J. OPERATOR THEORY
43(2000), 295–327

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POINTWISE UNITARY COACTIONS ON C^*-ALGEBRAS WITH CONTINUOUS TRACE

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Communicated by Norberto Salinas

Abstract. Let G be a second countable locally compact group, A a separable continuous trace C^*-algebra and δ a pointwise unitary coaction of G on A. It is shown that the crossed product $A \times_\delta G$ of (A, G, δ) has continuous trace and that the restriction map $\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$ is a proper G-bundle via the dual action of G on $(A \times_\delta G)^\wedge$. Further, $A \times_\delta G$ is isomorphic to the pull-back $\mathrm{Res}^* A$.

We obtain a characterization of continuous trace crossed products $A \times_{\alpha,\mathrm{r}} G$ by an action α of G on A: when α acts freely on $\widehat{A}$, the crossed product has continuous trace if and only if the action of G on $\widehat{A}$ is proper.

Keywords: *Continuous trace C^*-algebra, coaction, crossed product, pointwise unitary.*

MSC (2000): 46L55.

INTRODUCTION

Let G be a second countable locally compact group, A a separable continuous trace C^*-algebra and δ a coaction of G on A. The coaction δ is called pointwise unitary if, for every $\pi \in \widehat{A}$, there is a non-degenerate representation μ of $C_0(G)$ such that (π, μ) is a covariant representation. If G is abelian, then there is a natural one-to-one correspondence between the coactions of G and the strongly continuous actions of the dual group $\widehat{G}$, and the pointwise unitary coactions correspond to the pointwise unitary actions of $\widehat{G}$ (see [12]).

In [15], Olesen and Raeburn obtained a couple of quite remarkable results on pointwise unitary actions $\alpha : G \to \mathrm{Aut}(A)$ of an abelian group G on a continuous trace algebra A: They showed that the crossed product $A \times_\alpha G$ has Hausdorff spectrum and $(A \times_\alpha G)^\wedge$ is a proper $\widehat{G}$-bundle over $\widehat{A}$ with respect to the dual action $\widehat{\alpha}$. Moreover, they were also able to identify $A \times_\alpha G$ with the pull back $\mathrm{Res}^* A$ of A, where $\mathrm{Res} : (A \times_\alpha G)^\wedge \to \widehat{A}$ denotes the restriction map.

The main purpose of this paper is to provide complete non-abelian analogous of these results for pointwise unitary coactions: We show that the crossed product $A \times_\delta G$ has continuous trace such that $(A \times_\delta G)^\wedge$ is a proper G-bundle over $\widehat{A}$ with respect to the dual action $\widehat{\delta}$ of G on $A \times_\delta G$ (Theorem 3.6), and $A \times_\delta G$ is isomorphic to the pull-back $\mathrm{Res}^* A$ (Theorem 3.7).

As an application of their results, Olesen and Raeburn obtained a characterization of certain crossed products with continuous trace: If $\alpha : G \to \mathrm{Aut}(A)$ is an action of the abelian group G on the continuous trace algebra A such that the corresponding action of G on $\widehat{A}$ is free, then $A \times_\alpha G$ has continuous trace if and only if the action of G on $\widehat{A}$ is proper. The "if" direction has been known to be true for actions of arbitrary groups ([25]), but so far the converse direction has been an open problem. Similar to the ideas used by Olesen and Raeburn in the abelian case we use our results to close this gap (Theorem 4.7), where the assumption that G is abelian may be omitted in the above statement.

We would like to stress that the methods for the proofs of our key results, namely that $A \times_\delta G$ has continuous trace and $(A \times_\delta G)^\wedge$ is a proper G-bundle over $\widehat{A}$, differ quite substantially from the methods used in [15] for the abelian case. In order to explain these differences, recall first that an action $\alpha : G \to \mathrm{Aut}(A)$ is called *unitary* if there exists a strictly continuous homomorphism $u : G \to UM(A)$ such that $\alpha_s = \mathrm{Ad}\, u_s$ for all $s \in G$. Unitary actions are precisely those actions which are exterior equivalent to trivial actions. Further, α is called *locally unitary* if each $\pi \in \widehat{A}$ has an open neighborhood U such that α restricts to a unitary action on the corresponding ideal A_U of A. Similar definitions can be made for coactions (see Definition 2.3). Locally unitary actions (respectively coactions) are always pointwise unitary. If A has continuous trace, then a result of Rosenberg shows that every pointwise unitary action of a compactly generated abelian group on A is automatically locally unitary ([28]). Since every locally compact group has an open compactly generated subgroup, Olesen and Raeburn were able to divide the problem into a compactly generated step to which they were able to apply the existing results for locally unitary actions (see [23]), and a discrete step exploiting the fact that the dual group of a discrete group is compact.

Although similar results have been obtained for locally unitary coactions by Landstad et al. in [12], there did not exist an analogue of Rosenberg's result so far (in fact, we derive such an analogue as another application of our main results in Theorem 4.4 below). This problem made it necessary to look for an alternative approach. Such an approach was provided by Echterhoff in [3] when he investigated pointwise unitary actions of subgroup bundles on continuous trace C^*-algebras. He showed that such an action is "unitary on closures of subsequences of convergent sequences"; a localization property which turned out to be sufficient for proving analogues of the results of Olesen and Raeburn for coactions. Thus, we use some of Echterhoff's ideas to show that every pointwise unitary coaction has a similar property.

A further result concerns the exterior equivalence of coactions. Again, Olesen and Raeburn showed in [15] that, if G is abelian and A has continuous trace, then two pointwise unitary actions α and β of G on A are exterior equivalent if and only if the $\widehat{G}$-bundles $(A \times_\alpha G)^\wedge \to \widehat{A}$ and $(A \times_\beta G)^\wedge \to \widehat{A}$ are isomorphic. This uses the fact that the crossed product of a pointwise unitary action is isomorphic to the pull back $\mathrm{Res}^* A$ (see above). A similar result was obtained in [12] for locally unitary coactions. In the case of pointwise unitary coactions, we have that the exterior equivalence of two coactions implies that the corresponding G-bundles are isomorphic (this follows from [21], Proposition 2.8). The converse is true at least when G is a Lie group (Corollary 4.5). This is an immediate consequence of Theorem 4.4 and [12]. We believe that the converse is also true for arbitrary G.

