J. OPERATOR THEORY
42(1999), 189–220

DIRECT AND INVERSE PROBLEMS FOR A DAMPED STRING

VYACHESLAV N. PIVOVARCHIK

Communicated by Florian-Horia Vasilescu

Abstract. In this paper small transverse vibrations of a string of inhomogeneous stiffness in a damping medium with the left end fixed and the right end equipped with a concentrated mass are considered. By means of the Liouville transformation the corresponding differential equation is reduced to a Sturm–Liouville problem with parameter-dependent boundary conditions and parameter-dependent potential. This problem is considered as a spectral problem for the corresponding quadratic operator pencil. The inverse problem, i.e. the determination of the potential and the boundary conditions by the given spectrum and length of the string, is solved for weakly damped strings (having no purely imaginary eigenvalues). Uniqueness of the solution in an appropriate class is proved.

Keywords: *Inverse problem, Sturm-Liouville equation, damped string vibrations, operator pencil.*

MSC (2000): Primary 34A55; Secondary 34B24, 34L20.

1. INTRODUCTION

The boundary-value problem

$$\frac{\partial}{\partial s}\left(A(s)\frac{\partial u}{\partial s}\right)-\frac{\partial^2 u}{\partial t^2}-p\frac{\partial u}{\partial t}=0, \tag{1.1}$$

$$u(0,t)=0, \tag{1.2}$$

$$\left.\frac{\partial u}{\partial s}\right|_{s=l}+\nu\left.\frac{\partial u}{\partial t}\right|_{s=l}+\mu\left.\frac{\partial^2 u}{\partial t^2}\right|_{s=l}=0 \tag{1.3}$$

describes small transverse vibrations of a string of stiffness $A(s)$ and with a constant damping coefficient $p > 0$. Here $u(s,t)$ is the transverse displacement and $l > 0$ is the length of the string. The left end of the string is fixed and the right end is equipped with a ring of mass $\mu > 0$ moving in the direction orthogonal to the equilibrium position of the string. The damping coefficient of the ring is $\nu > 0$. Substituting $u(s,t) = v(\lambda, s)\mathrm{e}^{\mathrm{i}\lambda t}$ into (1.1)–(1.3) we obtain the system for the amplitude function $v(\lambda, s)$:

$$(A(s)v'(\lambda,s))' + \lambda^2 v(\lambda,s) - \mathrm{i}p\lambda v(\lambda,s) = 0, \tag{1.4}$$

$$v(\lambda,0) = 0, \tag{1.5}$$

$$v'(\lambda,l) + \mathrm{i}\nu\lambda v(\lambda,l) - \mu\lambda^2 v(\lambda,l) = 0. \tag{1.6}$$

The problem for a string with damping at the right end ($\nu > 0$, $p = 0$) and with the left end free ($v_s(s,t)|_{s=0} = 0$) was considered in [1], [7], [14], [15] and [21]. Basis properties of the eigenfunctions of such problems were discussed in [3] and [27]. The problem generated by equation (1.4) on a semiaxis with $p(s)$ depending on s was considered in [8], where an algorithm for the determination of the parameters if the spectrum is given was deduced. But the conditions on the spectral data were given in an implicit form. Inverse problems for $p(s)$ purely imaginary were solved for a semiaxis in [9] and for a finite interval in [5]. In the present paper we assume that $p = \text{const.} > 0$, $A(s) > 0$ for $s \in [0,l]$ and $A(s) \in W_2^2(0,l)$ and we investigate the spectrum of the problem (1.4)–(1.6). The question of interest is if it is enough to know the spectrum to determine all the parameters, i.e. the set $\{A(s), l, p, \nu, \mu\}$. The answer is negative due to the invariance of the problem (1.4)–(1.6) under the transformation $s' = rs$, $l' = rl$, $A'(s') = r^2 A(s)$, $\nu' = r^{-1}\nu$, $\mu' = r^{-1}\mu$, $p' = p$, where r is an arbitrary positive number. But it is shown in the present paper that the set $\{A(s), p, \nu, \mu\}$ may be found by the spectrum and the length l of the string for the case of a so-called weakly damped string, i.e. one having no purely imaginary eigenvalues. It is proved that then the solution is unique in an appropriate class.

2. AUXILIARY RESULTS

Assume that $A(s) > 0$ for $s \in [0, l]$ and $A(s) \in W_2^2(0, l)$. This enables us to apply the Liouville transformation [2]

$$x(s) = \int_0^s (A(s'))^{-\frac{1}{2}}\, \mathrm{d}s', \tag{2.1}$$

$$y(\lambda, x) = (A[x])^{\frac{1}{4}}\, v[\lambda, x]. \tag{2.2}$$

Here $v[\lambda, x] = v(\lambda, s(x))$ and $A[x] = A(s(x))$. Substituting (2.1) and (2.2) into (1.4)–(1.6) we obtain

$$y''(\lambda, x) + (\lambda^2 - \mathrm{i}\lambda p - q(x))y(\lambda, x) = 0, \tag{2.3}$$

$$y(\lambda, 0) = 0, \tag{2.4}$$

$$y'(\lambda, a) + (-m\lambda^2 + \mathrm{i}\alpha\lambda + \beta)y(\lambda, a) = 0, \tag{2.5}$$

where

$$q(x) = (A[x])^{-\frac{1}{4}}\, \frac{\mathrm{d}^2}{\mathrm{d}x^2}\, (A[x])^{\frac{1}{4}}\,, \tag{2.6}$$

$$m = \mu\, (A[a])^{\frac{1}{2}}\,, \tag{2.7}$$

$$\alpha = \nu\, (A[a])^{\frac{1}{2}}\,, \tag{2.8}$$

$$\beta = -\frac{1}{4}\, (A[a])^{-1}\, \frac{\mathrm{d}A[x]}{\mathrm{d}x}\bigg|_{x=a}\,, \tag{2.9}$$

$$a = \int_0^l (A(s))^{-\frac{1}{2}}\, \mathrm{d}s. \tag{2.10}$$

We identify the spectrum of the problem (1.4)–(1.6), i.e. the spectrum of the problem (2.3)–(2.5), with the spectrum of the operator pencil $\mathcal{L}(\mathrm{i}\lambda)$ defined by (A.8) (see the Appendix). An isolated eigenvalue λ_0 of finite algebraic multiplicity is said to be *normal* if the image $\operatorname{Im} \mathcal{L}(\lambda_0)$ is closed. This definition coincides with the definition introduced in [6] for the linear case.

Lemma 2.1. *The spectrum of the problem* (2.3)–(2.5) *consists only of normal eigenvalues which accumulate at infinity. All eigenvalues have geometric multiplicity one, i.e. each eigensubspace is one-dimensional.*

Proof. Equation (2.3) admits a unique solution satisfying (2.4) and the condition $y'(\lambda,0)=1$. This solution is of the form (cf. [20], p. 23)

$$S[\lambda,x]=S(\tau(\lambda),x)=\frac{\sin\tau(\lambda)x}{\tau(\lambda)}+\int\limits_0^x K(x,t)\frac{\sin\tau(\lambda)t}{\tau(\lambda)}\,\mathrm{d}t, \tag{2.11}$$

where $\tau(\lambda)=\sqrt{\lambda^2-\mathrm{i}p\lambda}$, $K(x,0)=0$ and $K(x,t)$ belongs to $W_2^1(0,a)$ as a function of each variable when the other variable is fixed. The set of the eigenvalues of the problem (2.3)–(2.5) coincides with the set of the zeros of the entire function

$$\chi(\lambda)=S'[\lambda,a]+(-m\lambda^2+\mathrm{i}\alpha\lambda+\beta)S[\lambda,a]. \tag{2.12}$$

Now the assertions of Lemma 2.1 follow. ∎

Let $\{\lambda_k\}$ be a countable set of complex not necessarily different points symmetric with respect to the imaginary axis. If a point λ_k occurs n times, then we call n the multiplicity of λ_k. Let us assume that the multiplicities of symmetrically located points coincide and the number of purely imaginary points (counted with multiplicities) is even. Then a way of numbering is called *proper* if

(i) $\operatorname{Re}\lambda_{k+1}\geqslant\operatorname{Re}\lambda_k$;
(ii) there exist two points of zero index $(\lambda_{+0},\lambda_{-0})$;
(iii) $\lambda_{-k}=-\overline{\lambda}_k$ for all not purely imaginary λ_k;
(iv) a point of multiplicity m is considered as m coinciding points.

This way of numbering may be arbitrary in other respects.

DEFINITION 2.2. (cf. [16]) An entire function $s(\lambda)$ of exponential type $\sigma>0$ is said to be of *sinus type* if

(i) all the zeros of $s(\lambda)$ lie in a horizontal strip $|\operatorname{Im}\lambda|<h<\infty$;
(ii) there exists a real number h_1 such that the inequalities

$$0<m\leqslant|s(\lambda)|\leqslant M<\infty$$

are valid for $|\operatorname{Im}\lambda|=h_1$;
(iii) the function $s(\lambda)$ is of the same type in the upper and the lower halfplane.

LEMMA 2.3. *Let $\chi(\lambda)$ be an entire function of exponential type $\leqslant a$ having the form*

$$\begin{aligned}\chi(\lambda)=B_0\big((\tau+\mathrm{i}B_1+B_2\tau^{-1}+\mathrm{i}B_3\tau^{-2})\sin\tau a\\ +(A_1+\mathrm{i}A_2\tau^{-1}+A_3\tau^{-2})\cos\tau a\big)\\ +\Psi_1(\tau)\delta_1(\tau)\tau^{-2}+\Psi_2(\tau)\delta_2(\tau)\tau^{-2},\end{aligned} \tag{2.13}$$

where $\tau = \sqrt{\lambda^2 - \mathrm{i}p\lambda}$, $p \in \mathbb{R}$, $A_k \in \mathbb{R}$ $(k = \overline{1,3})$, $B_k \in \mathbb{R}$, $(k = \overline{0,3})$, $A_1 \neq 0$, $B_0 \neq 0$, $\Psi_1(\tau) = \int\limits_0^a \mathrm{e}^{\mathrm{i}\tau x} f_1(x)\mathrm{d}x$, $\Psi_2(\tau) = \int\limits_0^a \mathrm{e}^{-\mathrm{i}\tau x} f_2(x)\mathrm{d}x$, $f_k \in L_2(0,a)$, *and* $\delta_k(\tau)$ $(k = 1,2)$ *are bounded functions. Then the properly numbered zeros* λ_n *of* $\chi(\lambda)$ *have the following asymptotics*

$$\lambda_n \underset{n\to\infty}{=} \frac{\pi n}{a} + \frac{\mathrm{i}p}{2} + \frac{p_1}{n} + \frac{\mathrm{i}p_2}{n^2} + \frac{p_3}{n^3} + \frac{b_n}{n^3} \tag{2.14}$$

where $\{b_n\} \in l_2$,

$$p_1 = -A_1\pi^{-1} - p^2 a(8\pi)^{-1}, \tag{2.15}$$

$$p_2 = a\pi^{-2}(B_1A_1 - A_2), \tag{2.16}$$

$$\begin{aligned} p_3 = {} & ap^2p_1(8\pi^2)^{-1} - a^3\pi^{-3}p^4(128)^{-1} + a^2\pi^{-3}(-A_3 + A_1B_2 \\ & - A_2B_1 + A_1(3B_1^2 + A_1^2) - a^{-1}A_1B_1). \end{aligned} \tag{2.17}$$

Proof. It is clear that the function $\chi(\lambda)(\lambda - \lambda_k)^{-1}$, where λ_k is any zero of $\chi(\lambda)$, is of sinus-type and it has an infinite set of zeros. The first step is to prove the equality

$$\lambda_n = \frac{\pi n}{a} + \frac{\mathrm{i}p}{2} + \mathrm{o}(1).$$

Let $\chi(\lambda_n) = 0$, $|\lambda_n| \underset{n\to\infty}{\longrightarrow} \infty$. If $|\mathrm{Im}\,\tau(\lambda_n)| \to \infty$, then

$$(B_0\tau(\lambda_n))^{-1}\chi(\lambda_n)\mathrm{e}^{-|\mathrm{Im}\,\tau(\lambda_n)|a} + \mathrm{o}(1) = 0.$$

