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THE METHOD OF LIMIT OPERATORS FOR ONE-DIMENSIONAL SINGULAR INTEGRALS WITH SLOWLY OSCILLATING DATA

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Abstract. One of the great challenges of the spectral theory of singular integral operators is a theory unifying the three "forces" which determine the local spectra: the oscillation of the Carleson curve, the oscillation of the Muckenhoupt weight, and the oscillation of the coefficients. In this paper we demonstrate how by employing the method of limit operators one can describe the spectra in case all data of the operator (the curve, the weight, and the coefficients) are slowly oscillating.

Keywords: *Singular integral, Toeplitz operator, pseudodifferential operator, slow oscillation.*

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1. INTRODUCTION

Given an oriented rectifiable simple arc Γ in the plane and a function f in $L^1(\Gamma)$, the Cauchy singular integral $S_\Gamma f$,

$$(S_\Gamma f)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma\setminus\Gamma(t,\varepsilon)} \frac{f(\tau)}{\tau - t}\,\mathrm{d}\tau, \quad \Gamma(t,\varepsilon) := \big\{\tau\in\Gamma : |\tau - t| < \varepsilon\big\},$$

exists for almost all $t \in \Gamma$. We study S_Γ as an operator on the weighted L^p space $L^p(\Gamma, w)$ with the norm

$$\|f\| := \bigg(\int\limits_\Gamma |f(\tau)|^p w(\tau)^p |\,\mathrm{d}\tau|\bigg)^{\frac{1}{p}}$$

where $1 < p < \infty$ and $w : \Gamma \to [0,\infty]$ is a measurable function such that $w^{-1}(\{0,\infty\})$ has measure zero. After a long development, which culminated with the work by Hunt, Muckenhoupt, Wheeden ([11]) and David ([8]), it became clear that S_Γ is a well-defined and bounded operator on $L^p(\Gamma, w)$ if and only if

$$\sup_{\varepsilon>0}\sup_{t\in\Gamma}\frac{1}{\varepsilon}\Bigg(\int_{\Gamma(t,\varepsilon)} w(\tau)^p|\mathrm{d}\tau|\Bigg)^{\frac{1}{p}}\Bigg(\int_{\Gamma(t,\varepsilon)} w(\tau)^{-q}|\mathrm{d}\tau|\Bigg)^{\frac{1}{q}} < \infty, \tag{1.1}$$

where $1/p + 1/q = 1$ (see also [9] and [3]). We write A_p for the set of all pairs (Γ, w) satisfying (1.1). Using Hölder's inequality, it is easily seen that (1.1) implies that

$$\sup_{\varepsilon>0}\ \sup_{t\in\Gamma}|\Gamma(t,\varepsilon)|/\varepsilon < \infty, \tag{1.2}$$

where $|\Gamma(t,\varepsilon)|$ stands for the Lebesgue (length) measure of $\Gamma(t,\varepsilon)$.

Condition (1.1) is called the Muckenhoupt condition. Condition (1.2) says that the measure μ defined on $\mathbb{C}$ by $\mu(D) :=$ length of $D\cap\Gamma$ is a Carleson measure, which means that $\sup\limits_{D}\mu(D)/\mathrm{diam}(D) < \infty$. In [9] and [3] the curves Γ satisfying (1.2) are therefore named Carleson curves. We follow this practice here. However, it should be noted that condition (1.2) already appeared in a 1935 paper by L.V. Ahlfors and that it has gained permanent attention since David's paper ([8]). Hence, the curves for which (1.2) holds are frequently also referred to as Ahlfors or Ahlfors-David curves.

The spectrum of the operator S_Γ on $L^p(\Gamma, w)$ has been known since the sixties from the work of Widom ([22]) and Gohberg, Krupnik (see [10]) in the case of nice curves Γ and nice weights w, it was found only in the nineties by Spitkovsky ([21]) for nice curves Γ and arbitrary Muckenhoupt weights w, and only very recently the spectrum was completely identified in [4] and [1] for general (composed) Carleson curves Γ and general Muckenhoupt weights w. The book [3] is an account of this development.

The spectrum and the essential spectrum of the operator S_Γ on $L^p(\Gamma, w)$ are defined by

$$\mathrm{sp}\, S_\Gamma = \{z\in\mathbb{C} : S_\Gamma - zI \text{ is not invertible on } L^p(\Gamma,w)\},$$
$$\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma = \{z\in\mathbb{C} : S_\Gamma - zI \text{ is not Fredholm on } L^p(\Gamma,w)\}.$$

One can show that $\mathrm{sp}\, S_\Gamma$ is the union of $\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$ and the set

$$\{z\in\mathbb{C} : S_\Gamma - zI \text{ is Fredholm of nonzero index on } L^p(\Gamma,w)\}.$$

Since index formulas for $S_\Gamma - zI$ are available (see, e.g., [3], Sections 10.2 to 10.4, for the details), in order to determine $\mathrm{sp}\, S_\Gamma$ it therefore suffices to identify $\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma$. Further, one can associate with each point $t \in \Gamma$ a so-called local spectrum $\mathrm{sp}_t\, S_\Gamma$, and standard localization theory implies that

$$\mathrm{sp}_{\mathrm{ess}}\, S_\Gamma = \bigcup_{t\in\Gamma} \mathrm{sp}_t\, S_\Gamma. \tag{1.3}$$

Thus, the local spectra gives us the essential spectrum and eventually also the spectrum. But the local spectra even do more: they tell us where the pieces of the essential spectrum come from.

It should be emphasized that $\mathrm{sp}_t\, S_\Gamma$ depends not only on the point t and the behavior of the curve Γ in a neighborhood of t but on all of the space $L^p(\Gamma, w)$ near t. In particular, $\mathrm{sp}_t\, S_\Gamma$ depends also on the value of p and on the behavior of the weight w in a neighborhood of t. If instead of S_Γ we consider operators of the form $aI + bS_\Gamma$, then $\mathrm{sp}(aI + bS_\Gamma)$ can again be found from $\mathrm{sp}_{\mathrm{ess}}(aI + bS_\Gamma)$ by means of an index formula, while the essential spectrum can be given in terms of local spectra as above:

$$\mathrm{sp}_{\mathrm{ess}}(aI + bS_\Gamma) = \bigcup_{t\in\Gamma} \mathrm{sp}_t(aI + bS_\Gamma). \tag{1.4}$$

In this more general case, the local spectrum $\mathrm{sp}_t(aI + bS_\Gamma)$ depends not only on p, on the behavior of the curve Γ and the weight w, but also on the coefficients a and b in a neighborhood of t.

We remark that representations like (1.3) and (1.4) are now standard tools in the theory of singular integral operators. The crucial point and the subject of this paper is the identification of the local spectra $\mathrm{sp}_t\, S_\Gamma$ and $\mathrm{sp}_t(aI + bS_\Gamma)$. If t is not an endpoint of the curve Γ, then $\mathrm{sp}_t\, S_\Gamma$ is the doubleton $\{-1, 1\}$. In [4] and [1], it was shown that if t is an endpoint of Γ, then $\mathrm{sp}_t\, S_\Gamma$ is always a so-called logarithmic leaf with a halo, which may degenerate to a pure logarithmic leaf, a spiralic horn, a horn, a circular arc, or a line segment between -1 and 1 in case Γ and/or w are sufficiently nice at t; also see [3].

The approach of [4] and [1] is based on Wiener-Hopf factorization and the use of submultiplicative functions. An entirely different approach makes use of Mellin convolutions and Mellin pseudodifferential operators and was elaborated in [19], [20] and [12], [15], [17]. In [5] and [6] we showed how Mellin techniques can be applied to large classes of Carleson curves and Muckenhoupt weights. In particular, in [6] we gave a new proof to part of the results of [4] and [1] and were able to interpret the main result of [4], namely, the emergence of leaves with a halo, in an alternative manner.

In this paper we show a third way to the understanding of the nature of the local spectra of S_Γ on $L^2(\Gamma, w)$: by employing the method of limit operators, we will *completely* localize the problem. Roughly speaking, we associate with S_Γ on $L^2(\Gamma, w)$ a family of limit operators $\{S_{\Gamma_\xi}\}_{\xi\in M_0(\mathrm{SO})}$ where S_{Γ_ξ} acts on $L^2(\Gamma_\xi, w_\xi)$, Γ_ξ is a pure logarithmic spiral and w_ξ is a pure power weight; by $M_0(\mathrm{SO})$ we denote the "interesting" part of the maximal ideal space of the C^*-algebra of slowly oscillating functions. The local spectra of S_{Γ_ξ} are logarithmic double spirals. Thus, we arrive at the conclusion that the interpretation of a leaf with a halo as a union of logarithmic double spirals is equivalent to the fact that the local spectrum of S_Γ is the union of the spectra of its limit operators.

As will be shown, the method of limit operators actually yields more. Namely, it also allows us to "freeze" slowly oscillating functions to constants and therefore implies Fredholm criteria for singular integral operators with slowly oscillating coefficients on slowly oscillating Carleson curves with slowly oscillating Muckenhoupt weights.

For the sake of simplicity, we confine ourselves to the Hilbert space case, i.e., in what follows we will assume that $p = 2$. Furthermore, we will also assume that the curves and weights are C^∞ everywhere except at the endpoints, although refined techniques would give the results under much less smoothness.

2. Mellin convolutions

Put $\mathbb{R}_+ := (0, \infty)$, $\mathrm{d}\mu(\rho) := \mathrm{d}\rho/\rho$, $X := L^2(\mathbb{R}_+, \mathrm{d}\mu)$. Note that $\mathbb{R}_+$ is a multiplicative group and that $\mathrm{d}\mu$ is the (normalized) invariant measure on $\mathbb{R}_+$. The Fourier transform associated with $(\mathbb{R}_+, \mathrm{d}\mu)$ is called the Mellin transform and is given by

$$M : L^2(\mathbb{R}_+, \mathrm{d}\mu) \to L^2(\mathbb{R}), \quad (Mf)(\lambda) = \int_{\mathbb{R}_+} f(\rho)\rho^{-\mathrm{i}\lambda} \frac{\mathrm{d}\rho}{\rho}.$$

The operator M is invertible,

$$M^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, \mathrm{d}\mu), \quad (M^{-1}g)(r) = \frac{1}{2\pi} \int_{\mathbb{R}} g(\lambda) r^{\mathrm{i}\lambda}\, \mathrm{d}\lambda.$$

For a function a, we denote by aI the operator of multiplication by a, and in case aI follows another operator, M say, we abbreviate aIM to aM. If $a \in L^\infty(\mathbb{R})$, then the operator $\mathrm{CO}(a) := M^{-1}aM$ is bounded on X. It is referred to as the Mellin convolution operator with the symbol a.