This paper is organized as follows. In the first section, we give the basic definitions. In Section 2, we introduce invariant ideals. Further, we recall the definitions of unitary, locally unitary and pointwise unitary coactions, and we show some basic properties of such coactions. In the third section we prove Proposition 3.4 which says that pointwise unitary coactions on C^*-algebras with continuous trace are "unitary on closures of subsequences of convergent sequences" (see the proposition for the correct meaning of this). We use this result in the proofs of Theorem 3.6 and Theorem 3.7. The fourth section gives applications of Theorem 3.6 and Theorem 3.7 as stated above. We prove our results for reduced coactions. However, in the appendix we show by using results of Quigg ([19]) that all our results also hold for full pointwise unitary coactions.

1. PRELIMINARIES

In this paper, G is always a locally compact group, $\lambda_G : G \to \mathcal{L}(L^2(G))$ denotes the left regular representation of G, and $C^*_{\mathrm{r}}(G)$ is the reduced group C^*-algebra of G which is the norm closure of $\lambda_G(L^1(G))$ in $\mathcal{L}(L^2(G))$. Further, $C_0(G)$ is the C^*-algebra of continuous functions on G vanishing at infinity. Since λ_G is a bounded strictly continuous $M(C^*_{\mathrm{r}}(G))$-valued function, we may regard it as a (unitary) element of $M(C_0(G) \otimes C^*_{\mathrm{r}}(G))$. We write W_G for λ_G whenever we have this interpretation in mind.

Let V,W be two subspaces in a common C^*-algebra. Then we define $VW := \overline{\mathrm{sp}}\{vw : v \in V, w \in W\}$. Now a $*$-homomorphism $\phi : A \to M(B)$ (A and B are C^*-algebras) is called *non-degenerate* if $\phi(A)B = B$. In this case, ϕ extends uniquely to a strictly continuous homomorphism on the multiplier algebra $M(A)$ (see [12], Lemma 1.1) which is also denoted by ϕ. Note that a representation π of a C^*-algebra A is non-degenerate (in the usual sense) if and only if π is non-degenerate as a homomorphism into $M(\mathcal{K}(\mathcal{H}_\pi))$.

For $s \in G$ and $f \in C_0(G)$, $\tau_s(f)$ and $\sigma_s(f)$ are the left and right translations of f, respectively. That is, $\tau_s(f)(t) = f(s^{-1}t)$ and $\sigma_s(f)(t) = f(ts)$ for all $t \in G$. We have $\tau_s \otimes \mathrm{id}(W_G) = (1 \otimes \lambda_G(s^{-1})) \cdot W_G$ and $\sigma_s \otimes \mathrm{id}(W_G) = W_G \cdot (1 \otimes \lambda_G(s))$.

The integrated form of the group homomorphism

$$s \mapsto \lambda_G(s) \otimes \lambda_G(s), \quad G \to M(C^*_{\mathrm{r}}(G) \otimes C^*_{\mathrm{r}}(G))$$

induces a non-degenerate homomorphism $\delta_G : C^*_{\mathrm{r}}(G) \to M(C^*_{\mathrm{r}}(G) \otimes C^*_{\mathrm{r}}(G))$ (cf. [12], Chapter 2). This endows $C^*_{\mathrm{r}}(G)$ with a comultiplication, that is $(\delta_G \otimes \mathrm{id}) \circ \delta_G = (\mathrm{id} \otimes \delta_G) \circ \delta_G$. Here and in the sequel, all tensor products are the minimal ones.

Recall from [12] that a *reduced coaction* of G on a C^*-algebra A is a non-degenerate injective $*$-homomorphism $\delta : A \to M(A \otimes C^*_{\mathrm{r}}(G))$ such that

$$\delta(A)(1 \otimes C^*_{\mathrm{r}}(G)) \subset A \otimes C^*_{\mathrm{r}}(G), \tag{1.1}$$

and

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta \text{ as maps of } A \text{ into } M(A \otimes C^*_{\mathrm{r}}(G) \otimes C^*_{\mathrm{r}}(G)). \tag{1.2}$$

We call (A, G, δ) a *cosystem*, and condition (1.2) is called the *coaction identity*. If equality holds in (1.1) (that is $\overline{\mathrm{sp}}\{\delta(a)(1 \otimes z) : a \in A,\ z \in C^*_{\mathrm{r}}(G)\} = A \otimes C^*_{\mathrm{r}}(G)$), then we say that δ is a *non-degenerate* coaction.

Before we introduce covariant representations and the crossed product of a cosystem (A, G, δ) let us recall the notion of slice maps. Let A and B be C^*-algebras, and let $f \in B^*$ (B^* is the dual space of B). The linear map

$$S_f : A \odot B \to A, \quad \sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i f(b_i)$$

extends to a bounded linear map of $M(A \otimes B)$ into $M(A)$ with $\|S_f\| = \|f\|$ (see [12], before Lemma 1.5). These maps are called *slice maps*. We have the following

Lemma 1.1. *Let $f \in B^*$, $m \in M(A \otimes B)$ and $a \in A$. Then*
(i) *$S_f : M(A \otimes B) \to M(A)$ is strictly continuous;*
(ii) *if $m \in M(A \otimes B)$ is such that $m(1 \otimes b), (1 \otimes b)m \in A \otimes B$ for all $b \in B$, then $S_f(m) \in A$;*
(iii) *we can factor $f = g_1 \cdot b_1 = b_2 \cdot g_2$ with $b_i \in B$, $g_i \in B^*$, and we have*

$$S_f(m) \cdot a = S_{g_1}(m(a \otimes b_1)) \quad \text{and} \quad a \cdot S_f(m) = S_{g_2}((a \otimes b_2)m)$$

for all $a \in A$ and $m \in M(A \otimes B)$. Here $g_1 \cdot b_1$ means $g_1 \cdot b_1(a) = g_1(ab_1)$ and $b_2 \cdot g_2(a) = g_2(b_2 a)$ for $a \in A$.

