M(0)\Gamma_0)$.

Definition 2.9. A generalized resolvent $\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}(S)$ is said to be from *the class* $\Omega_{\widetilde{\kappa}}^{\widetilde{k}}(S)$ if it admits a representation (2.1) with a minimal extension $\widetilde{A} \in \mathrm{Ext}_S^{\widetilde{k}}(-\infty, 0)$.

A description of the class $\Omega_{\widetilde{\kappa}}^{\widetilde{k}}(S)$ was given in [12] (see also [10] for the case $\operatorname{clos}\mathcal{D}(S) = \Pi$).

Theorem 2.10. *Let S be a simple symmetric operator with N^k-property in a Pontryagin space Π, $\kappa := \kappa^-(\Pi)$, $\mathcal{T}_- = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* such that $A_0 = S_{\rm F}$ and $A_1 = S_{\rm K}$, $\rho(A_0) \neq \emptyset$, and let $M(\lambda)$ be the corresponding Weyl function. Then the formula* (2.3) *establishes a one-to-one correspondence between the set of generalized resolvents $\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}^{\widetilde{k}}(S)$ holomorphic at the point λ and the set of pairs $\{\varphi, \psi\} \in \widetilde{N}_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$ such that* $\det(\psi(\lambda) + M(\lambda)\varphi(\lambda)) \neq 0$.

Proposition 2.11. *Under the assumptions of Theorem* 2.10 *the formula* (2.3) *establishes a one-to-one correspondence between the set of all regular extensions $\widetilde{A}_\tau$ of S with N^k-property and the set of all nonpositive selfadjoint linear relations $\tau = \{\varphi, \psi\} \in \widetilde{\mathcal{C}}(\mathbb{C}^n)$ such that* $\det(\psi + M(\lambda)\varphi) \not\equiv 0$. *For every collection* *of points z_j, $(j = 1, \dots, m)$, there is a regular selfadjoint extension $\widetilde{A}$ of S with N^k-property such that $z_j \in \rho(\widetilde{A})$ $(j = 1, \dots, m)$.*

Proof. The first statement is immediate from Theorem 2.10. Let the Weyl function $M(\lambda)$ of the operator S corresponding to the boundary triple $\mathcal{T}_-$ have the expansions (1.11) and let $\mathcal{Z}_j$ be the sets defined in Proposition 1.9. As follows from Lemma 1.8, the set $\mathcal{Z} = \bigcup_j \mathcal{Z}_j$ is nowhere dense in $\mathcal{S}_n$ and, therefore, there is a nonpositive matrix $B \in \mathcal{S}_n \setminus \mathcal{Z}$. In view of Theorem 2.10 the corresponding extension $\widetilde{A}_B = \ker(\Gamma_1 + B\Gamma_0)$ has N^k-property. The reasoning from Proposition 2.9 shows that $z_j \in \rho(\widetilde{A}_B)$ for all $j = 1, \dots, m$. ∎

Representation theory of symmetric operators. Let $\mathcal{L}$ be a nondegenerate subspace of Π and $P_{\mathcal{L}}$ be the Pontryagin space orthogonal projection from Π onto $\mathcal{L}$. A compressed resolvent $P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}|\mathcal{L}$ of the extension $\widetilde{A} = \widetilde{A}^*(\supset S)$ is said to be an $\mathcal{L}$-resolvent of the operator S. Let us remind some facts from the representation theory of M.G. Kreĭn (see [22], [28], [13], [9]) which are necessary for the description of $\mathcal{L}$-resolvents. A point $\lambda \in \widehat{\rho}(S)$ is said to be an $\mathcal{L}$-regular point of the operator S and is written as $\lambda \in \rho(S, \mathcal{L})$ if the following direct decomposition holds

$$\Pi = \mathcal{R}(A - \lambda) \dot{+} \mathcal{L}. \tag{2.12}$$

Similarly $\infty \in \rho(S, \mathcal{L})$ if $\mathcal{D}(S)$ is closed in Π and $\Pi = \mathcal{D}(S) \dot{+} \mathcal{L}$. Let us define two operator valued functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ holomorphic on $\rho(S, \mathcal{L})$. Let $\mathcal{P}(\lambda)$ $(\mathcal{P}(\infty))$ be a skew projection from Π onto $\mathcal{L}$ parallel to $\mathcal{R}(S - \lambda)$ $(\mathcal{D}(S))$ and let $\mathcal{Q}(\lambda)$ $(\mathcal{Q}(\infty))$ be defined by the equality

$$\mathcal{Q}(\lambda) = P_{\mathcal{L}}(S - \lambda)^{-1}(I - \mathcal{P}(\lambda)), \quad \mathcal{Q}(\infty) = P_{\mathcal{L}} S (I - \mathcal{P}(\infty)). \tag{2.13}$$

Let for all $l \in \mathcal{L}$ (see [9])

$$\widehat{\mathcal{P}}(\lambda)^* l := \{\mathcal{P}(\lambda)^* l, \overline{\lambda}\mathcal{P}(\lambda)^* l\}, \quad \widehat{\mathcal{Q}}(\lambda)^* l := \{\mathcal{Q}(\lambda)^* l, \overline{\lambda}\mathcal{Q}(\lambda)^* l + l\}. \tag{2.14}$$

Then $\widehat{\mathcal{P}}(\lambda)^* l$, $\widehat{\mathcal{Q}}(\lambda)^* l \in S^*$. Analogously, setting

$$\widehat{\mathcal{P}}(\infty)^* l := \{0, \mathcal{P}(\infty)^* l\}, \quad \widehat{\mathcal{Q}}(\infty)^* l := \{l, \mathcal{Q}(\infty)^* l\} \quad (l \in \mathcal{L}) \tag{2.15}$$

we obtain $\widehat{\mathcal{P}}(\infty)^* l$, $\widehat{\mathcal{Q}}(\infty)^* l \in S^*$. Indeed, $\mathcal{P}(\infty)^* l \perp \mathcal{D}(S)$ since

$$[\mathcal{P}(\infty)^* l, h] = [l, \mathcal{P}(\infty) h] = 0 \quad (\forall\, h \in \mathcal{D}(S)). \tag{2.16}$$

Further, for all $h \in \mathcal{D}(S)$ we obtain in view of (2.13)

$$[\mathcal{Q}(\infty)^* l, h] - [l, Sh] = [l, \mathcal{Q}(\infty) h] - [l, Sh] = 0. \tag{2.17}$$

Hence $\{l, \mathcal{Q}(\infty)^* l\} \in S^*$. If $\lambda \in \rho(S, \mathcal{L})$ then $\mathcal{N}_\lambda = \mathcal{P}(\overline{\lambda})^* \mathcal{L}$ (see [9]). Analogously, if $\infty \in \rho(S, \mathcal{L})$, then $\operatorname{mul} S^* = \mathcal{P}(\infty)^* \mathcal{L}$. Let us set $\rho_{\mathrm{s}}(S, \mathcal{L}) = \rho(S, \mathcal{L}) \cap \overline{\rho(S, \mathcal{L})}$.

Theorem 2.12. *Let $\mathcal{L}$ be a nondegenerate subspace of* Π *and* $\lambda \in \rho_{\rm s}(S,\mathcal{L})$. *Then the following direct decomposition holds*

$$S^* = S \dotplus \widehat{\mathcal{P}}(\lambda)^*\mathcal{L} \dotplus \widehat{\mathcal{Q}}(\lambda)^*\mathcal{L}. \tag{2.18}$$

Proof. In the case $\lambda \neq \infty$ the statement was proved in [9]. For all vectors $\{f,f'\} \in S^*$ we set $f_0 = (I - \mathcal{P}(\infty))f$, $l = \mathcal{P}(\infty)f$. Then

$$\{f,f'\} - \{f_0, Sf_0\} - \{l, \mathcal{Q}(\infty)^*l\} = \{0, f' - Sf_0 - \mathcal{Q}(\infty)^*l\} \in S^*.$$

Therefore, there exists $k \in \mathcal{L}$ such that $f' - Sf_0 - \mathcal{Q}(\infty)^*l = \mathcal{P}(\infty)^*k$. Thus we have

$$\{f,f'\} = \{f_0, Sf_0\} + \{l, \mathcal{Q}(\infty)^*l\} + \{0, \mathcal{P}(\infty)^*k\} \in S + \widehat{\mathcal{P}}(\infty)^*\mathcal{L} + \widehat{\mathcal{Q}}(\infty)^*\mathcal{L}.$$

The inverse inclusion is evident. ∎

Using (2.18) we determine a family of boundary triples for the relation S^* in the case when the set $\rho_{\rm s}(S,\mathcal{L})$ is not empty.

Proposition 2.13. *Let $\mathcal{L}$ be a positive subspace of* Π *and* $\infty \in \rho_{\rm s}(S,\mathcal{L})$. *Then the boundary triple* $\{\mathcal{L},\Gamma_0,\Gamma_1\}$ *of the relation* S^* *can be defined by the formulas*

$$\Gamma_0\widehat{f} = l_0 = \mathcal{P}(\infty)f, \quad \Gamma_1\widehat{f} = l_1 = P_{\mathcal{L}}f' - \mathcal{Q}(\infty)f, \tag{2.19}$$

where

$$\widehat{f} = \{f,f'\} = \{f_0, Sf_0\} + \widehat{\mathcal{P}}(\infty)^*l_1 + \widehat{\mathcal{Q}}(\infty)^*l_0 \quad (f_0 \in \mathcal{D}(S), l_0, l_1 \in \mathcal{L}).$$

Proof. Let $\widehat{g} = \{g_0, Sg_0\} + \widehat{\mathcal{P}}(\infty)^*k_1 + \widehat{\mathcal{Q}}(\infty)^*k_0$ $(g_0 \in \mathcal{D}(S);\ k_0, k_1 \in \mathcal{L})$. Making use of (2.16), (2.17), (2.18) we obtain

$$\begin{aligned}
[f',g] - [f,g'] &= [Sf_0 + \mathcal{P}(\infty)^*l_1 + \mathcal{Q}(\infty)^*l_0, g_0 + k_0] \\
&\qquad - [f_0 + l_0, Sg_0 + \mathcal{P}(\infty)^*k_1 + \mathcal{Q}(\infty)^*k_0] \\
&= [Sf_0, k_0] - [l_0, Sg_0] + (l_1, k_0) - (l_0, k_1) \\
&\qquad + [\mathcal{Q}(\infty)^*l_0, g_0] - [f_0, \mathcal{Q}(\infty)^*k_0] \\
&= (l_1, k_0) - (l_0, k_1).
\end{aligned}$$

Remark 2.14. If $a = \overline{a} \in \rho_{\rm s}(S,\mathcal{L})$ then one can define a boundary triple of S^* by the formulas (see [13], [9])

$$\Gamma_0\widehat{f} = l_0 = P_{\mathcal{L}}f - \mathcal{Q}(a)(f' - af), \quad \Gamma_1\widehat{f} = l_1 = \mathcal{P}(a)(f' - af), \tag{2.20}$$

where $\widehat{f} = \widehat{f}_0 + \widehat{\mathcal{P}}(a)^* l_0 + \widehat{\mathcal{Q}}(a)^* l_1$ ($\widehat{f}_0 \in S$; $l_0, l_1 \in \mathcal{L}$) (see [13], [9]).

RESOLVENT MATRIX. Setting in the resolvent formula (2.3)

$$\begin{aligned} a_{11}(\lambda) &= M(\lambda), & a_{12}(\lambda) &= \gamma(\lambda)^*|\mathcal{L}, \\ a_{21}(\lambda) &= P_{\mathcal{L}}\gamma(\lambda), & a_{22}(\lambda) &= P_{\mathcal{L}}(A_0 - \lambda)^{-1}|\mathcal{L}, \end{aligned} \tag{2.21}$$

and taking into account that the operators $a_{12}(\lambda)$, $a_{21}(\lambda)$ are invertible for $\lambda \in \rho(S, \mathcal{L})$ (see [9]) one obtains from (2.3) for $\lambda \in \rho(S, \mathcal{L}) \cap \rho(A_0) \cap \rho(\widetilde{A})$

$$\begin{aligned} &P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}|\mathcal{L} \\ &\quad = a_{22}(\lambda) - a_{21}\varphi(\lambda)(\psi(\lambda) + a_{11}(\lambda)\varphi(\lambda))^{-1}a_{12}(\lambda) \\ &\quad = a_{22}(\lambda) - a_{21}(\lambda)\varphi(\lambda)(a_{12}(\lambda)^{-1}\psi(\lambda) + a_{12}(\lambda)^{-1}a_{11}(\lambda)\varphi(\lambda))^{-1} \\ &\quad = (w_{11}(\lambda)\psi(\lambda) + w_{12}(\lambda)\varphi(\lambda))(w_{21}(\lambda)\psi(\lambda) + w_{22}(\lambda)\varphi(\lambda))^{-1}, \end{aligned} \tag{2.22}$$

where the matrices $w_{ij} \in \mathcal{B}(\mathcal{L})$ $(i, j = 1, 2)$ are defined by

$$\begin{aligned} w_{11}(\lambda) &= a_{22}(\lambda)a_{12}(\lambda)^{-1}, & w_{12}(\lambda) &= a_{22}(\lambda)a_{12}(\lambda)^{-1}a_{11}(\lambda) - a_{21}(\lambda), \\ w_{21}(\lambda) &= a_{12}(\lambda)^{-1}, & w_{22}(\lambda) &= a_{12}(\lambda)^{-1}a_{11}(\lambda). \end{aligned} \tag{2.23}$$

The matrix $W(\lambda) = (w_{ij}(\lambda))_{i,j=1}^2$ is said to be the resolvent matrix of the operator S corresponding to the scale space $\mathcal{L}$ and the boundary triple $\mathcal{T}$. A simple formula for the calculation of the resolvent matrix $W(\lambda)$ was given in [9] (see [13] for the Hilbert space case).

PROPOSITION 2.15. *Let $\mathcal{L}$ be a subspace of Π such that $\rho_s(S, \mathcal{L}) \neq \emptyset$, $\mathcal{T} = \{\mathcal{L}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^*, $G(\lambda) := \operatorname{col}\big(-\mathcal{Q}(\lambda), \mathcal{P}(\lambda)\big)$, $\widehat{G}(\lambda)^* := \big(-\widehat{\mathcal{Q}}(\lambda)^*, \widehat{\mathcal{P}}(\lambda)^*\big)$. Then the resolvent matrix $W(\lambda)$ is*

$$W(\lambda) = \big(\Gamma\widehat{G}(\lambda)^*\big)^* = \begin{pmatrix} -\Gamma_0\widehat{\mathcal{Q}}(\lambda)^* & \Gamma_0\widehat{\mathcal{P}}(\lambda)^* \\ -\Gamma_1\widehat{\mathcal{Q}}(\lambda)^* & \Gamma_1\widehat{\mathcal{P}}(\lambda)^* \end{pmatrix}^* \quad (\lambda \in \rho(S, \mathcal{L})) \tag{2.24}$$

and satisfies the equality

$$\begin{gathered} J_{2n} - W(\lambda)J_{2n}W(\mu)^* = (\lambda - \overline{\mu})G(\lambda)G(\mu)^*; \\ J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \lambda, \mu \in \rho(S, \mathcal{L}). \end{gathered} \tag{2.25}$$

REMARK 2.16. It follows from (2.25) that the kernels

$$\frac{J_{2n} - W(\lambda)J_{2n}W(\mu)^*}{\lambda - \overline{\mu}}, \quad W(\lambda, \mu) = \frac{J_{2n} - W(\mu)^*J_{2n}W(\lambda)}{\lambda - \overline{\mu}} \tag{2.26}$$

have κ negative squares. Let $\{\varphi,\psi\}$ be an N_k-pair and the pair $\{\widetilde{\varphi},\widetilde{\psi}\}$ be defined by

$$\begin{pmatrix}\widetilde{\psi}(\lambda)\\ \widetilde{\varphi}(\lambda)\end{pmatrix} = W(\lambda)\Phi(\lambda), \quad \Phi(\lambda) = \begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}.$$

Then it follows from (2.26) that the kernel

$$N_{\widetilde{\varphi},\widetilde{\psi}}(\lambda,\mu) = N_{\varphi,\psi}(\lambda,\mu) + \Phi(\mu)^* W(\lambda,\mu)\Phi(\lambda)$$

also has a finite number κ' ($\leqslant \kappa + k$) of negative squares.