This equality is false because $\lim\limits_{n\to\infty} |\sin\tau(\lambda_n)a|\mathrm{e}^{-|\mathrm{Im}\,\tau(\lambda_n)|a} = \frac{1}{2}$. Hence, there exists a number $M > 0$ such that $|\mathrm{Im}\,\tau(\lambda_n)| \leqslant M < \infty$. Then

$$B_0^{-1}(\tau(\lambda_n))^{-1}\chi(\lambda_n) = \sin\tau(\lambda_n)a + \mathrm{o}(1)$$

and

$$\tau(\lambda_n) = \frac{\pi n}{a} + \mathrm{o}(1)$$

or

$$\lambda_n = \frac{\pi n}{a} + \frac{\mathrm{i}p}{2} + \mathrm{o}(1).$$

Set $\tau_n := \sqrt{\lambda_n^2 - \mathrm{i}p\lambda_n}$ (we choose the branch of the root such that $\lambda_n - \sqrt{\lambda_n^2 - \mathrm{i}p\lambda_n} = \mathrm{O}(1)$), then

$$\begin{aligned} B_0^{-1}\tau_n^{-1}\chi(\lambda_n) = {} & \sin\tau_n a\left(1 + \mathrm{i}B_1\tau_n^{-1} + B_2\tau_n^{-2} + \mathrm{i}B_3\tau_n^{-3}\right) \\ & + \cos\tau_n a\left(A_1\tau_n^{-1} + \mathrm{i}A_2\tau_n^{-2} + A_3\tau_n^{-3}\right) + \widetilde{b}_n\tau_n^{-3} = 0 \end{aligned} \tag{2.18}$$

where $\left\{\widetilde{b}_n\right\} \in l_2$. Expressing $e^{2i\tau_n a}$ from (2.18) and substituting $\tau_n = \frac{\pi n}{a} + \triangle_n$ ($\triangle_n = o(1)$), we obtain

$$\begin{aligned} e^{2i\triangle_n a} = \Big[&- i\widetilde{b}_n\tau_n^{-3} + \Big(- \widetilde{b}_n^2\tau_n^{-6} + \Big(1 + i\,(B_1 + A_1)\,\tau_n^{-1} \\ &+ (B_2 - A_2)\,\tau_n^{-2} + i\,(B_3 + A_3)\,\tau_n^{-3}\Big)\Big(1 + i\,(B_1 - A_1)\,\tau_n^{-1} \\ &+ \,(B_2 + A_2)\,\tau_n^{-2} + i\,(B_3 - A_3)\,\tau_n^{-3}\Big)\Big)^{\frac{1}{2}}\Big]^2 \Big(1 + i\,(B_1 + A_1)\,\tau_n^{-1} \\ &+ (B_2 - A_2)\,\tau_n^{-2} + i\,(B_3 + A_3)\,\tau_n^{-3}\Big)^{-2}, \end{aligned}$$

and consequently

$$\begin{aligned} 2i\,\triangle_n\,a = \ln \Big[&1 + i\,(B_1 - A_1)\left(\pi n a^{-1} + \triangle_n\right)^{-1} \\ &+ (B_2 + A_2)\left(\pi n a^{-1} + \triangle_n\right)^{-2} \\ &+ i\,(B_3 - A_3)\left(\pi n a^{-1} + \triangle_n\right)^{-3}\Big] \\ &- \ln \Big[1 + i\,(B_1 + A_1)\left(\pi n a^{-1} + \triangle_n\right)^{-1} \\ &+ (B_2 - A_1)\left(\pi n a^{-1} + \triangle_n\right)^{-2} \\ &+ i\,(B_3 + A_3)\left(\pi n a^{-1} + \triangle_n\right)^{-3}\Big] + o(n^{-3}). \end{aligned} \tag{2.19}$$

Decomposition of the right hand side of (2.19) into a power series implies

$$\triangle_n = -\frac{A_1}{\pi n} + \triangle_n^{(1)}, \tag{2.20}$$

where $\triangle_n^{(1)} = o(n^{-1})$. Substitution of (2.20) into (2.19) yields

$$\triangle_n^{(1)} = \frac{i a^2}{\pi^2 n^2}\,(B_1 A_1 - A_2) + \triangle_n^{(2)}, \tag{2.21}$$

where $\triangle_n^{(2)} = O(n^{-3})$. Then substituting (2.21) and (2.20) into (2.19) we obtain

$$\triangle_n^{(2)} = \frac{a^2}{\pi^3 n^3}\left(-A_3 + A_1 B_2 - A_2 B_1 + A_1\left(3B_1^2 + A_1^2\right) - A_1 B_1 a^{-1}\right) + n^{-3} b_n^{(2)},$$

where $\{b_n^{(2)}\} \in l_2$. Now it is easy to get (2.14) using the formula $\lambda_n = \frac{ip}{2} + \sqrt{\tau_n^2 - \frac{p^2}{4}}$. ∎

3. THE DIRECT PROBLEM

In this paragraph we present the description of the spectrum of the problem (2.3)–(2.5) which is the same as that of problem (1.4)–(1.6).

THEOREM 3.1. *Let $A(s) \in W_2^4(0,l)$, $A(s) > 0$ $(s \in [0,l])$, $\mu > 0$, $\nu > 0$, $l > 0$, $p > 0$. Then the spectrum of* (2.3)–(2.5) *coincides with the set $\Lambda = \{\lambda_n\}$ satisfying the following conditions:*

(i) *the set Λ is symmetric with respect to the imaginary axis and symmetric points have the same multiplicities;*

(ii) *the number of purely imaginary points, counted with multiplicities, is even;*

(iii) $\operatorname{Im}\lambda_k > 0$ *for all k;*

(iv)(a) *if $\nu\mu^{-1} > p$, then all not purely imaginary and all multiple points are located in the strip $\frac{p}{2} < \operatorname{Im}\lambda_k < \frac{\nu}{2\mu}$;*

(iv)(b) *if $\nu\mu^{-1} < p$, then all not purely imaginary and all multiple points are located in the strip $\frac{\nu}{2\mu} < \operatorname{Im}\lambda_k < \frac{p}{2}$;*

(iv)(c) *if $\nu\mu^{-1} = p$, then all not purely imaginary points are located on the axis $\operatorname{Im}\lambda = \frac{p}{2}$ and if $\frac{\mathrm{i}p}{2} \in \Lambda$, then its multiplicity is equal to 2; all other points of Λ are simple;*

(v)(a) *if $\nu\mu^{-1} > p$, then for every $\lambda_k \in \left(0, \frac{\mathrm{i}p}{2}\right]$ there exists $\lambda_{-k} \in \left(\frac{\mathrm{i}p}{2}, \mathrm{i}\infty\right)$ such that $\operatorname{Im}(\lambda_k + \lambda_{-k}) > p$;*

(v)(b) *if $\nu\mu^{-1} < p$, then for every $\lambda_k \in \left[\frac{\mathrm{i}p}{2}, \mathrm{i}\,\infty\right)$ there exists $\lambda_{-k} \in \left(0, \frac{\mathrm{i}p}{2}\right)$ such that $\operatorname{Im}(\lambda_k + \lambda_{-k}) < p$;*

(v)(c) *if $\nu\mu^{-1} = p$, then Λ is symmetric with respect to the axis $\operatorname{Im}\lambda = \frac{p}{2}$;*

(vi) *under proper numbering*

$$\lambda_n \underset{n\to\infty}{=} \frac{\pi n}{a} + \frac{\mathrm{i}p}{2} + \frac{p_1}{n} + \frac{\mathrm{i}p_2}{n^2} + \frac{p_3}{n^3} + \frac{b_n}{n^3}, \tag{3.1}$$

where $a = \int\limits_0^l (A(s))^{-1/2}\,\mathrm{d}s$, $p_i \in \mathbb{R}$, $\{b_n\} \in l_2$, $p_2 \underset{\leqslant}{>} 0$ when $p \underset{\geqslant}{<} \nu\mu^{-1}$.

Proof. Assertion (i) follows from the symmetry of the problem, i.e. from the identity $v(-\overline{\lambda}, s) = \overline{v(\lambda, s)}$, where $v(\lambda, s)$ is a solution of (1.4)–(1.6). Assertion (ii) may be proved by applying Lemma A.8 to the pencil $\widetilde{L}(\mathrm{i}\lambda)$ defined by (A.7) (see Appendix). Assertion (iii) is a consequence of the results of [13] applied to the pencil $\widetilde{L}(\mathrm{i}\lambda)$. To prove assertions (iv)(a), (iv)(b), (iv)(c) it is sufficient to apply

Lemmas A.9 and A.10 to the pencil $\widetilde{L}(\mathrm{i}\lambda)$. To prove assertion (v)(a) we apply Theorem A.6 to the operator pencil

$$\widetilde{L}_1(\mathrm{i}\lambda) = \widetilde{L}\left(\mathrm{i}\lambda - \frac{p}{2}\right) = -\lambda^2\widetilde{M} + \mathrm{i}\lambda\left(\widetilde{K} - p\widetilde{M}\right) + \widetilde{A} + \frac{p^2}{4}\widetilde{M} - \frac{p}{2}\widetilde{K}$$

(see Appendix). To prove assertion (v)(b) we apply Theorem A.6 to the pencil $L_1(-\mathrm{i}\lambda)$. Now let $p = \nu\mu^{-1}$. Then $L(\lambda) = \tau^2\widetilde{M} + \widetilde{A}$. Hence, assertion (v)(c) follows from the self-adjointness of $\widetilde{A}$ and the symmetry of $\widetilde{M}$. Now in order to prove assertion (vi) we substitute (2.11) into (2.12) and obtain

$$\begin{aligned}\chi(\lambda) &= \cos\tau a + K(a,a)\frac{\sin\tau a}{\tau} + \int_0^a K_x(a,t)\frac{\sin\tau t}{\tau}\,\mathrm{d}t \\ &\quad + (-m\lambda^2 + \mathrm{i}\alpha\lambda + \beta)\left(\frac{\sin\tau a}{\tau} + \int_0^a K(a,t)\frac{\sin\tau t}{\tau}\,\mathrm{d}t\right).\end{aligned} \tag{3.2}$$

Due to $A(s) \in W_2^4(0,l)$ and $A(s) > 0$, definition (2.6) implies $q(x) \in W_2^2(0,a)$, and consequently ([20], p. 23) $K(x,t)$ has all partial derivatives up to third order inclusive. The partial derivatives of third order belong to $L_2(0,a)$ as functions of one variable when the other one is fixed, in particular $K_{ttt}(x,t)$. Hence, it is possible to integrate three times by parts in (3.2):

$$\begin{aligned}\chi(\lambda) &= \cos\tau a + K(a,a)\frac{\sin\tau a}{\tau} - K_x(a,a)\frac{\cos\tau a}{\tau^2} + \int_0^a K_{xt}(a,t)\frac{\cos\tau t}{\tau^2}\,\mathrm{d}t \\ &\quad + \big(-m\lambda^2 + \mathrm{i}\alpha\lambda + \beta\big)\Bigg(\frac{\sin\tau a}{\tau} - \frac{K(a,a)}{\tau^2}\cos\tau a \\ &\quad + K_t(a,a)\frac{\sin\tau a}{\tau^3} + K_{tt}(a,a)\frac{\cos\tau a}{\tau^4} - \int_0^a K_{ttt}(a,t)\frac{\cos\tau t}{\tau^4}\,\mathrm{d}t\Bigg).\end{aligned} \tag{3.3}$$

The function $\chi(\lambda)$ satisfies the conditions of Lemma 2.3 and consequently the zeros of $\chi(\lambda)$, numbered in a proper way, satisfy equality (2.14) with

$$p_1 = \frac{1}{\pi m} + \frac{K(a,a)}{\pi} - \frac{p^2 a}{8\pi}, \tag{3.4}$$

$$p_2 = \frac{a}{\pi^2 m}\left(\frac{\alpha}{m} - p\right), \tag{3.5}$$

$$\begin{aligned}p_3 &= \frac{p^2a^2p_1^2}{8\pi^2} - \frac{p^4a^3}{128\pi^3} + \frac{a^2}{\pi^3}\big(-A_3 + A_1B_2 - A_2B_1 \\ &\quad + A_1\left(3B_1^2 + A_1^2\right) - A_1B_1a^{-1}\big),\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&B_1 = p - \frac{\alpha}{m}, \\
&B_2 = -\frac{K(a,a)}{m} + K_t(a,a) - \frac{\beta}{m} - \frac{p}{2}\left(p - \frac{\alpha}{m}\right), \\
&B_3 = B_1\left(K_t(a,a) - \frac{p^2}{8}\right), \\
&A_1 = -\frac{1}{m} - K(a,a), \\
&A_2 = -K(a,a)\left(p - \frac{\alpha}{m}\right) \\
&A_3 = -\frac{K_x(a,a)}{m} - K_{tt}(a,a) + K(a,a)\left(\frac{\beta}{m} + \frac{p}{2}\left(p - \frac{\alpha}{m}\right)\right).
\end{aligned} \tag{3.7}
$$

So Theorem 3.1 is proved. ∎

REMARK 3.2. Purely imaginary eigenvalues λ_k and λ_{-k} mentioned in (v)(a) and (v)(b) of Theorem 3.1 may be chosen in such a way that each of the corresponding eigenfunctions $y_k(x)$ and $y_{-k}(x)$ of the problem (2.3)–(2.5) (and consequently each of the eigenfunctions $v_k(s)$ and $v_{-k}(s)$ of the problem (1.4)–(1.6)) has exactly k zeros in the interval $(0, a]$ (in $(0, l]$, respectively). This assertion may be proved in quite the same way as in [23].