Proof. See [12], Lemma 1.5. ∎

Now consider the special case where $B = C_{\mathrm{r}}^*(G)$. Let $B_{\mathrm{r}}(G)$ be the algebra of continuous coefficient functions for the representations of $C_{\mathrm{r}}^*(G)$. Then $B_{\mathrm{r}}(G)$ may be identified with $C_{\mathrm{r}}^*(G)^*$. Further, the Fourier algebra $A(G) \subset B_{\mathrm{r}}(G)$ is dense in $C_0(G)$, and $A(G)$ is the predual of the von Neumann algebra $C_{\mathrm{r}}^*(G)''$ (the bicommutant of $C_{\mathrm{r}}^*(G)$ in $\mathcal{L}(L^2(G))$, see [4]). Let $\Sigma : C_{\mathrm{r}}^*(G) \otimes C_{\mathrm{r}}^*(G) \to C_{\mathrm{r}}^*(G) \otimes C_{\mathrm{r}}^*(G)$ be the flip map. For a unitary $W \in M(A \otimes C_{\mathrm{r}}^*(G))$, let $W_{12} = W \otimes 1$ and $W_{13} = (\mathrm{id} \otimes \Sigma)(W \otimes 1)$.

Lemma 1.2. *There is a bijection between the set of unitary elements W of $M(A \otimes C_{\mathrm{r}}^*(G))$ which satisfy the corepresentation identity*

$$W_{12} W_{13} = (\mathrm{id} \otimes \delta_G)(W) \tag{1.3}$$

and the set of non-degenerate homomorphisms $\phi : C_0(G) \to M(A)$. This bijection is determined by

$$W = (\phi \otimes \mathrm{id})(W_G),$$

$$\phi(f) = S_f(W) \quad \text{for all } f \in A(G).$$

Especially, $S_f(W_G) = f$ for all $f \in A(G)$.

Proof. See [21], Lemma 1.2. ∎

Let (A, G, δ) be a cosystem, and let B be a C^*-algebra. A pair (π, μ) of non-degenerate homomorphisms $\pi : A \to M(B)$ and $\mu : C_0(G) \to M(B)$ such that

$$(\pi \otimes \mathrm{id})(\delta(a)) = (\mu \otimes \mathrm{id})(W_G)(\pi(a) \otimes 1)(\mu \otimes \mathrm{id})(W_G^*),$$

as elements of $M(B \otimes C^*_{\mathrm{r}}(G))$, is called a *covariant representation of* (A, G, δ) *in* $M(B)$. By Lemma 1.2, our definition of a covariant representation coincides with the definition given in [12], Chapter 3. Let $C^*(\pi, \mu) := \mathrm{sp}\,\{\pi(a)\mu(f) : a \in A, f \in C_0(G)\}$. It follows from [24], Lemma 2.10, that $C^*(\pi, \mu)$ is a C^*-algebra.

Let (π, μ) be a covariant representation of a cosystem (A, G, δ). We say that $(C^*(\pi, \mu), \pi, \mu)$ is a *crossed product* of (A, G, δ) if every representation (ρ, ν) of (A, G, δ) factors through (π, μ), i.e., if there is a non-degenerate homomorphism $\theta : C^*(\pi, \mu) \to C^*(\rho, \nu)$ such that $\theta \circ \pi = \rho$ and $\theta \circ \mu = \nu$. Let M be the representation of $C_0(G)$ as multiplication operators on $L^2(G)$. It follows from [12], Theorem 3.7, that $(C^*(\delta, 1 \otimes M), \delta, 1 \otimes M)$ is a crossed product for (A, G, δ). So there always exists a crossed product. Further, the crossed product is unique in the following sense: If $(C^*(\pi, \mu), \pi, \mu)$ and $(C^*(\rho, \nu), \rho, \nu)$ are crossed products for (A, G, δ), then there is an isomorphism $\theta : C^*(\pi, \mu) \to C^*(\rho, \nu)$ such that $\theta \circ \pi = \rho$ and $\theta \circ \mu = \nu$. We shall therefore refer to *the* crossed product and denote it by $(A \times_\delta G, j_A, j_{C_0(G)})$ or shortly $A \times_\delta G$. By [12], Chapter 2, we may regard $A \times_\delta G$ as a subalgebra of $M(A \otimes \mathcal{K}(L^2(G)))$. For any covariant representation (π, μ) of (A, G, δ) we let $\pi \times \mu$ denote the unique non-degenerate homomorphism of $A \times_\delta G$ such that $(\pi \times \mu) \circ j_A = \pi$ and $(\pi \times \mu) \circ j_{C_0(G)} = \mu$. Note that, since δ and $1 \otimes M$ are injective and $(\delta, 1 \otimes M)$ is a covariant representation of (A, G, δ), it follows that j_A and $j_{C_0(G)}$ are injective.

For every cosystem (A, G, δ), we can define an action of G on its crossed product in the following way: Let $(A \times_\delta G, j_A, j_{C_0(G)})$ be the crossed product of (A, G, δ). Then, by [12], p. 768, there exists an action $\widehat{\delta} : G \to \mathrm{Aut}(A \times_\delta G)$ such that

$$\widehat{\delta}_s(j_A(a) j_{C_0(G)}(f)) = j_A(a) j_{C_0(G)}(\sigma_s(f))$$

for all $a \in A$, $f \in C_0(G)$ and $s \in G$, where σ_s denotes the right translation by $s \in G$. The action $\widehat{\delta}$ is called the *dual action* of δ.