The following description of $\mathcal{L}$-resolvents of the operator S is implied by Theorem 2.3 and by the formulas (2.22), (2.24) (see [9]).

THEOREM 2.17. *Let S be a simple Pontryagin space symmetric operator, $\mathcal{L}$ be a positive subspace of Π such that $\rho_{\rm s}(S,\mathcal{L}) \neq \emptyset$, $\{\mathcal{L},\Gamma_0,\Gamma_1\}$ be a boundary triple of the relation S^*, and let $W(\lambda)$ be the corresponding resolvent matrix, $\lambda_0 \in \rho(A_0)\cap\rho(S,\mathcal{L})$, $\kappa = \kappa^-(\Pi)$. Then the formula (2.22) establishes a one-to-one correspondence between the set of $\mathcal{L}$-resolvents $P_{\mathcal{L}}\mathbf{R}_\lambda|\mathcal{L}$ ($\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}(S)$) holomorphic at the point λ_0 and the set of classes of equivalent $N_{\widetilde{\kappa}-\kappa}$-pairs $\{\varphi,\psi\}$ holomorphic at λ_0 such that*

$$\det\left(w_{21}(\lambda_0)\psi(\lambda_0) + w_{22}(\lambda_0)\varphi(\lambda_0)\right) \neq 0. \tag{2.27}$$

To prove this it remains to note that the matrix function $w_{21}(\lambda)$ is invertible for all $\lambda \in \rho(S,\mathcal{L})\cap\rho(A_0)$ and the corresponding Weyl function $M(\lambda)$ takes the form $M(\lambda) = w_{21}(\lambda)^{-1}w_{22}(\lambda)$. Therefore, the inequalities (2.4) and (2.27) are equivalent.

A description of $\mathcal{L}$-resolvents with $N^{\widetilde{k}}$-property is implied by Theorem 2.10.

THEOREM 2.18. *Let S be a simple Pontryagin space symmetric operator with N^k-property, $\mathcal{L}$ be a positive subspace of Π such that $\rho_{\rm s}(S,\mathcal{L}) \neq \emptyset$, $\mathcal{T}_- = \{\mathcal{L},\Gamma_0,\Gamma_1\}$ be a boundary triple for the relation S^* such that $A_0 = S_{\rm F}$, $A_1 = S_{\rm K}$, and let $W(\lambda) = (w_{ij}^-(\lambda))_{i,j=1}^2$ be the corresponding resolvent matrix, $\rho(A_0)\cap\rho(S,\mathcal{L}) \neq \emptyset$, $\kappa = \kappa^-(\Pi)$. Then the formula*

$$P_{\mathcal{L}}(\widetilde{A}-\lambda)^{-1}|\mathcal{L} = (w_{11}^-(\lambda)\psi(\lambda)+w_{12}^-(\lambda)\varphi(\lambda))(w_{21}^-(\lambda)\psi(\lambda)+w_{22}^-(\lambda)\varphi(\lambda))^{-1} \tag{2.28}$$

establishes a one-to-one correspondence between the set of all $\mathcal{L}$-resolvents $P_{\mathcal{L}}\mathbf{R}_\lambda|\mathcal{L}$ ($\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}^{\widetilde{k}}(S)$) holomorphic at the point λ_0 and the set of pairs $\{\varphi,\psi\} \in \widetilde{N}_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$ such that the nondegeneracy condition (2.27) holds.

REMARK 2.19. Let $\mathcal{T} = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ be an arbitrary boundary triple for the relation S^* related to the boundary triple $\mathcal{T}_-$ by the equality $\Gamma = U\Gamma_-$. It follows from (2.24) that the resolvent matrix $W(\lambda)$ associated with the scale space $\mathcal{L}$ and the boundary triple $\mathcal{T}$ is related to the matrix $W_-(\lambda)$ by the equality

$$W(\lambda) = W_-(\lambda)U^*. \tag{2.29}$$

Under the assumptions of Theorem 2.18, the formula (2.22) establishes a one-to-one correspondence between the set of all $\mathcal{L}$-resolvents $P_{\mathcal{L}}\mathbf{R}_\lambda|\mathcal{L}$ ($\mathbf{R}_\lambda \in \Omega_{\widetilde{\kappa}}^{\widetilde{k}}(S)$) holomorphic at the point λ_0 and the set of pairs $\{\varphi, \psi\}$ represented in the form $\{\psi, \varphi\} = U\{\widetilde{\psi}, \widetilde{\varphi}\}$ where $\{\widetilde{\varphi}, \widetilde{\psi}\} \in \widetilde{N}_{\widetilde{\kappa-\kappa}}^{-(\widetilde{k-k})}$ and such that the nondegeneracy condition (2.27) holds.

3. INTERPOLATION IN THE GENERALIZED NEVANLINNA CLASS

PROBLEM (IP$_\kappa$). Given are $n, m \in \mathbb{N}$; $\kappa, r_j \in \mathbb{Z}_+$, $z_j \in \mathbb{C} \setminus \mathbb{R}$ ($j = 1, \dots, m$) and $n \times 1$-matrices V_{jp}, W_{jp} ($j = 1, \dots, m$; $p = 0, 1, \dots, r_j$), find an N_κ-pair $\{A(\lambda), B(\lambda)\}$ holomorphic at the points z_j such that the following equalities hold

$$\sum_{k=0}^{r_j} V_{jk}^*(\lambda - z_j)^k B(\lambda) = \sum_{k=0}^{r_j} W_{jk}^*(\lambda - z_j)^k A(\lambda) + \mathrm{O}\left((\lambda - z_j)^{r_j+1}\right) \tag{3.1}$$

$(\lambda \to z_j;\ j = 1, \dots, m)$.

Let $r = \sum_{j=1}^{m} (r_j + 1)$. The $n \times r$-matrices

$$V = (V_1, \dots, V_m), \quad W = (W_1, \dots, W_m), \tag{3.2}$$

where $V_j = (V_{j0}, \dots, V_{jr_j})$, $W_j = (W_{j0}, \dots, W_{jr_j})$ are called the data of the Problem (IP$_\kappa$). Let Z be the $r \times r$-matrix which is the direct sum of $(r_j+1) \times (r_j+1)$-Jordan boxes

$$Z_j = \begin{pmatrix} \overline{z}_j & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & \overline{z}_j \end{pmatrix} \quad (j = 1, \dots, m). \tag{3.3}$$

Two $m \times n$-matrix functions $\mathcal{K}(\lambda)$, $\mathcal{L}(\lambda)$ with rows $\mathcal{K}_j(\lambda)$, $\mathcal{L}_j(\lambda)$ defined and locally holomorphic on $\mathbb{C} \setminus \mathbb{R}$ are said to interpolate the data V, W of the Problem (IP$_\kappa$) if (see [3])

$$\frac{1}{p!}\mathcal{K}_j^{(p)}(z_j) = V_{jq}^*, \quad \frac{1}{p!}\mathcal{L}_j^{(p)}(z_j) = W_{jp}^*, \tag{3.4}$$

($j = 1,\dots,m$; $p = 0,\dots,r_j$). Associated to each pair $\mathcal{K}(\lambda),\mathcal{L}(\lambda)$ is the $r\times r$-matrix $\mathcal{R}_{\mathcal{K}\mathcal{L}} = \big((\mathcal{R}_{\mathcal{K}\mathcal{L}})_{ij}^{pq}\big)$ defined by

$$(3.5)\quad (\mathcal{R}_{\mathcal{K}\mathcal{L}})_{ij}^{pq} = \mathrm{D}_l^p \mathrm{D}_{\overline{\lambda}}^q \left.\frac{\mathcal{L}_i(l)\mathcal{K}_j(\lambda)^* - \mathcal{K}_i(l)\mathcal{L}_j(\lambda)^*}{l-\overline{\lambda}}\right|_{l=z_i,\,\lambda=z_j} \qquad \Big(\mathrm{D}_l^p = \frac{1}{p!}\frac{\mathrm{d}^p}{\mathrm{d}l^p}\Big).$$

Let us remind the following statement from [3].

Proposition 3.1. *Suppose that the matrices $\mathcal{K}(\lambda)$, $\mathcal{L}(\lambda)$ interpolate the data of Problem* (IP_κ). *Then the matrix $\mathbf{P} = \mathcal{R}_{\mathcal{K}\mathcal{L}}$ is a solution of the Lyapunov equation*

$$(3.6)\qquad \mathbf{P}Z - Z^*\mathbf{P} = V^*W - W^*V$$

if and only if for all indices i,j such that $z_i = \overline{z}_j$ the following consistency conditions hold

$$(3.7)\qquad \mathrm{D}_\lambda^p\left(\mathcal{L}_i(\lambda)\mathcal{K}_j(\overline{\lambda})^* - \mathcal{K}_i(\lambda)\mathcal{L}_j(\overline{\lambda})^*\right)\big|_{\lambda=z_i} = 0 \quad (0\leqslant p\leqslant \min\{r_i,r_j\}).$$

One can rewrite the equalities (3.1) in the form

$$(3.8)\qquad \mathrm{D}_\lambda^p\left(\mathcal{K}_i(\lambda)B(\lambda) - \mathcal{L}_i(\lambda)A(\lambda)\right)\big|_{\lambda=z_i} = 0 \quad (i=1,\dots,m;\ 0\leqslant p\leqslant r_i),$$

where $\mathcal{L}(\lambda),\mathcal{K}(\lambda)$ is any pair of matrices which interpolate the data V,W.

Let the pair $\{A(\lambda),B(\lambda)\}$ be a solution of Problem (IP_κ) such that

$$(3.9)\qquad \det\left(B(z_j) + z_jA(z_j)\right) \neq 0 \quad (j=1,\dots,m).$$

Associated to the pair $\{A(\lambda),B(\lambda)\}$ is an equivalent pair $\{\widehat{A}(\lambda),\widehat{B}(\lambda)\}$

$$(3.10)\qquad \widehat{A}(\lambda) = A(\lambda)\big(B(\lambda)+\lambda A(\lambda)\big)^{-1},\quad \widehat{B}(\lambda) = B(\lambda)\big(B(\lambda)+\lambda A(\lambda)\big)^{-1}.$$

Corollary 3.2. *Let the pair $\{A(\lambda),B(\lambda)\}$ be a solution of Problem* (IP_κ) *which satisfies* (3.9) *and the pair $\widetilde{\mathcal{K}}(\lambda)$, $\widetilde{\mathcal{L}}(\lambda)$ be defined by*

$$(3.11)\qquad \widetilde{\mathcal{K}}(\lambda) = (\lambda\mathcal{K}(\lambda)+\mathcal{L}(\lambda))\widehat{A}(\lambda),\quad \widetilde{\mathcal{L}}(\lambda) = (\lambda\mathcal{K}(\lambda)+\mathcal{L}(\lambda))\widehat{B}(\lambda),$$

where $\mathcal{K}(\lambda),\mathcal{L}(\lambda)$ is any pair which interpolates the data V,W. Then the $r\times r$-matrix $\big((\mathcal{R}_{\widetilde{\mathcal{K}},\widetilde{\mathcal{L}}})_{ij}^{pq}\big)$ is a solution of the Lyapunov equation (3.6).

Proof. The matrices $\widetilde{\mathcal{K}}(\lambda),\widetilde{\mathcal{L}}(\lambda)$ interpolate the data V,W. Indeed

$$(3.12)\qquad \begin{aligned} &\big(\mathcal{K}(\lambda) - \widetilde{\mathcal{K}}\,(\lambda)\big)^{(p)}\Big|_{\lambda=z_i} \\ &\quad = \big((\mathcal{K}_i(\lambda)B(\lambda) - \mathcal{L}_i(\lambda)A(\lambda))(\lambda A(\lambda)+B(\lambda))^{-1}\big)^{(p)}\Big|_{\lambda=z_i} = 0; \end{aligned}$$

$$\begin{aligned}(3.13)\qquad &\left(\mathcal{L}(\lambda)-\widetilde{\mathcal{L}}\,(\lambda)\right)^{(p)}\Big|_{\lambda=z_i}\\ &= \left((\mathcal{L}_i(\lambda)A(\lambda)-\mathcal{K}_i(\lambda)B(\lambda))\lambda(\lambda A(\lambda)+B(\lambda))^{-1}\right)^{(p)}\Big|_{\lambda=z_i}=0\end{aligned}$$

for all $i=1,\dots,m$; $p=0,1,\dots,r_i$ and satisfy the equation

$$\widetilde{\mathcal{L}}(\lambda)\widetilde{\mathcal{K}}(\overline{\lambda})-\widetilde{\mathcal{K}}(\lambda)\widetilde{\mathcal{L}}(\overline{\lambda})^*=0$$

is a neighborhood of the set $\{z_j\}_{j=1}^m$. It follows from Proposition 3.1 that the matrix $\mathbf{P}_{AB}$ satisfies the Lyapunov equation (3.6). ∎

The matrix $\mathbf{P}_{AB}=\left((\mathcal{R}_{\widetilde{\mathcal{K}},\widetilde{\mathcal{L}}})_{ij}^{pq}\right)$ is said to be the Pick matrix corresponding to the pair $\{A,B\}$ and Problem (IP$_\kappa$).