REMARK 3.3. Problem (2.3)–(2.5) with $p = 0$ was considered in [25].

DEFINITION 3.4. A smooth string ($A(s) > 0$, $A(s) \in W_2^2(0, l)$) is said to be *weakly damped* if for every $v(s) \in W_2^2(0, l)$ with $v(0) = 0$ and $v(s)$ not equal to 0 identically the following inequality holds

$$
\left(p\int_0^l |v|^2\,\mathrm{d}s + \nu A(l)|v(l)|^2\right)^2 < 4\left(\int_0^l |v|^2\,\mathrm{d}s + \mu A(l)|v(l)|^2\right)\int_0^l A(s)|v'|^2\,\mathrm{d}s.
$$

REMARK 3.5. It is easy to verify that a string is weakly damped if and only if the corresponding operator pencil $\widetilde{L}(\lambda)$ (see (A.7)) or, which is the same, the operator pencil $\mathcal{L}(\lambda)$ (see (A.8)) is weakly damped ([13]). The pencil $\widetilde{L}(\lambda)$ is weakly damped if and only if it has no real eigenvalues ([13]). Hence the string is weakly damped if and only if its spectrum has no purely imaginary points.

4. INVERSE STURM-LIOUVILLE PROBLEM WITH A POTENTIAL AND BOUNDARY CONDITIONS DEPENDING ON THE SPECTRAL PARAMETER

We consider here the inverse problem for (2.3)–(2.5) assuming the string to be weakly damped. First we prove the inverse assertion to Lemma 2.3.

LEMMA 4.1. *Let* $\Lambda = \{\lambda_k\}$ *be a sequence satisfying the conditions* (i), (ii), (iii) *in Theorem* 3.1 *and having the asymptotics* (3.1) *with* $a > 0$ *and* $p > 0$. *Then the entire function*

$$\chi(\lambda) = \lim_{n\to\infty} \prod_{k=+0}^{n} \left(1 - \frac{\lambda}{\lambda_k}\right) \prod_{k=-n}^{-0} \left(1 - \frac{\lambda}{\lambda_k}\right) \tag{4.1}$$

may be represented in the form (2.13).

Proof. Using (3.1) we can find

$$a = \lim_{n\to\infty} \frac{\pi n}{\lambda_n}, \tag{4.2}$$

$$p = -2\mathrm{i} \lim_{n\to\infty} \left(\lambda_n - \frac{\pi n}{a}\right), \tag{4.3}$$

$$p_1 = \lim_{n\to\infty} n \left(\lambda_n - \frac{\pi n}{a} - \frac{\mathrm{i}p}{2}\right), \tag{4.4}$$

$$p_2 = -\mathrm{i} \lim_{n\to\infty} n^2 \left(\lambda_n - \frac{\pi n}{a} - \frac{\mathrm{i}p}{2} - \frac{p_1}{n}\right), \tag{4.5}$$

$$p_3 = \lim_{n\to\infty} n^3 \left(\lambda_n - \frac{\pi n}{a} - \frac{\mathrm{i}p}{2} - \frac{p_1}{n} - \frac{\mathrm{i}p_2}{n^2}\right). \tag{4.6}$$

Let us consider an auxiliary problem denoted by $(2.3^{(1)})$–$(2.5^{(1)})$, i.e. the problem (2.3)–(2.5) with a obtained by (4.2), p obtained by (4.3) and with real $q(x) = q^{(1)}(x) \in W_2^2(0,a)$ such that $\frac{1}{2}\int_0^a q^{(1)}(x)\,\mathrm{d}x \neq -p_1 - \frac{p^2 a}{8\pi}$ ($q^{(1)}(x)$ is arbitrary in other respects). Set $m^{(1)} := \left(\frac{p^2 a}{8\pi} + p_1 + \frac{1}{2}\int_0^a q^{(1)}(x)\,\mathrm{d}x\right)^{-1}$ and $\alpha^{(1)} := m^{(1)}\big(p + \frac{m^{(1)}\pi^2}{a} p_2\big)$ and let $\beta^{(1)}$ be the solution of the equation

$$\begin{aligned} p_3 = \frac{p^2 a p_1}{8\pi^2} - \frac{p^4 a^3}{128\pi^3} + \frac{a^2}{\pi^3}\Bigg(&-\frac{A_1^{(1)} B_1^{(1)}}{a} + A_1^{(1)}\big(3{B_1^{(1)}}^2 + {A_1^{(1)}}^2\big) \\ &- A_2^{(1)} B_1^{(1)} + A_1^{(1)} B_2^{(1)} - A_3^{(1)}\Bigg), \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
B_1^{(1)} &= p - \frac{\alpha^{(1)}}{m^{(1)}}, \\
B_2^{(1)} &= -\frac{K^{(1)}(a,a)}{m^{(1)}} + K_t^{(1)}(a,a) - \frac{\beta}{m^{(1)}} - \frac{p}{2}\left(p - \frac{\alpha^{(1)}}{m^{(1)}}\right), \\
B_3^{(1)} &= B_1^{(1)}\left(K_t^{(1)}(a,a) - \frac{p^2}{8}\right), \\
A_1^{(1)} &= -\frac{1}{m^{(1)}} - K^{(1)}(a,a), \\
A_2^{(1)} &= -K^{(1)}(a,a)\left(p - \frac{\alpha^{(1)}}{m^{(1)}}\right), \\
A_3^{(1)} &= \frac{-K_x^{(1)}(a,a)}{m^{(1)}} - K_{tt}^{(1)}(a,a) + K^{(1)}(a,a)\left(\frac{\beta}{m^{(1)}} + \frac{p}{2}\left(p - \frac{\alpha^{(1)}}{m^{(1)}}\right)\right).
\end{aligned} \tag{4.8}
$$

Here $K^{(1)}(x,t)$ is the kernel of the integral representation (2.11) for the case of $q(x) = q^{(1)}(x)$, i.e.

$$
\begin{aligned}
\chi^{(1)}(\lambda) = \cos\tau a &+ K^{(1)}(a,a)\frac{\sin\tau a}{\tau} - K_x^{(1)}(a,a)\frac{\cos\tau a}{\tau^2} \\
&+ \int_0^a K_{xt}^{(1)}(a,t)\frac{\cos\tau t}{\tau^2}\,\mathrm{d}t + \left(-m^{(1)}\lambda^2 + \mathrm{i}\alpha^{(1)}\lambda + \beta^{(1)}\right)\cdot \\
&\cdot\left(\frac{\sin\tau a}{\tau} - K^{(1)}(a,a)\frac{\cos\tau a}{\tau^2} + K_t^{(1)}(a,a)\frac{\sin\tau a}{\tau^3}\right. \\
&\left. + K_{tt}^{(1)}(a,a)\frac{\cos\tau a}{\tau^4} - \int_0^a K_{ttt}^{(1)}(a,t)\frac{\cos\tau a}{\tau^4}\,\mathrm{d}t\right).
\end{aligned} \tag{4.9}
$$

Under proper numbering, the zeros $\lambda_n^{(1)}$ of the function $\chi^{(1)}(\lambda)$ have the following asymptotics

$$
\lambda_n^{(1)} = \frac{\pi n}{a} + \frac{\mathrm{i}p}{2} + \frac{p_1}{n} + \frac{\mathrm{i}p_2}{n^2} + \frac{p_3}{n^3} + \frac{b_n^{(1)}}{n^3}, \tag{4.10}
$$

where $\left\{b_n^{(1)}\right\} \in l_2$.

Then

$$
\chi^{(1)}(\lambda) = C\lambda^{2r}\lim_{n\to\infty}\prod_{-n}^{n}\left(1 - \frac{\lambda}{\lambda_k^{(1)}}\right). \tag{4.11}
$$

Here the nonnegative even number $2r$ is the multiplicity of $\lambda = 0$ considered as a zero of $\chi^{(1)}(\lambda)$. We include the factors corresponding to all $\lambda_k^{(1)} \neq 0$ $(k = \pm r, \pm(r+1), \ldots)$ into the infinite product in (4.11). Let $\lambda_j^{(1)} \neq 0$ be a zero of $\chi^{(1)}(\lambda)$. Set

$$\chi_1^{(1)}(\lambda) := \left(1 - \frac{\lambda}{\lambda_j^{(1)}}\right)^{-1} \chi^{(1)}(\lambda).$$

Due to Lemma 2.3 all zeros of $\chi_1^{(1)}(\lambda)$ are located in the strip $|\operatorname{Im}\lambda| < \sup_k |\operatorname{Im}\lambda_k^{(1)}|$ and for λ belonging to the axis $\operatorname{Im}\lambda = \sup_k |\operatorname{Im}\lambda_k| + c$ $(c > 0)$ the inequalities

$$0 < \inf |\chi_1^{(1)}(\lambda)|, \quad \sup |\chi_1^{(1)}(\lambda)| < \infty$$

hold. So the function $\chi_1^{(1)}(\lambda)$ is of sinus-type. Formula (4.10) compared with (3.1) yields

$$\lambda_n = \lambda_n^{(1)} + \frac{\widetilde{b}_n}{\left(\lambda_n^{(1)}\right)^3}, \quad \text{for } \lambda_n^{(1)} \neq 0,$$

where $\{\widetilde{b}_n\} \in l_2$. Then using the results of [17], Lemma 5, we have

$$\chi(\lambda)\left(1 - \frac{\lambda}{\lambda_j}\right)^{-1} = \chi_1^{(1)}(\lambda)\left(T_0 + \frac{T_1}{\lambda} + \frac{T_2}{\lambda^2} + \frac{T_3}{\lambda^3}\right) + \frac{\varphi(\lambda)}{\lambda^3},$$

where $T_k \in \mathbb{C}$ $(k = \overline{0,3})$, $\varphi(\lambda)$ is an entire function of exponential type $\leqslant a$ belonging to $L_2(-\infty, \infty)$ when $\lambda \in \mathbb{R}$. Hence

$$\begin{aligned}
\chi(\lambda) &= \chi^{(1)}(\lambda)\frac{\lambda_j^{(1)}}{\lambda_j}\left(1 + \frac{\lambda_j - \lambda_j^{(1)}}{\lambda_j^{(1)} - \lambda}\right)\left(T_0 + \frac{T_1}{\lambda} + \frac{T_2}{\lambda^2} + \frac{T_3}{\lambda^3}\right) \\
&\quad + \frac{\Psi(\lambda)}{\lambda^3}\left(1 - \frac{\lambda}{\lambda_j}\right) \\
&= \chi^{(1)}(\lambda)\left(T_{01} + \frac{T_{11}}{\lambda} + \frac{T_{21}}{\lambda^2} + \frac{T_{31}}{\lambda^3}\right) + \frac{\Psi_1(\lambda)}{\lambda^3}
\end{aligned} \tag{4.12}$$

where $T_{k1} \in \mathbb{C}$ and $\Psi_1(\lambda)$ is an entire function of exponential type $\leqslant a$ belonging to $L_2(-\infty, \infty)$. Substituting (4.9) into (4.12) we obtain (2.13). ∎

ASSUMPTION I. The set $\Lambda = \{\lambda_k\}$ of complex numbers, where no λ_k is purely imaginary, satisfies conditions (i) and (ii) in Theorem 3.1.