The dual action $\widehat{\delta}$ induces an action of G on $(A \times_\delta G)^\wedge$ by $s \cdot (\pi \times \mu) := (\pi \times \mu) \circ \widehat{\delta}_{s^{-1}}$. On the generators, we have

$$\begin{aligned}(\pi \times \mu)(\widehat{\delta}_{s^{-1}}(j_A(a) j_{C_0(G)}(f))) &= (\pi \times \mu)(j_A(a) j_{C_0(G)}(\sigma_{s^{-1}}(f))) \\ &= (\pi \times (\mu \circ \sigma_{s^{-1}}))(j_A(a) j_{C_0(G)}(f)).\end{aligned}$$

Thus, $s \cdot (\pi \times \mu) = \pi \times (\mu \circ \sigma_{s^{-1}})$.

2. INVARIANT IDEALS AND POINTWISE UNITARY COACTIONS

In this section, we start with the definition of invariant ideals of a cosystem (A, G, δ). We do this similarly to the definition given for full coactions in [14]. We then define unitary, locally unitary and pointwise unitary coactions. Invariant ideals for reduced coactions do not behave as well as invariant ideals for full coactions (see Remark 2.2 (iii)). However, in the third and fourth chapters, we only deal with pointwise unitary coactions with A having continuous trace, and in this case everything works well (see Remark 2.4 (ii)).

Definition 2.1. Let (A, G, δ) be a non-degenerate cosystem. A closed ideal I of A is called δ-*invariant* if

$$\delta(I)(1 \otimes C^*_{\mathrm{r}}(G)) = I \otimes C^*_{\mathrm{r}}(G). \tag{2.1}$$

Remark 2.2. (i) Let (A, G, δ) be a non-degenerate cosystem, and I be a δ-invariant ideal of A. Let $\theta_I : A \to M(I)$ be the map defined by $\theta_I(a)b = ab$ and $b\theta_I(a) = ba$ for $a \in A$ and $b \in I$. We have $\delta(I)(A \otimes C^*_{\mathrm{r}}(G)) \subset I \otimes C^*_{\mathrm{r}}(G)$. By taking adjoints we see that also $(A \otimes C^*_{\mathrm{r}}(G))\delta(I) \subset I \otimes C^*_{\mathrm{r}}(G)$. Thus, it follows from [12], Lemma 1.4, that $(\theta_I \otimes \mathrm{id}) : M(A \otimes C^*_{\mathrm{r}}(G)) \to M(I \otimes C^*_{\mathrm{r}}(G))$ restricts to an isomorphism of $\delta(I)$ into $M(I \otimes C^*_{\mathrm{r}}(G))$. In particular, the map

$$((\theta_I \otimes \mathrm{id}) \circ \delta)|I : I \to M(I \otimes C^*_{\mathrm{r}}(G))$$

is injective. In fact, $((\theta_I \otimes \mathrm{id}) \circ \delta)|I$ is a coaction on I. To see this, observe that (as in the proof of [14], Proposition 2.1) (2.1) implies that $((\theta_I \otimes \mathrm{id}) \circ \delta)|I$ is non-degenerate as a homomorphism into $M(I \otimes C^*_{\mathrm{r}}(G))$. Then it follows from [12], Remark 4.2 (2), that $((\theta_I \otimes \mathrm{id}) \circ \delta)|I$ is a non-degenerate coaction on I. In the sequel, we shall write δ_I for $((\theta_I \otimes \mathrm{id}) \circ \delta)|I$.

(ii) The above remark says that an ideal I of A which is δ-invariant in the sense of Definition 2.1 is also δ-invariant in the sense of [12], Chapter 4. The converse is also true: Assume that I is δ-invariant in the sense of [12], Chapter 4. Then $\delta(I)(1 \otimes C^*_{\mathrm{r}}(G)) \subset I \otimes C^*_{\mathrm{r}}(G)$, and δ_I is a coaction on I. Since δ is a non-degenerate coaction, it follows from [12], Remark 4.2 (1), that δ_I is also a non-degenerate coaction. Thus,

$$(\theta_I \otimes \mathrm{id})(\delta(I)(1 \otimes C^*_{\mathrm{r}}(G))) = (\delta_I(I)(1 \otimes C^*_{\mathrm{r}}(G)) = I \otimes C^*_{\mathrm{r}}(G).$$

Since $\delta(I)(1 \otimes C^*_{\mathrm{r}}(G)) \subset I \otimes C^*_{\mathrm{r}}(G)$ and since $\theta_I \otimes \mathrm{id}$ leaves $I \otimes C^*_{\mathrm{r}}(G) \subset A \otimes C^*_{\mathrm{r}}(G)$ fixed, we see that I is δ-invariant.

(iii) Let I be a δ-invariant ideal and $q : A \to A/I$ the quotient map. Define

$$\delta^I : A/I \to M(A/I \otimes C^*_{\mathrm{r}}(G)), \quad q(a) \mapsto (q \otimes \mathrm{id})(\delta(a)).$$

By [12], Lemma 4.6, δ^I is a non-degenerate homomorphism which satisfies (1.1) and the coaction identity (1.2). So δ^I satisfies all the requirements for a coaction, except possibly injectivity. If G is amenable, then δ^I is injective, and it is a non-degenerate coaction on A/I. This uses the fact that $1 \in B_{\mathrm{r}}(G)$ for amenable G. For a non-degenerate full cosystem (A, G, δ), May Nilsen showed that if I is δ-invariant in the sense of [14], then δ^I is a non-degenerate coaction of G on A/I for arbitrary G (see [14], Proposition 2.2).