Remark 3.3. It follows from the symmetry condition (iii) of Definition 1.1 that $\widehat{A}(\lambda)=\widehat{A}(\overline{\lambda})^*$, $\widehat{B}(\lambda)=\widehat{B}(\overline{\lambda})^*$. Making use of these equalities one can rewrite the matrix $\mathbf{P}_{AB}$ in the form

$$(3.14)\qquad (\mathbf{P}_{AB})_{ij}^{pq}=\mathrm{D}_l^p\mathrm{D}_{\overline{\lambda}}^q\left[\Phi_i(l)\frac{B(\bar{l})^*A(\overline{\lambda})-A(\bar{l})^*B(\overline{\lambda})}{l-\overline{\lambda}}\Phi_j(\lambda)^*\right]\Big|_{l=z_i,\,\lambda=z_j},$$

where

$$(3.15)\qquad \Phi_i(l)=(l\mathcal{K}_i(l)+\mathcal{L}_i(l))(B(\bar{l})^*+lA(\bar{l})^*)^{-1}.$$

Proposition 3.4. *Let an N_κ-pair $\{A(\lambda),B(\lambda)\}$ be a solution of Problem* (IP$_\kappa$)*, X be a positive matrix and let the conditions* (3.9) *be satisfied both for $\{A(\lambda),B(\lambda)\}$ and $\{X^{1/2}A(\lambda),X^{-1/2}B(\lambda)\}$. Then $\mathbf{P}_{AB}=\mathbf{P}_{X^{1/2}A,X^{-1/2}B}$.*

Proof. The matrix $\widetilde{\mathbf{P}}=\mathbf{P}_{X^{1/2}A,X^{-1/2}B}$ associated with the pair $\{X^{1/2}A(\lambda),$ $X^{-1/2}B(\lambda)\}$ and Problem $(\widetilde{\mathrm{IP}_\kappa})$ can be written in the form

$$(3.16)\qquad \widetilde{\mathbf{P}}_{ij}^{pq}=\mathrm{D}_l^p\mathrm{D}_{\overline{\lambda}}^q\left[\widetilde{\Phi}_i(l)\frac{B(\bar{l})^*A(\overline{\lambda})-A(\bar{l})^*B(\overline{\lambda})}{l-\overline{\lambda}}\widetilde{\Phi}_j(\lambda)^*\right]\Big|_{l=z_i,\,\lambda=z_j},$$

where

$$(3.17)\qquad \widetilde{\Phi}_i(l)=(l\mathcal{K}_i(l)X+\mathcal{L}_i(l))(B(\bar{l})^*+lA(\bar{l})^*X)^{-1}.$$

To prove the equality $\mathbf{P}_{AB}=\widetilde{\mathbf{P}}$ it is sufficient to show that

$$(3.18)\qquad \mathrm{D}_l^p\;\Phi_i(l)|_{l=z_i}=\mathrm{D}_l^p\;\widetilde{\Phi}_i(l)\Big|_{l=z_i}\qquad (i=1,\dots,m;\;p=0,1,\dots,r_i).$$

Indeed, it follows from (3.13) that

$$\mathrm{D}_l^p\left[(l\mathcal{K}_i(l)+\mathcal{L}_i(l))\widehat{B}(l)\Psi(l)\right]\Big|_{l=z_i}=\mathrm{D}_l^p\left[\mathcal{L}_i(l))\Psi(l)\right]\big|_{l=z_i} \tag{3.19}$$

for each matrix function $\Psi(l)$ holomorphic at $l=z_i$. From (3.19) and the identity

$$(B(\bar{l})^*+lA(\bar{l})^*)^{-1}=\left[X+\widehat{B}(l)(I-X)\right](B(\bar{l})^*+lA(\bar{l})^*X)^{-1}$$

it follows that

$$\begin{aligned}\mathrm{D}_l^p\Phi_i(l)\Big|_{l=z_i}&=\mathrm{D}_l^p\Big\{(l\mathcal{K}_i(l)+\mathcal{L}_i(l))X(B(\bar{l})^*+lA(\bar{l})^*X)^{-1}\\&\qquad+(l\mathcal{K}_i(l)+\mathcal{L}_i(l))\widehat{B}(l)(I-X)(B(\bar{l})^*+lA(\bar{l})^*X)^{-1}\Big\}\Big|_{l=z_i}\\&=\mathrm{D}_l^p\left\{(l\mathcal{K}_i(l)X+\mathcal{L}_i(l)X+\mathcal{L}_i(l)(I-X))(B(\bar{l})^*+lA(\bar{l})^*X)^{-1}\right\}\big|_{l=z_i}\\&=\mathrm{D}_l^p\left\{(l\mathcal{K}_i(l)X+\mathcal{L}_i(l))(B(\bar{l})^*+lA(\bar{l})^*X)^{-1}\right\}\big|_{l=z_i}\\&=\mathrm{D}_l^p\,\widetilde{\Phi}_i(l)\Big|_{l=z_i}.\quad\blacksquare\end{aligned}$$

Definition 3.5. Let $\{A(\lambda),B(\lambda)\}$ be a solution of Problem (IP_κ), which not necessarily satisfies (3.9) and let X be a positive $n\times n$-matrix such that the pair $\{X^{1/2}A,X^{-1/2}B\}$ satisfies (3.9). Then the matrix $\mathbf{P}_{AB}:=\mathbf{P}$ defined by the formulas (3.16) is said to be the *Pick matrix corresponding to the pair* $\{A,B\}$ *and Problem* (IP_κ). In turn, the pair $\{A(\lambda),B(\lambda)\}$ is said to be *associated with the Pick matrix* $\mathbf{P}$.

The definition is correct by Proposition 3.4. The cases where some columns V_{j0} or W_{j0} of the data matrices V,W vanish are not excluded if $\kappa\neq 0$ (see Example 4.11 below). If, for example, $V_{10}=0$ (or $W_{10}=0$) then it follows from (3.6) that $P_{11}^{00}(\bar{z}_1-z_1)=0$ and, therefore, $P_{11}^{00}=0$. This is impossible if $\mathbf{P}$ is a positive nondegenerate matrix. It is natural to assume, however, that at least one of the columns V_{j0} or W_{j0} is not trivial. Moreover, we assume that the data matrices V,W,Z satisfy the assumption

$$\bigcap_{j=0}^{\infty}\ker(W+VZ)Z^j=\{0\}. \tag{3.20}$$

Now we are in a position to formulate Problem $\mathrm{IP}_\kappa(V,W,Z,\mathbf{P})$.

PROBLEM $\mathrm{IP}_\kappa(V,W,Z,\mathbf{P})$. Given are:

(i) points $z_1,\dots,z_m\in\mathbb{C}\setminus\mathbb{R}$; integers $n,m\in\mathbb{N}$, $r_1,\dots,r_m,\kappa\in\mathbb{Z}_+$, $r=\sum_{j=1}^{m}(r_j+1)$;

(ii) $n\times r$-matrices V,W of the form (3.2) such that the assumption (3.20) is fulfilled;

(iii) $m\times n$-matrix functions $\mathcal{K}(\lambda),\mathcal{L}(\lambda)$ defined and locally holomorphic on $\mathbb{C}\setminus\mathbb{R}$ which interpolate the data V,W;

(iv) A Hermitian solution $\mathbf{P}$ of the Lyapunov equation (3.6).

Find: N_κ-pair $\{A(\lambda),B(\lambda)\}$ holomorphic at the points z_j $(j=1,\dots,m)$ such that the equations (3.8) hold and $\mathbf{P}_{AB}=\mathbf{P}$.

We shall consider the case where the Pick matrix $\mathbf{P}$ is nondegenerate. The indefinite degenerate Nevanlinna-Pick problem was investigated in [33].

THE MODEL OPERATOR. The operator approach to Problem IP_0 $(V,W,Z,\mathbf{P})$ was elaborated in [3]. We apply the model operator from [3] to the case $\kappa\neq 0$. Let $\mathcal{B}$ be the linear space of formal sums

$$f(t)=f_2+E(t)f_1\quad (f_1\in\mathbb{C}^r,\ f_2\in\mathbb{C}^n,\ E(t)=(W+tV)(Z-t)^{-1}),\tag{3.21}$$

equipped with the inner product

$$[f,g]=(\mathbf{P}f_1,g_1)_r+(f_2,g_2)_n\quad (g=g_2+E(t)g_1).\tag{3.22}$$

If the Pick matrix $\mathbf{P}$ is nondegenerate then the space $\mathcal{B}$ is a Pontryagin space of the negative index $\mathrm{sq}_-(\mathbf{P})$.

LEMMA 3.6. *The mapping $\mathcal{I}:\{f_1,f_2\}\mapsto f(t)=f_2+E(t)f_1$ from $\mathbb{C}^n\oplus\mathbb{C}^r$ into $\mathcal{B}$ is an isomorphism iff the assumption* (3.20) *holds.*

Proof. It follows from the equality

$$\begin{aligned} f(t)&=f_2+(W+Vt)(Z-t)^{-1}f_1=(f_2-Vf_1)+(W+VZ)(Z-t)^{-1}f_1\\ &=(f_2-Vf_1)-(W+VZ)\sum_{j=0}^{\infty}Z^jt^{-(j+1)}f_1\end{aligned}\tag{3.23}$$

that $f(t)\equiv 0$ if and only if $f_2-Vf_1=0$ and $f_1\in\ker(W+VZ)Z^j$ for all $j\geqslant 0$.

Assume that the hypothesis (3.20) holds and $f(t)\equiv 0$. Then it follows from (3.23) that $f_1=f_2=0$. Conversely, assume that $f_1\in\bigcap_{j=0}^{\infty}\ker(W+VZ)Z^j$, $(f_1\neq 0)$ and let $f(t)=Vf_1+(W+tV)(Z-t)^{-1}f_1$. Then

$$f(t)=-(W+VZ)\sum_{j=0}^{\infty}Z^jt^{-(j+1)}f_1\equiv 0,$$

while $f_1\neq 0$. Therefore, the mapping $\mathcal{I}$ has a nontrivial kernel. ∎

Due to Lemma 3.6 one can consider the model space $\mathcal{B}$ as a space of rational functions. Let us consider the multiplication S in $\mathcal{B}$ which corresponds to the linear manifold

$$
\begin{aligned}
S &= \{\{f(t), tf(t)\} \mid f(t), tf(t) \in H\} \\
&= \{\{Vf + E(t)f, -Wf + E(t)Zf\} \mid f \in \mathbb{C}^r\}.
\end{aligned}
\tag{3.24}
$$

Proposition 3.7. *S is a symmetric operator and the point spectrum of S is empty.*

Proof. The first statement is implied by the Lyapunov equation (3.6). Assume that $\lambda \in \sigma_{\mathrm{p}}(S)$. Then it follows from the relation

$$
S - \lambda = \{\{Vf + E(t)f, -(W + \lambda V)f + E(t)(Z - \lambda)f\} \mid f \in \mathbb{C}^r\}
$$

that there is $f \neq 0$ such that $(Z - \lambda)f = 0$, $(W + \lambda V)f = 0$. This implies $f \in \bigcap\limits_{j=0}^{\infty} \ker(W + VZ)Z^j$, which contradicts the assumption (3.20). ∎

Moreover, as is shown below (see Proposition 4.1) S is a simple symmetric operator in $\mathcal{B}$ with deficiency indices $n_{\pm}(S) = n$. Let G be the embedding mapping $G : \mathbb{C}^n \subset \mathcal{B}$. Then G^* is the projection from $\mathcal{B}$ onto $\mathbb{C}^n$ and $G^*G = I_{\mathbb{C}^n}$. Let e_{jp} be a standart basis in $\mathbb{C}^r$, $(j = 1, \ldots, m;\ p = 0, \ldots, r_j)$.

Proposition 3.8. *Let $\widetilde{A}$ be a selfadjoint extension of the relation S with the exit in a bigger Pontryagin space $\widetilde{\Pi} \supset \mathcal{B}$ such that $z_i \in \rho(\widetilde{A})$ $(i = 1, \ldots, m)$. Then $\widetilde{A}$ is a minimal extension of S iff $\widetilde{A}$ is G-minimal.*

Proof. In order to prove G-minimality of the extension $\widetilde{A}$ it is sufficient to prove the inclusion

$$
\mathcal{B} \subset \Pi(G, \widetilde{A}) := \text{c.l.s.}\{Gh + (\widetilde{A} - \lambda)^{-1}Gf \mid f, h \in \mathbb{C}^n,\ \lambda \in \rho(\widetilde{A})\}.
\tag{3.25}
$$

Indeed, $\mathbb{C}^n \subset \Pi(G, \widetilde{A})$. For all $i = 1, \ldots, m$, we have

$$
-E(t)e_{i,0} = \left\{(S - \overline{z}_i)^{-1}(W + \overline{z}_i V) + V\right\} e_{i,0} \in \Pi(G, \widetilde{A}).
$$

Analogously, for all $i = 1, \ldots, m$, $p = 0, \ldots, r_i$, we obtain

$$
-E(t)e_{i,p} = \sum_{k=0}^{p} (\widetilde{A} - \overline{z}_i)^{-k}\left\{(\widetilde{A} - \overline{z}_i)^{-1}(W + \overline{z}_i V) + V\right\}e_{i,p-k} \in \Pi(G, \widetilde{A}).
\tag{3.26}
$$

This proves the inclusion (3.25). The inverse statement is evident. ∎

As shown in [30], Problem $\mathrm{IP}_0(V, W, Z, \mathbf{P})$ is solvable if and only if the matrix $\mathbf{P}$ is nonnegative. In the next theorem the condition $\mathrm{sq}_-(\mathbf{P}) \leqslant \kappa$ is proved to be necessary and sufficient for Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$ to be solvable in the class $\widetilde{N}_\kappa$.

THEOREM 3.9. *Let* $\mathbf{P}$ *be nondegenerate.*

(i) *If* $sq_-(\mathbf{P}) \leqslant \kappa$ *then for each minimal selfadjoint extension* $\widetilde{A}$ *of the relation* S *with the exit in a bigger Pontryagin space* $\widetilde{\Pi} \supset \mathcal{B}$ *such that* $\kappa^-(\widetilde{\Pi}) = \kappa$ *and* $z_j \in \rho(\widetilde{A})$ $(j = 1, \dots, m)$, *the* N_κ*-pair defined by the equality*

$$\{A(\lambda), B(\lambda)\} = \{-G^*(\widetilde{A} - \lambda)^{-1}G, I + \lambda G^*(\widetilde{A} - \lambda)^{-1}G\} \quad (\lambda \in \rho(\widetilde{A})) \tag{3.27}$$

is a solution of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$.

(ii) *Conversely, let the* N_κ*-pair* $\{A(\lambda), B(\lambda)\}$ *be a solution of Problem* $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$. *Then* $sq_-(\mathbf{P}) \leqslant \kappa$. *If, additionally,* $B(\lambda) + \lambda A(\lambda) \equiv I$ *then the pair* $\{A(\lambda), B(\lambda)\}$ *has a representation* (3.27), *where* $\widetilde{A}$ *is a minimal selfadjoint extension of the relation* S *such that* $z_j \in \rho(\widetilde{A})$ $(j = 1, \dots, m)$.

Proof. (i) Let the N_κ-pair $\{A(\lambda), B(\lambda)\}$ admit a representation (3.27) in which $\widetilde{A}$ is a minimal selfadjoint extension of S in a Pontryagin space $\widetilde{\Pi} \supset \mathcal{B}$ and $z_j \in \rho(\widetilde{A})$ $(j = 1, \dots, m)$. Then the following equality holds

$$\begin{aligned}
&\mathrm{D}^p_\lambda \{\, \mathcal{K}_i(\lambda) + (\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))\, G^* R_\lambda \,\} \,|_{\lambda = z_i} \\
&\quad = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathcal{K}_i(\lambda) + (\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))G^* R_\lambda}{(\lambda - z_i)^{p+1}} \, \mathrm{d}\lambda \\
&\quad = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \sum_{k=0}^{p-1} \frac{\mathcal{K}_i(\lambda)G^* + (z_i\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))G^* R_{z_i}}{(\lambda - z_i)^{p-k+1}} R^k_{z_i} \\
&\qquad + \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathcal{K}_i(\lambda)G^* + (\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))G^* R_\lambda}{\lambda - z_i} R^p_{z_i} \, \mathrm{d}\lambda \\
&\quad = \sum_{k=0}^{p} \left\{ V^*_{i,p-k} G^* + (z_i V^*_{i,p-k} + W^*_{i,p-k}) G^* R_{z_i} \right\} R^k_{z_i}.
\end{aligned} \tag{3.28}$$

Here $R_\lambda = (\widetilde{A} - \lambda)^{-1}$ and $\gamma = \partial B(z_i, \varepsilon)$ where $B(z_i, \varepsilon) = \{\lambda : |\lambda - z_i| \leqslant \varepsilon\} \subset \rho(\widetilde{A})$. Setting $\varphi_{ip} = -E(t)e_{ip}$ $(i = 1, \dots, m;\ p = 0, \dots, r_i - 1)$ we obtain from (3.26) and (3.28)

$$\begin{aligned}
&\mathrm{D}^p_\lambda \{\, \mathcal{K}_i(\lambda)B(\lambda) - \mathcal{L}_i(\lambda)A(\lambda) \,\}|_{\lambda = z_i} \\
&\quad = \mathrm{D}^p_\lambda \{\mathcal{K}_i(\lambda) - (\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))A(\lambda)\} \,|_{\lambda = z_i} \\
&\quad = \left(G^* \sum_{k=0}^{p} R^k_{\overline{z}_i} \{R_{\overline{z}_i} G(W + \overline{z}_i V) + GV\} e_{i,p-k} \right)^* = (G^* \varphi_{ip})^* = 0.
\end{aligned} \tag{3.29}$$

Therefore, the pair $\{A(\lambda), B(\lambda)\}$ satisfies the equations (3.8).