Let $\mathcal{B}(X)$ stand for the C^*-algebra of all bounded linear operators on X. An operator $B \in \mathcal{B}(X)$ is said to be locally invertible at the origin if there exist a function $\varphi \in C(\mathbb{R}_+)$ which is identically 1 on $(0,\varepsilon)$ for some $\varepsilon > 0$ and operators $D', D'' \in \mathcal{B}(X)$ such that $\varphi B D' = \varphi I$ and $D'' B \varphi I = \varphi I$. It is easily seen that the local invertibility of B at the origin is equivalent to the existence of a function $\psi \in C(\mathbb{R}_+)$ which is 1 on $(0,\varepsilon)$ for some $\varepsilon > 0$, of two operators $E', E'' \in \mathcal{B}(X)$, and of two compact operators K', K'' on X such that $\psi B E' = \psi I + K'$ and $E'' B \psi I = \psi I + K''$. The local spectrum $\operatorname{sp}_0 B$ is defined as the set of all $z \in \mathbb{C}$ for which $B - zI$ is not locally invertible at the origin.

The following result is well known.

Theorem 2.1. *If* $a \in L^\infty(\mathbb{R})$ *then* $\operatorname{sp} \mathrm{CO}(a) = \operatorname{sp}_0 \mathrm{CO}(a) = \mathcal{R}(a)$, *where* $\mathcal{R}(a)$ *is the essential range of* a, *i.e., the spectrum of* a *as an element of the* C^**-algebra* $L^\infty(\mathbb{R})$.

3. LOGARITHMIC SPIRALS AND POWER WEIGHTS

For $\delta \in \mathbb{R}$, the curve

$$G_\delta := \{\tau = r\mathrm{e}^{\mathrm{i}\delta \log r} : r > 0\} \tag{3.1}$$

is a logarithmic spiral. It is easily seen that G_δ is a Carleson curve. Given $\gamma \in \mathbb{R}$, we define the power weight u_γ by $u_\gamma(\tau) = |\tau|^\gamma$. Thus, if $\tau = r\mathrm{e}^{\mathrm{i}\delta \log r}$, then

$$u_\gamma(\tau) = r^\gamma = \mathrm{e}^{\gamma \log r}. \tag{3.2}$$

One can show that u_γ is a Muckenhoupt weight on G_δ, i.e., $(G_\delta, u_\gamma) \in A_2$, if and only if $-1/2 < \gamma < 1/2$ (see, e.g., [3], Theorem 2.2).

The map $\Phi_{\delta,\gamma}$ defined by

$$(\Phi_{\delta,\gamma} f)(r) = |1 + \mathrm{i}\delta|^{1/2}\, r^{\gamma+1/2} f(r\mathrm{e}^{\mathrm{i}\delta \log r}), \quad r \in \mathbb{R}_+, \tag{3.3}$$

is an isometric isomorphism of $L^2(G_\delta, u_\gamma)$ onto $X = L^2(\mathbb{R}_+, \mathrm{d}\mu)$. Let $-1/2 < \gamma < 1/2$ and let S_{G_δ, u_γ} denote the Cauchy singular integral operator S_{G_δ} as an element of $\mathcal{B}(L^2(G_\delta, u_\gamma))$. A straightforward computation yields

$$(\Psi_{\delta,\gamma}(S_{G_\delta,u_\gamma})g)(r) := (\Phi_{\delta,\gamma} S_{G_\delta,u_\gamma} \Phi_{\delta,\gamma}^{-1} g)(r) = \int_{\mathbb{R}_+} k\left(\frac{r}{\rho}\right) g(\rho) \frac{\mathrm{d}\rho}{\rho}, \quad g \in X,$$

where

$$k(\rho) = \frac{1 + \mathrm{i}\delta}{\pi \mathrm{i}} \frac{\rho^{\gamma+1/2}}{1 - \rho^{1+\mathrm{i}\delta}}.$$

The Mellin transform of k is given by

$$(3.4)\qquad (Mk)(\lambda) = \sigma(\delta,\gamma,\lambda) := \coth\left(\pi\frac{\lambda + \mathrm{i}(\gamma+1/2)}{1+\mathrm{i}\delta}\right), \quad \lambda \in \mathbb{R}$$

(see [5]), and hence we can write

$$(3.5)\qquad \Psi_{\delta,\gamma}(S_{G_\delta,u_\gamma}) = \mathrm{CO}(\sigma_{\delta,\gamma}), \quad \sigma_{\delta,\gamma}(\lambda) := \sigma(\delta,\gamma,\lambda).$$

Thus, $\Psi_{\delta,\gamma}(S_{G_\delta,u_\gamma})$ is the Mellin convolution operator with the symbol $\sigma_{\delta,\gamma}$.

To determine the essential range of $\sigma_{\delta,\gamma}$, notice first that

$$\coth \pi z = M_{-1,1}(\mathrm{e}^{2\pi z}), \qquad \text{where} \quad M_{-1,1}(\zeta) := \frac{\zeta+1}{\zeta-1}.$$

As λ traces out $\mathbb{R}$, the point $(\lambda+\mathrm{i}(\gamma+1/2))/(1+\mathrm{i}\delta)$ traverses the straight line

$$(3.6)\qquad \{z = x + \mathrm{i}y :\ y = -\delta x + (\gamma+1/2)\},$$

and the map $z \mapsto \mathrm{e}^{2\pi z}$ maps this line to the logarithmic spiral $\mathrm{e}^{2\pi\mathrm{i}(\gamma+1/2)}G_{-\delta}$. Finally, the Möbius transform $M_{-1,1}$ maps this logarithmic spiral to what we call a logarithmic double spiral between -1 and 1. The points $\lambda = \pm\infty$ are mapped into ± 1. Thus,

$$(3.7)\qquad \mathcal{R}(\sigma_{\delta,\gamma}) = \mathcal{S}_{\delta,\gamma} := M_{-1,1}(\mathrm{e}^{2\pi\mathrm{i}(\gamma+1/2)}G_{-\delta}) \cup \{-1,1\}.$$

An operator $A \in \mathcal{B}(L^2(G_\delta,u_\gamma))$ is called *locally invertible at the origin* if there are a function $\varphi \in C(G_\delta)$ which is identically 1 on $\{r\mathrm{e}^{\mathrm{i}\delta\log r} : r \in (0,\varepsilon)\}$ and operators $D', D'' \in \mathcal{B}(L^2(G_\delta,u_\gamma))$ such that $\varphi AD' = \varphi I$ and $D''A\varphi = \varphi I$. *The local spectrum* $\mathrm{sp}_0\, A$ is the set of all $z \in \mathbb{C}$ such that $A - zI$ is not locally invertible at the origin. Clearly, $\mathrm{sp}_0\, A = \mathrm{sp}_0\, \Psi_{\delta,\gamma}(A)$. From what was said above and from Theorem 2.1 we get the following result.

Theorem 3.1. *If $\delta \in \mathbb{R}$ and $\gamma \in (-1/2,1/2)$ then*

$$\mathrm{sp}\, S_{G_\delta,u_\gamma} = \mathrm{sp}_0\, S_{G_\delta,u_\gamma} = \mathcal{S}_{\delta,\gamma}.$$

We remark that the condition $\gamma \in (-1/2,1/2)$ guarantees that the straight line (3.6) intersects the imaginary axis strictly between 0 and i. Therefore the singularities $k\mathrm{i}$ $(k \in \mathbb{Z})$ of $\coth \pi z$ are not met.

4. SLOW OSCILLATION

Slowly oscillating functions. Given an open subset I of $\mathbb{R}$, we denote by $C_{\mathrm{b}}^{\infty}(I)$ the set of all infinitely differentiable functions f on I which are bounded on I together with all their derivatives $f^{(n)}$ $(n \geqslant 1)$. A function $F \in C_{\mathrm{b}}^{\infty}(\mathbb{R})$ is called slowly oscillating at $+\infty$ if $F'(x) \to 0$ as $x \to +\infty$. We remark that if $F \in C_{\mathrm{b}}^{\infty}(\mathbb{R})$ and $F'(x) \to 0$ as $x \to +\infty$, then automatically $F^{(n)}(x) \to 0$ as $x \to +\infty$ for all $n \geqslant 1$. A function $a \in C_{\mathrm{b}}^{\infty}(\mathbb{R}_+)$ is said to be slowly oscillating at 0 if the function $F \in C_{\mathrm{b}}^{\infty}(\mathbb{R})$ defined by $F(x) = a(\mathrm{e}^{-x})$ is slowly oscillating at $+\infty$. Substituting $r = \mathrm{e}^{-x}$, it is easily seen that $a \in C_{\mathrm{b}}^{\infty}(\mathbb{R}_+)$ is slowly oscillating at 0 if and only if

$$\sup_{r\in\mathbb{R}_+} |(rD_r)^j a(r)| < \infty \quad \text{for all } j \geqslant 0, \tag{4.1}$$

$$\lim_{r\to 0} |ra'(r)| = 0. \tag{4.2}$$

Again notice that (4.1) and (4.2) imply that actually

$$\lim_{r\to 0} |(rD_r)^j a(r)| = 0 \quad \text{for all } j \geqslant 1.$$

To have a sufficient supply of examples, we mention that if $f \in C_{\mathrm{b}}^{\infty}(\mathbb{R})$ and $a \in C_{\mathrm{b}}^{\infty}(\mathbb{R}_+)$ equals $f(\log(-\log r))$ for all sufficiently small $r > 0$, then a is slowly oscillating at 0.

We let SO^{∞} stand for the set of all functions in $C_{\mathrm{b}}^{\infty}(\mathbb{R}_+)$ which are slowly oscillating at 0 and constant on $[1,\infty)$. Let SO be the closure of SO^{∞} in $L^{\infty}(\mathbb{R}_+)$. Thus, SO is a C^*-subalgebra of $L^{\infty}(\mathbb{R}_+)$. We denote by $M(\mathrm{SO})$ the maximal ideal space of SO and by $M_0(\mathrm{SO})$ the set (fiber) of all $\xi \in M(\mathrm{SO})$ with the property that $\xi(\varphi) = \varphi(0)$ whenever $\varphi \in C_{\mathrm{b}}^{\infty}(\mathbb{R}_+)$ has a finite limit at 0. For $r \in \mathbb{R}_+$, define $\delta_r \in M(\mathrm{SO})$ by $\delta_r(a) = a(r)$. Since functions in SO are constant on $[1,\infty)$, the identification of r and δ_r allows us to regard $(0,1]$ as a subset of $M(\mathrm{SO})$. Let $\mathrm{clos}_{\mathrm{SO}^*}(0,1]$ denote the weak-star closure of $(0,1]$ in SO^*, the dual space of SO. Thus, a functional $\xi \in \mathrm{SO}^*$ belongs to $\mathrm{clos}_{\mathrm{SO}^*}(0,1]$ if and only if for every $\varepsilon > 0$ and every finite collection $a_1, \ldots, a_N$ of functions in SO there exists an $r \in (0,1]$ such that

$$|\xi(a_k) - a_k(r)| < \varepsilon \quad \text{for all } k \in \{1,\ldots,N\}.$$

Proposition 4.1. *We have* $M_0(\mathrm{SO}) = \big(\mathrm{clos}_{\mathrm{SO}^*}(0,1]\big) \setminus (0,1]$.