(iv) Let $I \subset J$ be δ-invariant ideals of A such that δ^I and δ^J are coactions on A/I and A/J, respectively. Then $J/I \subset A/I$ is δ^I-invariant, and $(\delta^I)^{J/I} = \delta^J$. To see this, let $q_I : A \to A/I$, $q_J : A \to A/J$ and $q_{J/I} : A/I \to A/J$ be the quotient maps. Let $a \in J$ and $z \in C^*_{\mathrm{r}}(G)$. Then

$$\begin{aligned} q_I(a) \otimes z = q_I \otimes \mathrm{id}(a \otimes z) &\approx q_I \otimes \mathrm{id}\Big(\sum \delta(a_i)(1 \otimes z_i)\Big) \\ &= \sum \delta^I(q_I(a_i))(1 \otimes z_i) \in \delta^I(J/I)(1 \otimes C^*_{\mathrm{r}}(G)), \end{aligned}$$

$a_i \in J$, $z_i \in C^*_{\mathrm{r}}(G)$, and

$$\begin{aligned} \delta^I(q_I(a))(1 \otimes z) &= q_I \otimes \mathrm{id}(\delta(a)(1 \otimes z)) \\ &\approx q_I \otimes \mathrm{id}\Big(\sum b_i \otimes w_i\Big), \\ &= \sum q_I(b_i) \otimes w_i \in (J/I) \otimes C^*_{\mathrm{r}}(G) \end{aligned}$$

($b_i \in J$, $w_i \in C^*_{\mathrm{r}}(G)$). It follows that $\delta^I(J/I)(1 \otimes C^*_{\mathrm{r}}(G)) = (J/I) \otimes C^*_{\mathrm{r}}(G)$, and J/I is δ^I-invariant. For the second statement, let $a \in J$. Then

$$\begin{aligned} (\delta^I)^{J/I}(q_{J/I}(q_I(a))) &= (q_{J/I} \otimes \mathrm{id})(\delta^I(q_I(a))) = ((q_{J/I} \circ q_I) \otimes \mathrm{id})(\delta(a)) \\ &= (q_J \otimes \mathrm{id})(\delta(a)) = \delta^J(q_J(a)). \end{aligned}$$

Thus, $(\delta^I)^{J/I} = \delta^J$.

Let A be a C^*-algebra, and let $\phi : C_0(G) \to M(A)$ be a non-degenerate homomorphism. The arguments used in the proof of [19], Lemma 1.13 (which is stated for full coactions) show that the map $\delta : A \to M(A \otimes C^*_{\mathrm{r}}(G))$ given by

$$\delta(a) := (\phi \otimes \mathrm{id})(W_G)(a \otimes 1)(\phi \otimes \mathrm{id})(W_G^*)$$

for $a \in A$ is a non-degenerate coaction of G on A. Such a coaction is called a *unitary* coaction, and we say that ϕ *implements* δ.

If (A, G, δ) is an arbitrary cosystem, then any covariant representation (π, μ) in $M(\pi(A))$ induces a unitary coaction δ^π on $\pi(A)$ defined by

$$\delta^\pi(\pi(a)) := (\pi \otimes \mathrm{id})(\delta(a))$$

for all $a \in A$. The coaction δ^π is implemented by μ.

DEFINITION 2.3. Let A be a liminal C^*-algebra with Hausdorff spectrum, and let δ be a coaction of G on A. Then

(i) δ is called *locally unitary* if each $\pi \in \widehat{A}$ has an open neighborhood N of π such that the corresponding ideal I with $\widehat{I} = N$ is δ-invariant and such that δ_I is a unitary coaction on I;

(ii) δ is called *pointwise unitary* if, for every $\pi \in \widehat{A}$, there is a non-degenerate representation μ of $C_0(G)$ such that (π, μ) is a covariant representation of (A, G, δ).

REMARK 2.4. Let A be as in the preceding definition.

(i) Every unitary coaction on A is locally unitary. Also, every locally unitary coaction on A is pointwise unitary. Indeed, let $\pi \in \widehat{A}$ and I an ideal of A such that $\pi \in \widehat{I}$ and δ_I is a unitary coaction on I which is implemented by $\phi : C_0(G) \to M(I)$. Then $(\pi, \pi \circ \phi)$ is a covariant representation of (A, G, δ).

(ii) If (A, G, δ) is a pointwise unitary cosystem, then, by [12], Theorem 5.3 (2), every closed ideal I of A is δ-invariant, and δ^I is a non-degenerate coaction.

(iii) Landstad et al. gave in [12] a definition of locally unitary coactions which worked with closed neighbourhoods, so that the coactions on the corresponding quotients are unitary. We find it more suitable to work with open neighbourhoods. Our definition of locally unitary coactions is equivalent to the definition given in [12]. To see this we realize A as the section algebra $\Gamma_0(E)$ of a C^*-bundle E over $\widehat{A}$. Suppose that δ is locally unitary in our sense. Let $\pi \in \widehat{A}$ and $N \subset \widehat{A}$ be an open neighbourhood of π such that δ restricts to a unitary coaction on $I := \Gamma_0(E|N)$. Let $K \subset N$ be a compact neighbourhood of π and $J = \bigcap\{\ker \rho : \rho \in K\}$. Since $\widehat{A}$ is Hausdorff, K is closed. Thus, $A/J = \Gamma_0(E|K)$ and we obtain a homomorphism from $C_0(G)$ into A/J which implements the coaction δ^J on A/J. The converse works in a similar way.

In [12], Landstad et al. required in their definition of pointwise unitary coactions that the coaction has to be non-degenerate. The next proposition shows that this is not necessary. For a cosystem (A, G, δ) and $g \in A(G)$, we define $\delta_g : A \to A$ by $\delta_g(a) := S_g(\delta(a))$ (that δ_g maps A into A follows from (1.1) and Lemma 1.1). The closure of the set $\delta_{A(G)}(A) := \{\delta_g(a) : a \in A,\, g \in A(G)\}$ is a C^*-subalgebra of A ([10], Lemma 2). By [10], Theorem 5, a cosystem (A, G, δ) is non-degenerate if and only if $\overline{\delta_{A(G)}(A)} = A$.