Using (3.29) one can rewrite the equality (3.28) in the form

$$\begin{aligned}
&\mathrm{D}_\lambda^p \{\mathcal{K}_i(\lambda) + (\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))G^* R_\lambda\}\big|_{\lambda=z_i} \\
&\qquad = \mathrm{D}_\lambda^p \{(\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))G^* R(\lambda)(I-G)\}\big|_{\lambda=z_i}.
\end{aligned} \tag{3.30}$$

It follows from (3.26), (3.28) and (3.30) that

$$\begin{aligned}
P_{ij}^{pq} &= [E(t)e_{j,q}, E(t)e_{i,p}] \\
&= \mathrm{D}_\lambda^p \mathrm{D}_{\overline{\mu}}^q \{(\lambda\mathcal{K}_i(\lambda)+\mathcal{L}_i(\lambda))G^* R_\lambda(I-GG^*)R_{\overline{\mu}}(\mu\mathcal{K}_j(\mu)+\mathcal{L}_j(\mu))^*\}\Big|_{\substack{\lambda=z_i\\ \mu=\overline{z}_j}} \\
&= \mathrm{D}_\lambda^p \mathrm{D}_{\overline{\mu}}^q \Big\{(\lambda\mathcal{K}_i(\lambda) + \mathcal{L}_i(\lambda))\frac{B(\lambda)A(\mu)^* - A(\lambda)B(\mu)^*}{\lambda-\overline{\mu}}\cdot \\
&\qquad \cdot (\mu\mathcal{K}_j(\mu) + \mathcal{L}_j(\mu))^*\Big\}\Big|_{\substack{\lambda=z_i\\ \mu=\overline{z}_j}} \\
&= (\mathbf{P}_{AB})_{ij}^{pq}.
\end{aligned} \tag{3.31}$$

Thus the pair $\{A(\lambda), B(\lambda)\}$ is a solution of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$.

(ii) Conversely, let the N_κ-pair $\{A(\lambda), B(\lambda)\}$ be a solution of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$ such that $B(\lambda)+\lambda A(\lambda) \equiv I$ in a domain $\mathcal{O}$. Then in view of Theorem 2.4 there exists a $\widetilde{G}$-minimal selfadjoint relation $\widetilde{A}$ in $\widetilde{\Pi} = \mathbb{C}^n \oplus \Pi_\kappa$ (where $\widetilde{G} : \mathbb{C}^n \subset \widetilde{\Pi}$ is an embedding operator) such that the following representation holds

$$\{A(\lambda), B(\lambda)\} = \{-\widetilde{G}^*(\widetilde{A}-\lambda)^{-1}\widetilde{G},\, I + \lambda\widetilde{G}^*(\widetilde{A}-\lambda)^{-1}\widetilde{G}\} \tag{3.32}$$

and $\mathcal{O} \subset \rho(\widetilde{A})$. Let us define a linear operator U from $\mathcal{B}$ into $\widetilde{\Pi}$ by

$$\begin{aligned}
U\varphi_{ip} &= \sum_{k=0}^{p}(\widetilde{A}-\overline{z}_i)^{-k}((\widetilde{A}-\overline{z}_i)^{-1}\widetilde{G}(W+\overline{z}_i V) + \widetilde{G}V)e_{ip-k} \\
&\qquad\qquad (i = 1, \ldots, m;\ p = 0, \ldots, r_i); \\
UGf &= \widetilde{G}f \quad \forall\, f \in \mathbb{C}^n.
\end{aligned} \tag{3.33}$$

It follows from (3.32), (3.29) and (3.31) that

$$[U\varphi_{ip}, U\varphi_{jq}] = P_{ij}^{pq} = [\varphi_{ip}, \varphi_{jq}] \quad (i, j = 1, \ldots, m;\ p = 0, \ldots, r_i;\ q = 0, \ldots, r_j),$$

$$[U\varphi_{ip}, \widetilde{G}f] = 0 = [\varphi_{ip}, Gf], \quad [\widetilde{G}f, \widetilde{G}f] = [Gf, Gf] \quad (\forall\, f \in \mathbb{C}^n).$$

Hence, the operator U is isometric from $\mathcal{B}$ into $\widetilde{\Pi}$. The inclusion $A = USU^{-1} (\subset \widetilde{A})$ is implied by the relations (3.26), (3.33). One can identify now S with the operator

A and, therefore, $\widetilde{A}$ is a minimal extension of S. The inequality $\mathrm{sq}_-(\mathbf{P}) \leqslant \kappa$ follows from the representation (3.31).

Suppose now that $\{A(\lambda), B(\lambda)\}$ is a solution of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$ and the matrix $B(\lambda) + \lambda A(\lambda)$ is degenerate for some z_j $(j = 1, \dots, m)$. Then by Lemma 2.6 there is a nonnegative invertible matrix X such that $\det(B(z_j) + z_j X A(z_j)) \neq 0$ for all $j = 1, \dots, m$. The pair $\{X^{1/2}A(\lambda), X^{-1/2}B(\lambda)\}$ is a solution of Problem $\mathrm{IP}_\kappa(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$ with the data matrix $\{\widetilde{V}, \widetilde{W}\} = \{X^{1/2}V, X^{-1/2}W\}$. Therefore, $\mathrm{sq}_-(\mathbf{P}) \leqslant \kappa$. ∎

COROLLARY 3.10. *If $\mathbf{P}$ is nondegenerate and $\kappa := sq_-(\mathbf{P}) \leqslant \widetilde{\kappa}$ then Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ has a solution.*

Proof. In the case where $\widetilde{\kappa} = \kappa$ it follows from Proposition 1.9 that there is a selfadjoint extension A_0 of the operator S such that $z_j \in \rho(A_0)$ $(j = 1, \dots, m)$. The solvability of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$ is implied by Theorem 3.9.

In the case where $\widetilde{\kappa} > \kappa$ we consider a boundary triple $\mathcal{T} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* such that $A_0 = \ker \Gamma_0$. Let $M(\lambda)$ be the corresponding Weyl function and let $\tau(\lambda)$ be any function from the class $N_{\widetilde{\kappa}-\kappa}$ holomorphic at the points z_j and such that $\det \tau(z_j) \neq 0$ $(j = 1, \dots, m)$. Then the polynomials $\Delta_j(\varepsilon) = \det(\varepsilon\tau(z_j) + M(z_j))$ have a finite number of zeros. Let us choose $\varepsilon > 0$ such that $\Delta_j(\varepsilon) \neq 0$ for all $j = 1, \dots, m$. It follows from Theorem 3.2. that there is a minimal selfadjoint extension $\widetilde{A} \supset S$ in a Pontryagin space $\widetilde{\Pi}$ such that $\kappa^-(\widetilde{\Pi}) = \widetilde{\kappa}$ and $z_j \in \rho(\widetilde{A})$ for all $j = 1, \dots, m$. Now the statement is implied by Theorem 3.9 ∎

PROPOSITION 3.11. *Suppose that for some choice of κ numbers $\lambda_j \in \mathbb{C}_+$ $(\lambda_j \neq z_k, j = 1, \dots, \kappa;\ k = 1, \dots, m)$ the following condition holds*

$$\operatorname{rank}(V^* : (Z^* - \lambda_1)^{-1}V^* : \cdots : (Z^* - \lambda_\kappa)^{-1}V^*) = (\kappa + 1)n. \tag{3.34}$$

Then all solutions of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$ are functions.

Proof. Suppose that there exists a solution $\{A(\lambda), B(\lambda)\}$ of Problem $\mathrm{IP}_\kappa(V, W, Z, \mathbf{P})$ such that for some points $\lambda_j \in \mathbb{C}_+$ and vectors $(0 \neq) f_j \in \mathbb{C}^n$ $(j = 1, \dots, \kappa + 1)$ the following relations are fulfilled

$$A(\lambda_j)f_j = 0, \quad B(\lambda_j)f_j \neq 0 \quad (j = 1, \dots, \kappa + 1). \tag{3.35}$$

As usual, we may take $B(\lambda) + \lambda A(\lambda) = I$. Then, in view of Theorem 3.9, there exist a Pontryagin space $\widetilde{\Pi} \supset \mathcal{B}$, a selfadjoint extension $\widetilde{A}$ of S and an embedding operator $G : \mathbb{C}^n \subset \widetilde{\Pi}$ such that the equality (3.27) holds. Let us set $h_j = (\widetilde{A} - \lambda_j)^{-1}Gf_j$ $(j = 1, \dots, \kappa + 1)$. Then

$$\{h_j, Gf_j + \lambda_j h_j\} \in \widetilde{A}, \quad G^*h_j = 0 \quad (\forall\, j = 1, \dots, \kappa + 1). \tag{3.36}$$

Since $0 = [Gf_j + \lambda_j h_j, h_k] - [h_j, Gf_k + \lambda_k h_k] = (\lambda_j - \overline{\lambda_k})[h_j, h_k]$ $(j, k = 1, \ldots, \kappa+1)$ the linear space l.s.$\{h_j \mid j = 1, \ldots, \kappa+1\}$ is neutral and, therefore, the vectors h_j are linearly depending. Let (for definiteness) $h_{k+1} = \sum\limits_{j=1}^{\kappa} \alpha_j h_j$. Setting $f_0 = f_{\kappa+1} - \sum\limits_{j=1}^{\kappa} \alpha_j f_j$, $f'_j = (\lambda_{\kappa+1} - \lambda_j)\alpha_j f_j$ we obtain from (3.36) $\Big\{0, Gf_0 + \sum\limits_{j=1}^{\kappa} (\widetilde{A} - \lambda_j)^{-1} G f'_j \Big\} \in \widetilde{A}$. Making use of the relation

$$E(t) - E(\lambda) = (S - \lambda)(V + E(t))(Z - \lambda)^{-1}, \tag{3.37}$$

we obtain

$$\begin{aligned} 0 &= (f_0, Vx)_n + \sum_{j=1}^{\kappa} [(\widetilde{A} - \lambda_j)^{-1} G f'_j, E(t)x] \\ &= (f_0, Vx)_n + \sum_{j=1}^{\kappa} [f'_j, (\widetilde{A} - \overline{\lambda}_j)^{-1}(E(t)x - E(\overline{\lambda}_j)x)] \\ &= (f_0, Vx)_n + \sum_{j=1}^{\kappa} [f'_j, V(Z - \overline{\lambda}_j)^{-1}x + E(t)(Z - \overline{\lambda}_j)^{-1}x] \\ &= (V^* f_0, x)_r + \sum_{j=1}^{\kappa} ((Z^* - \lambda_j)^{-1} V^* f'_j, x)_r \end{aligned}$$

for all $Vx + E(t)x \in \mathcal{D}(S)$ $(x \in \mathbb{C}^r)$. Thus we have the equality $V^* f_0 + \sum\limits_{j=1}^{\kappa} (Z^* - \lambda_j)^{-1} V^* f'_j = 0$. In view of (3.34) this implies $f_0 = f'_1 = \cdots = f'_\kappa = 0$. Hence we obtain the equalities $\alpha_j f_j = 0$ $(j = 1, \ldots, \kappa)$, $f_{\kappa+1} = 0$ which contradict the assumption $f_{\kappa+1} \neq 0$. ∎

4. SOLUTION MATRIX

In what follows we suppose that the Pick matrix $\mathbf{P}$ of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ is nondegenerate and $\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}$. In this case S is a symmetric operator in $\mathcal{B}$ with deficiency indices (n, n) and one can find explicit formulas for all the objects of the representation theory which are connected with the operator S.

PROPOSITION 4.1. *Let $\mathbf{P}$ be a nondegenerate symmetric matrix. Then:*

(i) *the space $\mathcal{B}$ is a reproducing kernel space with the reproducing kernel*

$$K(t, \lambda) = I + E(t)\mathbf{P}^{-1}E(\lambda)^* \quad (\lambda \in \mathbb{C} \setminus \{\overline{z}_1, \ldots, \overline{z}_m\}); \tag{4.1}$$

(ii) *the defect subspace* $\mathcal{N}_\lambda$ *of the operator* S *takes a form* $\mathcal{N}_\lambda = \{K(t,\overline{\lambda})h : h \in \mathbb{C}^n\}$*;*

(iii) S *is a simple symmetric operator.*

Proof. (i) The first statement is implied by the following equality

$$\begin{aligned}[f(t), K(t,\lambda)h] &= (f_2, h)_n + (\mathbf{P}f_1, \mathbf{P}^{-1}E(\lambda)^*h)_r \\ &= h^*(f_2 + E(\lambda)f_1) = h^*f(\lambda)\end{aligned} \tag{4.2}$$

where $h \in \mathbb{C}^n$ and $f(t)$ is a vector function of the form (3.21).

(ii) It follows from (4.1) that $K(t,\overline{\lambda})h \in \mathcal{N}_\lambda$ for all $h \in \mathbb{C}^n$ since

$$[(t-\overline{\lambda})f(t), K(t,\overline{\lambda})h] = h^*(\overline{\lambda}-\overline{\lambda})f(\overline{\lambda}) = 0$$

for all $f(t) \in \mathcal{D}(S)$. Therefore, $\{K(t,\overline{\lambda}) \mid h \in \mathbb{C}^n\} \subset \mathcal{N}_\lambda$ for all $\lambda \in \mathbb{C} \setminus \{\overline{z}_1,\ldots,\overline{z}_m\}$. As follows from Lemma 3.6, the subspace $\{K(t,\overline{\lambda})h : h \in \mathbb{C}^n\}$ is n-dimensional. It coincides with $\mathcal{N}_\lambda$ since $\dim \mathcal{N}_\lambda = n$.

(iii) Assume that $f \in \mathcal{B}$ and f is orthogonal to $K(t,\overline{\lambda})h$ for all $\lambda \in \mathbb{C} \setminus \{\overline{z}_1,\ldots,\overline{z}_n\}$ and $h \in \mathbf{C}^m$. Then it follows from (4.2) that $h^*f(\lambda) \equiv 0$. Hence $f(\lambda) \equiv 0$. ∎

PROPOSITION 4.2. *Let a subspace* $\mathcal{L} = G\mathbb{C}^n \subset \mathcal{B}$ *be a scale subspace of the operator* S*. Then:*

(i) $\rho(S,\mathcal{L}) = \mathbb{C} \setminus \{\overline{z}_1,\ldots,\overline{z}_m\}$ *and the operator functions* $\mathcal{P}(\lambda)$ *and* $\mathcal{Q}(\lambda)$ *take the form*

$$\mathcal{P}(\lambda)f = f(\lambda), \quad \mathcal{Q}(\lambda)f = V(Z-\lambda)^{-1}f_1 \quad (\forall\, \lambda \in \rho(S,\mathcal{L});\ f \in \mathcal{B}), \tag{4.3}$$

$$\mathcal{P}(\infty)f = f_2 - Vf_1 = f(\infty), \quad \mathcal{Q}(\infty)f = -Wf_1, \tag{4.4}$$

where f *is a vector function of the form* (3.21)*;*

(ii) *the adjoint operators* $\mathcal{P}(\lambda)^*$ *and* $\mathcal{Q}(\lambda)^*$ *are calculated by*

$$\mathcal{P}(\lambda)^*h = K(t,\lambda)h, \quad \mathcal{Q}(\lambda)^*h = E(t)\mathbf{P}^{-1}(Z^*-\overline{\lambda})^{-1}V^*h, \tag{4.5}$$

$$\mathcal{P}(\infty)^*h = K(t,\infty)h = I - E(t)\mathbf{P}^{-1}V^*, \quad \mathcal{Q}(\infty)^*h = -E(t)\mathbf{P}^{-1}W^*h; \tag{4.6}$$

(iii) *the adjoint linear relation* S^* *has the representation*

$$S^* = \left\{\widehat{f} = \{f,f'\} = \{f_0, Sf_0\} + \widehat{\mathcal{P}}(\infty)^*l_1 + \widehat{\mathcal{Q}}(\infty)^*l_0 : l_0, l_1 \in \mathbb{C}^n, f_0 \in \mathcal{D}(S)\right\} \tag{4.7}$$

and a boundary triple for S^* can be defined by

$$\Gamma_0\widehat{f} = l_0 = f(\infty), \quad \Gamma_1\widehat{f} = l_1 = P_{\mathcal{L}}f' - \mathcal{Q}(\infty)f; \tag{4.8}$$

(iv) *The resolvent matrix $W_\infty(\lambda)$ of the operator S corresponding to the boundary triple* (4.8) *takes the form*

$$W_\infty(\lambda) = \begin{pmatrix} V(Z-\lambda)^{-1}\mathbf{P}^{-1}V^* & -I - V(Z-\lambda)^{-1}\mathbf{P}^{-1}W^* \\ I-(W+\lambda V)(Z-\lambda)^{-1}\mathbf{P}^{-1}V^* & \lambda + (W+\lambda V)(Z-\lambda)^{-1}\mathbf{P}^{-1}W^* \end{pmatrix}. \tag{4.9}$$

Proof. (i) It follows from the identity (3.37) that for all $\lambda \in \mathbb{C}\setminus\{\overline{z}_1,\dots,\overline{z}_m\}$ the vector function $f(t) = f_2 + E(t)f_1$ can be decomposed as

$$f(t) - f(\lambda) = E(t)f_1 - E(\lambda)f_1 = (S-\lambda)(V+E(t))(Z-\lambda)^{-1}f_1 \in \mathcal{R}(S-\lambda).$$

This yields the equalities (4.3). Similarly, it follows from the relation $f(t) = (Vf_1 + E(t)f_1) + (f_2 - Vf_1)$ that $\infty \in \rho(S,\mathcal{L})$ and the equalities (4.4) hold.