This result is well known. It can be proved by standard arguments, similar to those in [7], proof of Proposition 3.29. A simple application of the Banach-Alaoglu theorem shows that Proposition 4.1 can be restated as follows.

Proposition 4.2. *Let $a_1, \dots, a_N$ be any finite collection of functions in* SO. *If $\xi \in M_0(\mathrm{SO})$, then there exists a sequence $\{r_n\} \subset \mathbb{R}_+$ such that $r_n \to 0$ and*

$$\xi(a_k) = \lim_{n\to\infty} a_k(r_n) \quad \text{for all } k \in \{1, \dots, N\}. \tag{4.3}$$

Conversely, if $\{r_n\} \subset \mathbb{R}_+$ is a sequence such that $r_n \to 0$ and the limits $a_k(r_n)$ exist for all k, then there is a $\xi \in M_0(\mathrm{SO})$ such that (4.3) *holds.*

Using a diagonal process, one can easily derive the following result from Proposition 4.2.

Corollary 4.3. *Let $\{a_k\}_{k=1}^{\infty}$ be a countable subset of* SO. *If $\xi \in M_0(\mathrm{SO})$, then there exists a sequence $\{r_n\} \subset \mathbb{R}_+$ such that $r_n \to 0$ and*

$$\xi(a_k) = \lim_{n\to\infty} a_k(r_n) \quad \text{for all } k \in \{1, 2, \dots\}.$$

In what follows we write $a(\xi) := \xi(a)$.

Slowly oscillating curves. Let Γ be an unbounded oriented simple arc with the starting point t. We say that Γ is *slowly oscillating* at t if

$$\Gamma = \left\{\tau = t + r\mathrm{e}^{\mathrm{i}\theta(r)} : r \in \mathbb{R}_+\right\} \tag{4.4}$$

where θ is a real-valued function in $C^\infty(\mathbb{R}_+)$ and $r\theta'(r)$ is slowly oscillating at 0. As we are only interested in the behavior of Γ near its endpoint t, we will henceforth without loss of generality assume that θ is constant on $[1, \infty)$. Recall that slow oscillation of $r\theta'(r)$ at 0 means that

$$\sup_{r\in\mathbb{R}_+} |(rD_r)^j \theta(r)| < \infty \quad \text{for all } j \geqslant 1, \tag{4.5}$$

$$\lim_{r\to 0} |(rD_r)^2 \theta(r)| = 0. \tag{4.6}$$

Condition (4.4) says that Γ may be parametrized by the distance to the starting point t. Note that $\theta(r)$ may be unbounded as $r \to 0$. Since

$$|\mathrm{d}\tau| = \sqrt{1 + (r\theta'(r))^2}\, \mathrm{d}r,$$

condition (4.5) for $j = 1$ ensures that Γ is a Carleson curve.

For example, if $g \in C_{\mathrm{b}}^\infty(\mathbb{R})$ and if the function $\theta \in C^\infty(\mathbb{R}_+)$ is equal to $g(\log(-\log r)) \log r$ for all sufficiently small $r > 0$ and constant on $[1, \infty)$, then θ satisfies (4.5) and (4.6). In case $g(x) = \delta$ for all $x \in \mathbb{R}_+$, the beginning piece of the curve (4.4) is the beginning piece of the logarithmic spiral G_δ considered in Section 3.

Slowly oscillating weights. Let Γ be a slowly oscillating curve as above. We call a function $w : \Gamma \to (0, +\infty)$ a *slowly oscillating weight* at the point t if

$$w(t + r\mathrm{e}^{\mathrm{i}\theta(r)}) = \mathrm{e}^{v(r)}, \quad r \in \mathbb{R}_+, \tag{4.7}$$

where $v \in C^\infty(\mathbb{R}_+)$ and $rv'(r)$ is slowly oscillating at the point 0:

$$\sup_{r\in\mathbb{R}_+} |(rD_r)^j v(r)| < \infty \quad \text{for all } j \geqslant 1, \tag{4.8}$$

$$\lim_{r\to 0} |(rD_r)^2 v(r)| = 0. \tag{4.9}$$

We will again without loss of generality assume that v is constant on $[1.1, \infty)$. One can show (see, e.g., [2], Lemma 4.2 and [3], Theorem 2.36) that the Muckenhoupt condition (1.1) (with $p = q = 2$) is satisfied if and only if

$$-\frac{1}{2} < \liminf_{r\to 0} rv'(r) \leqslant \limsup_{r\to 0} rv'(r) < \frac{1}{2}. \tag{4.10}$$

If $v \in C^\infty(\mathbb{R}_+)$ is constant on $[1, \infty)$ and $v(r) = h(\log(-\log r)) \log r$ for all sufficiently small $r > 0$ with some $h \in C_{\mathrm{b}}^\infty(\mathbb{R})$ then (4.8) and (4.9) are obviously in force, while (4.10) is equivalent to the inequalities

$$-\frac{1}{2} < \liminf_{x\to+\infty} \big(h(x) + h'(x)\big) \leqslant \limsup_{x\to+\infty} \big(h(x) + h'(x)\big) < \frac{1}{2}.$$

In the case $h(x) = \gamma$, the weight (4.7) coincides for small $r > 0$ with the power weight r^γ we encountered in Section 3.

We denote by A_2^0 the set of all pairs $(\Gamma, w) \in A_2$ in which Γ is a slowly oscillating curve and w is a slowly oscillating weight. Thus, $(\Gamma, w) \in A_2^0$ means that (4.4) to (4.10) are true with functions θ and v in $C^\infty(\mathbb{R}_+)$ which are constant on $[1, \infty)$.

Slowly oscillating coefficients. Let $(\Gamma, w) \in A_2^0$. We denote by $\mathrm{SO}^\infty(\Gamma)$ and $\mathrm{SO}(\Gamma)$ the set of all functions $c_\Gamma : \Gamma \to \mathbb{C}$ such that

$$c_\Gamma(t + r\mathrm{e}^{\mathrm{i}\theta(r)}) = c(r), \quad r \in \mathbb{R}_+,$$

where $c \in \mathrm{SO}^\infty$ and $c \in \mathrm{SO}$, respectively. Notice that functions in $\mathrm{SO}^\infty(\Gamma)$ and $\mathrm{SO}(\Gamma)$ are constant outside the unit disk centered at the starting point t of Γ.

Finally, let $\mathcal{C}_{\Gamma,w}$ denote the smallest closed subalgebra of $\mathcal{B}(L^2(\Gamma,w))$ containing the set

$$\big\{c_\Gamma I : c_\Gamma \in \mathrm{SO}^\infty(\Gamma)\big\} \cup \{S_\Gamma\} \cup \{S_\Gamma^*\}, \tag{4.11}$$

where S_Γ^* is the adjoint of the Cauchy singular integral operator S_Γ. Obviously, $\mathcal{C}_{\Gamma,w}$ is a C^*-subalgebra of $\mathcal{B}(L^2(\Gamma,w))$, and replacing in (4.11) the set $\mathrm{SO}^\infty(\Gamma)$ by $\mathrm{SO}(\Gamma)$ would not change $\mathcal{C}_{\Gamma,w}$.

The local spectrum $\mathrm{sp}_t\, A$ of an operator $A \in \mathcal{B}(L^2(\Gamma,w))$ at the point t is defined as in Section 3, where we considered the special case $\Gamma = G_\delta$ and $w = u_\gamma$. Our aim is to determine $\mathrm{sp}_t\, A$ for $A \in \mathcal{C}_{\Gamma,w}$. We emphasize once more that it is the search for the *local* spectra $\mathrm{sp}_t\, A$ which allows us to assume without loss of generality that the functions θ and v as well as the coefficients c_Γ are constant outside some disk centered at the point t.

5. Mellin pseudodifferential operators

In Section 3 we transformed the singular integral on a logarithmic spiral with a power weight into a Mellin convolution operator. Singular integrals on general curves with general weights lead to pseudodifferential operators. These are best understood in case the curve and the weight (and the coefficients) are slowly oscillating.

In this section we compile a few well known results on Mellin pseudodifferential operators with slowly oscillating symbols. Proofs can be found in [5], [6], [12]–[16].

We denote by $\mathcal{E}$ the set of all functions $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ for which

$$\sup_{(r,\lambda)\in\mathbb{R}_+\times\mathbb{R}} |(r\partial_r)^j \partial_\lambda^l a(r,\lambda)| < \infty \quad \text{for all } j \geqslant 0,\ l \geqslant 0.$$

Let $C_0^\infty(\mathbb{R}_+)$ be the functions in $C^\infty(\mathbb{R}_+)$ with compact support.

Theorem 5.1. *If a is in $\mathcal{E}$ then the Mellin pseudodifferential operator $\mathrm{OP}(a) = a(r, D_r)$ defined for $f \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral*

$$\big(\mathrm{OP}(a)f\big)(r) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\lambda \int_{\mathbb{R}_+} a(r,\lambda)\left(\frac{r}{\rho}\right)^{\mathrm{i}\lambda} f(\rho)\frac{\mathrm{d}\rho}{\rho}, \quad r \in \mathbb{R}_+,$$

extends to a bounded operator on $X = L^2(\mathbb{R}_+, \mathrm{d}\mu)$.

Thus $\mathrm{OP}(a) \in \mathcal{B}(X)$ for every $a \in \mathcal{E}$.

The set $\widetilde{\mathcal{E}}$ of all slowly oscillating functions on $\mathbb{R}_+ \times \mathbb{R}$ is the subset of $\mathcal{E}$ consisting of the functions a satisfying

$$\lim_{r\to 0} \sup_{\lambda\in\mathbb{R}} |(r\partial_r)^j \partial_\lambda^l a(r,\lambda)| = 0 \quad \text{for all } j \geqslant 1,\, l \geqslant 0.$$

We denote by $\mathcal{E}_0$ the functions $q \in \mathcal{E}$ for which

$$\lim_{r\to 0} \sup_{\lambda\in\mathbb{R}} |(r\partial_r)^j \partial_\lambda^l q(r,\lambda)| = 0 \quad \text{for all } j \geqslant 0,\, l \geqslant 0.$$

Put

$$\mathrm{OP}\,\widetilde{\mathcal{E}} := \big\{\mathrm{OP}(a) : a \in \widetilde{\mathcal{E}}\big\}, \quad \mathrm{OP}\,\mathcal{E}_0 := \big\{\mathrm{OP}(q) : q \in \mathcal{E}_0\big\},$$

and let $\widetilde{\mathcal{A}}$ and $\mathcal{J}_0$ be the closures of $\mathrm{OP}\,\widetilde{\mathcal{E}}$ and $\mathrm{OP}\,\mathcal{E}_0$ in $\mathcal{B}(X)$, respectively. Finally, let $C_{\mathrm{b}}(\mathbb{R}_+ \times \mathbb{R})$ stand for the bounded and continuous functions on $\mathbb{R}_+ \times \mathbb{R}$ and denote by $\mathcal{I}_0$ the closed ideal of $C_{\mathrm{b}}(\mathbb{R}_+ \times \mathbb{R})$ constituted by the functions p for which $p(r,\lambda) \to 0$ as $r \to 0$ uniformly with respect to $\lambda \in \mathbb{R}$.