PROPOSITION 2.5. *Let A be a liminal C^*-algebra with Hausdorff spectrum. Then every pointwise unitary (hence also every locally unitary) coaction is automatically non-degenerate.*

Proof. Since δ is pointwise unitary, there exists, for every $\pi \in \widehat{A}$, a non-degenerate representation μ of $C_0(G)$ such that (π, μ) is a covariant representation.

Then μ implements a unitary coaction δ^π on $\pi(A)$. Since unitary coactions are non-degenerate, this implies that

$$\pi(\overline{\delta_{A(G)}(A)}) = \overline{\delta^\pi_{A(G)}(\pi(A))} = \pi(A) \tag{2.2}$$

for all $\pi \in \widehat{A}$. Let $f \in C_0(\widehat{A})$. By [12], Theorem 5.3 (1), $\delta(f) = f \otimes 1$, where we identify f with its image under the Dauns-Hofmann isomorphism. Thus, $f \cdot \delta_g(a) = \delta_g(fa)$ for all $f \in C_0(\widehat{A})$, $g \in A(G)$ and $a \in A$. That is, $\overline{\delta_{A(G)}(A)}$ is closed under multiplication by elements of $C_0(\widehat{A})$. Now it follows from (2.2) and [2], Lemma 10.5.3 that $\overline{\delta_{A(G)}(A)} = A$, and δ is non-degenerate. ∎

For the next proposition, recall that σ denotes the right translation on $C_0(G)$.

Proposition 2.6. *Let δ be a unitary coaction of G on A implemented by the homomorphism $\phi : C_0(G) \to M(A)$. Let $W := (\phi \otimes \mathrm{id})(W_G)$. Then, if we regard $A \times_\delta G$, $A \otimes C_0(G)$ and $A \otimes C^*_{\mathrm{r}}(G)$ as subalgebras of $M(A \otimes \mathcal{K}(L^2(G)))$, the map $\operatorname{Ad} W^* : A \times_\delta G \to A \otimes C_0(G)$ is an isomorphism. Especially, $A \times_\delta G$ has continuous trace if and only if A has continuous trace. Furthermore, $\operatorname{Ad} W^*$ induces a G-homeomorphism between $\widehat{A} \times G = (A \otimes C_0(G))^\wedge$ and $(A \times_\delta G)^\wedge$, which is given by $(\pi, s) \mapsto \pi \times (\pi \circ \phi \circ \sigma_{s^{-1}})$.*

Proof. That $\operatorname{Ad} W^* : A \times_\delta G \to A \otimes C_0(G)$ is an isomorphism follows from [12], Corollary 2.10. By [30], Theorem 2 (a), this implies that $A \times_\delta G$ has continuous trace if and only if A has continuous trace.

It follows from the first part of the proof of [12], Theorem 5.9, that $\operatorname{Ad} W^*$ induces the G-homeomorphism

$$\widehat{A} \times G \to (A \times_\delta G)^\wedge, \quad (\pi, s) \mapsto (\pi \times (\pi \circ \phi)) \circ \widehat{\delta}_{s^{-1}}.$$

Now $(\pi \times (\pi \circ \phi)) \circ \widehat{\delta}_{s^{-1}} = \pi \times (\pi \circ \phi \circ \sigma_{s^{-1}})$, and the result follows. ∎

Except for the openness the next proposition was shown in [12], Theorem 5.5. Our proof works with C^*-bundles and uses a result of Nilsen ([14], Theorem 4.3).

Proposition 2.7. *Let A be a liminal C^*-algebra with Hausdorff spectrum, and let δ be a pointwise unitary coaction on A. Then the map*

$$\operatorname{Res} : (A \times_\delta G)^\wedge \to \widehat{A}, \quad \pi \times \mu \mapsto \pi$$

is well defined, and it is a continuous, open surjection.

Proof. By Lemma A.5, we may identify $A \times_\delta G$ with $\Gamma_0(E')$, where E' is a C^*-bundle over $\widehat{A}$ with fibers $A_\rho \times_{\delta_\rho} G$ such that A_ρ is an elementary algebra and δ_ρ is a unitary coaction of G on A_ρ.

Let $\pi\times\mu$ be an irreducible representation of $A\times_\delta G$. By [2], Theorem 10.4.3, there is a $\rho\in\widehat{A}$ such that $\pi\times\mu\in(A_\rho\times_{\delta_\rho}G)^\wedge$. Let $\phi: C_0(G)\to M(A_\rho)$ be a nondegenerate homomorphism which implements δ_ρ. By Proposition 2.6 and since A_ρ is an elementary algebra, there is an $s\in G$ such that $\pi\times\mu\cong\rho\times(\rho\circ\phi\circ\sigma_s)$. Thus, $\pi\cong\rho$ and π is irreducible. It then follows from Lee's Theorem ([13], Theorem 4) that

$$\mathrm{Res}:(A\times_\delta G)^\wedge\to\widehat{A},\quad \pi\times\mu\mapsto\pi$$

is open and continuous. ∎

Suppose that (A,G,δ) is a cosystem. Let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on a Hilbert space $\mathcal{H}$, and let $\Sigma: C^*_{\mathrm{r}}(G)\otimes\mathcal{K}(\mathcal{H})\to\mathcal{K}(\mathcal{H})\otimes C^*_{\mathrm{r}}(G)$ denote the flip map. Then $\delta^{\mathrm{s}}=(\mathrm{id}_A\otimes\Sigma)\circ(\delta\otimes\mathrm{id}_{\mathcal{K}})$ is a coaction of G on $A\otimes\mathcal{K}(\mathcal{H})$. We call the cosystem $(A\otimes\mathcal{K}(\mathcal{H}),G,\delta^{\mathrm{s}})$ the *stabilization* of (A,G,δ).