(ii) It follows from (4.2) and (4.3) that

$$\begin{aligned}
[\mathcal{P}(\lambda)^*h, f] &= (h, f(\lambda))_n = [K(t,\lambda)h, f(t)];\\
[\mathcal{Q}(\lambda)^*h, f] &= (h, V(Z-\lambda)^{-1}f_1)_n = ((Z^*-\overline{\lambda})^{-1}V^*h, f_1)_r\\
&= [E(t)\mathbf{P}^{-1}(Z^*-\overline{\lambda})^{-1}V^*h, f_1].
\end{aligned}$$

This proves the equalities (4.5). Analogously, the equalities (4.6) are implied by (4.2) and (4.4).

(iii) Since $\infty \in \rho(S,\mathcal{L})$, the representation (4.7) and the formulas (4.8) are direct corollaries of Theorem 2.12 and Proposition 2.13.

(iv) Using the relation $\widehat{\mathcal{P}}(\lambda)^*h = \{K(t,\lambda)h, \overline{\lambda}K(t,\lambda)h\}$ and the formulas (4.5), (4.8), one obtains for $\lambda \in \rho(S,\mathcal{L}), h \in \mathbb{C}^n$

$$\begin{aligned}
\Gamma_0\widehat{\mathcal{P}}(\lambda)^*h &= \mathcal{P}(\infty)K(t,\lambda)h = K(\infty,\lambda)h\\
&= h - V\mathbf{P}^{-1}(Z^*-\overline{\lambda})^{-1}(W^*+\overline{\lambda}V^*)h;
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\Gamma_1\widehat{\mathcal{P}}(\lambda)^*h &= P_{\mathcal{L}}(\overline{\lambda}K(t,\lambda)h) - \mathcal{Q}(\infty)K(t,\lambda)h\\
&= \overline{\lambda}h + W\mathbf{P}^{-1}(Z^*-\overline{\lambda})^{-1}(W^*+\overline{\lambda}V^*)h.
\end{aligned} \tag{4.11}$$

On account on the formulas (4.8), (4.4) and (4.5), the application of the operators Γ_0, Γ_1 to the vector function $\widehat{\mathcal{Q}}(\lambda)^*h = \{\mathcal{Q}(\lambda)^*h, \overline{\lambda}\mathcal{Q}(\lambda)^*h + h\}$ yields

$$\Gamma_0\widehat{\mathcal{Q}}(\lambda)^*h = \mathcal{P}(\infty)\mathcal{Q}(\overline{\lambda})^*h = -V\mathbf{P}^{-1}(Z^*-\overline{\lambda})^{-1}V^*h, \tag{4.12}$$

$$\Gamma_1\widehat{\mathcal{Q}}(\lambda)^*h = P_{\mathcal{L}}(\overline{\lambda}\mathcal{Q}(\lambda)^*h+h) - \mathcal{Q}(\infty)\mathcal{Q}(\lambda)^*h = h + W\mathbf{P}^{-1}(Z^*-\overline{\lambda})^{-1}V^*h. \tag{4.13}$$

Therefore, the formula (4.9) is implied by the relations (4.10)–(4.13) and (2.24). ∎

Let us define a matrix function $\Omega_\infty(\lambda)$ by the equality

$$
\begin{aligned}
\Omega_\infty(\lambda) &= \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W_\infty(\lambda) = I_{2n} + \begin{pmatrix} W \\ V \end{pmatrix} (Z-\lambda)^{-1}\mathbf{P}^{-1} \begin{pmatrix} W \\ V \end{pmatrix}^* J_{2n}, \\
J_{2n} &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\end{aligned} \tag{4.14}
$$

REMARK 4.3. An arbitrary resolvent matrix $W(\lambda)$ of the operator S is related to the matrix $W_\infty(\lambda)$ by the equality $W(\lambda) = W_\infty(\lambda)U^*$ where U is a J_{2n}-unitary matrix (see Remark 2.19). One can find the explicit formula for the resolvent matrix $W_a(\lambda)$ $(a = \overline{a})$ of the operator S corresponding to the boundary triple (2.20). The corresponding solution matrix $\Omega_a(\lambda)$ takes the form

$$
\Omega_a(\lambda) = \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W_a(\lambda) = I_{2n} + (\lambda - a) \begin{pmatrix} W \\ V \end{pmatrix} (Z-\lambda)^{-1}\mathbf{P}^{-1}(Z-a)^{-*} \begin{pmatrix} W \\ V \end{pmatrix}^* J_{2n}.
$$

In the case $\kappa = 0$, the resolvent matrices $W_a(\lambda)$ and $W_\infty(\lambda)$ were found in [3] using a different method.

LEMMA 4.4. *Let the Pick matrix $\mathbf{P}$ be nondegenerate, $\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}$ and let $W(\lambda)$ be a resolvent matrix of the operator S such that the matrix $w_{21}(\lambda)^{-1}$ has the holomorphic continuation at the points z_j $(j = 1, \ldots, m)$. Then the formula*

$$
\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W(\lambda) \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} w_{21}(\lambda)^{-1} \tag{4.15}
$$

establishes a one-to-one correspondence between the set of those solutions $\{A(\lambda), B(\lambda)\}$ of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ which satisfy the assumptions

$$
B(\lambda) + \lambda A(\lambda) = I \tag{4.16}
$$

and the set of $N_{\widetilde{\kappa}-\kappa}$-pairs $\{\varphi(\lambda), \psi(\lambda)\}$ such that $z_j \in \rho(\varphi, \psi)$ $(j = 1, \ldots, m)$ and

$$
\psi(\lambda) + w_{21}(\lambda)^{-1} w_{22}(\lambda)\varphi(\lambda) = I. \tag{4.17}
$$

Proof. Let the $N_{\widetilde{\kappa}}$-pair $\{A(\lambda), B(\lambda)\}$ be a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ which satisfies the assumptions (4.16). In view of Theorem 3.9 there exists a self-adjoint extension $\widetilde{A}$ of the operator S such that $z_j \in \rho(\widetilde{A})$ $(j = 1, \ldots, m)$ and the following equality holds

$$
\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} G^*(\widetilde{A}-\lambda)^{-1}G \\ I \end{pmatrix} \quad (\lambda \in \rho(\widetilde{A})). \tag{4.18}
$$

Here G is the embedding operator $G : \mathbb{C}^n \subset \widetilde{\Pi}$. Let the Weyl function $M(\lambda) = w_{21}(\lambda)^{-1}w_{22}(\lambda)$ of S correspond to a selfadjoint extension A_0. It follows from (2.21) and (2.23) that $z_j \in \rho(A_0)$ $(j = 1, \dots, m)$. In view of Theorems 2.3 and 2.17 there exists an $N_{\widetilde{\kappa}-\kappa}$-pair $\{\widetilde{\varphi}, \widetilde{\psi}\}$ holomorphic at the points z_j $(j = 1, \dots, m)$ such that

$$\det\left(\widetilde{\psi}(z_j) + M(z_j)\widetilde{\varphi}(z_j)\right) \neq 0 \quad (j = 1, \dots, m). \tag{4.19}$$

and the following equality holds for all $\lambda \in \rho(\widetilde{A}) \cap \rho(S, \mathcal{L}) \cap \rho(A_0)$

$$\begin{pmatrix} G^*(\widetilde{A} - \lambda)^{-1}G \\ I \end{pmatrix} = W(\lambda) \begin{pmatrix} \widetilde{\psi}(\lambda) \\ \widetilde{\varphi}(\lambda) \end{pmatrix} \left(w_{21}(\lambda)\widetilde{\psi}(\lambda) + w_{22}(\lambda)\widetilde{\varphi}(\lambda)\right)^{-1}. \tag{4.20}$$

It follows from (4.18) and (4.20) that

$$\begin{aligned} \begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} &= \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W(\lambda) \begin{pmatrix} \widetilde{\psi}(\lambda) \\ \widetilde{\varphi}(\lambda) \end{pmatrix} \left(w_{21}(\lambda)\widetilde{\psi}(\lambda) + w_{22}(\lambda)\widetilde{\varphi}(\lambda)\right)^{-1} \\ &= \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W(\lambda) \begin{pmatrix} \widetilde{\psi}(\lambda) \\ \widetilde{\varphi}(\lambda) \end{pmatrix} \left(\widetilde{\psi}(\lambda) + M(\lambda)\widetilde{\varphi}(\lambda)\right)^{-1} w_{21}(\lambda)^{-1} \\ &= \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W(\lambda) \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} w_{21}(\lambda)^{-1}, \end{aligned} \tag{4.21}$$

where the matrix functions

$$\begin{aligned} \psi(\lambda) &= \widetilde{\psi}(\lambda)\left(\widetilde{\psi}(\lambda) + M(\lambda)\widetilde{\varphi}(\lambda)\right)^{-1}, \\ \varphi(\lambda) &= \widetilde{\varphi}(\lambda)\left(\widetilde{\psi}(\lambda) + M(\lambda)\widetilde{\varphi}(\lambda)\right)^{-1} \end{aligned} \tag{4.22}$$

are holomorphic at the points z_j $(j = 1, \dots, m)$. The condition (4.17) is implied by (4.22).

Conversely, let $\{\varphi, \psi\}$ be an $N_{\widetilde{\kappa}-\kappa}$-pair holomorphic at the points z_j $(j = 1, \dots, m)$ which satisfies the conditions (4.17) and let a pair $\{A(\lambda), B(\lambda)\}$ be defined by the equality (4.15). It follows from (4.15) that

$$\begin{pmatrix} 0 & -I \\ I & \lambda \end{pmatrix} \begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = W(\lambda) \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} w_{21}(\lambda)^{-1}. \tag{4.23}$$

This implies

$$\begin{aligned} B(\lambda) + \lambda A(\lambda) &= \left(w_{21}(\lambda)\psi(\lambda) + w_{22}(\lambda)\varphi(\lambda)\right) w_{21}(\lambda)^{-1} \\ &= w_{21}(\lambda) \left(\psi(\lambda) + M(\lambda)\varphi(\lambda)\right) w_{21}(\lambda)^{-1} = I. \end{aligned}$$

In accordance with Theorem 2.17 there exists a minimal selfadjoint extension $\widetilde{A}$ of the operator S such that $z_j \in \rho(\widetilde{A})$ $(j = 1, \dots, m)$ and the following equality holds

$$\begin{pmatrix} \widetilde{G}^*(\widetilde{A}-\lambda)^{-1}\widetilde{G} \\ I \end{pmatrix} = W(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}(w_{21}(\lambda)\psi(\lambda)+w_{22}(\lambda)\varphi(\lambda))^{-1}$$
$$= W(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}w_{21}(\lambda)^{-1}.$$

According to Theorem 3.9, the pair $\{A(\lambda), B(\lambda)\}$ defined by the equality (4.15) is a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$. ∎

THEOREM 4.5. *Let the matrix* $\mathbf{P}$ *be nondegenerate and* $\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}$. *Then the formula*

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \Omega_\infty(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}, \tag{4.24}$$

establishes a one-to-one correspondence between the set of all the solutions $\{A(\lambda), B(\lambda)\}$ *of Problem* $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ *and the set of* $N_{\widetilde{\kappa}-\kappa}$*-pairs* $\{\varphi(\lambda), \psi(\lambda)\}$ *holomorphic at the points* z_j $(j = 1, \dots, m)$ *such that:*

(i) *the matrix* $\Omega_\infty(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}$ *is holomorphic at the points* z_j $(j = 1, \dots, m)$*;*

(ii) $\mathrm{rank}\,\Omega_\infty(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}\Big|_{\lambda=z_j} = n$, $(j = 1, \dots, m)$.

Proof. Step 1. Let an N_κ-pair $\{A(\lambda), B(\lambda)\}$ be a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ such that $B(\lambda) + \lambda A(\lambda) \equiv I$. Then in view of Theorem 3.9 there exists a selfadjoint extension $\widetilde{A}$ of the operator S in a bigger space $\widetilde{\Pi}$ such that $z_j \in \rho(\widetilde{A})$ $(j = 1, \dots, m)$ and the following equality holds

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} G^*(\widetilde{A}-\lambda)^{-1}G \\ I \end{pmatrix} \quad (\lambda \in \rho(\widetilde{A})). \tag{4.25}$$

Here G is the embedding operator $G : \mathbb{C}^n \subset \widetilde{\Pi}$.