ASSUMPTION II. Being properly numbered, the sequence $\{\lambda_k\}$ satisfies equation (3.1) with $a > 0$ and $p > 0$.

Let the function $\chi(\lambda)$ be defined by (4.1) where $\Lambda = \{\lambda_k\}$ satisfies the Assumptions I and II. Then, according to Lemma 4.1, it is of the form (2.13). Hence, we can find the constants involved. Set

$$\theta_n = \frac{\mathrm{i}p}{2} + \sqrt{\left(\frac{\frac{\pi}{2} + 2\pi n}{a}\right)^2 - \frac{p^2}{4}}$$

($\operatorname{Re}\theta_n > 0$ for n large enough). Then using (2.13) we obtain

$$B_0 = \lim_{n\to\infty}\left(\chi(\theta_n)\frac{a}{2\pi n}\right), \tag{4.13}$$

$$B_1 = -\mathrm{i}\lim_{n\to\infty}\left(B_0^{-1}\chi(\theta_n) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right)\right), \tag{4.14}$$

$$B_2 = \lim_{n\to\infty}\frac{2\pi n}{a}\left(B_0^{-1}\chi(\theta_n) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right) - \mathrm{i}B_1\right), \tag{4.15}$$

$$B_3 = -\mathrm{i}\lim_{n\to\infty}\left(\frac{2\pi n}{a}\right)^2\left(B_0^{-1}\chi(\theta_n) - \frac{1}{a}\left(2\pi n+\frac{\pi}{2}\right) - \mathrm{i}B_1 - \frac{aB_2}{2\pi n+\frac{\pi}{2}}\right). \tag{4.16}$$

Set

$$\xi_n = \frac{\mathrm{i}p}{2} + \sqrt{\left(\frac{2\pi n}{a}\right)^2 - \frac{p^2}{4}},$$

($\operatorname{Re}\xi_n > 0$ for n large enough). Then

$$A_1 = B_0^{-1}\lim_{n\to\infty}\chi(\xi_n), \tag{4.17}$$

$$A_2 = -\mathrm{i}\lim_{n\to\infty}\frac{2\pi n}{a}\left(B_0^{-1}\chi(\xi_n) - A_1\right), \tag{4.18}$$

$$A_3 = \lim_{n\to\infty}\left(\frac{2\pi n}{a}\right)^2\left(B_0^{-1}\chi(\xi_n) - A_1 - \frac{\mathrm{i}aA_2}{2\pi n}\right). \tag{4.19}$$

ASSUMPTION III(a). Let $\operatorname{Im}\lambda_k > \frac{p}{2}$ for all $\lambda_k \in \Lambda$.

ASSUMPTION III(b). Let $\operatorname{Im}\lambda_k \in \left(0, \frac{p}{2}\right)$ for all $\lambda_k \in \Lambda$.

ASSUMPTION III(c). Let $\operatorname{Im}\lambda_k = \frac{p}{2}$ for all $\lambda_k \in \Lambda$.

LEMMA 4.2. *Under the Assumptions* I, II, III(a) (III(b)) *the constants given by* (4.13), (4.14) *satisfy the inequalities* $B_0 < 0$, $B_1 > 0$ ($B_0 < 0$, $B_1 < 0$).

Proof. The function $\chi(\lambda)$ defined by (4.1) may be represented as follows

$$
\begin{aligned}
\chi(\lambda) &= \frac{\sin\tau a \lim\limits_{n\to\infty}\prod\limits_{+0}^{n}\left(1-\frac{\lambda}{\lambda_k}\right)\prod\limits_{-n}^{-0}\left(1-\frac{\lambda}{\lambda_k}\right)}{a\tau\prod\limits_{k=1}^{\infty}\left(1-\frac{\tau^2a^2}{\pi^2k^2}\right)} \\
&= \frac{\tau\sin\tau a}{a}\left(\frac{1}{\tau^2}+\frac{1}{\lambda_0\lambda_{-0}}-\frac{\lambda(\lambda_0+\lambda_{-0}-\mathrm{i}p)}{\tau^2\lambda_0\lambda_{-0}}\right) \\
&\quad\times\lim_{n\to\infty}\prod_{k=1}^{n}\frac{\left(\frac{1}{\tau^2}+\frac{1}{\lambda_k\lambda_{-k}}-\frac{\lambda(\lambda_k+\lambda_{-k}-\mathrm{i}p)}{\tau^2\lambda_k\lambda_{-k}}\right)}{\left(\frac{1}{\tau^2}-\frac{a^2}{\pi^2k^2}\right)}
\end{aligned}
\tag{4.20}
$$

where $\tau=\sqrt{\lambda^2-\mathrm{i}p\lambda}$ and a is given by (4.2). Substituting now $\lambda=\theta_n$, $\tau(\theta_n)=\frac{1}{a}\left(2\pi n+\frac{\pi}{2}\right)$ into (4.20) we obtain

$$
\begin{aligned}
\chi(\theta_n) &= \frac{\tau_n}{a}\left(\frac{1}{\tau_n^2}+\frac{1}{\lambda_0\lambda_{-0}}-\frac{\theta_n}{\tau_n^2}\left(\frac{\lambda_0+\lambda_{-0}-\mathrm{i}p}{\lambda_0\lambda_{-0}}\right)\right) \\
&\quad\times\lim_{j\to\infty}\prod_{k=1}^{j}\frac{\left(\frac{1}{\tau_n^2}+\frac{1}{\lambda_k\lambda_{-k}}-\frac{\theta_n}{\tau_n^2}\left(\frac{\lambda_k+\lambda_{-k}-\mathrm{i}p}{\lambda_k\lambda_{-k}}\right)\right)}{\left(\frac{1}{\tau_n^2}-\frac{a^2}{\pi^2k^2}\right)}
\end{aligned}
$$

and consequently

$$
B_0=\lim_{n\to\infty}\frac{\chi(\theta_n)}{\tau_n}=\frac{1}{a\lambda_0\lambda_{-0}}\ \lim_{j\to\infty}\prod_{k=1}^{j}\frac{\pi^2k^2}{(-\lambda_k\lambda_{-k}a^2)}. \tag{4.21}
$$

Due to the symmetry of the problem, $\lambda_k\lambda_{-k}<0$ for all $k\in\mathbb{N}\cup\{0\}$. Hence, (4.21) implies $B_0<0$. Using (4.21) we obtain from (4.14):

$$
B_1=-\mathrm{i}\lim_{n\to\infty}\left(B_0^{-1}\chi(\theta_n)-\tau_n\right)=\mathrm{i}\sum_{k=0}^{\infty}\left(\lambda_k+\lambda_{-k}-\mathrm{i}p\right).
$$

Assumption III(a), III(b) implies $\operatorname{Im}(\lambda_k+\lambda_{-k}-\mathrm{i}p)>0$ ($\operatorname{Im}(\lambda_k+\lambda_{-k}-\mathrm{i}p)<0$) and consequently $B_1<0$ ($B_1>0$). ∎

LEMMA 4.3. *The Assumptions* I, II, III(a) (III(b)) *imply* $A_1B_1>A_2$ ($A_1B_1<A_2$)*, where* A_1, A_2, B_1 *are defined by* (4.14), (4.17), (4.18).

Proof. We apply Lemma 2.3 to the function $\chi(\lambda)$ and obtain $B_1A_1-A_2=\frac{\pi^2}{a}p_2$. Then the Assumption III(a) (III(b)) implies $p_2>0$ ($p_2<0$). ∎

Set

$$m = B_1 \left(A_2 - A_1 B_1\right)^{-1}, \tag{4.22}$$

$$\alpha = m\left(p - B_1\right), \tag{4.23}$$

$$\beta = B_2 m + 8^{-1} p^2 m + 2^{-1} p(\alpha - mp) + A_2 B_1^{-1} - m B_3 B_1^{-1}. \tag{4.24}$$

Lemmas 4.2 and 4.3 imply the following corollaries.

COROLLARY 4.4. *Under Assumptions* I, II, *and* III(a) *or* III(b) *the inequality* $m > 0$ *holds.*

COROLLARY 4.5. *Under Assumptions* I, II, III(a) (III(b)) *the inequality* $\alpha > mp$ $(\alpha < mp)$ *holds.*

Set

$$g_1(\tau) \stackrel{\text{def}}{=} \frac{-m\left(\chi\left(\frac{\mathrm{i}p}{2} + \sqrt{\tau^2 - \frac{p^2}{4}}\right) - \chi\left(\frac{\mathrm{i}p}{2} - \sqrt{\tau^2 - \frac{p^2}{4}}\right)\right)}{2B_0 \mathrm{i}(\alpha - mp)\sqrt{\tau^2 - \frac{p^2}{4}}}, \tag{4.25}$$

$$\begin{aligned} g_2(\tau) \stackrel{\text{def}}{=} & -\frac{m}{B_0}\, \chi\left(\frac{\mathrm{i}p}{2} + \sqrt{\tau^2 - \frac{p^2}{4}}\right) \\ & + \left(m\tau^2 - \mathrm{i}(\alpha - mp)\sqrt{\tau^2 - \frac{p^2}{4}} - \beta + \frac{p}{2}(\alpha - mp)\right) g_1(\tau). \end{aligned} \tag{4.26}$$

LEMMA 4.6. *Under Assumptions* I, II *and* III(a) *or* III(b), $g_1(\tau)$ *and* $g_2(\tau)$ *are entire functions of* τ *and admit the representations*

$$g_1(\tau) = \frac{\sin \tau a}{\tau} + \frac{A_2}{B_1}\frac{\cos \tau a}{\tau^2} + \frac{B_3}{B_1}\frac{\sin \tau a}{\tau^3} + \frac{\Psi_1(\tau)}{\tau^3} \tag{4.27}$$

$$g_2(\tau) = \cos \tau a - \frac{A_2}{B_1}\frac{\sin \tau a}{\tau} + \frac{\Psi_2(\tau)}{\tau}, \tag{4.28}$$

where $\Psi_k(\tau)$ $(k = 1, 2)$ *are entire functions of exponential type* $\leqslant a$ *belonging to* $L_2(-\infty, \infty)$.

Proof. It follows from (4.25) and (4.26) that $g_1(\tau)$ and $g_2(\tau)$ are entire functions of τ. Substituting (2.13) into (4.25) and (4.26) we obtain (4.27) and (4.28). ∎

COROLLARY 4.7. *The zeros ν_n and μ_n of the functions $g_1(\tau)$ and $g_2(\tau)$ have the following asymptotics*

$$\nu_n = \frac{\pi n}{a} - \frac{A_2}{\pi B_1 n} + \frac{b_{1n}}{n},$$
$$\mu_n = \frac{\pi(n-\frac{1}{2})}{a} - \frac{A_2}{\pi B_1 n} + \frac{b_{2n}}{n},$$

where $n \in \mathbb{N}$, $\{b_{kn}\}_{n=1}^{\infty} \in l_2$, $k = 1, 2$.

Proof. Applying Lemma 3.4.2 of [20], p. 225 to the functions $g_1(\tau)$ and $g_2(\tau)$ we get the assertion. ∎

Consider the auxiliary problem

$$y'' + \left(\lambda^2 - \mathrm{i}\lambda p - q_2\right) y = 0, \tag{4.29}$$
$$y(\lambda, 0) = 0, \tag{4.30}$$
$$y'(\lambda, a) + (-m\lambda^2 + \mathrm{i}mp\lambda + \beta) y(\lambda, a) = 0, \tag{4.31}$$

where $q_2 = \text{constant} > 0$.