Theorem 5.2. (i) *If $a, b \in \widetilde{\mathcal{E}}$ then $\mathrm{OP}(a)\mathrm{OP}(b) - \mathrm{OP}(ab) \in \mathcal{J}_0$.*

(ii) *If $a \in \widetilde{\mathcal{E}}$ then $\big(\mathrm{OP}(a)\big)^* - \mathrm{OP}(\overline{a}) \in \mathcal{J}_0$.*

(iii) *If $A, B \in \widetilde{\mathcal{A}}$ and $A - B \in \mathcal{J}_0$ then $\mathrm{sp}_0\, A = \mathrm{sp}_0\, B$.*

(iv) *$\widetilde{\mathcal{A}}$ is a C^*-subalgebra of $\mathcal{B}(X)$ and $\mathcal{J}_0$ is a closed two-sided (selfadjoint) ideal of $\widetilde{\mathcal{A}}$. The algebra $\widetilde{\mathcal{A}}/\mathcal{J}_0$ is commutative and if $B \in \widetilde{\mathcal{A}}$ then $\mathrm{sp}_0\, B$ coincides with the spectrum of $B + \mathcal{J}_0$ in $\widetilde{\mathcal{A}}/\mathcal{J}_0$.*

(v) *If $B \in \widetilde{\mathcal{A}}$ is the uniform limit of $\mathrm{OP}(a_n) \in \mathrm{OP}\widetilde{\mathcal{E}}$, then the cosets $a_n + \mathcal{I}_0$ converge in $C_{\mathrm{b}}(\mathbb{R}_+ \times \mathbb{R})/\mathcal{I}_0$ to a coset $a + \mathcal{I}_0$ and*

$$\mathrm{sp}_0\, B = \Big\{\mu \in \mathbb{C} : \lim_{\varepsilon\to 0} \inf_{(r,\lambda)\in(0,\varepsilon)\times\mathbb{R}} |a(r,\lambda) - \mu| = 0\Big\}.$$

Now suppose $(\Gamma, w) \in A_2^0$. The map Φ given by

$$(\Phi f)(r) = |1 + \mathrm{i}r\theta'(r)|^{1/2} \mathrm{e}^{v(r)} r^{1/2} f(t + r\mathrm{e}^{\mathrm{i}\theta(r)}), \quad r \in \mathbb{R}_+,$$

is an isometric isomorphism of $L^2(\Gamma, w)$ onto $X = L^2(\mathbb{R}_+, \mathrm{d}\mu)$. Consider the map

$$\Psi : \mathcal{B}(L^2(\Gamma, w)) \to \mathcal{B}(X), \quad A \mapsto \Phi A \Phi^{-1}.$$

The following theorem reveals our interest in Mellin pseudodifferential operators with slowly oscillating symbols.

Theorem 5.3. *Let* $(\Gamma, w) \in A_2^0$. *If* $c_\Gamma \in \mathrm{SO}(\Gamma)$ *then*

$$\Psi(c_\Gamma I) = cI.$$

For the Cauchy singular integral operator $S_{\Gamma,w} \in \mathcal{B}(L^2(\Gamma, w))$ *we have*

$$\Psi(S_{\Gamma,w}) = \mathrm{OP}(\sigma_{\Gamma,w}) + \mathrm{OP}(q)$$

where

$$\sigma_{\Gamma,w}(r,\lambda) := \coth\left(\pi \frac{\lambda + \mathrm{i}(rv'(r) + 1/2)}{1 + \mathrm{i}r\theta'(r)}\right) \quad \textit{for } (r,\lambda) \in \mathbb{R}_+ \times \mathbb{R};$$

the function $\sigma_{\Gamma,w}$ *is a function in* $\widetilde{\mathcal{E}}$ *and* q *belongs to* $\mathcal{E}_0$.

It is clear that $\Psi(AB) = \Psi(A)\Psi(B)$. Furthermore, if $c \in \mathrm{SO}^\infty$ then obviously $cI = \mathrm{OP}(a)$ with $a(r,\lambda) = c(r)$ and thus $a \in \widetilde{\mathcal{E}}$. Theorems 5.2 and 5.3 therefore imply that $\Psi(A) \in \widetilde{\mathcal{A}}$ for every $A \in \mathcal{C}_{\Gamma,w}$. In summary, we arrive at the following.

Corollary 5.4. *If* $(\Gamma, w) \in A_2^0$, *then the map* $\widetilde{\Psi}$ *defined by*

$$\widetilde{\Psi} : \mathcal{C}_{\Gamma,w} \to \widetilde{\mathcal{A}}/\mathcal{J}_0, \quad A \mapsto \Psi(A) + \mathcal{J}_0$$

is a well-defined C^**-algebra homomorphism and*

$$\mathrm{sp}_t\, A = \mathrm{sp}_0\, \Psi(A) = \mathrm{sp}\, \widetilde{\Psi}(A) \quad \textit{for every } A \in \mathcal{C}_{\Gamma,w}. \tag{5.1}$$

6. LIMIT OPERATORS OF PSEUDODIFFERENTIAL OPERATORS

For $y \in \mathbb{R}_+$, the (multiplicative) shift operator V_y is defined on the space $X = L^2(\mathbb{R}_+, \mathrm{d}\mu)$ by

$$(V_y f)(r) = f(r/y), \quad r \in \mathbb{R}_+.$$

Given $\mu \in \mathbb{R}$, we define the multiplication operator E_μ by

$$(E_\mu f)(r) = r^{\mathrm{i}\mu} f(r), \quad r \in \mathbb{R}_+.$$

Put $W_{(y,\mu)} = V_y E_\mu$. Clearly, $W_{(y,\mu)}$ is a unitary operator on $L^2(\mathbb{R}_+, \mathrm{d}\mu)$ and $W_{(y,\mu)}^{-1} = E_{-\mu} V_{1/y}$. A straightforward computation shows that if $B = \mathrm{OP}(a)$ with $a \in \mathcal{E}$ then

$$W_{(y,\mu)}^{-1} B W_{(y,\mu)} = \mathrm{OP}\big(a_{(y,\mu)}\big) \quad \text{with} \quad a_{(y,\mu)}(r,\lambda) = a(ry, \lambda + \mu). \tag{6.1}$$

Now fix $B \in \mathcal{B}(X)$. A sequence $h = \{h_n\}_{n=1}^{\infty} = \{(y_n, \mu_n)\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathbb{R}$ is referred to as a *test sequence* if $y_n \to 0$ as $n \to \infty$. Let h be a test sequence. An operator $B_h \in \mathcal{B}(X)$ is called *a limit operator of B with respect to h* if

$$\lim_{n\to\infty} \|(W_{h_n}^{-1} B W_{h_n} - B_h)\mathrm{OP}(\varphi)\| = 0$$

and

$$\lim_{n\to\infty} \|\mathrm{OP}(\varphi)(W_{h_n}^{-1} B W_{h_n} - B_h)\| = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, i.e., every $\varphi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ with compact support. It is easily seen that if B has a limit operator B_h with respect to h, then B_h is uniquely determined (which justifies the notation B_h).

If B has a limit operator B_h with respect to h and if g is a subsequence of h, then the limit operator B_g with respect to g also exists and coincides with the operator B_h.

The collection of all limit operators B_h of an operator $B \in \mathcal{B}(X)$ is called its *local operator symbol* (at the origin) and is denoted by $\mathrm{Lim}_0\, B$. Here are a few simple properties of limit operators (see, e.g., [18] and the references therein).

Proposition 6.1. *Let $h \subset \mathbb{R}_+ \times \mathbb{R}$ be a test sequence.*

(i) *If $B \in \mathcal{B}(X)$ and B_h exists then $\|B_h\| \leqslant \|B\|$.*

(ii) *If $A, B \in \mathcal{B}(X)$, $\alpha \in \mathbb{C}$, and if the limit operators A_h, B_h exist, then $(\alpha A)_h$, $(A+B)_h$, $(AB)_h$, $(A^*)_h$ also exist and*

$$(\alpha A)_h = \alpha A_h, \quad (A+B)_h = A_h + B_h, \quad (AB)_h = A_h B_h, \quad (A^*)_h = (A_h)^*.$$

(iii) *If $A, A^{(m)} \in \mathcal{B}(X)$, if $A_h^{(m)}$ exist, and if $\|A - A^{(m)}\| \to 0$ as $m \to \infty$, then A_h exists and $\|A_h - A_h^{(m)}\| \to 0$ as $m \to \infty$.*

Let $C_{\mathrm{b}}(\mathbb{R})$ denote the bounded continuous functions on $\mathbb{R}$ and recall that $C_{\mathrm{b}}^\infty(\mathbb{R})$ stands for the infinitely differentiable functions on $\mathbb{R}$ which are bounded together with all their derivatives. Also recall that $\mathrm{CO}(a)$ is the Mellin convolution operator $M^{-1}aM$ (Section 2) and put

$$\mathrm{CO}\, C_{\mathrm{b}} := \big\{\mathrm{CO}(a) : a \in C_{\mathrm{b}}(\mathbb{R})\big\}, \quad \mathrm{CO}\, C_{\mathrm{b}}^\infty := \big\{\mathrm{CO}(a) : a \in C_{\mathrm{b}}^\infty(\mathbb{R})\big\}.$$

Theorem 6.2. (i) *If $B \in \widetilde{\mathcal{A}}$ then every test sequence $h \subset \mathbb{R}_+ \times \mathbb{R}$ contains a subsequence g such that B_g exists.*

(ii) *If $B = \mathrm{OP}(a) \in \mathrm{OP}\,\widetilde{\mathcal{E}}$, $h = \{(y_n, \mu_n)\} \subset \mathbb{R}_+ \times \mathbb{R}$ is a test sequence, and B_h exists, then $B_h = \mathrm{CO}(a_h) \in \mathrm{CO}C_{\mathrm{b}}^\infty$ and*

$$a_h(\lambda) = \lim_{n\to\infty} a(y_n, \lambda + \mu_n) \quad \textit{for } \lambda \in \mathbb{R}, \tag{6.2}$$

the convergence being uniform on compact subsets of $\mathbb{R}$.

(iii) *If* $B \in \widetilde{\mathcal{A}}$, $h \subset \mathbb{R}_+ \times \mathbb{R}$ *is a test sequence, and* B_h *exists, then* $B_h = \mathrm{CO}(a_h) \in \mathrm{CO}\, C_{\mathrm{b}}$ *and* (6.2) *holds with* a *as in Theorem* 5.2 (v), *again uniformly on compact subsets of* $\mathbb{R}$.

(iv) *If* $B \in \mathcal{J}_0$ *then* $B_h = 0$ *for every test sequence* h.