In view of Proposition 1.9 there is a selfadjoint extension A_0 of the operator S in the space Π such that $z_j \in \rho(A_0)$ $(j = 1, \dots, m)$. Let $\mathcal{T} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the relation S^* such that $\ker\Gamma_0 = A_0$ and let $M(\lambda)$ and $W(\lambda)$ be the Weyl function and the resolvent matrix corresponding to the boundary triple $\mathcal{T}$ and the scale space $\mathcal{L} = \mathbb{C}^n$. Then the matrix functions

$$w_{21}(\lambda)^{-1} = \gamma(\overline{\lambda})^*|\mathcal{L}, \quad w_{21}(\lambda)^{-1}w_{22}(\lambda) = M(\lambda)$$

are holomorphic at the points z_j $(j = 1, \ldots, m)$ and the resolvent matrix $W(\lambda)$ is related to the matrix $W_\infty(\lambda)$ by the equality $W(\lambda) = W_\infty(\lambda)U^*$, where U is a J_{2n}-unitary matrix in $\mathcal{H} \oplus \mathcal{H}$. In view of Lemma 4.4 there exists an $N_{\widetilde{\kappa-\kappa}}$-pair $\{\widetilde{\varphi}(\lambda), \widetilde{\psi}(\lambda)\}$ such that $z_j \in \rho(\widetilde{\varphi}(\lambda), \widetilde{\psi}(\lambda))$ and

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda & I \\ -I & 0 \end{pmatrix} W(\lambda) \begin{pmatrix} \widetilde{\psi}(\lambda) \\ \widetilde{\varphi}(\lambda) \end{pmatrix} w_{21}(\lambda)^{-1} = \Omega_\infty(\lambda) \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix},$$

where

$$\begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} = U^* \begin{pmatrix} \widetilde{\psi}(\lambda) w_{21}(\lambda)^{-1} \\ \widetilde{\varphi}(\lambda) w_{21}(\lambda)^{-1} \end{pmatrix}.$$

Step 2. Let the pair $\{A(\lambda), B(\lambda)\}$) be an arbitrary solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$. In view of Lemma 2.6 there is a positive matrix X such that

$$\det(B(z_j) + z_j X A(z_j)) \neq 0 \quad (j = 1, \ldots, m).$$

Alongside with Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ let us consider Problem $\mathrm{IP}_{\widetilde{\kappa}}(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$ with the data $\{\widetilde{V}, \widetilde{W}\} = \{X^{1/2}V, X^{-1/2}W\}$ and let us set

$$\tag{4.26} \begin{gathered} \begin{pmatrix} \widetilde{B}(\lambda) \\ \widetilde{A}(\lambda) \end{pmatrix} = \widetilde{X} \begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} \left(X^{-1/2} B(\lambda) + \lambda X^{1/2} A(\lambda) \right)^{-1}, \\ \widetilde{X} = \begin{pmatrix} X^{-1/2} & 0 \\ 0 & X^{1/2} \end{pmatrix}. \end{gathered}$$

The pair $\{\widetilde{A}(\lambda), \widetilde{B}(\lambda)\}$) is a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$ and $\widetilde{B}(\lambda) + \lambda \widetilde{A}(\lambda) \equiv I$. As was shown in *Step* 1, the following equality holds

$$\tag{4.27} \begin{pmatrix} \widetilde{B}(\lambda) \\ \widetilde{A}(\lambda) \end{pmatrix} = \widetilde{\Omega}_\infty(\lambda) \begin{pmatrix} \widetilde{\psi}(\lambda) \\ \widetilde{\varphi}(\lambda) \end{pmatrix},$$

where $\{\widetilde{\varphi}(\lambda), \widetilde{\psi}(\lambda)\}$ is an $N_{\widetilde{\kappa-\kappa}}$-pair holomorphic at the points z_j $(j = 1, \ldots, m)$ and $\widetilde{\Omega}_\infty(\lambda)$ is a solution matrix of Problem $\mathrm{IP}_{\widetilde{\kappa}}(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$ related to $\Omega_\infty(\lambda)$ by

$$\tag{4.28} \widetilde{\Omega}_\infty(\lambda) = \widetilde{X} \Omega_\infty(\lambda) \widetilde{X}^{-1}.$$

The equalities (4.26), (4.27) and (4.28) yield the equality (4.24) with

$$\tag{4.29} \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} = \widetilde{X}^{-1} \begin{pmatrix} \widetilde{\psi}(\lambda) \\ \widetilde{\varphi}(\lambda) \end{pmatrix} \left(X^{-1/2} B(\lambda) + \lambda X^{1/2} A(\lambda) \right).$$

Step 3. Conversely, let the $N_{\widetilde{\kappa-\kappa}}$-pair $\{\varphi, \psi\}$ be holomorphic at the points z_j $(j = 1, \ldots, m)$ and satisfy the assumptions (i), (ii). As follows from Remark 2.16, the pair $\{A(\lambda), B(\lambda)\}$ defined by the equality (4.24) is also a generalized Nevanlinna pair. In view of Lemma 2.6 there is a nonnegative invertible matrix X such that the matrices $X^{-1/2}B(z_j) + z_j X^{1/2} A(z_j)$ are invertible for all $j = 1, \dots, m$. Let us consider Problem $IP_{\widetilde{\kappa}}(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$ with the data $\{\widetilde{V}, \widetilde{W}\} = \{X^{1/2}V, X^{-1/2}W\}$ and the corresponding resolvent matrix $\widetilde{W}_\infty(\lambda)$ and the solution matrix $\widetilde{\Omega}_\infty(\lambda)$. Using the equalities (4.24), (4.28), we obtain

$$\begin{aligned}\widetilde{W}_\infty(\lambda)\widetilde{X}\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix} &= \begin{pmatrix}0 & -I\\ I & \lambda\end{pmatrix}\widetilde{X}\Omega_\infty(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}\\ &= \begin{pmatrix}-X^{1/2}A(\lambda)\\ X^{-1/2}B(\lambda)+\lambda X^{1/2}A(\lambda)\end{pmatrix}.\end{aligned} \tag{4.30}$$

Let a pair $\{\widetilde{\varphi}(\lambda), \widetilde{\psi}(\lambda)\}$ be defined by $\operatorname{col}\{\widetilde{\varphi}(\lambda), \widetilde{\psi}(\lambda)\} = \widetilde{X}\operatorname{col}\{\varphi(\lambda), \psi(\lambda)\}$. It follows from (4.30) that the matrix

$$\Delta(\lambda) = \widetilde{w}_{21}(\lambda)\widetilde{\psi}(\lambda) + \widetilde{w}_{22}(\lambda)\widetilde{\varphi}(\lambda) = X^{-1/2}(B(\lambda) + \lambda X A(\lambda))$$

has a holomorphic continuation at the points z_j and $\Delta(z_j)$ are invertible for all $j = 1, \dots, m$.

Let S_X be the model operator corresponding to Problem $IP_{\widetilde{\kappa}}(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$. In accordance with Theorem 2.17, there exists a $\widetilde{G}$-minimal selfadjoint extension $\widetilde{A}_X$ of the operator S_X such that the following equality holds

$$\begin{pmatrix}\widetilde{G}^*(\widetilde{A}_X-\lambda)^{-1}\widetilde{G}\\ I\end{pmatrix} = \widetilde{W}_\infty(\lambda)\begin{pmatrix}\widetilde{\psi}(\lambda)\\ \widetilde{\varphi}(\lambda)\end{pmatrix}\Delta(\lambda)^{-1}, \quad \lambda \in \rho(\widetilde{A}_X) \cap \rho(S_X, \mathcal{L}). \tag{4.31}$$

It follows from (4.24) and (4.31) that

$$\begin{aligned}\begin{pmatrix}\lambda & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\widetilde{G}^*(\widetilde{A}-\lambda)^{-1}\widetilde{G}\\ I\end{pmatrix} &= \begin{pmatrix}\lambda & I\\ -I & 0\end{pmatrix}\widetilde{W}_\infty(\lambda)\widetilde{X}\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}\Delta(\lambda)^{-1}\\ &= \widetilde{X}\Omega_\infty(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix}\Delta(\lambda)^{-1} = \widetilde{X}\begin{pmatrix}B(\lambda)\\ A(\lambda)\end{pmatrix}\Delta(\lambda)^{-1}.\end{aligned}$$

This implies that the $\mathcal{L}$-resolvent $\widetilde{G}^*(\widetilde{A}-\lambda)^{-1}\widetilde{G}$ is holomorphic at the points z_j ($j = 1, \dots, m$). In view of Remark 2.2, this implies that $z_j \in \rho(\widetilde{A})$. As follows from Theorem 3.9, the pair $\{A(\lambda), B(\lambda)\}$ is a solution of Problem $IP_{\widetilde{\kappa}}(\widetilde{V}, \widetilde{W}, Z, \mathbf{P})$ and, therefore, the pair $\{A(\lambda), B(\lambda)\}$ is a solution of Problem $IP_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$. ∎

REMARK 4.6. If the two-sided interpolation Problem $IP_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ has symmetric interpolation points, that is $z_j = \overline{z}_k$ for some $j, k = 1, \dots, m$. this parametrization is not satisfactory since the behaviour of the parameter $\{\varphi, \psi\}$ at the points z_j is not specified. Namely, the matrix function $w_{21}(\lambda)$ is not holomorphic at the point z_j and the assumption $z_j \in \rho(\varphi, \psi)$ is not enough for the condition (i) of Theorem 4.5 to be satisfied at the point z_j.

One-sided interpolation problem. In the case where the interpolation Problem $\mathrm{IP}_{\widetilde{\kappa}}(V,W,Z,\mathbf{P})$ has no symmetric interpolation points one can simplify the statement of Theorem 4.5. In particular, this happens if Problem $\mathrm{IP}_{\widetilde{\kappa}}(V,W,Z,\mathbf{P})$ is a one-sided interpolation problem that is all the points z_j are in the upper halfplane $\mathbb{C}_+$.

In this case the Lyapunov equation has a unique solution (see [3]) $\mathbf{P}=(P_{ij}^{pq})$

$$P_{ij}^{pq} = \mathrm{D}_\lambda^p \mathrm{D}_{\overline{\mu}}^q \left.\frac{\mathcal{L}_i(\lambda)\mathcal{K}_j(\mu)^* - \mathcal{K}_i(\lambda)\mathcal{L}_j(\mu)^*}{\lambda-\overline{\mu}}\right|_{\lambda=z_i,\mu=z_j} \tag{4.32}$$

called the Pick matrix of Problem $\mathrm{IP}_\kappa(V,W,Z) := \mathrm{IP}_\kappa(V,W,Z,\mathbf{P})$. The solution matrix $\Omega_\infty(\lambda)$ of the one-sided interpolation problem is holomorphic at the points z_j $(j=1,\dots,m)$ and the hypothesis (i) of Theorem 4.5 becomes superfluous. Moreover, it follows from (4.25), (4.29) that $z_j \in \rho(\varphi,\psi)$ for all the pairs $\{\varphi,\psi\}$ which parametrize the set of solutions of Problem $\mathrm{IP}_\kappa(V,W,Z)$.

Proposition 4.7. *Let Problem* $\mathrm{IP}_{\widetilde{\kappa}}(V,W,Z)$ *has no symmetric interpolation points and the Pick matrix* $\mathbf{P}$ *be nondegenerate and* $\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}$. *Then the formula* (4.24) *establishes a one-to-one correspondence between the set of all solutions* $\{A(\lambda),B(\lambda)\}$ *of Problem* $\mathrm{IP}_{\widetilde{\kappa}}(V,W,Z)$ *and the set of* $N_{\widetilde{\kappa}-\kappa}$*-pairs* $\{\varphi(\lambda),\psi(\lambda)\}$ *such that* $z_j\in\rho(\varphi,\psi)$ $(j=1,\dots,m)$ *and*

$$\operatorname{rank}\Omega_\infty(z_j)\begin{pmatrix}\psi(z_j)\\ \varphi(z_j)\end{pmatrix} = n \quad (j=1,\dots,m). \tag{4.33}$$

An $N_{\widetilde{\kappa}-\kappa}$-pair $\{\varphi(\lambda),\psi(\lambda)\}$ is said to be an *excluded pair* in the parametrization (4.24) of the solutions of Problem $\mathrm{IP}_\kappa(V,W,Z)$ if the conditions (4.33) are not fulfilled. The excluded parameters of the indefinite Nevanlinna–Pick problem were investigated in [14], [2]. Subject to a signature of defect subspaces of the model operator S, the case where Problem $\mathrm{IP}_\kappa(V,W,Z)$ has no excluded parameters is possible. This observation from [14] is based on a general fact from the extension theory of symmetric operators in Pontryagin spaces ([23]) and can be generalized to the case of Problem $\mathrm{IP}_\kappa(V,W,Z)$. Let us denote by Δ_+ the set of the points $\lambda\in\widehat{\rho}(S)$ such that the defect subspace $\mathcal{N}_\lambda$ is positive.

Theorem 4.8. ([23]) *Let* S *be a simple symmetric operator in a Pontryagin space. A point* $\lambda\in\widehat{\rho}(S)$ *is not an eigenvalue of a regular selfadjoint extension of* S *if and omly if* $\lambda\in\Delta_+$.

Let us rewrite the condition $\lambda\in\Delta_+$ for the model operator S of Problem $\mathrm{IP}_\kappa(V,W,Z)$ in terms of the reproducing kernel $K(\lambda,\mu)$. In view of Proposition 4.1 and the equality

$$[K(t,\overline{\lambda})h, K(t,\overline{\lambda})h)] = h^*K(\overline{\lambda},\overline{\lambda})h \quad (\lambda\in\rho(S,\mathcal{L}),\, h\in\mathbb{C}^n)$$

the condition $\lambda \in \rho(S, \mathcal{L}) \cap \Delta_+$ is equivalent to the positivity of the matrix $K(\overline{\lambda}, \overline{\lambda})$. In the case where $\lambda = z_j$, the condition $z_j \in \Delta_+$ is equivalent to the condition $\overline{z}_j \in \Delta_+$ which, in turn, is equivalent to the condition $K(z_j, z_j) > 0$ $(j = 1, \dots, m)$. An application of Theorem 4.8 to the model operator S yields the following statement.

PROPOSITION 4.9. *Let Problem* $\mathrm{IP}_\kappa(V, W, Z)$ *has no symmetric interpolation points, the Pick matrix* $\mathbf{P}$ *be nondegenerate,* $\kappa = \mathrm{sq}_-(\mathbf{P})$ *and let the matrices* $K(z_j, z_j)$ *be positive for all* $j = 1, \dots, m$. *Then Problem* $\mathrm{IP}_\kappa(V, W, Z)$ *has no excluded parameters.*

Proof. Let A_0 be a selfadjoint extension of the operator S corresponding to the boundary triple $\mathcal{T}_\infty$. In view of Theorem 4.8, $z_j \in \rho(A_0)$ for all $j = 1, \dots, m$. It follows also from Theorem 4.8 and Theorem 2.17 that the matrix $w_{21}(\lambda)\psi(\lambda) + w_{22}(\lambda)\varphi(\lambda)$ is invertible at the points z_j $(j = 1, \dots, m)$ for every N_0-pair $\{\varphi, \psi\}$ and, therefore, the assumptions (ii) of Theorem 4.5 are fulfilled. Moreover, it follows from the identity $B(z_j) + z_j A(z_j) = w_{21}(z_j)\psi(z_j) + w_{22}(z_j)\varphi(z_j)$ that all solutions of Problem $\mathrm{IP}_\kappa(V, W, Z)$ satisfy the assumptions $\det(B(z_j) + z_j A(z_j)) \neq 0$, $j = 1, \dots, m$. ∎

EXAMPLE 4.10. Let us consider Problem $\mathrm{IP}_1(V, W, Z)$ with the data matrices

$$V = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 2\mathrm{i} & 3 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & -2\mathrm{i} \end{pmatrix}.$$

Then $\mathbf{P} = \begin{pmatrix} -2 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$ and the solution matrix $\Omega_\infty(\lambda)$ takes the form

$$\Omega_\infty(\lambda) = \frac{1}{(\lambda+\mathrm{i})(\lambda+2\mathrm{i})} \begin{pmatrix} \lambda^2+1 & -3\mathrm{i}(\lambda+\mathrm{i}) & -6\lambda & 0 \\ 0 & (\lambda+\mathrm{i})(\lambda+2\mathrm{i}) & 0 & 0 \\ 0 & 0 & \lambda^2+4 & 0 \\ 0 & 0 & -3\mathrm{i}(\lambda+2\mathrm{i}) & (\lambda+\mathrm{i})(\lambda+2\mathrm{i}) \end{pmatrix}.$$

One can check that the parameter $\{0, I\}$ is excluded at the point i. Let us note also that the kernel matrix $K(\lambda, \lambda)$ is not positive at the point $z = \mathrm{i}$: $K(\mathrm{i}, \mathrm{i}) = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 1 \end{pmatrix}$.

Let us consider a special case of Problem $\mathrm{IP}_\kappa(V, W, Z)$ where there are no multiple points and the matrices V_j coincide with the identity matrices I_n $(j = 1, \dots, m)$.