LEMMA 4.8. *For arbitrary $p > 0$, $a > 0$, $\beta \in \mathbb{R}$ it is possible to choose $q_2 > 0$ such that the operator A_1 defined by* (A.9) (see Appendix) *satisfies the condition $A_1 \geqslant \left(\frac{p}{2} + \varepsilon\right) I$, $(\varepsilon > 0)$.*

Proof. Evidently,

$$\begin{aligned}\left(\left(A_1 - \left(\frac{p}{2} + \varepsilon\right) I\right) Y, Y\right) &= -\int_0^a y''\overline{y}\,\mathrm{d}x + q_2 \int_0^a |y|^2\,\mathrm{d}x + y'(a)\overline{y(a)} + \beta|y(a)|^2 \\ &\quad - \left(\frac{p}{2} + \varepsilon\right)|y(a)|^2 - \left(\frac{p}{2} + \varepsilon\right)\int_0^a |y|^2\,\mathrm{d}x \\ &= \int_0^a |y'|^2\,\mathrm{d}x + \left(q_2 - \frac{p}{2} - \varepsilon\right)\int_0^a |y|^2\,\mathrm{d}x \\ &\quad + \left(\beta - \frac{p}{2} - \varepsilon\right)|y(a)|^2.\end{aligned}$$

As $y(0) = 0$ we obtain

$$|y(a)|^2 = \int_0^a (y'\overline{y} + y\overline{y}')\,\mathrm{d}x \leqslant 2\int_0^a |y'||y|\,\mathrm{d}x \leqslant \frac{1}{c^2}\int_0^a |y'|^2\,\mathrm{d}x + c^2 \int_0^a |y|^2\,\mathrm{d}x$$

with any real $c \neq 0$. Choosing $c^2 > \left|\beta - \frac{p}{2} - \varepsilon\right|$, $q_2 > \frac{p}{2} + \varepsilon + \left|\beta - \frac{p}{2} - \varepsilon\right| c^2 > \frac{p}{2} + \left(\beta - \frac{p}{2} - \varepsilon\right)^2 + \varepsilon$ we obtain

$$\begin{aligned}\left(-\frac{p}{2} + \beta - \varepsilon\right) |y(a)|^2 &\leqslant \left|\beta - \varepsilon - \frac{p}{2}\right| \int\limits_0^a (y'\overline{y} + y\overline{y}')\,\mathrm{d}x \\ &\leqslant \left|\beta - \varepsilon - \frac{p}{2}\right| c^{-2} \int\limits_0^a |y'|^2\,\mathrm{d}x + \left|\beta - \varepsilon - \frac{p}{2}\right| c^2 \int\limits_0^a |y|^2\,\mathrm{d}x \\ &< \int\limits_0^a |y'|^2\,\mathrm{d}x + \left(q_2 - \frac{p}{2} - \varepsilon\right) \int\limits_0^a |y|^2\,\mathrm{d}x.\end{aligned}$$

The lemma is proved. ∎

It is possible to rewrite the system (4.29)–(4.31) as follows

$$y'' + \tau^2 y - q_2 y = 0, \tag{4.32}$$

$$y(\tau, 0) = 0, \tag{4.33}$$

$$y'(\tau, a) + (-m\tau^2 + \beta) y(\tau, a) = 0. \tag{4.34}$$

We assume in what follows q_2 to be so large that $A_1 \geqslant \left(\frac{p}{2} + \varepsilon\right) I$. Denote by $\tau_n^{(2)}$ the eigenvalues of the problem (4.32)–(4.34) and by $\lambda_n^{(2)}$ those of (4.29)–(4.31). It is clear that $\tau_n^{(2)} = \sqrt{\lambda_n^{(2)2} - \mathrm{i}p\lambda_n^{(2)}}$. The set $\left\{\tau_n^{(2)}\right\}$ is real and symmetric with respect to the origin.

Set $S_2(\tau, x) = \frac{\sin\sqrt{\tau^2 - q_2}x}{\sqrt{\tau^2 - q_2}}$ and $S_2'(\tau, x) = \cos\sqrt{\tau^2 - q_2}x$. Denote by $\nu_n^{(2)}$ the zeros of $S_2(\tau, a)$ and by $\mu_n^{(2)}$ the zeros of $S_2'(\tau, a)$ $(\mu_{-n} = -\mu_n,\ \nu_{-n} = -\nu_n)$. Evidently, $\nu_n = \sqrt{\frac{\pi^2 n^2}{a^2} + q_2}$ and $\mu_n = \sqrt{\left(\pi n - \frac{\pi}{2}\right)^2 a^{-2} + q_2}$ and

$$0 < \mu_1^{(2)} < \nu_1^{(2)} < \cdots < \mu_n^{(2)} < \nu_n^{(2)} < \mu_{n+1}^{(2)} < \nu_{n+1}^{(2)} < \cdots. \tag{4.35}$$

Introduce the function

$$\chi(\lambda, \eta) = \lim_{n\to\infty} \prod_{+0}^{n} \left(1 - \frac{\lambda}{\lambda_k^{(2)} + \eta(\lambda_k - \lambda_k^{(2)})}\right) \prod_{-n}^{-0} \left(1 - \frac{\lambda}{\lambda_k^{(2)} + \eta(\lambda_k - \lambda_k^{(2)})}\right). \tag{4.36}$$

Denote by $\lambda_k(\eta) = \lambda_k^{(2)} + \eta(\lambda_k - \lambda_k^{(2)})$ the zeros of $\chi(\lambda, \eta)$. Under Assumption I $\operatorname{Re}\lambda_k \neq 0$ and by Lemma 4.8 $\operatorname{Re}\lambda_k^{(2)} \neq 0$. Therefore $\operatorname{Re}\lambda_k(\eta) \neq 0$ for any $\eta \in [0, 1]$.

LEMMA 4.9. *There exists a neighbourhood $\Omega \subset \mathbb{C}$ of the interval $[0,1]$ such that the function $\chi(\lambda,\eta)$ is an entire function of λ at any fixed $\eta \in \Omega$ and $\chi(\lambda,\eta)$ is holomorphic on Ω at any fixed $\lambda \in \mathbb{C}$.*

Proof. Applying Theorem 3.1 to the problem (4.32)–(4.34) we obtain

$$\lambda_n^{(2)} = \frac{\pi n}{a} + \frac{\mathrm{i}p}{2} + \mathrm{o}(1) \tag{4.37}$$

and consequently

$$\lambda_n - \lambda_n^{(2)} = \mathrm{o}(1). \tag{4.38}$$

Hence, the limit in (4.36) exists and $\chi(\lambda,\eta)$ is an entire function of λ at any fixed $\eta \in \Omega$. Now if $\lambda \neq \lambda_k^{(2)}$ we rewrite (4.36) as follows

$$\chi(\lambda,\eta) = \lim_{n\to\infty} \prod_0^n \frac{(\lambda_k^{(2)} - \lambda)(\lambda_{-k}^{(2)} - \lambda)\left(1 + \eta \frac{\lambda_k - \lambda_k^{(2)}}{\lambda_k^{(2)} - \lambda}\right)\left(1 + \eta \frac{\lambda_{-k} - \lambda_{-k}^{(2)}}{\lambda_{-k}^{(2)} - \lambda}\right)}{\lambda_k^{(2)}\lambda_{-k}^{(2)}\left(1 + \eta \frac{\lambda_k - \lambda_k^{(2)}}{\lambda_k^{(2)}}\right)\left(1 + \eta \frac{\lambda_{-k} - \lambda_{-k}^{(2)}}{\lambda_{-k}^{(2)}}\right)}. \tag{4.39}$$

If $\lambda = \lambda_k^{(2)}$ then the corresponding factor is of the form $\eta(\lambda_k - \lambda_k^{(2)})$. Due to (4.37), (4.38) it follows from (4.39) that $\chi(\lambda,\eta)$ is holomorphic on Ω at any fixed $\lambda \in \mathbb{C}$. ∎

LEMMA 4.10. *Under Assumptions* I, II *and* III(a) *or* III(b) *for any $\eta \in (0,1]$ there exist the limits*

$$B_0(\eta) = \lim_{n\to\infty}\left(\chi(\theta_n,\eta)\frac{a}{2\pi n}\right) \neq 0, \tag{4.40}$$

$$B_1(\eta) = -\mathrm{i}\lim_{n\to\infty}\left(B_0^{-1}(\eta)\chi(\theta_n,\eta) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right)\right), \tag{4.41}$$

$$B_2(\eta) = \lim_{n\to\infty}\frac{2\pi n}{a}\left(\chi(\theta_n,\eta)B_0^{-1}(\eta) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right) - \mathrm{i}B_1(\eta)\right), \tag{4.42}$$

$$\begin{aligned} B_3(\eta) = -\mathrm{i}\lim_{n\to\infty}\left(\frac{2\pi n}{a}\right)^2 \Bigg(& B_0^{-1}(\eta)\chi(\theta_n,\eta) \\ & - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right) - \mathrm{i}B_1(\eta) - \frac{aB_2(\eta)}{2\pi n + \frac{\pi}{2}}\Bigg), \end{aligned} \tag{4.43}$$

$$A_1(\eta) = B_0^{-1}(\eta)\lim_{n\to\infty}\chi(\theta_n,\eta), \tag{4.44}$$

$$A_2(\eta) = -\mathrm{i}\lim_{n\to\infty}\frac{2\pi n}{a}\left(B_0^{-1}(\eta)\chi(\xi_n,\eta) - A_1(\eta)\right), \tag{4.45}$$

$$A_3(\eta) = \lim_{n\to\infty}\left(\left(\frac{2\pi n}{a}\right)^2\left(B_0^{-1}(\eta)\chi(\xi_n,\eta) - A_1(\eta) - \frac{\mathrm{i}aA_2(\eta)}{2\pi n}\right)\right), \tag{4.46}$$

where

$$\theta_n = \frac{\mathrm{i}p}{2} + \sqrt{\left(\frac{\frac{\pi}{2} + 2\pi n}{a}\right)^2 - \frac{p^2}{4}}, \quad \xi_n = \frac{\mathrm{i}p}{2} + \sqrt{\left(\frac{2\pi n}{a}\right)^2 - \frac{p^2}{4}}.$$

Proof. The function $\chi(\lambda, \eta)$ satisfies all the conditions of Lemma 4.1 for any $\eta \in (0, 1]$. Thus $\chi(\lambda, \eta)$ is of the form (2.13) with A_k and B_k depending on η. Then the assertions of the lemma follow. ∎

Remark 4.11. From the proof of Lemma 5 of [17] it follows that $A_k(\eta)$ and $B_k(\eta)$ are analytic in some complex neighbourhood of the interval $[0, 1]$.

Corollary 4.12. *Under Assumptions* I, II *and* III(a) (III(b)) *the coefficients $B_0(\eta)$ and $B_1(\eta)$, defined by* (4.40), (4.41), *satisfy the inequalities $B_0(\eta) < 0$, $B_1(\eta) > 0$, ($B_0(\eta) < 0$, $B_1(\eta) < 0$) for all $\eta \in (0, 1]$.*

Proof. The function $\chi(\lambda, \eta)$ satisfies the conditions of Lemma 4.2. Therefore the assertion of Corollary 4.12 follows. ∎

The proof of the next corollary is analogous.

Corollary 4.13. *Under the Assumptions* I, II, III(a) (III(b)) *the coefficients $A_1(\eta), A_2(\eta), B_1(\eta)$, defined by* (4.41), (4.44), (4.45), *satisfy the inequalities $A_1(\eta)B_1(\eta) > A_2(\eta)$ ($A_1(\eta)B_1(\eta) < A_2(\eta)$) for any $\eta \in (0, 1]$.*

Set

$$\begin{aligned}
m(\eta) &= B_1(\eta)\left(A_2(\eta) - A_1(\eta)B_1(\eta)\right)^{-1}, \\
\alpha(\eta) &= m(\eta)\left(p - B_1(\eta)\right), \\
\beta(\eta) &= \left(B_2(\eta) - \frac{B_3(\eta)}{B_1(\eta)} + \frac{p^2}{8}\right) m(\eta) + \frac{p}{2}(\alpha(\eta) - m(\eta)p) + \frac{A_2(\eta)}{B_1(\eta)}, \\
g_1(\tau, \eta) &:= \frac{-m(\eta)\left(\chi\left(\frac{\mathrm{i}p}{2} + \sqrt{\tau^2 - \frac{p^2}{4}}, \eta\right) - \chi\left(\frac{\mathrm{i}p}{2} - \sqrt{\tau^2 - \frac{p^2}{4}}, \eta\right)\right)}{2\mathrm{i}B_0(\eta)(\alpha(\eta) - m(\eta)p)\sqrt{\tau^2 - \frac{p^2}{4}}}, \\
g_2(\tau, \eta) &:= \frac{-m(\eta)}{B_0(\eta)}\chi\left(\frac{\mathrm{i}p}{2} + \sqrt{\tau^2 - \frac{p^2}{4}}, \eta\right) \\
&\quad + \left(m\tau^2 - \mathrm{i}(\alpha - mp)\sqrt{\tau^2 - \frac{p^2}{4}} - \beta + \frac{p}{2}(\alpha - mp)\right) g_1(\tau, \eta).
\end{aligned}$$

Denote by $\nu_n(\eta)$ and $\mu_n(\eta)$ the zeros of $g_1(\tau, \eta)$ and $g_2(\tau, \eta)$, respectively ($\nu_{-n}(\eta) = -\nu_n(\eta)$, $\mu_{-n}(\eta) = -\mu_n(\eta)$). It is clear that $\nu_n(0) = \nu_n^{(2)}$ and $\mu_n(0) = \mu_n^{(2)}$.