Proof. (i) Denote by $\mathrm{Li}(X)$ the set of all operators $B \in \mathcal{B}(X)$ with the property that every test sequence $h \subset \mathbb{R}_+ \times \mathbb{R}$ has a subsequence g such that B_g exists. We have to prove that $\widetilde{\mathcal{A}} \subset \mathrm{Li}(X)$. One can show that $\mathrm{Li}(X)$ is a C^*-subalgebra of $\mathcal{B}(X)$ (see, e.g., [18], Proposition 2). Thus, it suffices to verify that $\mathrm{OP}\,\widetilde{\mathcal{E}} \subset \mathrm{Li}(X)$. So let $a \in \widetilde{\mathcal{E}}$ and let $h = \{(y_n, \mu_n)\} \subset \mathbb{R}_+ \times \mathbb{R}$ be a test sequence. The sequence $\{a_n\} \subset C^\infty(\mathbb{R}_+ \times \mathbb{R})$ given by $a_n(r, \lambda) = a(y_n r, \lambda + \mu_n)$ is bounded in the space $C^\infty(\mathbb{R}_+ \times \mathbb{R})$, i.e., a_n and all its partial derivatives are uniformly bounded on compact subsets of $\mathbb{R}_+ \times \mathbb{R}$. Hence, by Montel's property, the sequence $\{a_n\}$ contains a subsequence $\{a_{n_k}\}$ which converges in $C^\infty(\mathbb{R}_+ \times \mathbb{R})$. Obviously, the limit function b belongs to $\mathcal{E}$. Taking into account (6.1), it is easily seen that $\mathrm{OP}(b)$ is the limit operator of $\mathrm{OP}(a)$ with respect to $\{h_{n_k}\}$.

(ii) From the proof of part (i) we know that $B_h = \mathrm{OP}(b)$ where $b \in \mathcal{E}$ is given by

$$b(r, \lambda) = \lim_{n \to \infty} a(y_n r, \lambda + \mu_n), \tag{6.3}$$

the convergence being uniform on compact subsets of $\mathbb{R}_+ \times \mathbb{R}$. Since $b \in \mathcal{E}$ we have $b(r, \cdot\,) \in C_{\mathrm{b}}^\infty(\mathbb{R})$ for each $r \in \mathbb{R}_+$. It remains to show that b is independent of r. Let $0 < r < s < \infty$. As $a \in \widetilde{\mathcal{E}}$, it follows that

$$\delta_k := \sup \left\{ |(\rho \partial_\rho) a(\rho, \lambda)| : \rho \in \bigcup_{n \geqslant k} (y_n r, y_n s),\ \lambda \in \mathbb{R} \right\} \to 0$$

as $k \to \infty$. We have

$$\begin{aligned} |a(y_n s, \lambda + \mu_n) - a(y_n r, \lambda + \mu_n)| &= \left| \int_{y_n r}^{y_n s} (\rho \partial_\rho) a(\rho, \lambda + \mu_n) \frac{\mathrm{d}\rho}{\rho} \right| \\ &\leqslant \delta_k \log \frac{y_n s}{y_n r} = \delta_k \log \frac{s}{r} = \mathrm{o}(1) \end{aligned}$$

as $k \to \infty$, and hence $b(s, \lambda) = b(r, \lambda)$. Letting $r = 1$ in (6.3) we arrive at (6.2).

(iii) This follows from parts (i) and (ii) and from Proposition 6.1 (iii).

(iv) This is again immediate from parts (i) and (ii) and from Proposition 6.1 (iii). ∎

The importance of the operator symbol $\mathrm{Lim}_0\, B$ is uncovered by the following result.

Theorem 6.3. *If $B \in \widetilde{\mathcal{A}}$ then*

$$\mathrm{sp}_0\, B = \bigcup_{B_h \in \mathrm{Lim}_0\, B} \mathrm{sp} B_h.$$

Proof. It suffices to prove that $0 \notin \mathrm{sp}_0\, B$ if and only if $0 \notin \mathrm{sp} B_h$ for all $B_h \in \mathrm{Lim}_0\, B$. Given $B \in \widetilde{\mathcal{A}}$, let $a \in C_{\mathrm{b}}(\mathbb{R}_+ \times \mathbb{R})$ be as in Theorem 5.2 (v) and define

$$f_a(\varepsilon) := \inf_{\lambda \in \mathbb{R}} |a(\varepsilon, \lambda)| \quad (\varepsilon > 0).$$

If $0 \notin \mathrm{sp}_0\, B$, then $\liminf_{\varepsilon \to 0} f_a(\varepsilon) =: \delta > 0$ by Theorem 5.2 (v). Let $h = \{(y_n, \mu_n)\}$ be a test sequence and suppose B_h exists. Then $f_a(y_n) > \delta/2$ for all sufficiently large n and hence, by virtue of Theorem 6.2 (iii), $\inf_{\lambda \in \mathbb{R}} |a_h(\lambda)| \geqslant \delta/2$. This proves that $0 \notin \mathrm{sp} B_h$ (Theorems 6.2 (iii) and 2.1).

Now let us suppose that $0 \notin \mathrm{sp} B_h$ for all $B_h \in \mathrm{Lim}_0\, B$ but, contrary to what we want, let us assume that $\liminf_{\varepsilon \to 0} f_a(\varepsilon) = 0$ (recall Theorem 5.2 (v)). Then there is a test sequence $h = \{(y_n, \mu_n)\}$ such that

$$a(y_n, \mu_n) \to 0. \tag{6.4}$$

Pick a subsequence $g = \{(y_{n_k}, \mu_{n_k})\}$ of h such that B_g exists. By Theorem 6.2 (iii), we have $B_g = \mathrm{CO}(b_g)$ with

$$b_g(\lambda) = \lim_{n \to \infty} a(y_{n_k}, \lambda + \mu_{n_k}). \tag{6.5}$$

Since $0 \notin \mathrm{sp} B_g$, it follows from Theorem 2.1 that $\inf_{\lambda \in \mathbb{R}} |b_g(\lambda)| > 0$. But this is impossible, because (6.4) and (6.5) give $b_g(0) = 0$. This contradiction completes the proof. ∎

7. LIMIT OPERATORS OF SINGULAR INTEGRAL OPERATORS

Let $(\Gamma, w) \in A_2^0$ be as in Section 4. If $A \in \mathcal{C}_{\Gamma,w}$ then $\Psi(A) \in \widetilde{\mathcal{A}}$ due to the remark after Theorem 5.3. From Theorem 6.2 (iii) we therefore deduce that

$$\mathrm{Lim}_0\, \Psi(A) \subset \mathrm{CO}\, C_{\mathrm{b}},$$

i.e., all limit operators of $\Psi(A)$ are Mellin convolutions with $C_{\mathrm{b}}(\mathbb{R})$ symbols. Since the local spectra of Mellin convolutions are available from Theorem 2.1, we can use Theorem 6.3 and (5.1) in order to compute $\mathrm{sp}_t\, A$.

In what follows we write test sequences $h \subset \mathbb{R}_+ \times \mathbb{R}$ in the form $h = \{(r_n, \mu_n)\}_{n=1}^{\infty}$. Recall that $\Psi_{\delta,\gamma}(A) := \Phi_{\delta,\gamma} A \Phi_{\delta,\gamma}^{-1}$ with $\Phi_{\delta,\gamma}$ as in (3.3).

Proposition 7.1. *Let $c_\Gamma \in \mathrm{SO}(\Gamma)$ and let $h = \{(r_n, \mu_n)\} \subset \mathbb{R}_+ \times \mathbb{R}$ be a test sequence. The limit operator $[\Psi(c_\Gamma I)]_h$ exists if and only if the limit $\alpha := \lim\limits_{n\to\infty} c(r_n)$ exists. In that case*

$$[\Psi(c_\Gamma I)]_h = \Psi_{\delta,\gamma}(\alpha I) = \alpha I$$

where αI is the operator of multiplication by the constant α.

Proof. Immediate from Theorem 5.3. ∎

Proposition 7.2. *Let $S_{\Gamma,w}$ be the Cauchy singular integral operator on the space $L^2(\Gamma, w)$ and let θ and v be given by (4.4) and (4.7). Suppose $h = \{(r_n, \mu_n)\} \subset \mathbb{R}_+ \times \mathbb{R}$ is a test sequence. The limit operator $[\Psi(S_{\Gamma,w})]_h$ exists if and only if the two limits*

$$\delta := \lim_{n\to\infty} r_n \theta'(r_n), \quad \gamma := \lim_{n\to\infty} r_n v'(r_n) \tag{7.1}$$

exist and if μ_n converges to some point $\mu \in \mathbb{R} \cup \{\pm\infty\}$. In that case

$$\big[\Psi(S_{\Gamma,w})\big]_h = \begin{cases} E_{-\mu} \Psi_{\delta,\gamma}(S_{G_\delta, u_\gamma}) E_\mu & \text{for } \mu \in \mathbb{R}, \\ \pm I & \text{for } \mu \in \{\pm\infty\}; \end{cases}$$

where G_δ is the logarithmic spiral (3.1), u_γ is the power weight (3.2), and E_μ is multiplication by $r^{\mathrm{i}\mu}$.

Proof. Combining Theorems 5.3 and 6.2 we see that the existence of the limit operator is equivalent to the existence of the two limits (7.1) and of a point $\mu \in \mathbb{R} \cup \{\pm\infty\}$ such that $\mu_n \to \mu$. If $\mu \in \mathbb{R}$, then $\big[\Psi(S_{\Gamma,w})\big]_h = \mathrm{CO}(\sigma_{\delta,\gamma,\mu})$, where $\sigma_{\delta,\gamma,\mu}(\lambda) := \sigma_{\delta,\gamma}(\lambda + \mu)$ and $\sigma_{\delta,\gamma}$ is given by (3.5). Due to (6.1), $\mathrm{CO}(\sigma_{\delta,\gamma,\mu}) = E_{-\mu} \mathrm{CO}(\sigma_{\delta,\gamma}) E_\mu$, and from (3.5) we infer that $\mathrm{CO}(\sigma_{\delta,\gamma}) = \Psi_{\delta,\gamma}(S_{G_\delta, u_\gamma})$. Finally, it is easily seen that $[\Psi(S_{\Gamma,w})]_h = \pm I$ if $\mu = \pm\infty$. ∎

8. LOCAL SPECTRA OF SINGULAR INTEGRAL OPERATORS

Throughout what follows, we let $S_{\Gamma,w}$ stand for the Cauchy singular integral operator as an element of $\mathcal{B}(L^2(\Gamma, w))$ and we define θ and w by (4.4) and (4.7). Recall that t is the starting point of Γ.

Theorem 8.1. *Let* $(\Gamma, w) \in A_2^0$. *Denote by* $\mathcal{P}$ *the set of the partial limits of the map*

$$(0, \varepsilon) \to \mathbb{R} \times (-1/2, 1/2), \quad r \mapsto \big(r\theta'(r), rv'(r)\big)$$

as $r \to 0$. *Then*

$$\operatorname{sp}_t S_{\Gamma,w} = \bigcup_{(\delta,\gamma)\in\mathcal{P}} \operatorname{sp} S_{G_\delta,u_\gamma} = \bigcup_{(\delta,\gamma)\in\mathcal{P}} \mathcal{S}_{\delta,\gamma}$$

where $\mathcal{S}_{\delta,\gamma}$ *is the logarithmic double spiral* (3.7).