LEMMA 2.8. *Let (A,G,δ) be a cosystem, and let $(A\otimes\mathcal{K}(\mathcal{H}),G,\delta^{\mathrm{s}})$ be its stabilization. Then:*

(i) *A closed ideal I of A is δ-invariant if and only if $I\otimes\mathcal{K}(\mathcal{H})$ is δ^{s}-invariant. Furthermore, δ^I is a coaction if and only if $(\delta^{\mathrm{s}})^{I\otimes\mathcal{K}}$ is coaction. In this case we have $(\delta^I)^{\mathrm{s}}=(\delta^{\mathrm{s}})^{I\otimes\mathcal{K}}$.*

(ii) *δ is a unitary coaction if and only if δ^{s} is a unitary coaction.*

(iii) *Suppose that A is a liminal C^*-algebra with Hausdorff spectrum. Then δ is pointwise (locally) unitary if and only if δ^{s} is pointwise (locally) unitary.*

Proof. (i) Let I be a δ-invariant ideal. Then

$$\begin{aligned}\delta^{\mathrm{s}}(I\otimes\mathcal{K}(\mathcal{H}))(1\otimes 1\otimes C^*_{\mathrm{r}}(G)) &= (\mathrm{id}\otimes\Sigma)((\delta(I)\otimes\mathcal{K}(\mathcal{H}))(1\otimes C^*_{\mathrm{r}}(G)\otimes 1))\\ &= (\mathrm{id}\otimes\Sigma)(I\otimes C^*_{\mathrm{r}}(G)\otimes\mathcal{K}(\mathcal{H}))\\ &= I\otimes\mathcal{K}(\mathcal{H})\otimes C^*_{\mathrm{r}}(G),\end{aligned}$$

and $I\otimes\mathcal{K}$ is δ^{s}-invariant. The same equation shows that the converse is also true. Now we show that $(\delta^{\mathrm{s}})^{I\otimes\mathcal{K}}=(\delta^I)^{\mathrm{s}}$ for all δ-invariant ideals of A. Let I be δ-invariant, and let $q: A\to A/I$ be the quotient map. Then

$$\begin{aligned}(\delta^{\mathrm{s}})^{I\otimes\mathcal{K}}((q\otimes\mathrm{id})(a\otimes k)) &= (q\otimes\mathrm{id}\otimes\mathrm{id})(\delta^{\mathrm{s}}(a\otimes k))\\ &= (q\otimes\mathrm{id}\otimes\mathrm{id})((\mathrm{id}\otimes\Sigma)(\delta(a)\otimes k))\\ &= (\mathrm{id}\otimes\Sigma)((q\otimes\mathrm{id})(\delta(a)\otimes k))\\ &= (\mathrm{id}\otimes\Sigma)(\delta^I(q(a))\otimes k)=(\delta^I)^{\mathrm{s}}(q(a)\otimes k).\end{aligned}$$

Thus, $(\delta^I)^{\mathrm{s}}=(\delta^{\mathrm{s}})^{I\otimes\mathcal{K}}$, and δ^I is a coaction if and only if $(\delta^{\mathrm{s}})^{I\otimes\mathcal{K}}$ is coaction.

(ii) Let δ be a unitary coaction implemented by $\phi : C_0(G) \to M(A)$. Then δ^{s} is implemented by $\phi \otimes 1 : C_0(G) \to M(A \otimes \mathcal{K}(\mathcal{H}))$. So this direction is clear. For the converse direction, suppose that A acts faithfully and non-degenerately on a Hilbert space $\mathcal{H}_1$, so that we can regard $A \otimes \mathcal{K}(\mathcal{H})$ and $M(A \otimes \mathcal{K}(\mathcal{H}))$ as subalgebras of $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H})$. Let $\phi : C_0(G) \to M(A \otimes \mathcal{K}(\mathcal{H}))$ be a homomorphism which implements δ^{s}, that is

$$\delta^{\mathrm{s}}(a \otimes k) = \phi \otimes \mathrm{id}(W_G)(a \otimes k \otimes 1)\phi \otimes \mathrm{id}(W_G^*). \tag{2.3}$$

Since δ is non-degenerate as a homomorphism, this implies that

$$1_{\mathcal{H}_1} \otimes k \otimes 1 = \phi \otimes \mathrm{id}(W_G)(1_{\mathcal{H}_1} \otimes k \otimes 1)\phi \otimes \mathrm{id}(W_G^*),$$

which is equivalent to

$$(1_{\mathcal{H}_1} \otimes k \otimes 1)\phi \otimes \mathrm{id}(W_G) = \phi \otimes \mathrm{id}(W_G)(1_{\mathcal{H}_1} \otimes k \otimes 1).$$

Slicing yields (see Lemma 1.2)

$$(1_{\mathcal{H}_1} \otimes k)\phi(f) = \phi(f)(1_{\mathcal{H}_1} \otimes k)$$

for all $f \in C_0(G)$ and $k \in \mathcal{K}(\mathcal{H})$. It follows that ϕ maps into $(\mathbb{C} \otimes \mathcal{L}(\mathcal{H}))'$, the commutant of the von Neumann tensor product $\mathbb{C} \otimes \mathcal{L}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H})$. But $(\mathbb{C} \otimes \mathcal{L}(\mathcal{H}))' = \mathcal{L}(\mathcal{H}_1) \otimes \mathbb{C}$ by [29], Proposition IV 1.9. Thus, ϕ maps into $M(A \otimes \mathcal{K}(\mathcal{H})) \cap \mathcal{L}(\mathcal{H}_1) \otimes \mathbb{C}$, that is, for every $f \in C_0(G)$, $\phi(f)$ has the form $b \otimes 1$ for some $b \in M(A)$. Thus, $\phi = \psi \otimes 1$, where $\psi : C_0(G) \to M(A)$ is a non-degenerate homomorphism. That ψ implements δ, follows from (2.3).