LEMMA 4.14. *Under Assumptions* I, II *and* III(a) *or* III(b) *for any* $\eta \in [0,1]$, $\nu_n(\eta) \neq \mu_k(\eta)$ *is valid, and if* $n \neq \pm k$, *then* $\nu_n(\eta) \neq \nu_k(\eta)$, $\mu_n(\eta) \neq \mu_k(\eta)$.

Proof. For given n and small enough $\eta > 0$ the zeros $\nu_n(\eta)$ and $\mu_n(\eta)$ are real due to (4.35) and to the symmetry of the problem. They can lose analyticity with respect to η only when they collide. Hence, if real $\nu_n(\eta_0) = \mu_k(\eta_0)$ at $\eta_0 \in (0,1]$, then $g_1(\mu_k(\eta_0), \eta_0) = g_2(\mu_k(\eta_0), \nu_0) = 0$, and consequently $\chi\Big(\frac{\mathrm{i}p}{2} + \sqrt{\mu_k^2(\eta_0) - \frac{p^2}{4}}, \eta_0\Big) = 0$. This means that there exists a $\lambda_j(\eta_0) = \frac{\mathrm{i}p}{2} + \sqrt{\mu_k^2(\eta_0) - \frac{p^2}{4}}$ with $\operatorname{Im}\lambda_j(\eta_0) = \frac{p}{2}$, which contradicts III(a) or III(b). Hence, $\nu_n(\eta) \neq \mu_k(\eta)$. The proofs of the other assertions are analogous. ∎

COROLLARY 4.15. *For any* $\eta \in [0,1]$

$$\mu_1^2(\eta) < \nu_1^2(\eta) < \cdots < \mu_n^2(\eta) < \nu_n^2(\eta) < \mu_{n+1}^2(\eta) < \nu_{n+1}^2(\eta) < \cdots.$$

Denote by $\mathcal{N}^+$ ($\mathcal{N}^-$) the class of sets $\{q(x), a, p, m, \alpha, \beta\}$ satisfying the following conditions:

(1) $a > 0$, $m > 0$, $\alpha > mp > 0$, $(0 < \alpha < mp)$, $\beta \in \mathbb{R}$, $q(x) \in L_2(0,a)$ is real;

(2) the operator A acting in $L_2(0,a)$ defined by $q(x)$ and β according to the formulae

$$\begin{aligned} Ay &= -y'' + q(x)y, \\ D(A) &= \big\{y : y \in W_2^2(0,a),\, y(0) = y'(a) + \beta y(a) = 0\big\} \end{aligned} \tag{4.47}$$

is strictly positive;

(3) the operator pencil $\mathcal{L}(\lambda)$ defined by (A.8) (see Appendix) is weakly damped.

THEOREM 4.16. *Let* $\Lambda = \{\lambda_k\}$ *be a set of complex numbers satisfying Assumptions* I, II, III(a). *Then there exists a unique set* $\{q(x), a, p, m, \alpha, \beta\}$ *from* $\mathcal{N}^+$ *such that* Λ *is the spectrum of the problem* (2.3)–(2.5) *with* $\{q(x), a, p, m, \alpha, \beta\}$.

Proof. Under the conditions of Theorem 4.16 there exist the limits (4.2) and (4.3). Hence, we can find $a > 0$ and $p > 0$, and the function $\chi(\lambda)$ by (4.1). Now find the constants B_k and A_k from (4.13)–(4.19), m from (4.22), α from (4.23), and β from (4.24). By Corollary 4.4, $m > 0$ and by Corollary 4.5, $\alpha > mp > 0$. Then we construct the functions $g_1(\tau)$ and $g_2(\tau)$ using (4.25) and (4.26). By Corollaries 4.7 and 4.15 the sets $\{\nu_k\}$ and $\{\mu_k\}$ of the zeros of $g_1(\tau)$ and $g_2(\tau)$ satisfy all the conditions of Theorem 3.4.1 of [20]. Hence, there exists a unique real $q(x) \in L_2(0,a)$ such that the corresponding Sturm-Liouville problems

$$\tau^2 y + y'' - q(x)y = 0, \tag{4.48}$$

$$y(0) = y(a) = 0 \tag{4.49}$$

and

$$\tau^2 y + y'' - q(x)y = 0, \tag{4.48$'$}$$

$$y(0) = y'(a) = 0 \tag{4.49$'$}$$

have the spectra $\{\nu_k\}$ and $\{\mu_k\}$, correspondingly.

We can construct $q(x)$ in the following way (cf. [20]). Without loss of generality let us assume that $\mu_1^2 > 0$. The function

$$e(\tau) = \mathrm{e}^{-\mathrm{i}\tau a}\left(g_2(\tau) + \mathrm{i}\tau g_1(\tau)\right)$$

is the so-called Jost function of the corresponding Sturm-Liouville problem prolonged on the semiaxis. In our case this function has no zeros in the closed lower halfplane. Introduce the so-called S-matrix ([20])

$$S(\tau) = \frac{e(\tau)}{e(-\tau)}$$

and the function

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1 - S(\tau))\mathrm{e}^{\mathrm{i}\tau x}\,\mathrm{d}\tau.$$

The Marchenko integral equation

$$K(x,t) + F(x+t) + \int_{x}^{\infty} K(x,s)F(s+t)\,\mathrm{d}s = 0$$

possesses a unique solution $K(x,t)$ and

$$\widetilde{q}(x) = -2\,\frac{\mathrm{d}K(x,x)}{\mathrm{d}x}$$

is the potential of the Sturm–Liouville problem on the semiaxis prolonged. We have to prove now that the projection $q(x) = \widetilde{q}(x)$ $(x \in [0,a])$ is the unknown function (potential) we are looking for.

Now if we solve the direct problems (4.48), (4.49) and (4.48$'$), (4.49$'$) with $q(x)$ obtained, we can find $g_1(\tau)$ and $g_2(\tau)$ coinciding with the ones defined by (4.25) and (4.26). Then the function

$$\chi(\lambda) = -\frac{B_0}{m}\left(g_2(\tau) + (-m\lambda^2 + \mathrm{i}\alpha\lambda + \beta)g_1(\tau)\right)$$

coincides with the one obtained from the set Λ when substituted into (4.1). Hence, the set of zeros of $\chi(\lambda)$ coincides with Λ. Then it follows from [24] that the operator A defined by (4.47) is strictly positive. The pencil $\mathcal{L}(\lambda)$ is weakly damped because $\operatorname{Re}\lambda_k \neq 0$ for all k. Theorem 4.16 is proved. ∎

THEOREM 4.17. *Let* $\Lambda = \{\lambda_k\}$ *be a set of complex numbers satisfying Assumptions* I, II, III(b) *and let* $B_1 < p$, *where* B_1 *is defined by* (4.14). *Then there exists a unique set* $\{q(x), a, p, m, \alpha, \beta\}$ *from* $\mathcal{N}^-$ *such that* Λ *is the spectrum of the problem* (2.3)–(2.5) *with the parameters* $\{q(x), a, p, m, \alpha, \beta\}$.

The proof of this theorem is quite the same as that of Theorem 4.16.

REMARK 4.18. Under Assumptions I, II, III(c) there exists a set $\{q(x), a, p, m, \alpha = mp, \beta\}$ corresponding to the spectrum Λ, but the set is not unique. In this case we need two spectra to solve the inverse problem and the situation is the same as in [19] (see also [18]). Such inverse problem will be considered in a forthcoming publication.

5. INVERSE PROBLEM FOR A STRING

Now consider the problem of the determination of the set of the parameters $\{A(s), p, \mu, \nu\}$ by given Λ and $l > 0$. Denote by $\mathcal{A}_l^+$, ($\mathcal{A}_l^-$) the class of sets $\{A(s), p, \mu, \nu\}$ such that $A(s) \in W_2^2(0, l)$, $A(s) > 0$ for $s \in [0, l]$, $p > 0$, $\mu > 0$, $\nu > p\mu$ ($0 < \nu < p\mu$) and such that the corresponding string is weakly damped (see Definition 3.4).

THEOREM 5.1. *Let* $\Lambda = \{\lambda_k\}$ *be a set of complex numbers satisfying the conditions of Theorem* 4.16 (*Theorem* 4.17). *Then for any* $l > 0$ *there exists a unique set* $\{A(s), p, \mu, \nu\}$ *from* $\mathcal{A}_l^+$ *(from* $\mathcal{A}_l^-$) *such that the spectrum of the corresponding problem* (1.4)–(1.6) *coincides with* Λ.

Proof. It is enough to prove that the formulae (2.6)–(2.10) perform a one–to–one correspondence between the class $\mathcal{A}_l^+$ ($\mathcal{A}_l^-$) and the class of sets $\{q(x), a, m, \alpha, p, \beta, l\}$ where $\{q(x), a, m, \alpha, p, \beta\} \in \mathcal{N}^+(\mathcal{N}^-)$ and $l > 0$. Rewrite (2.6), (2.9) as follows

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(A[x]\big)^{\frac{1}{4}} - q(x)\big(A[x]\big)^{\frac{1}{4}} = 0,$$

$$\left.\frac{\mathrm{d}A[x]}{\mathrm{d}x}\right|_{x=a} + 4\beta A[a] = 0.$$

Consider the Cauchy problem

$$f'' - q(x)f = 0 \tag{5.1}$$

$$f'(a) = -\beta \tag{5.2}$$

$$f(a) = 1. \tag{5.3}$$

As $q(x) \in L_2(0,a)$ the problem (5.1)–(5.3) possesses a unique solution $f(x) \in W_2^2(0,a)$ and $f(x) > 0$ for $x \in [0,a]$ because $A \gg 0$ [10]. It is clear that

$$A[x] = C\big(f(x)\big)^4.$$

The constant C may be found from the equation

$$\int_0^a \big(A[x]\big)^{\frac{1}{2}}\,\mathrm{d}x = l$$

which is a consequence of (2.1).

Thus

$$C = l^2\bigg(\int_0^a \big(f(x)\big)^2\,\mathrm{d}x\bigg)^{-2}$$

and

$$A[x] = l^2\bigg(\int_0^a \big(f(x)\big)^2\,\mathrm{d}x\bigg)^{-2}\big(f(x)\big)^4. \tag{5.4}$$

Then (2.7) and (2.8) yield

$$\mu = m\big(A[a]\big)^{-\frac{1}{2}} > 0 \tag{5.5}$$

$$\nu = \alpha\big(A[a]\big)^{-\frac{1}{2}} > 0. \tag{5.6}$$

In order to find $A(s)$ we use the equation

$$\int_0^x \big(A[x']\big)^{\frac{1}{2}}\,\mathrm{d}x' = s \tag{5.7}$$

which follows from (2.1). We can find $x = x(s)$ using (5.7) and can find $A(s)$ as $A[x(s)]$. Hence,

$$A(s) = l^2\bigg(\int_0^a \big(f(x)\big)^2\,\mathrm{d}x\bigg)^{-2}\big(f(x(s))\big)^4.$$

It is clear that $A(s) \in W_2^2(0,l)$ and $A(s) > 0$. By (5.5), (5.6) and Theorem 4.16 (4.17) we find $\{A(s),\mu,p,\nu\} \in \mathcal{A}_l^+(\mathcal{A}_l^-)$.