Proof. From (5.1) we know that $\operatorname{sp}_t S_{\Gamma,w} = \operatorname{sp}_0 \Psi(S_{\Gamma,w})$, and Theorems 5.3 and 6.3 imply that the latter set equals

$$\bigcup_h \operatorname{sp}\big[\Psi(S_{\Gamma,w})\big]_h, \tag{8.1}$$

the union over all h for which the limit operator exists. Proposition 7.2 shows that (8.1) is equal to

$$\bigcup_{(\delta,\gamma)\in\mathcal{P}} \operatorname{sp} S_{G_\delta,u_\gamma},$$

and Theorem 3.1 completes the proof. ∎

The previous theorem identifies the local spectrum of $S_{\Gamma,w}$ as a union of logarithmic double spirals. Actually it does more. Namely, it tells us why a logarithmic double spiral $\mathcal{S}_{\delta,\gamma}$ is contained in the local spectrum: because S_{G_δ,u_γ} is a limit operator of $S_{\Gamma,w}$ in the sense of Section 7.

One can show (see [6]), that

$$\bigcup_{(\delta,\gamma)\in\mathcal{P}} \mathcal{S}_{\delta,\gamma} = \bigcup_{(\delta,\gamma)\in\operatorname{conv}\mathcal{P}} \mathcal{S}_{\delta,\gamma}$$

where $\operatorname{conv}\mathcal{P}$ is the convex hull of $\mathcal{P}$. Moreover, given any nonempty compact convex set $\mathcal{N} \subset \mathbb{R} \times (-1/2, 1/2)$, there exists a pair $(\Gamma, w) \in A_2^0$ such that $\mathcal{N} = \operatorname{conv}\mathcal{P}$ (again see [6]). Finally, we remark that any possible local spectrum of a bounded Cauchy singular integral is attained at a slowly oscillating curve with a slowly oscillating weight: given $(\Gamma, w) \in A_2$, there exists a pair $(\Gamma_0, w_0) \in A_2^0$ such that

$$\operatorname{sp}_t S_{\Gamma,w} = \operatorname{sp}_t S_{\Gamma_0,w_0}$$

(see [4] and [6]).

Theorem 8.2. *Let* $(\Gamma, w) \in A_2^0$ *and* $a_\Gamma, b_\Gamma \in \mathrm{SO}(\Gamma)$. *Put* $A := a_\Gamma I + b_\Gamma S_{\Gamma,w}$ *and denote by* $\mathcal{P}_A$ *the set of the partial limits of the map*

$$(0, \varepsilon) \to \mathbb{C}^2 \times \mathbb{R} \times (-1/2, 1/2), \quad r \mapsto \big(a(r), b(r), r\theta'(r), rv'(r)\big)$$

as $r \to 0$. *Then*

$$\mathrm{sp}_t A = \bigcup_{(\alpha,\beta,\delta,\gamma)\in\mathcal{P}_A} \mathrm{sp}(\alpha I + \beta S_{G_\delta, u_\gamma}) = \bigcup_{(\alpha,\beta,\delta,\gamma)\in\mathcal{P}_A} (\alpha + \beta \mathcal{S}_{\delta,\gamma}).$$

Proof. Taking into account Theorem 5.3, this follows as in the proof of Theorem 8.1 from Theorem 6.3 in conjunction with Propositions 7.1 and 7.2. ∎

Put $T(r) := r\theta'(r)$ and $V(r) := rv'(r)$. Proposition 4.2 implies that if $(\alpha, \beta, \delta, \gamma) \in \mathcal{P}_A$, then there is a $\xi \in M_0(\mathrm{SO})$ such that

$$a(\xi) = \alpha, \quad b(\xi) = \beta, \quad T(\xi) = \delta, \quad V(\xi) = \gamma. \tag{8.2}$$

Conversely, again by Proposition 4.2, given any $\xi \in M_0(\mathrm{SO})$ there exists a point $(\alpha, \beta, \delta, \gamma) \in \mathcal{P}_A$ such that (8.2) holds. With the notations

$$S_{\xi\theta'(\xi), \xi v'(\xi)} := S_{G_{T(\xi)}, u_{V(\xi)}}, \quad \mathcal{S}_{\xi\theta'(\xi),\xi v'(\xi)} := \mathcal{S}_{T(\xi),V(\xi)}$$

we can therefore restate Theorem 8.2 as follows.

Theorem 8.3. *If* $(\Gamma, w) \in A_2^0$ *and* $a_\Gamma, b_\Gamma \in \mathrm{SO}(\Gamma)$, *then*

$$\begin{aligned}\mathrm{sp}_t(a_\Gamma I + b_\Gamma S_{\Gamma,w}) &= \bigcup_{\xi\in M_0(\mathrm{SO})} \mathrm{sp}\Big(a(\xi)I + b(\xi) S_{\xi\theta'(\xi),\xi v'(\xi)}\Big) \\ &= \bigcup_{\xi\in M_0(\mathrm{SO})} \Big(a(\xi) + b(\xi)\mathcal{S}_{\xi\theta'(\xi),\xi v'(\xi)}\Big).\end{aligned}$$

The following Theorem 8.4 essentially generalizes Theorems 8.1 to 8.3. We remark that the proofs of Theorems 8.1 to 8.3 are based on Theorem 6.3, while the proof of Theorem 8.4 does not make use of Theorem 6.3, although this proof has recourse to the concept of limit operators. Thus, we could omit Theorems 6.3 and 8.1 to 8.3 entirely and could confine ourselves to only Theorem 8.4, but we think that Theorem 8.4 is more transparent in conjunction with its special cases covered by Theorems 8.1 to 8.3.

Let $\mathcal{D}_{\Gamma,w}$ be the smallest closed subalgebra of $\mathcal{B}(L^2(\Gamma, w))$ containing the multiplications by constants and the operators $S_{\Gamma,w}$ and $S^*_{\Gamma,w}$. For $\xi \in M_0(\mathrm{SO})$, put

$$\mathcal{D}_\xi := \mathcal{D}_{\xi\theta'(\xi),\xi v'(\xi)} := \mathcal{D}_{G_{T(\xi)}, u_{V(\xi)}}.$$

THEOREM 8.4. *Let* $(\Gamma, w) \in A_2^0$. *For each* $\xi \in M_0(\mathrm{SO})$ *there exists a* C^**-algebra homomorphism*

$$\sigma_\xi : \mathcal{C}_{\Gamma,w} \to \mathcal{D}_\xi$$

such that

$$\sigma_\xi(a_\Gamma I) = a(\xi) \quad (a_\Gamma \in \mathrm{SO}(\Gamma)), \tag{8.3}$$

$$\sigma_\xi(S_{\Gamma,w}) = S_{\xi\theta'(\xi),\xi v'(\xi)} \tag{8.4}$$

and

$$\mathrm{sp}_t A = \bigcup_{\xi \in M_0(\mathrm{SO})} \mathrm{sp}\, \sigma_\xi(A) \quad \textit{for every } A \in \mathcal{C}_{\Gamma,w}. \tag{8.5}$$

Further, for each $\xi \in M_0(\mathrm{SO})$ *there is a* C^**-algebra homomorphism*

$$\mathrm{Sym}_\xi : \mathcal{C}_{\Gamma,w} \to C_{\mathrm{b}}^\infty(\mathbb{R})$$

such that

$$(\mathrm{Sym}_\xi\, a_\Gamma I)(\lambda) = a(\xi) \quad (a_\Gamma \in \mathrm{SO}(\Gamma)), \tag{8.6}$$

$$(\mathrm{Sym}_\xi\, S_{\Gamma,w})(\lambda) = \coth\left(\pi \frac{\lambda + \mathrm{i}(\xi v'(\xi) + 1/2)}{1 + \mathrm{i}\xi\theta'(\xi)}\right) \tag{8.7}$$

and

$$\mathrm{sp}_t A = \bigcup_{\xi \in M_0(\mathrm{SO})} \mathcal{R}(\mathrm{Sym}_\xi\, A) \quad \textit{for every } A \in \mathcal{C}_{\Gamma,w}. \tag{8.8}$$

Proof. Let $A \in \mathcal{C}_{\Gamma,w}$. From Corollary 5.4 we know that $\Psi(A) \in \widetilde{\mathcal{A}}$ and

$$\mathrm{sp}_t A = \mathrm{sp}\,\big(\Psi(A) + \mathcal{J}_0\big). \tag{8.9}$$

The commutative C^*-algebra $\widetilde{\mathcal{A}}/\mathcal{J}_0$ contains the commutative C^*-algebra $Z := \Psi(\mathrm{SO}(\Gamma))/\mathcal{J}_0$, and the maximal ideal space of the algebra Z may be identified with $M_0(\mathrm{SO})$. For $\xi \in M_0(\mathrm{SO})$ denote by $\mathcal{J}_\xi^0$ the smallest closed two-sided ideal of $\widetilde{\mathcal{A}}/\mathcal{J}_0$ containing $\big\{\Psi(f_\Gamma I) + \mathcal{J}_0 : f_\Gamma \in \mathrm{SO}(\Gamma),\ f(\xi) = 0\big\}$. The local principle of Allan and Douglas (see [7], Theorem 1.34 or [3], Theorem 8.2) implies that

$$\mathrm{sp}\big(\Psi(A) + \mathcal{J}_0\big) = \bigcup_{\xi \in M_0(\mathrm{SO})} \mathrm{sp}\Big(\big(\Psi(A) + \mathcal{J}_0\big) + \mathcal{J}_\xi^0\Big), \tag{8.10}$$

the spectra on the right taken in $(\widetilde{\mathcal{A}}/\mathcal{J}_0)/\mathcal{J}_\xi^0$. Combining (8.9) and (8.10) we get

$$\mathrm{sp}_t(A) = \bigcup_{\xi \in M_0(\mathrm{SO})} \mathrm{sp}\Big(\big(\Psi(A) + \mathcal{J}_0\big) + \mathcal{J}_\xi^0\Big). \tag{8.11}$$