NEVANLINNA-PICK PROBLEM. *Given:* $n, m \in \mathbb{N}$, $z_j \in \mathbb{C} \setminus \mathbb{R}$, $W_j \in \mathcal{B}(\mathbb{C}^n)$ $(j = 1, \dots, m)$. *Find:* $n \times n$*-matrix function* $F(\lambda) \in N_{\widetilde{\kappa}}(\mathbb{C}^n)$ *such that* $F(z_j) = W_j$ $(j = 1, \dots, m)$.

The data of the problem are block-matrices

$$V = (I_n, \dots, I_n), \quad W = (W_1^*, \dots, W_m^*), \quad Z = \operatorname{diag}\,(\overline{z}_j I_n)_{j=1}^m.$$

The Pick matrix $\mathbf{P}$ has the block form $\mathbf{P} = (P_{jk})_{j,k=1}^m$ where

$$P_{jk} = \frac{W_j - W_k^*}{z_j - \overline{z}_k}.$$

Let us write the solution matrix and the inverse to the Pick matrix in the block form $\Omega(\lambda) = (\Omega_{jk}(\lambda))_{j,k=1}^2$, $\mathbf{P}^{-1} = (\pi_{jk})_{j,k=1}^m$ where Ω_{jk}, π_{jk} are $n \times n$-matrices.

COROLLARY 4.11. *Let $\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}$. Then the formula*

$$F(\lambda) = (\Omega_{11}(\lambda)\psi(\lambda) + \Omega_{12}(\lambda)\varphi(\lambda))(\Omega_{21}(\lambda)\psi(\lambda) + \Omega_{22}(\lambda)\varphi(\lambda))^{-1} \tag{4.34}$$

describes all solutions of the Nevanlinna–Pick problem $\mathrm{IP}_\kappa(V, W, Z)$ *when* $\{\varphi, \psi\}$ *ranges over the class* $\widetilde{N}_{\widetilde{\kappa}-\kappa}$ *and satisfies the conditions*

$$\det(\Omega_{21}(z_j)\psi(z_j) + \Omega_{22}(z_j)\varphi(z_j)) \neq 0 \quad (j = 1, \dots, m). \tag{4.35}$$

It follows from Proposition 3.11 that the conditions (4.33) and (4.35) are equivalent in this case. The formula like (4.34) was obtained first by D. Arov, V. Adamyan and M. Kreĭn in [1] where this problem in the scalar generalized Schur class was reduced to the Schur–Takagi problem. The matrix case was investigated in [20].

One can simplify the assumptions $K(z_j, z_j) > 0$ of Proposition 4.9. Indeed, in this case the defect subspaces $\mathcal{N}_{z_j}$ and $\mathcal{N}_{\overline{z}_j}$ have the same signature. Namely,

$$\begin{aligned} \mathcal{N}_{z_j} &= \operatorname{span}\{K(t, \overline{z}_j)h : h \in \mathbb{C}^n\}, \\ \mathcal{N}_{\overline{z}_j} &= E(t)\mathbf{P}^{-1}\mathcal{H}_j, \\ \mathcal{H}_j &= \operatorname{span}\{e_{jn+1}, \dots, e_{(j+1)n}\}, \end{aligned}$$

where e_j is nm-vector with the entries δ_{kl}, $(k, l = 1, \dots, m{\cdot}n)$. Let $h = (0, \dots, 0, h_j, 0, \dots, 0) \in \mathcal{H}_j$. Then it follows from the equality

$$[E(t)\mathbf{P}^{-1}h, E(t)\mathbf{P}^{-1}h] = h^*\mathbf{P}^{-1}h = h_j^*\pi_{jj}h_j$$

that the positivity of the matrix $K(z_j, z_j)$ is equivalent to the positivity of the matrix π_{jj} for all $j = 1, \dots, m$ and, therefore, the following analog of Proposition 4.9 holds.

COROLLARY 4.12. *Let the Pick matrix* $\mathbf{P}$ *of Nevanlinna–Pick Problem* $\mathrm{IP}_\kappa(V,W,Z)$ *be nondegenerate and* $\kappa = \mathrm{sq}_-(\mathbf{P})$. *If* z_j *belongs to* Δ_+ *or, equivalently,* $\pi_{jj} > 0$ *for all* $j = 1,\dots,m$, *then Problem* $\mathrm{IP}_\kappa(V,W,Z)$ *has no excluded parameters.*

The statement of Corollary 4.12 was proved in [14]. Moreover, as was shown in [14], the scalar Nevanlinna–Pick problem has a unique excluded parameter at the point z_j in the case where $\pi_{jj} = 0$ and infinitely many excluded parameters if $\pi_{jj} < 0$.

5. INTERPOLATION IN THE GENERALIZED STIELTJES CLASSES

PROBLEM $\mathrm{IP}^{\pm\widetilde{k}}_{\widetilde{\kappa}}(V,W,Z,\mathbf{P})$. Given are:

(i) $n,m \in \mathbb{N}$, $r_j, \widetilde{\kappa}, \widetilde{k} \in \mathbb{Z}_+$ $z_j \in \mathbb{C}\setminus\mathbb{R}$ $(j = 1,\dots,m)$;

(ii) $n \times r$-matrices V,W of the form (3.2) such that the assumptions (3.20) are fulfilled;

(iii) $m \times n$-matrix functions $\mathcal{K}(\lambda), \mathcal{L}(\lambda)$ defined and locally holomorphic on $\mathbb{C}\setminus\mathbb{R}$ which interpolate the data V,W;

(iv) A Hermitian solution $\mathbf{P}$ of the Lyapunov equation (3.6).

Find: $N^{\pm\widetilde{k}}_{\widetilde{\kappa}}$-pair $\{A(\lambda),B(\lambda)\}$ holomorphic at the points z_j $(j = 1,\dots,m)$ such that the equations (3.8) hold and $\mathbf{P}_{AB} = \mathbf{P}$.

Let $r\times r$-matrix $\mathbf{P}_-$ be defined by $\mathbf{P}_- := \mathbf{P}Z - V^*W$. A solvability criterion for Problem $\mathrm{IP}^{-\widetilde{k}}_{\widetilde{\kappa}}(V,W,Z,\mathbf{P})$ is implied immediately by Theorem 3.9.

THEOREM 5.1. *Let the matrices* $\mathbf{P}$, $\mathbf{P}_-$ *be nondegenerate. If Problem* $\mathrm{IP}^{-\widetilde{k}}_{\widetilde{\kappa}}(V,W,Z,\mathbf{P})$ *is solvable the following inequalities hold*

$$\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}, \quad k := \mathrm{sq}_-(\mathbf{P}_-) \leqslant \widetilde{k}. \tag{5.1}$$

If (5.1) *is fulfilled then the formula* (3.27) *establishes a one-to-one correspondence between the set of solutions of Problem* $\mathrm{IP}^{-\widetilde{k}}_{\widetilde{\kappa}}(V,W,Z,\mathbf{P})$ *such that* $B(\lambda)+\lambda A(\lambda) \equiv I$ *and the set of* $\mathcal{L}$*-resolvents* $G^*\mathbf{R}_\lambda G$ $(\mathbf{R}_\lambda \in \Omega^{\widetilde{k}}_{\widetilde{\kappa}}(S))$ *of the relation* S *holomorphic at the points* z_j $(j = 1,\dots,m)$.

Proof. Let the pair $\{A(\lambda),B(\lambda)\}$ be a solution of the Problem $\mathrm{IP}^{-\widetilde{k}}_{\widetilde{\kappa}}(V,W,Z)$ such that $B(\lambda)+\lambda A(\lambda) \equiv I$. In view of Theorem 3.9, it admits the representation

$$\{A(\lambda),B(\lambda)\} = \{-G^*(\widetilde{A}-\lambda)^{-1}G, \quad I + \lambda G^*(\widetilde{A}-\lambda)^{-1}G\}, \tag{5.2}$$

where $\widetilde{A}$ is a G-minimal selfadjoint extension of the model relation S. It follows from (5.2) that for all $\lambda \in \rho(\widetilde{A})$, $f \in \mathbb{C}^n$, the following equality holds

$$\operatorname{Im}(B(\lambda)f, \lambda A(\lambda)f) = \operatorname{Im}[\lambda R_\lambda Gf, Gf] \quad (R_\lambda = (\widetilde{A}-\lambda)^{-1}). \tag{5.3}$$

Using the Hilbert identity one obtains from (5.3) for all $\lambda \in \mathbb{C}_+$

$$\operatorname{Im}(B(\lambda)f, \lambda A(\lambda)f) = \operatorname{Im}\lambda\,[(I+\lambda R_\lambda)Gf, R_\lambda Gf]\,. \tag{5.4}$$

In view of the G-minimality of $\widetilde{A}$, the last equality yields the equivalence: $\{\lambda A(\lambda), B(\lambda)\}$ is an $N_{\widetilde{k}}$-pair if and only if the extension $\widetilde{A}$ has $N^{\widetilde{k}}$-property. Thus the statement of Theorem 5.1 is implied by Theorem 3.9 and by the equality (5.4). ∎

PROPOSITION 5.2. *Let the matrices* $\mathbf{P}$ *and* $\mathbf{P}_-$ *be nondegenerate and let* $\mathcal{T}_\infty = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ *and* $\mathcal{T}_- = \{\mathbb{C}^n, \Gamma_0^-, \Gamma_1^-\}$ *be the boundary triples defined by the equality* (4.8) *and by*

$$\begin{pmatrix}\Gamma_0^-\\ \Gamma_1^-\end{pmatrix} = U_-\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}, \quad U_- = \begin{pmatrix} I_n & 0\\ -W\mathbf{P}_-^{-1}W^* & I_n\end{pmatrix}. \tag{5.5}$$

Then:

(i) *the model operator* S *has* N^k*-property where* $k = \mathrm{sq}_-(\mathbf{P}_-)$*;*

(ii) *the extensions* $S_0^- := \ker\Gamma_0^-$ *and* $S_1^- := \ker\Gamma_1^-$ *coincide with* S_{F} *and* S_{K} *respectively;*

(iii) *the resolvent matrix* $W_-(\lambda)$ *of the operator* S *corresponding to the boundary triple* $\mathcal{T}_-$ *is related to* $W_\infty(\lambda)$ *by the equality* $W_-(\lambda) = \big(w_{ij}^-(\lambda)\big)_{i,j=1}^2 = W_\infty(\lambda)U_-^*$.

Proof. The first statement is implied by the equality

$$(Sf(t), f(t)) = -(Wf, Vf) + (\mathbf{P}Zf, f) = (\mathbf{P}_- f, f)$$

where $f(t) = Vf + E(t)f$, $f \in \mathbb{C}^r$. In view of (4.10), (4.11), the Weyl function $M(\lambda)$ corresponding to the boundary triple $\mathcal{T}_\infty$ takes the form

$$\begin{aligned} M(\lambda) = {}&\big[\lambda + W\mathbf{P}^{-1}(Z^*-\lambda)^{-1}(W^*+\lambda V^*)\big]\cdot\\ &\cdot\big[I - V\mathbf{P}^{-1}(Z^*-\lambda)^{-1}(W^*+\lambda V^*)\big]^{-1}. \end{aligned} \tag{5.6}$$

Since the matrix $\mathbf{P}_- = \mathbf{P}Z - V^*W = Z^*\mathbf{P} - W^*V$ is invertible, it follows from the equivalence $0 \in \rho(I_r - W^*V\mathbf{P}^{-1}Z^{-*}) \Leftrightarrow 0 \in \rho(I_n - V\mathbf{P}^{-1}Z^{-*}W^*)$ that $M(0) \in \mathcal{B}(\mathbb{C}^n)$, namely

$$M(0) = W\mathbf{P}^{-1}Z^{-*}W^*(I - V\mathbf{P}^{-1}Z^{-*}W^*)^{-1} = W\mathbf{P}_-^{-1}W^*.$$

As follows from (5.5), the Weyl function corresponding to the boundary triple $\mathcal{T}_-$ is related to $M(\lambda)$ by the equality $M_-(\lambda) = M(\lambda) - W\mathbf{P}_-^{-1}W^* = M(\lambda) - M(0)$ and, therefore, $M_-(0) = 0$. In view of Proposition 2.8 this implies the equality $\ker\Gamma_1^- = S_{\mathrm{K}}$.

The third statement is implied by (5.5), and Remark 2.19. ∎

COROLLARY 5.3. *Let the matrices* $\mathbf{P}, \mathbf{P}_-$ *be nondegenerate and* $\kappa := \mathrm{sq}_-(\mathbf{P}) \leqslant \widetilde{\kappa}$, $k := \mathrm{sq}_-(\mathbf{P}_-)$. *Then Problem* $\mathrm{IP}_{\widetilde{\kappa}}^{-k}(V, W, Z, \mathbf{P})$ *is solvable.*

Proof. As was mentioned in Proposition 5.2, the operator S has N^k-property. In view of Proposition 2.11, there is a selfadjoint extension $\widetilde{A}$ of S with N^k-property such that $z_j \in \rho(\widetilde{A})$ $(j = 1, \ldots, m)$. The solvability of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{-k}(V, W, Z, \mathbf{P})$ is implied now by Theorem 5.1. ∎

THEOREM 5.4. *Let the matrices* $\mathbf{P}$ *and* $\mathbf{P}_-$ *be nondegenerate and let the inequalities* (5.1) *hold. Then the formula*

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \Omega_-(\lambda) \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} \quad \big(\Omega_-(\lambda) = \Omega_\infty(\lambda) U_-^*\big) \tag{5.7}$$

establishes a one-to-one correspondence between the set of all solutions of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V, W, Z, \mathbf{P})$ *and the set of* $N_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$*-pairs* $\{\varphi(\lambda), \psi(\lambda)\}$ *holomorphic at the points* z_j $(j = 1, \ldots, m)$ *and such that the assumptions* (i), (ii) *of Theorem* 4.5 *are fulfilled.*

Proof. Let the $N_{\widetilde{\kappa}}$-pair $\{A(\lambda), B(\lambda)\}$ be a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}(V, W, Z, \mathbf{P})$ which is related to an $N_{\widetilde{\kappa}-\kappa}$-pair $\{\varphi_1(\lambda), \psi_1(\lambda)\}$ by the equality

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \Omega_\infty(\lambda) \begin{pmatrix} \psi_1(\lambda) \\ \varphi_1(\lambda) \end{pmatrix}. \tag{5.8}$$

One can rewrite the equality (5.8) in the form (5.7) where the $N_{\widetilde{\kappa}-\kappa}$-pair $\{\varphi(\lambda), \psi(\lambda)\}$ is defined by

$$\begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix} = U_-^{-*} \begin{pmatrix} \psi_1(\lambda) \\ \varphi_1(\lambda) \end{pmatrix}$$

which is also holomorphic at the points z_j $(j = 1, \ldots, m)$.

In the case where the pair $\{A(\lambda), B(\lambda)\}$ satisfies the conditions (4.16), it admits a representation (4.25), where $\widetilde{A}$ is a minimal selfadjoint extension of the operator S such that $z_j \in \rho(\widetilde{A})$ for all $j = 1, \ldots, m$ (see Theorem 3.9). It follows from (5.7) and (4.25) that

$$\begin{pmatrix} G^*(\widetilde{A} - \lambda)^{-1} G \\ I \end{pmatrix} = W_-(\lambda) \begin{pmatrix} \psi(\lambda) \\ \varphi(\lambda) \end{pmatrix}. \tag{5.9}$$

Let now $\{A(\lambda), B(\lambda)\} \in \widetilde{N}_{\widetilde{\kappa}}^{-\widetilde{k}}$. Then it follows from Theorem 5.1 that the extension $\widetilde{A}$ has $N^{\widetilde{k}}$-property. Using Theorem 2.18 and Proposition 5.2 one obtains from (5.9) that $\{\varphi(\lambda), \psi(\lambda)\} \in \widetilde{N}_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$. In the case where the conditions (4.16) do not hold, it is enough to find a positive matrix X satisfying (4.16) and to repeat the reasonings from the proof of Theorem 4.5 (Step 2).