The theorem is proved. ∎

6. APPENDIX

Consider the quadratic operator pencil

$$L(\lambda) = \lambda^2 M + \lambda K + A$$

with the domain $\mathrm{D}(L(\lambda)) = \mathrm{D}(A)$ acting in a separable Hilbert space H. The space H is a complexification of a real Hilbert space and the operators M, K, A are real. Consequently, the spectrum of $L(\lambda)$ is symmetric with respect to the real axis. Let the operators satisfy the conditions: $M \gg 0$, $K \geqslant 0$, $A^* = A \geqslant -\kappa I$, $\kappa > 0$ and suppose that there exists a number $\kappa_1 > \kappa$ such that $(A + \kappa_1 I)^{-1}$ is compact. Here I is the identity operator, M and K are bounded. For the sake of simplicity let us assume that the geometric multiplicity, i.e. the dimension of the eigensubspace, of every eigenvalue of the pencil

$$L(\lambda, \eta) = \lambda^2 M + \lambda \eta K + A \tag{A.1}$$

at any $\eta \in [0, 1]$ is equal to 1. This condition is satisfied by the pencils associated with the problems (1.4)–(1.6), (2.3)–(2.5). The spectrum of the pencil $L(\lambda, \eta)$ consists of normal eigenvalues only ([11]).

LEMMA A.1. *If λ_k is a nonreal or a multiple eigenvalue of $L(\lambda)$, then* $\operatorname{Re}\lambda_k \in [-\zeta_2, 0]$, *where* $\zeta_2 = \frac{1}{2} \sup\limits_{y \neq 0} \frac{(Ky,y)}{(My,y)}$.

Proof. Let λ_k be an eigenvalue and y_k the corresponding eigenvector. Then $(L(\lambda_k) y_k, y_k) = 0$ and

$$2\operatorname{Re}\lambda_k \operatorname{Im}\lambda_k\, (My_k, y_k) + \operatorname{Im}\lambda_k\, (Ky_k, y_k) = 0. \tag{A.2}$$

If $\operatorname{Im}\lambda_k \neq 0$, then (A.2) implies

$$\operatorname{Re}\lambda_k = -\frac{(Ky_k, y_k)}{2\,(My_k, y_k)}. \tag{A.3}$$

Thus the first assertion of the lemma follows.

Let $y_k^{(1)}$ be the first associated vector of the chain, then

$$L(\lambda_k) y_k^{(1)} + L'(\lambda_k) y_k = 0$$

and if $\operatorname{Im}\lambda_k = 0$, then

$$\begin{aligned}(L(\lambda_k) y_k^{(1)}, y_k) + (L'(\lambda_k) y_k, y_k) &= (y_k^{(1)}, L(\lambda_k) y_k) + (L'(\lambda_k) y_k, y_k) \\ &= 2\lambda_k (My_k, y_k) + (Ky_k, y_k) = 0\end{aligned}$$

and λ_k satisfies (A.3). ∎

Lemma A.2. *The real eigenvalues of* $L(\lambda)$ *are located on the interval* $\Big[-\zeta_2-\sqrt{\zeta_2^2+|a|},\sqrt{|a|}\Big]$, *where* $a=\inf\limits_{y\neq 0}\frac{(Ay,y)}{(My,y)}$.

Proof. Let λ_k be a real eigenvalue, then

$$\lambda_k^2(My_k,y_k)+\lambda_k(Ky_k,y_k)+(Ay_k,y_k)=0$$

and

$$\lambda_k=-\frac{(Ky_k,y_k)}{2(My_k,y_k)}\pm\left(\frac{(Ky_k,y_k)^2}{4(My_k,y_k)^2}-\frac{(Ay_k,y_k)}{(My_k,y_k)}\right)^{\frac{1}{2}}.$$

Then the assertion of the lemma follows. ∎

Consider the pencil $L(\lambda,\eta)$ defined by (A.1). It is clear that $L(\lambda,0)=\lambda^2I+A$ and $L(\lambda,1)=L(\lambda)$.

Lemma A.3. *Let* $\lambda=0$ *be an eigenvalue of* A. *Then* $\lambda=0$ *is an eigenvalue of* $L(\lambda,\eta)$ *for any* $\eta\in[0,1]$ *and*

(i) *if* $\operatorname{Ker}A\cap\operatorname{Ker}K=\{0\}$, *then the algebraic multiplicity of* $\lambda=0$ *is equal to* 2 *for* $\eta=0$ *and is equal to* 1 *for* $\eta\in(0,1]$;

(ii) *if* $\dim(\operatorname{Ker}A\cap\operatorname{Ker}K)=1$, *then the algebraic multiplicity of* $\lambda=0$ *is equal to* 2 *for all* $\eta\in[0,1]$.

Proof. Let $y\neq 0$ and $Ay=0$. Then $L(0,0)y=0$ and

$$\left.\frac{\partial L(\lambda,0)}{\partial\lambda}\right|_{\lambda=0}y+L(0,0)y=0.$$

That means that the first associated vector of the chain may be chosen equal to y. Due to the identity $\frac{\partial^2L(\lambda,0)}{\partial\lambda^2}=2M$, we obtain for the second vector y_2 of the chain

$$\frac{1}{2}\left.\frac{\partial^2L(\lambda,0)}{\partial\lambda^2}\right|_{\lambda=0}y+\left.\frac{\partial L(\lambda,0)}{\partial\lambda}\right|_{\lambda=0}y+L(0,0)y_2=0,$$

i.e.

$$My+Ay_2=0$$

and consequently

$$(My,y)+(Ay_2,y)=(My,y)+(y_2,Ay)=0.$$

The last equality is false because $Ay=0$ and $y\neq 0$. It means that if $\eta=0$ and $\lambda=0$ is an eigenvalue, then its algebraic multiplicity is equal to 2. Now let

$\eta > 0$, then $\lambda = 0$ remains an eigenvalue of $L(\lambda, \eta)$. Let y be a corresponding eigenvector and y_1 the first associated vector of the chain. Then

$$L(0,\eta)y_1 + \left.\frac{\partial L(\lambda,\eta)}{\partial \lambda}\right|_{\lambda=0} y = 0,$$

i.e.

$$Ay_1 + \eta K y = 0.$$

If $Ky = 0$, then $y_1 = y$. If $Ky \neq 0$, then $(Ay_1, y) + \eta(Ky, y) = \eta(Ky, y) = 0$ which is impossible for $\eta > 0$. The absence of the second associated vector of the chain may be proved analogously. ∎

It was proved in [22] that the spectrum of $L(\lambda)$ located in the right half-plane is real and semisimple (i.e. associated vectors are absent).

LEMMA A.4. *Let $K > 0$ and let $\theta_0 \in \mathbb{R}\backslash\{0\}$ be an eigenvalue of the operator-function $Q(\lambda,\theta) = I + \theta(\lambda^{-1}K^{-\frac{1}{2}}AK^{-\frac{1}{2}} + \lambda K^{-\frac{1}{2}}MK^{-\frac{1}{2}})$ at $\lambda = \lambda_0 \in \mathbb{R}\backslash\{0\}$. Then this eigenvalue is holomorphic in some neighbourhood $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ $(\varepsilon > 0)$:*

$$\theta(\lambda) = \theta_0 + \sum_{k=p}^{\infty} b_k(\lambda - \lambda_0)^k, \tag{A.4}$$

where $p \in \mathbb{N}$, $b_p \in \mathbb{R}\backslash\{0\}$.

Proof. The spectrum of $Q(\lambda_0, \theta)$ in the domain $\mathbb{C}\backslash\{0\}$ consists only of normal eigenvalues since $Q(\lambda_0,\theta)$ is a relatively compact perturbation of $Q_0(\lambda_0,\theta) = I + \theta\lambda_0^{-1}K^{-\frac{1}{2}}AK^{-\frac{1}{2}}$. The geometric multiplicity of any eigenvalue of $Q(\lambda_0,\theta)$ is equal to 1 because it coincides with that of the corresponding eigenvalue of $L(\lambda_0,\theta) = \lambda_0 K^{\frac{1}{2}}Q(\lambda_0,\theta)K^{\frac{1}{2}}$. Now it is possible to apply the Rellich-Nagy theorem ([26]). ∎

In what follows we use the method of [12].

LEMMA A.5. *Let λ_0 be a negative eigenvalue of $L(\lambda,\eta_0)$, $(\eta_0 > 0)$, then in some neighbourhood of (λ_0,η_0) $\{(\lambda,\eta) : |\lambda - \lambda_0| < \varepsilon,\ |\eta - \eta_0| < \delta,\ \varepsilon > 0,\ \delta > 0\}$ all the eigenvalues are given by the formula*

$$\lambda_j(\eta) = \lambda_0 + \sum_{k=1}^{\infty} \beta_k\big((\eta - \eta_0)_j^{\frac{1}{r}}\big)^k, \quad j = \overline{1,r}, \tag{A.5}$$

where $\beta_1 \neq 0$ is real or purely imaginary, $(\eta - \eta_0)_j^{\frac{1}{r}}$, $(j = \overline{1,r})$ means all the branches of the root.

Proof. After the inversion of (A.4) with $\eta = \theta^{-1}$ we obtain (A.5). ∎

Theorem A.6. *Let $K > 0$, then for every nonnegative eigenvalue λ_k of $L(\lambda)$ there exists an eigenvalue λ_{-k} such that*

$$\lambda_k + \lambda_{-k} < 0.$$

Proof. The eigenvalues of $L(\lambda, \eta)$ are piecewise holomorphic functions of η. They may lose analyticity only when they collide. This follows from [24] (the generalization of the results of [4] for unbounded operators). Positive eigenvalues being holomorphic functions of $\eta > 0$ [24], they move left along the real axis when η grows. We identify λ_{-k} as the one satisfying the condition $\lambda_{-k}(0) = -\lambda_k(0)$. For small enough $\eta > 0$ and $\lambda_{-k}(\eta)$ such that $\lambda_{-k}(0) < 0$ the formula

$$\lambda'_{-k}(\eta) = \frac{-\lambda_{-k}(\eta)\,(Ky_{-k}(\eta), y_{-k}(\eta))}{2\lambda_{-k}(\eta)\big\|M^{\frac{1}{2}}y_{-k}(\eta)\big\|^2 + \eta\,(Ky_{-k}(\eta), y_{-k}(\eta))} \tag{A.6}$$

implies $\lambda'_{-k}(\eta) \leqslant 0$. Let $\lambda_k(0) = 0$, then Lemma A.3 implies $\lambda_{-k}(0) = 0$ and $\lambda_{-k}(\eta) \neq 0$ for any $\eta \in (0, \varepsilon)$ where ε is some positive number. The equality $\operatorname{Im}\lambda_{-k}(\eta) = 0$ for small enough $\eta > 0$ follows from the independence of the total algebraic multiplicity of the positive spectrum on $\eta > 0$ ([22], [24]). Hence, $\lambda_{-k}(\eta) < 0$ for $\eta \in (0, \varepsilon)$, $(\varepsilon > 0)$. The assertion of Theorem A.6 is proved for sufficiently small $\eta > 0$. While $\eta > 0$ grows, $\lambda'_{-k}(\eta)$ may change its sign only when the denominator of the right hand side of (A.6) vanishes, i.e. under collision. If a collision takes place on the negative halfaxis, then the eigenvalues behave according to the formula (A.5). Collisions occur to be of three types. The first one is with r odd (see (A.5)). In this case we identify the eigenvalue negative after the collision as the one which was negative before the collision. A collision is said to be of the second type if r is even and $\beta_1 \neq 0$ is purely imaginary. In that case two new negative eigenvalues move in opposite directions. The third type collisions have even r and real $\beta_1 \neq 0$. Let $\lambda_{-k}(\eta)$ take part in a collision of the third type at $\eta = \eta_0 \in (0, 1]$. Then a collision of the second type indeed happened at some $\eta \in (0, \eta_0)$ in some $\lambda_1 \in (-\infty, \lambda_{-k}(\eta_0))$. The eigenvalue appeared after this collision and moves to the left, we identify as $\lambda_{-k}(\eta)$. ∎

Remark A.7. Theorem A.6 remains true in the case when the geometric multiplicity of the eigenvalues is not equal to 1. In this case we have to consider an eigenvalue of geometric multiplicity m as m coinciding simple eigenvalues.