For $\xi \in M_0(\mathrm{SO})$, consider the map

$$\sigma_\xi : \{a_\Gamma I : a_\Gamma \in \mathrm{SO}(\Gamma)\} \cup \{S_\Gamma\} \cup \{S_\Gamma^*\} \to \mathcal{D}_\xi$$

given by (8.3), (8.4), and $\sigma_\xi(S_\Gamma^*) = (S_{\xi\theta'(\xi),\xi v'(\xi)})^*$. We claim that σ_ξ extends to a C^*-algebra homomorphism of $\mathcal{C}_{\Gamma,w}$ into $\mathcal{D}_\xi$. To see this, let $A \in \mathcal{C}_{\Gamma,w}$ and $A = \lim A_m$ where the operators A_m are of the form

$$A_m = \sum_j \prod_k B_{jk}, \quad B_{jk} \in \{a_\Gamma I : a_\Gamma \in \mathrm{SO}(\Gamma)\} \cup \{S_\Gamma\} \cup \{S_\Gamma^*\}, \tag{8.12}$$

the sum and the products finite. Corollary 4.3 implies that there exists a sequence $\{r_n\} \subset \mathbb{R}_+$ such that

$$r_n \to 0, \quad r_n\theta'(r_n) \to \xi\theta'(\xi), \quad r_n v'(r_n) \to \xi v'(\xi),$$

and $a(r_n) \to a(\xi)$ for all $a_\Gamma \in \mathrm{SO}(\Gamma)$ which actually occur in the representations (8.12) of the operators A_m $(m = 1, 2, \ldots)$. Consider the test sequence $h = \{(r_n, 0)\}$. By virtue of Propositions 7.1, 7.2, and 6.1 (ii), the limit operator $[\Psi(A_m)]_h$ exists and

$$\big[\Psi(A_m)\big]_h = \Psi^{(\xi)}(\sigma_\xi(A_m)) \tag{8.13}$$

where $\Psi^{(\xi)} := \Psi_{\delta,\gamma}$ with $\delta = \xi\theta'(\xi)$, $\gamma = \xi v'(\xi)$ and $\sigma_\xi(A_m) \in \mathcal{D}_\xi$ is given by

$$\sigma_\xi(A_m) = \sum_j \prod_k \sigma_\xi(B_{jk}).$$

By (8.13) and Proposition 6.1 (i),

$$\|\sigma_\xi(A_m)\| = \|\Psi^{(\xi)}(\sigma_\xi(A_m))\| = \|\,[\Psi(A_m)]_h\,\| \leqslant \|A_m\|,$$

whence $\sigma_\xi(A) = \lim \sigma_\xi(A_m)$. Consequently, σ_ξ extends to a C^*-algebra homomorphism $\sigma_\xi : \mathcal{C}_{\Gamma,w} \to \mathcal{D}_\xi$. Moreover, since $[\Psi(A)]_h = \lim[\Psi(A_m)]_h$ due to Proposition 6.1 (iii), we obtain from (8.13) that

$$\big[\Psi(A)\big]_h = \Psi^{(\xi)}(\sigma_\xi(A)). \tag{8.14}$$

We now prove that if $A \in \mathcal{C}_{\Gamma,w}$ and $\xi \in M_0(\mathrm{SO})$, then

$$\operatorname{sp}\Big(\big(\Psi(A) + \mathcal{J}_0\big) + \mathcal{J}_\xi^0\Big) = \operatorname{sp} \Psi^{(\xi)}(\sigma_\xi(A)). \tag{8.15}$$

We have $0 \notin \operatorname{sp}((\Psi(A)) + \mathcal{J}_0) + \mathcal{J}_\xi^0)$ if and only if there exist $B', B'' \in \mathcal{C}_{\Gamma,w}$, $E', E'' \in \widetilde{\mathcal{A}}$, $Q', Q'' \in \mathcal{J}_0$, $f_\Gamma, g_\Gamma \in \mathrm{SO}(\Gamma)$ such that $f(\xi) = g(\xi) = 0$ and

$$\Psi(B')\Psi(A) - I = Q' + fE', \quad \Psi(A)\Psi(B'') - I = Q'' + gE'' \tag{8.16}$$

(also see [3], Proposition 8.6). We may, without loss of generality, assume that the test sequence $h = \{(r_n, 0)\}$ is chosen so that $f(r_n) \to f(\xi)$ and $g(r_n) \to g(\xi)$. Furthermore, we may also, without loss of generality, assume that the limit operators $[\Psi(B')]_h$ and $[\Psi(B'')]_h$ exist (recall Theorem 6.2 (i)). In view of Theorem 6.2, the equalities (8.16) are equivalent to the conditions

$$\big[\Psi(B')\big]_h \big[\Psi(A)\big]_h = I, \quad \big[\Psi(A)\big]_h \big[\Psi(B'')\big]_h = I,$$

which, together with (8.14), proves (8.15).

From (8.15) we see that (8.11) can be written in the form

$$\operatorname{sp}_t A = \bigcup_{\xi \in M_0(\mathrm{SO})} \operatorname{sp} \Psi^{(\xi)}(\sigma_\xi(A)), \tag{8.17}$$

which gives (8.5) immediately.

The definition of σ_ξ and Theorem 6.2 show that $\Psi^{(\xi)}(\sigma_\xi(A))$ belongs to $\mathrm{CO}C_{\mathrm{b}}$. Hence, there is a unique function $\operatorname{Sym}_\xi A$ in C_{b} such that

$$\Psi^{(\xi)}(\sigma_\xi(A)) = \mathrm{CO}(\operatorname{Sym}_\xi A). \tag{8.18}$$

It can be readily checked that the map $A \mapsto \operatorname{Sym}_\xi A$ is a C^*-algebra homomorphism and that (8.6) and (8.7) hold. Equality (8.8) is a straightforward consequence of (8.17), (8.18), and Theorem 2.1. ∎

In case A is of the form

$$A = \sum_j \prod_k B_{jk}, \quad B_{jk} \in \big\{a_\Gamma I : a_\Gamma \in \mathrm{SO}(\Gamma)\big\} \cup \{S_\Gamma\} \cup \{S_\Gamma^*\},$$

the sum and the products finite, Theorem 8.4 can be reformulated in the language of partial limits. For example, if

$$A = aI + b\, S_{\Gamma,w}\, cI + d\, S_{\Gamma,w}^*\, eI$$

with $a, b, c, d, e \in \mathrm{SO}(\Gamma)$, then

$$\mathrm{sp}_t\, A = \bigcup \mathcal{R}(\widetilde{a} + \widetilde{b}\widetilde{c}\,\sigma_{\delta,\gamma} + \widetilde{d}\widetilde{e}\,\overline{\sigma}_{\delta\gamma})$$

where the union is over all partial limits

$$(\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}, \widetilde{e}, \delta, \gamma) \in \mathbb{C}^5 \times \mathbb{R} \times (-1/2, 1/2)$$

of the map

$$r \mapsto \big(a(r), b(r), c(r), d(r), e(r), r\theta'(r), rv'(r)\big)$$

as $r \to 0$ and where $\mathcal{R}(\widetilde{a} + \widetilde{b}\widetilde{c}\,\sigma_{\delta,\gamma} + \widetilde{d}\widetilde{e}\,\overline{\sigma}_{\delta\gamma})$ is the set

$$\bigcup_{\lambda \in \mathbb{R} \cup \{\pm\infty\}} \left(\widetilde{a} + \widetilde{b}\widetilde{c} \coth\left(\pi \frac{\lambda + \mathrm{i}(\gamma + 1/2)}{1 + \mathrm{i}\delta}\right) + \widetilde{d}\widetilde{e} \coth\left(\pi \frac{\lambda - \mathrm{i}(\gamma + 1/2)}{1 - \mathrm{i}\delta}\right)\right).$$

The following special case of Theorem 8.4 can be regarded as the analogue of ([1], Theorem 4.2) = ([3], Theorem 9.27) for slowly oscillating coefficients.

Theorem 8.5. *Let* $(\Gamma, w) \in A_2^0$ *and put*

$$\mathcal{L}_t(\Gamma, w) := \bigcup_{\xi \in M_0(\mathrm{SO})} \big(\{\xi\} \times \mathcal{S}_{\xi\theta'(\xi), \xi v'(\xi)}\big).$$

For each $(\xi, \mu) \in \mathcal{L}_t(\Gamma, w)$ *there exists a* C^**-algebra homomorphism*

$$\mathrm{Sym}_{\xi,\mu} : \mathcal{C}_{\Gamma,w} \to \mathbb{C}$$

such that

$$\mathrm{Sym}_{\xi,\mu}\, a_\Gamma I = a(\xi) \quad \big(a_\Gamma \in \mathrm{SO}(\Gamma)\big), \quad \mathrm{Sym}_{\xi,\mu}\, S_{\Gamma,w} = \mu,$$

and

$$\mathrm{sp}_t A = \bigcup_{(\xi,\mu) \in \mathcal{L}_t(\Gamma,w)} \mathrm{Sym}_{\xi,\mu}\, A \quad \textit{for every } A \in \mathcal{C}_{\Gamma,w}.$$

9. SINGULAR INTEGRAL OPERATORS ON COMPACT SIMPLE ARCS

Let now Γ be a compact oriented rectifiable simple arc and let w be a weight on Γ. Suppose $(\Gamma, w) \in A_2$.

An operator $A \in \mathcal{B}(L^2(\Gamma, w))$ is called *Fredholm* if it is invertible modulo compact operators. The essential spectrum of A is the set

$$\mathrm{sp}_{\mathrm{ess}} A := \{z \in \mathbb{C} : A - zI \text{ is not Fredholm}\}.$$

The operator A is said to be *locally invertible at a point* $\tau \in \Gamma$ if there are operators $D', D'' \in \mathcal{B}(L^2(\Gamma, w))$ and a function $\varphi \in C(\Gamma)$ which is identically 1 in an open neighborhood of τ such that $\varphi A D' = \varphi I$ and $D'' A \varphi = \varphi I$. The local spectrum of A at $\tau \in \Gamma$ is defined as

$$\mathrm{sp}_\tau\, A := \{z \in \mathbb{C} : A - zI \text{ is not locally invertible at } \tau\}.$$

We write $(\Gamma, w) \in A_2^{\mathrm{SO}}$ if $(\Gamma, w) \in A_2$, if the curve and the weight are nice everywhere except at the endpoints, and if the curve and the weight are slowly oscillating at the endpoints. In more precise (but, unfortunately, rather technical) language this means the following. The arc Γ has two endpoints, a starting point t_1 and a terminating point t_2. Pick any point $t_0 \in \Gamma \setminus \{t_1, t_2\}$ and denote by Γ_1 and Γ_2 the subarcs of Γ between t_1 and t_0 and between t_0 and t_2, respectively. For $k = 1, 2$, suppose

$$\begin{aligned} &\Gamma_k = \{t_k + r e^{i\theta_k(r)} : r \in (0, s_k)\}, \\ &w(t_k + r e^{i\theta_k(r)}) = e^{v_k(r)} \quad \text{for } r \in (0, s_k). \end{aligned}$$

Without loss of generality assume that $s_1, s_2 \in (0, 1)$. We write $(\Gamma, w) \in A_2^{\mathrm{SO}}$ if the arcs Γ_k and the weights $w|\Gamma_k$ can be extended to unbounded arcs $\widetilde{\Gamma}_k$ and weights $\widetilde{w}_k$ on $\widetilde{\Gamma}_k$ such that $(\widetilde{\Gamma}_k, \widetilde{w}_k) \in A_2^0$. In other terms, $(\Gamma, w) \in A_2^{\mathrm{SO}}$ if and only if there are functions $\widetilde{\theta}_k$ and $\widetilde{v}_k$ satisfying the conditions of Section 4 such that

$$\theta_k(r) = \widetilde{\theta}_k(r) \quad \text{and} \quad v_k(r) = \widetilde{v}_k(r) \quad \text{for } r \in (0, s_k).$$

In what follows we omit the tilde, i.e., we assume that θ_k and v_k are the restrictions to $(0, s_k)$ of functions θ_k and v_k subject to the conditions listed in Section 4.