Conversely, let the $N_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$-pair $\{\varphi(\lambda),\psi(\lambda)\}$ be holomorphic at the points z_j, $(j=1,\dots,m)$. Then the operator $\widetilde{A}$ from (5.9) has $N^{\widetilde{k}}$-property and by virtue of Theorem 5.1 $\{A(\lambda),B(\lambda)\}\in N_{\widetilde{\kappa}}^{-\widetilde{k}}$. ∎

Using Theorem 5.4 and the equivalence

$$\{A(\lambda),B(\lambda)\}\in \widetilde{N}_\kappa^{+k} \Leftrightarrow \{-B(\lambda),A(\lambda)\}\in \widetilde{N}_\kappa^{-k}$$

we obtain the following description of solutions of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{+\widetilde{k}}(V,W,Z,\mathbf{P})$. Let us set

$$\mathbf{P}_+ := \mathbf{P}Z+W^*V,\quad U_+=\begin{pmatrix} I_n & V\mathbf{P}_+^{-1}V^* \\ 0 & I_n\end{pmatrix},\quad \Omega_+(\lambda)=\Omega_\infty(\lambda)U_+^*.$$

Corollary 5.5. *Let the matrices $\mathbf{P},\mathbf{P}_+$ be nondegenerate. If Problem $\mathrm{IP}_{\widetilde{\kappa}}^{+\widetilde{k}}(V,W,Z,\mathbf{P})$ is solvable then the following inequalities hold*

$$\kappa := \mathrm{sq}_-(\mathbf{P})\leqslant \widetilde{\kappa},\quad k:=\mathrm{sq}_-(\mathbf{P}_+)\leqslant \widetilde{k}. \tag{5.10}$$

The formula

$$\begin{pmatrix} B(\lambda)\\ A(\lambda)\end{pmatrix}=\Omega_+(\lambda)\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix} \tag{5.11}$$

establishes a one-to-one correspondence between the set of all solutions of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{+\widetilde{k}}(V,W,Z,\mathbf{P})$ and the set of $N_{\widetilde{\kappa}-\kappa}^{+(\widetilde{k}-k)}$-pairs $\{\varphi,\psi\}$ holomorphic at the points z_j $(j=1,\dots,m)$ and such that the assumptions (i), (ii) *of Theorem* 4.5 *are fulfilled.*

Proof. Let the pair $\{A(\lambda),B(\lambda)\}\in\widetilde{N}_{\widetilde{\kappa}}^{+\widetilde{k}}$ be a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{+\widetilde{k}}(V,W,Z,\mathbf{P})$. Then the pair $\{-B(\lambda),A(\lambda)\}$ belongs to the class $\widetilde{N}_{\widetilde{\kappa}}^{-\widetilde{k}}$ and by Theorem 5.4 it admits the representation

$$\begin{pmatrix} A(\lambda)\\ -B(\lambda)\end{pmatrix}=\left[I+\begin{pmatrix} V\\ -W\end{pmatrix}(Z-\lambda)^{-1}\mathbf{P}^{-1}\begin{pmatrix} V\\ -W\end{pmatrix}^*\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\right]\cdot$$
$$\cdot\begin{pmatrix} I_n & 0\\ -V\mathbf{P}_+^{-1}V^* & I_n\end{pmatrix}^*\begin{pmatrix}\psi(\lambda)\\ \varphi(\lambda)\end{pmatrix},$$

where $\{\varphi(\lambda),\psi(\lambda)\}\in\widetilde{N}_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$. Multiplying this equality by J_{2n}^* from the left one obtains the following formula

$$\begin{pmatrix} B(\lambda)\\ A(\lambda)\end{pmatrix}=\Omega_\infty(\lambda)U_+^*\begin{pmatrix}-\varphi(\lambda)\\ \psi(\lambda)\end{pmatrix}=\Omega_+(\lambda)\begin{pmatrix}\widetilde{\psi}(\lambda)\\ \widetilde{\varphi}(\lambda)\end{pmatrix},$$

which establishes a one-to-one correspondence between the set of all solutions of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{+\widetilde{k}}(V,W,Z,\mathbf{P})$ and the set of pairs $\{\widetilde{\varphi},\widetilde{\psi}\}=\{\psi,-\varphi\}\in\widetilde{N}_{\widetilde{\kappa}-\kappa}^{+(\widetilde{k}-k)}$ holomorphic at the points z_j $(j=1,\dots,m)$ and such that the assumptions (i), (ii) of Theorem 4.5 are fulfilled. ∎

ONE-SIDED INTERPOLATION IN GENERALIZED STIELTJES CLASSES. Let all the interpolation points z_j be in $\mathbb{C}_+$. Then the description of all the solutions of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V,W,Z)$ takes the following form.

PROPOSITION 5.6. *Let the matrices* $\mathbf{P}$ *and* $\mathbf{P}_-$ *be nondegenerate and let the inequalities* (5.1) *hold. Then the formula* (5.7) *establishes a one-to-one correspondence between the set of all solutions of Problem* $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V,W,Z)$ *and the set of* $N_{\widetilde{\kappa}-\kappa}^{-(\widetilde{k}-k)}$*-pairs* $\{\varphi(\lambda),\psi(\lambda)\}$ *such that* $z_j\in\rho(\varphi(\lambda),\psi(\lambda))$ $(j=1,\dots,m)$ *and*

$$\operatorname{rank}\Omega_-(z_j)\begin{pmatrix}\psi(z_j)\\ \varphi(z_j)\end{pmatrix}=n\quad (j=1,\dots,m). \tag{5.12}$$

Alongside with Problem $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V,W,Z)$ let us consider Problem $\mathrm{IP}_{\widetilde{k}}^{+\widetilde{\kappa}}(\widetilde{V},\widetilde{W},Z)$ where $\widetilde{V}=VZ$, $\widetilde{W}=W$. Let $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{P}}_+=\widetilde{\mathbf{P}}Z+\widetilde{W}^*\widetilde{V}$ and $\widetilde{\Omega}_+(\lambda)$ be the Pick matrices and the solution matrix (respectively) associated with Problem $\mathrm{IP}_{\widetilde{k}}^{+\widetilde{\kappa}}(\widetilde{V},\widetilde{W},Z)$. The straightforward calculations show that

$$\widetilde{\mathbf{P}}=\mathbf{P}_-,\quad \widetilde{\mathbf{P}}_+=Z^*\mathbf{P}Z,\quad \begin{pmatrix}I_n & 0\\ 0 & \lambda I_n\end{pmatrix}\Omega_-(\lambda)=\widetilde{\Omega}_+(\lambda)\begin{pmatrix}I_n & 0\\ 0 & \lambda I_n\end{pmatrix}. \tag{5.13}$$

LEMMA 5.7. *The pair* $\{A(\lambda),B(\lambda)\}$ *is a solution of Problem* $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V,W,Z)$ *if and only if the pair* $\{\lambda A(\lambda),B(\lambda)\}$ *is a solution of Problem* $\mathrm{IP}_{\widetilde{k}}^{+\widetilde{\kappa}}(VZ,W,Z)$.

Proof. Let us note first that the following equivalence holds

$$\{A(\lambda),B(\lambda)\}\in\widetilde{N}_{\widetilde{\kappa}}^{-\widetilde{k}}\Leftrightarrow\{\lambda A(\lambda),B(\lambda)\}\in\widetilde{N}_{\widetilde{k}}^{\widetilde{\kappa}}.$$

Let the pair $\{A(\lambda),B(\lambda)\}$ be a solution of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V,W,Z)$ and a pair $\mathcal{K}(\lambda),\mathcal{L}(\lambda)$ interpolate the data V,W. Then the pair $\lambda\mathcal{K}(\lambda),\mathcal{L}(\lambda)$ interpolates the data $\widetilde{V}=VZ,W$. Indeed, it follows from (3.4) that

$$\begin{aligned}\mathrm{D}_\lambda^{(p)}\left(\lambda\mathcal{K}_i(\lambda)\right)\big|_{\lambda=z_i}&=\frac{1}{p!}\left(\lambda(\mathcal{K}_i(\lambda))^{(p)}+p(\mathcal{K}_i(\lambda))^{(p-1)}\right)\Big|_{\lambda=z_i}\\ &=\left(\lambda\mathrm{D}_\lambda^{p}\mathcal{K}_i(\lambda)+\mathrm{D}_\lambda^{p-1}\mathcal{K}_i(\lambda)\right)\Big|_{\lambda=z_i}=z_iV_{ip}^*+V_{i,p-1}^*\end{aligned}$$

where $z_iV_{ip}^*+V_{i,p-1}^*$ are the rows of the matrix Z^*V^*. The equalities (where $i=1,\dots,m$; $p=0,1,\dots,r_i$; $V_{i,-1}=0$)

$$\mathrm{D}_\lambda^{(p)}\left(\lambda\mathcal{K}_i(\lambda)B(\lambda)-\mathcal{L}_i(\lambda)\lambda A(\lambda)\right)\big|_{\lambda=z_i}=0 \tag{5.14}$$

implied by (3.8) show that the pair $\{\lambda A(\lambda),B(\lambda)\}$ is a solution of Problem $\mathrm{IP}_{\widetilde{k}}^{+\widetilde{\kappa}}(VZ,W,Z)$. ∎

Corollary 5.8. *Let the matrices* $\mathbf{P}$ *and* $\mathbf{P}_+$ *be nondegenerate and let the inequalities* (5.10) *hold. Then the formula* (5.11) *establishes a one-to-one correspondence between the set of all solutions of Problem* $\mathrm{IP}_{\widetilde{\kappa}}^{\pm\widetilde{k}}(V,W,Z)$ *and the set of* $N_{\widetilde{\kappa}-\kappa}^{+(\widetilde{k}-k)}$*-pairs* $\{\varphi(\lambda),\psi(\lambda)\}$ *such that* $z_j\in\rho(\varphi(\lambda),\psi(\lambda))$ $(j=1,\dots,m)$ *and*

$$\operatorname{rank}\Omega_+(z_j)\begin{pmatrix}\psi(z_j)\\ \varphi(z_j)\end{pmatrix}=n\quad (j=1,\dots,m). \tag{5.15}$$

Let us state some sufficient conditions for Problem $\mathrm{IP}_{\widetilde{\kappa}}^{\pm\widetilde{k}}(V,W,Z)$ to have no excluded parameters in terms of reproducing kernels. Let $K(t,\lambda)$ be the reproducing kernel of Problem $\mathrm{IP}_{\widetilde{\kappa}}^{-\widetilde{k}}(V,W,Z)$ and let $K_-(t,\lambda)$ be the reproducing kernel of Problem $\mathrm{IP}_{\widetilde{k}}^{+\widetilde{\kappa}}(VZ,W,Z)$

$$K_-(t,\lambda)=I+E_-(t)\mathbf{P}_-^{-1}E_-(\lambda)^*,\quad E_-(t)=(W+tVZ)(Z-t)^{-1}.$$

Proposition 5.9. *Let the Pick matrices* $\mathbf{P},\mathbf{P}_-$ *of Problem* $\mathrm{IP}_\kappa^{-k}(V,W,Z)$ *be nondegenerate,*

$$\kappa=\operatorname{sq}_-(\mathbf{P}),\quad k=\operatorname{sq}_-(\mathbf{P}_-)$$

and let at least one of the matrices $K(z_j,z_j)$ *or* $K_-(z_j,z_j)$ *be positive for all* $j=1,\dots,m$*. Then Problem* $\mathrm{IP}_\kappa^{-k}(V,W,Z)$ *has no excluded parameters.*

Proof. Let $K(z_j,z_j)>0$. Then it follows from Proposition 4.9 that Problem $\mathrm{IP}_\kappa(V,W,Z)$ and, therefore, Problem $\mathrm{IP}_\kappa^{-k}(V,W,Z)$ have no excluded parameters at the point z_j.

In the case where $K_-(z_j,z_j)>0$, one should consider Problem $\mathrm{IP}_k^{+\kappa}(VZ,W,Z)$, where the reproducing kernel coincides with $K_-(t,\lambda)$. As was shown above, this problem has no excluded parameters at the point z_j and all its solutions $\{\lambda A(\lambda),B(\lambda)\}$ are described by the formula

$$\begin{pmatrix}B(\lambda)\\ \lambda A(\lambda)\end{pmatrix}=\widetilde{\Omega}_+(\lambda)\begin{pmatrix}\psi_+(\lambda)\\ \varphi_+(\lambda)\end{pmatrix}\quad \{\varphi_+(\lambda),\psi_+(\lambda)\}\in\widetilde{N}_0^{+0}. \tag{5.16}$$

It follows from (5.16) and (5.13) that Problem $\mathrm{IP}_\kappa^{-k}(V,W,Z)$ also has no excluded parameters at the point z_j and its solutions are described by the formula (5.7) where $\{\varphi(\lambda),\psi(\lambda)\}=\{\varphi_+(\lambda),\lambda\psi_+(\lambda)\}\in\widetilde{N}_0^{-0}$. ∎

A similar statement for Problem $\mathrm{IP}_\kappa^{+k}(V,W,Z)$ is also true after the replacement of $\mathbf{P}_-$ by $\mathbf{P}_+$ and of $K_-(\lambda,\lambda)$ by $K_+(\lambda,\lambda)$.

EEXAMPLE 5.10. Let us consider Problem $\mathrm{IP}_1^{-0}(V,W,Z)$ with the data matrices $V=\mathrm{col}\,\{1-\mathrm{i},0\}$, $W=\mathrm{col}\,\{\mathrm{i},0\}$, $Z=-\mathrm{i}$. Then $\mathbf{P}=-1$, $\mathbf{P}_-=1$ and the solution matrix $\Omega_-(\lambda)$ takes the form

$$\Omega_-(\lambda)=\frac{1}{\lambda+\mathrm{i}}\begin{pmatrix}\lambda+1 & 0 & -\lambda & 0\\ 0 & \lambda+\mathrm{i} & 0 & 0\\ -2 & 0 & \lambda+1 & 0\\ 0 & 0 & 0 & \lambda+\mathrm{i}\end{pmatrix}.$$

Here $K(\mathrm{i},\mathrm{i})=\begin{pmatrix}-1/4 & 0\\ 0 & 1\end{pmatrix}$ and $K_-(\mathrm{i},\mathrm{i})=\begin{pmatrix}5/4 & 0\\ 0 & 1\end{pmatrix}>0$. It follows from Proposition 5.9 that Problem $\mathrm{IP}_1^{-0}(V,W,Z)$ has no excluded parameters. However, Problem $\mathrm{IP}_1^{-1}(V,W,Z)$ with the same data has infinitely many excluded parameters (for example $\{\varphi,\psi\}=\left\{2I_2,\begin{pmatrix}\lambda+1 & 0\\ 0 & \lambda\end{pmatrix}\right\}\in N_0^{-1}$).

REMARK 5.11. The Nevanlinna–Pick interpolation problem in the Stieltjes class $S:=N_0^{+0}$ has been considered in [17] by the methods of the J-theory of V.P. Potapov. The Nevanlinna–Pick problem in the generalized Stieltjes classes N_κ^{+k} was studied recently in [2] where the set of excluded parameters of this problem was thoroughly investigated. In particular, there was given a description of this set in the scalar case. The statement of Proposition 5.9 generalizes to the case of Problem $\mathrm{IP}_\kappa^{-k}(V,W,Z)$ the corresponding statement from [2], where the assumption for the reproducing kernels to be positive in the interpolation points is transformed into the assumption for the diagonal blocks of the inverses of the Pick matrices to be positive (cf. Corollary 4.12). The general tangential and bitangential interpolation problems for Stieltjes functions has been investigated in [32], [4], [7].

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