Lemma A.8. *The total algebraic multiplicity of the real spectrum of $L(\lambda)$ is even.*

Proof. Consider the auxiliary operator pencil $L_0(\lambda) = \lambda^2 M + A$. As $L_0(-\lambda) = L_0(\lambda)$ the spectrum of $L_0(\lambda)$ is symmetric with respect to the origin and, consequently, symmetric with respect to the real and to the imaginary axes. If $\lambda_k = 0$ then $\lambda_{-k} = 0$ (see Lemma A.3). Hence the total algebraic multiplicity of the real spectrum of $L_0(\lambda)$ is even. When $\eta \in (0, 1]$ grows, the real spectrum of $L(\lambda, \eta)$ remains in a finite domain (Lemma A.2), so the total algebraic multiplicity of the real spectrum of $L(\lambda, \eta)$ may be changed if eigenvalues come onto the real axis from the complex plane. But collisions of all three types change (if they do) the total algebraic multiplicity of the real spectrum by an even number because of the symmetry of the problem. ∎

Consider the operator pencils

$$\widetilde{L}(\lambda) = \lambda^2 \widetilde{M} + \lambda \widetilde{K} + \widetilde{A}, \quad (\mathrm{D}(\widetilde{L}(\lambda)) = \mathrm{D}(\widetilde{A})) \tag{A.7}$$

acting in $L_2(0, l) \oplus \mathbb{C}$ where

$$\mathrm{D}(\widetilde{A}) = \left\{ \begin{pmatrix} v(s) \\ v(l) \end{pmatrix} : v(s) \in W_2^2(0, l), v(0) = 0 \right\}$$

$$\widetilde{A}Y = \widetilde{A} \begin{pmatrix} v(s) \\ v(l) \end{pmatrix} = \begin{pmatrix} -(A(s)v'(s))' \\ A(l)v'(l) \end{pmatrix},$$

$$\widetilde{K} = \begin{pmatrix} pI & 0 \\ 0 & A(l)\nu I \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} I & 0 \\ 0 & A(l)\mu I \end{pmatrix},$$

and

$$\mathcal{L}(\lambda) = \lambda^2 M_1 + \lambda K_1 + A_1 \tag{$A.8$}$$

acting in $L_2(0, a) \oplus \mathbb{C}$, where

$$\mathrm{D}(\mathcal{L}) = \mathrm{D}(A_1) = \left\{ \begin{pmatrix} y(x) \\ y(a) \end{pmatrix} : y(x) \in W_2^2(0, a),\ y(0) = 0 \right\}$$

$$A_1 \begin{pmatrix} y(x) \\ y(a) \end{pmatrix} = \begin{pmatrix} -y'' + q(x)y \\ y'(a) + \beta y(a) \end{pmatrix} \tag{A.9}$$

and

$$K_1 = \begin{pmatrix} pI & 0 \\ 0 & \alpha I \end{pmatrix}, \quad M_1 = \begin{pmatrix} I & 0 \\ 0 & mI \end{pmatrix}.$$

The spectrum of $\mathcal{L}(\lambda)$ coincides with the spectrum of $\widetilde{L}(\lambda)$. It is easy to check that $\widetilde{A}^* = \widetilde{A} \gg 0$, $\widetilde{K} \gg 0$, $\widetilde{M} \gg 0$. The spectrum of $\widetilde{L}(\lambda)$ is located in the open left half-plane ([13]). Consider the pencil

$$\widetilde{L}_1(\lambda) = \widetilde{L}\left(\lambda - \frac{p}{2}\right) = \lambda^2 \widetilde{M} + \lambda(\widetilde{K} - p\widetilde{M}) + \frac{p^2}{4}\widetilde{M} - \frac{p}{2}\widetilde{K} + \widetilde{A}.$$

Let $\frac{\nu}{\mu} > p$, i.e. $\widetilde{K} > p\widetilde{M}$, then applying Lemma A.1 to the pencil $\widetilde{L}_1(\lambda)$ we obtain that all nonreal and all multiple eigenvalues are located in the strip $-\frac{1}{2}\left(\frac{\nu}{\mu} - p\right) \leqslant \operatorname{Re}\lambda \leqslant 0$ and all multiple and nonreal eigenvalues of $\widetilde{L}(\lambda)$ lie in the strip $-\frac{\nu}{2\mu} \leqslant \operatorname{Re}\lambda \leqslant -\frac{p}{2}$.

Lemma A.9. *Let $\frac{\nu}{\mu} > p$, then all nonreal and all multiple eigenvalues of $\widetilde{L}(\lambda)$ are located in the strip $-\frac{\nu}{2\mu} < \operatorname{Re}\lambda_k < -\frac{p}{2}$.*

Proof. Let $\operatorname{Im}\lambda_k \neq 0$, $\operatorname{Re}\lambda_k = -\frac{p}{2}$, then (A.2) yields

$$\frac{p\int_0^l |v_k(s)|^2\,\mathrm{d}s + A(l)\nu|v_k(l)|^2}{2\left(\int_0^l |v_k(s)|^2\,\mathrm{d}s + A(l)\mu|v_k(l)|^2\right)} = \frac{p}{2}$$

and, consequently, $v_k(l) = 0$. In this case

$$\lambda_k^2 \widetilde{M} Y_k + \lambda_k \widetilde{K} Y_k + \widetilde{A} Y_k = 0,$$

i.e.

$$\lambda_k^2 \begin{pmatrix} I & 0 \\ 0 & A(l)\mu I \end{pmatrix}\begin{pmatrix} v_k(s) \\ v_k(l) \end{pmatrix} + \lambda_k \begin{pmatrix} pI & 0 \\ 0 & A(l)\mu I \end{pmatrix}\begin{pmatrix} v_k(s) \\ v_k(l) \end{pmatrix} + \begin{pmatrix} -(A(s)v_k'(s))' \\ v_k'(l) \end{pmatrix} = 0$$

and, consequently, $v_k(l) = v_k'(l) = 0$ which is impossible. Now let $\operatorname{Im}\lambda_k \neq 0$, $\operatorname{Re}\lambda_k = -\frac{\nu}{2\mu}$, then

$$\frac{p\int_0^l |v_k(s)|^2\,\mathrm{d}s + A(l)\nu|v_k(l)|^2}{2\left(\int_0^l |v_k(s)|^2\,\mathrm{d}s + A(l)\mu|v_k(l)|^2\right)} = \frac{\nu}{2\mu}$$

and, consequently, $\int_0^l |v_k(s)|^2\,\mathrm{d}s = 0$. Hence $|v_k(l)|^2 = \int_0^l |v_k(s)|^2\,\mathrm{d}s = 0$ and $Y_k = 0$ what is impossible. It is easy to prove using Lemma A.1 that all multiple eigenvalues are located in the strip $-\frac{\nu}{2\mu} < \operatorname{Re}\lambda_k < -\frac{p}{2}$. ∎

LEMMA A.10. *If $p = \frac{\nu}{\mu}$, then all nonreal eigenvalues of $\widetilde{L}(\lambda)$ are located on the axis* $\operatorname{Re}\lambda = -\frac{p}{2}$. *The only possible multiple eigenvalue (of multiplicity 2) is* $\lambda = -\frac{p}{2}$.

To prove Lemma A.10 it is sufficient to apply Lemma A.3 to the pencil $L_0(\tau)$, where $\tau = \sqrt{\lambda^2 + \lambda p}$.

It should be mentioned that another approach to inverse problems of such type was developed in [28].

Acknowledgements. It is a pleasure to thank Professor V. Adamyan for his interest and perceptive comments. The author is grateful to the unknown referee for helpful criticism which improved the paper. The research described in this publication was made possible in part by Award No. UM1-298 of the Government of Ukraine and U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union.

REFERENCES

1. D.Z. AROV, Realization of a canonical system with a dissipative boundary condition at one end of the segment in terms of the coefficient of dynamical compliance, [Russian], *Sibirsk. Mat. Zh.* **16**(1975), 440–463.
2. R. COURANT, D. HILBERT, *Methods of Mathematical Physics*, vol. 1, Interscience, New York 1953.
3. S. COX, E. ZUAZUA, The rate at which energy decays in a damped string, *Comm. Partial Differential Equations* **19**(1994), 213–243.
4. V.M. ENI, On stability of analytic operator-function's root number and on perturbation of its characteristic values and eigenvectors, [Russian], *Dokl. Akad. Nauk SSSR* **173**(1967), 1251–1254.
5. M.G. GASYMOV, G.S. GUSEYNOV, Determination of the diffusion operator, by spectral data, [Russian], *Dokl. Akad. Nauk Azerbaĭdzhana* **37**(1981), 19–23.
6. I.C. GOHBERG, M.G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Amer. Math. Soc., Providence 1969.
7. S.V. HRUSCEV, The Regge problem for strings, unconditionally convergent eigenfunction expansions and unconditional bases of exponentials in $L_2(-T, T)$, *J. Operator Theory* **14**(1985), 67–85.
8. M. JAULENT, Inverse scattering problems in absorbing media, *J. Math. Phys.* **17** (1976), 1351–1360.
9. M. JAULENT, C. JEAN, Inverse problem for the one-dimensional Schrödinger equation with energy-dependent potential, I, II, *Ann. Inst. H. Poincaré Phys. Théor.* **25**(1976), 105–137.

10. E. Kamke, Differentialgleichungen Lösungsmethoden and Lösungen. I, Leipzig, 1950.

11. M.V. Keldysh, On completeness of eigenfunctions of some classes of nonselfadjoint operators, [Russian], *Uspekhi Mat. Nauk* **26**(1971), 15–41.

12. A.G. Kostyuchenko, M.B. Orazov, The problem of oscillations of an elastic half-cylinder and related self-adjoint quadratic pencil, [Russian], *Trudy Sem. Petrovsk.* **6**(1981), 97–147.

13. M.G. Krein, H. Langer, On some mathematical principles in the linear theory of damped oscillations of continua. II, *Integral Equations Operator Theory*, **1**(1978), 539–566.

14. M.G. Krein, A.A. Nudelman, On direct and inverse problems for frequencies of boundary dissipation of inhomogeneous string, [Russian], *Dokl. Akad. Nauk SSSR*, **247**(1979), 1046–1049.

15. M.G. Krein, A.A. Nudelman, On some spectral properties of an inhomogeneous string with dissipative boundary condition, [Russian], *J. Operator Theory* **22**(1989), 369–395.

16. B.Ja. Levin, On bases of exponential functions in L_2, [Russian], *Zapiski Fiz.-Mat. Facult. Kharkov Univ. and Kharkov Mat. Obsch.* **27**(1961), 39–48.

17. B.Ja. Levin, I.V. Ostrovskii, On small perturbations of the set of the roots of sinus-type function, [Russian], *Izv. Akad. Nauk SSSR Ser. Mat.* **43**(1979), 87–110.

18. B.M. Levitan, Inverse Sturm–Liouville problems, [Russian], Moscow, Nauka 1984, p. 240.

19. B.M. Levitan, M.G. Gasymov, Determination of differential equation by two spectra, [Russian], *Uspekhi Mat. Nauk* **19**(1964), 3–63.

20. V.A. Marchenko, Sturm–Liouville Operators and Applications, [Russian], Kiev, Naukova Dumka 1977, 331.

21. N.K. Nikolskii, B.S. Pavlov, S.V. Hruscev, Unconditional bases of exponentials and generating kernels, Complex Analysis and Spectral Theory, Lecture Notes Math., vol. 864, Springer, New York, 1981, 214–335.

22. V.N. Pivovarchik, On eigenvalues of a quadratic operator pencil, [Russian], *Funktsional. Anal. i Prilozhen.* **25**(1989), 80–81.

23. V.N. Pivovarchik, On discrete spectrum of one boundary problem, [Russian], *Sibirsk Mat. Zh.* **31**(1990), 182–186.

24. V.N. Pivovarchik, On positive spectra of one class of polynomial operator pencils, *Integral Equations Operator Theory*, **19**(1994), 314–326.

25. V.N. Pivovarchik, Inverse problem for a smooth string with damping at one end, *J. Operator Theory* **38**(1997), 243–263.

26. F. Riesz, B. Sz.-Nagy, *Lecons d'analyse fonctionelle*, Academiai Kiadv, Budapest 1952.

27. M.A. Shubov, Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equation of nonhomogeneous damped string, *Integral Equations Operator Theory* **25**(1996), 289–328.

28. R.M. WOHLERS, Lumped and distributed passive networks, Academic Press, New York 1969.

VYACHESLAV N. PIVOVARCHIK
Department of Higher Mathematics
Odessa State Academy of Civil
Engineering and Architecture
Didrihson str., 4
2700029, Odessa
UKRAINE

E-mail: v.pivovarchik@paco.net

Received September 9, 1997; revised October 2, 1998.