In the same vein, we denote by $\mathrm{SO}^\infty_{t_1, t_2}(\Gamma)$ the functions in $C_{\mathrm{b}}^\infty(\Gamma \setminus \{t_1, t_2\})$ whose restrictions to Γ_k $(k = 1, 2)$ can be extended to functions in $\mathrm{SO}^\infty(\widetilde{\Gamma}_k)$. Thus, $a_\Gamma \in \mathrm{SO}^\infty_{t_1, t_2}(\Gamma)$ if and only if $a_\Gamma \in C_{\mathrm{b}}^\infty(\Gamma \setminus \{t_1, t_2\})$ and

$$a_\Gamma(t_k + r e^{i\theta_k(r)}) = a_k(r), \quad r \in (0, s_k), \quad k = 1, 2, \tag{9.1}$$

where $a_k|(0,s_k) = \widetilde{a}_k|(0,s_k)$ and $\widetilde{a}_k \in \mathrm{SO}^\infty$.

Finally, let $\mathcal{C}_{\Gamma,w}(t_1,t_2)$ stand for the closure in $\mathcal{B}(L^2(\Gamma,w))$ of the set of all operators

$$\sum_j \prod_k B_{jk}, \quad B_{jk} \in \Big\{ a_\Gamma I : a_\Gamma \in \mathrm{SO}^\infty_{t_1,t_2}(\Gamma) \Big\} \cup \{S_\Gamma\} \cup \{S^*_\Gamma\},$$

the sum and the products finite.

Theorem 9.1. *If $(\Gamma,w) \in A_2^{\mathrm{SO}}$ and $A \in \mathcal{C}_{\Gamma,w}(t_1,t_2)$ then*

$$\mathrm{sp}_{\mathrm{ess}}\, A = \Bigg(\bigcup_{\tau \in \Gamma \setminus \{t_1,t_2\}} \mathrm{sp}_\tau\, A \Bigg) \cup \mathrm{sp}_{t_1} A \cup \mathrm{sp}_{t_2} A.$$

This theorem follows from standard local principles (by Simonenko, Gohberg and Krupnik, or Allan and Douglas). See, e.g., [3], Sections 8.1 and 8.2.

Theorem 8.4 gives us $\mathrm{sp}_{t_1} A$ at the starting point t_1 of Γ. Changing the orientation of Γ and replacing S_Γ by $-S_\Gamma$, we can employ Theorem 8.4 to get $\mathrm{sp}_{t_2} A$ at the endpoint t_2. Finally, if $\tau \in \Gamma \setminus \{t_1,t_2\}$, we put

$$\sigma_\tau(a_\Gamma I) := \begin{pmatrix} a_\Gamma(\tau) & 0 \\ 0 & a_\Gamma(\tau) \end{pmatrix}, \quad \sigma_\tau(S_\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can show that σ_τ extends to a C^*-algebra homomorphism of $\mathcal{C}_{\Gamma,w}$ into $\mathbb{C}^{2\times 2}$ and that $\mathrm{sp}_\tau A = \mathrm{sp}\,\sigma_\tau(A)$, i.e., that $\mathrm{sp}_\tau A$ is the set of the eigenvalues of the 2×2 matrix $\sigma_\tau(A)$. Thus, under the hypotheses of Theorem 9.1, we can completely describe $\mathrm{sp}_{\mathrm{ess}} A$.

As an example, we state the final result for the operator of Theorem 8.2.

Theorem 9.2. *Let $(\Gamma,w) \in A_2^{\mathrm{SO}}$. Denote by $\mathrm{SO}_{t_1,t_2}(\Gamma)$ the closure of $\mathrm{SO}^\infty_{t_1,t_2}(\Gamma)$ in $L^\infty(\Gamma)$ and suppose $a_\Gamma, b_\Gamma \in \mathrm{SO}_{t_1,t_2}(\Gamma)$. Define θ_k, v_k, a_k, b_k $(k = 1,2)$ as above. Put $A = a_\Gamma I + b_\Gamma S_{\Gamma,w}$ and let $\mathcal{P}_A^1$ and $\mathcal{P}_A^2$, respectively, be the partial limits of the maps*

$$(0,\varepsilon) \to \mathbb{C}^2 \times \mathbb{R} \times (-1/2,1/2), \quad r \mapsto \big(a_1(r), b_1(r), r\theta_1'(r), rv_1'(r)\big)$$

and

$$(0,\varepsilon) \to \mathbb{C}^2 \times \mathbb{R} \times (-1/2,1/2), \quad r \mapsto \big(a_2(r), b_2(r), r\theta_2'(r), rv_2'(r)\big).$$

Then $\mathrm{sp}_{\mathrm{ess}} A$ equals

$$\bigcup_{(\alpha,\beta,\delta,\gamma)\in\mathcal{P}_A^1} (\alpha + \beta\,\mathcal{S}_{\delta,\gamma}) \cup \bigcup_{(\alpha,\beta,\delta,\gamma)\in\mathcal{P}_A^2} (\alpha - \beta\,\mathcal{S}_{\delta,\gamma})$$
$$\cup \bigcup_{\tau \in \Gamma\setminus\{t_1,t_2\}} \big\{ a_\Gamma(\tau) + b_\Gamma(\tau),\, a_\Gamma(\tau) - b_\Gamma(\tau) \big\}.$$

10. SINGULAR INTEGRAL OPERATORS ON COMPOSED CURVES

The results of the previous section can be extended to singular integral operators on composed curves without cusps, because such curves locally look like a star and singular integral operators on stars can be transformed into pseudodifferential operators with matrix-valued symbols on the line (see, e.g., [19] and [5]). We confine ourselves to stating the final result for operators as in Theorem 8.2.

Suppose $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$ where

$$\Gamma_k := \left\{\tau = t + r\mathrm{e}^{\mathrm{i}(\theta(r)+\theta_k(r))} : r \in \mathbb{R}_+\right\}$$

are unbounded oriented simple arcs with the common point t. Assume further that there are numbers

$$0 \leqslant m_1 < M_1 < m_2 < M_2 < \cdots < m_N < M_N < 2\pi$$

such that $m_k < \theta_k(r) < M_k$ for all k. Let a weight $w : \Gamma \to (0, +\infty)$ be given by

$$w(\tau) = \mathrm{e}^{v(|\tau - t|)}, \quad \tau \in \Gamma.$$

We assume that

$$r\theta'(r), r\theta_1'(r), \ldots, r\theta_N'(r), rv'(r)$$

are C^∞ on $(0, \infty)$, constant on $[1, \infty)$, and slowly oscillating at the origin and that (4.10) is satisfied. This guarantees that S_Γ is bounded on $L^2(\Gamma, w)$. Note that the boundedness of the functions θ_k together with the requirement that $r\theta_k'(r)$ be slowly oscillating at the origin implies that the θ_k themselves are slowly oscillating at the origin. It follows in particular that actually $r\theta_k'(r) \to 0$ as $r \to 0$.

Define $\nu_\delta : \mathbb{C} \setminus \mathrm{i}\mathbb{Z} \to \mathbb{C}$ by

$$\nu_0(z) := \coth(\pi z), \quad \nu_\delta(z) := \frac{\mathrm{e}^{(\pi-\delta)z}}{\sinh(\pi z)} \quad \text{for } \delta \in (0, 2\pi).$$

Put $\varepsilon_k := 1$ if t is the starting point of Γ_k and $\varepsilon_k := -1$ in case t is the terminating point of Γ_k. Given

$$\gamma \in (-1/2, 1/2), \quad \delta \in \mathbb{R}, \quad \Delta = (\Delta_1, \ldots, \Delta_N) \in [0, 2\pi)^N,$$

we define the function $\mu_{\delta,\gamma}$ by

$$\mu_{\delta,\gamma} : \mathbb{R} \cup \{\pm\infty\} \to \mathbb{C}, \quad \mu_{\delta,\gamma}(\lambda) := \frac{\lambda + \mathrm{i}(\gamma + 1/2)}{1 + \mathrm{i}\delta}$$

and the matrix-function $\sigma_{\delta,\Delta,\gamma}$ by

$$\sigma_{\delta,\Delta,\gamma} = \big(\sigma^{(j,k)}_{\delta,\Delta,\gamma}\big)^N_{j,k=1} : \mathbb{R} \cup \{\pm\infty\} \to \mathbb{C}^{N\times N},$$

$$\sigma^{(j,k)}_{\delta,\Delta,\gamma}(\lambda) := \begin{cases} \varepsilon_k\, \nu_{2\pi+\Delta_j-\Delta_k}\big(\mu_{\delta,\gamma}(\lambda)\big) & \text{if } j < k; \\ \varepsilon_k\, \nu_0\big(\mu_{\delta,\gamma}(\lambda)\big) & \text{if } j = k; \\ \varepsilon_k\, \nu_{\Delta_j-\Delta_k}\big(\mu_{\delta,\gamma}(\lambda)\big) & \text{if } j > k. \end{cases}$$

We write $a_\Gamma \in \mathrm{SO}(\Gamma)$ if $a_\Gamma|\Gamma_k \in \mathrm{SO}(\Gamma_k)$ for every k, i.e., if

$$a_\Gamma\big(t + r e^{\mathrm{i}(\theta(r)+\theta_k(r))}\big) = a_k(r)$$

and $a_k \in \mathrm{SO}$. Let

$$a(r) := (a_1(r), \dots, a_N(r)), \quad \Theta(r) := (\theta_1(r), \dots, \theta_N(r)).$$

Consider $A = a_\Gamma I + b_\Gamma S_{\Gamma,w}$ where Γ and w are as above and a_Γ, b_Γ belong to $\mathrm{SO}(\Gamma)$. Denote by $\mathcal{P}_A$ the set of all partial limits of the map

$$\mathbb{R}_+ \to \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{R} \times [0, 2\pi)^N \times (-1/2, 1/2),$$

$$r \mapsto \big(a(r), b(r), r\theta'(r), \Theta(r), rv'(r)\big).$$

Then

$$\mathrm{sp}_t A = \bigcup_{\lambda\in\mathbb{R}\cup\{\pm\infty\}} \; \bigcup_{(\alpha,\beta,\delta,\Delta,\gamma)\in\mathcal{P}_A} \mathrm{sp}\big(\mathrm{diag}(\alpha) + \mathrm{diag}(\beta)\, \sigma_{\delta,\Delta,\gamma}(\lambda)\big),$$

where, for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$, we denote by $\mathrm{diag}(\alpha)$ the diagonal matrix whose entries are $\alpha_1, \dots, \alpha_N$ and where $\mathrm{diag}(\beta)$ is defined analogously. Note that the spectra on the right are the eigenvalues of $N \times N$ matrices.

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