(iii) This follows from (i), (ii) and Remark 2.4 (iii). ∎

3. POINTWISE UNITARY COACTIONS ON C^*-ALGEBRAS WITH CONTINUOUS TRACE

A cosystem (A, G, δ) is called *separable* if A is separable and G is second countable. Note that the crossed product of a separable cosystem is separable. Let (A, G, δ) be a separable pointwise unitary cosystem such that A has continuous trace. In this section, we show that the crossed product $A \times_\delta G$ has continuous trace, and the dual action of G on $(A \times_\delta G)^\wedge$ is free and proper (Theorem 3.6). Further, $A \times_\delta G$ is isomorphic to the pull-back $\mathrm{Res}^* A$, where $\mathrm{Res}^* : (A \times_\delta G)^\wedge \to \widehat{A}$ is the restriction map (Theorem 3.7). The key result used for the proofs of these two theorems is Proposition 3.4 which says that, if (A, G, δ) is as above, then δ is "unitary on subsequences of convergent sequences" (see Proposition 3.4 for the correct statement of this). Before we prove Proposition 3.4, we have to prove some lemmas.

Lemma 3.1. *Let $\mathcal{H}$ be a separable Hilbert space, and let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(\mathcal{H})$ be a sequence of unitary operators on $\mathcal{H}$ such that $u_n T u_n^* \to T$ in norm for all $T \in \mathcal{K}(\mathcal{H})$. Then there is a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{T}$ such that $\lambda_n u_n \to 1_{\mathcal{H}}$ strongly.*

Proof. We equip $\mathcal{U}(\mathcal{H})$ with the strong operator topology and $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$ with the topology of pointwise convergence. The map $\mathcal{U}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{K}(\mathcal{H}))$, $u \mapsto \mathrm{Ad}\, u$ factors through an isomorphism between $\mathcal{PU}(\mathcal{H}) = \mathcal{U}(\mathcal{H})/\mathbb{T}$ and $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$. Since $\mathbb{T}$ is a compact Lie group, it follows from [16], Corollary 3, that the quotient map $\mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ has local sections. Thus, there is an open neighbourhood V of $\mathrm{id}_{\mathcal{K}(\mathcal{H})}$ in $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$ and a continuous function $\gamma : V \to \mathcal{U}(\mathcal{H})$ such that $\mathrm{Ad} \circ \gamma = \mathrm{id}_V$ and $\gamma(\mathrm{id}_{\mathcal{K}(\mathcal{H})}) = 1_{\mathcal{H}}$.

Now let $(u_n) \subset \mathcal{U}(\mathcal{H})$ be a sequence such that $\mathrm{Ad}\, u_n \to \mathrm{id}_{\mathcal{K}(\mathcal{H})}$ in $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$. We may suppose that $\mathrm{Ad}\, u_n \in V$ for all $n \in \mathbb{N}$. Then we have

$$\mathrm{Ad}(\gamma(\mathrm{Ad}\, u_n)) = \mathrm{Ad}\, u_n \quad \forall\, n \in \mathbb{N}$$

and $\gamma(\mathrm{Ad}\, u_n) \to 1_{\mathcal{H}}$ strongly. Let $v_n = \gamma(\mathrm{Ad}\, u_n)$. Since $\mathrm{Ad}\, v_n = \mathrm{Ad}\, u_n$, there is a $\lambda_n \in \mathbb{T}$ such that $v_n = \lambda_n u_n$, which completes the proof. ∎

For a C^*-algebra A and a Hilbert space $\mathcal{H}$, we define $\mathrm{Rep}(A, \mathcal{H})$ to be the set of all non-degenerate representations of A in $\mathcal{H}$. $\mathrm{Rep}(A, \mathcal{H})$ is equipped with the topology of strong pointwise convergence. In [2], Chapter 3.5, Dixmier defined $\mathrm{Rep}(A, \mathcal{H})$ to be the set of all (possibly degenerate) representations of A in $\mathcal{H}$. However, for technical reasons (see, for example, the proof of Lemma 3.2), we shall admit only non-degenerate representations. Further, we let $\mathrm{Irr}(A, \mathcal{H}) := \{\pi \in \mathrm{Rep}(A, \mathcal{H}) : \pi \text{ is irreducible}\}$, equipped with the topoloy induced by the topology of $\mathrm{Rep}(A, \mathcal{H})$. By [2], 3.5.8, the canonical map

$$\mathrm{Irr}(A, \mathcal{H}) \to \widehat{A}, \quad \pi \mapsto [\pi]$$

is continuous and open onto its image.

Lemma 3.2. *Let A be a separable C^*-algebra and $\mathcal{H}$ a separable Hilbert space. For $\pi \in \mathrm{Rep}(A, \mathcal{H})$, let $\widetilde{\pi}$ be the unique extension of π to $M(A)$. Then the map*

$$\pi \mapsto \widetilde{\pi}, \quad \mathrm{Rep}(A, \mathcal{H}) \to \mathrm{Rep}(M(A), \mathcal{H})$$

is continuous.

Proof. Let $(\pi_n)_{n\in\mathbb{N}} \subset \mathrm{Rep}(A, \mathcal{H})$ be a sequence with $\pi_n \to \pi \in \mathrm{Rep}(A, \mathcal{H})$ and let $b \in M(A)$, $a \in A$ and $\eta \in \mathcal{H}$. Then we have

$$\begin{aligned}
&\|\widetilde{\pi}_n(b)\pi(a)\eta - \widetilde{\pi}(b)\pi(a)\eta\| \\
&\qquad \leqslant \|\widetilde{\pi}_n(b)\pi(a)\eta - \widetilde{\pi}_n(b)\pi_n(a)\eta\| + \|\widetilde{\pi}_n(b)\pi_n(a)\eta - \widetilde{\pi}(b)\pi(a)\eta\| \\
&\qquad \leqslant \|b\|\, \|\pi(a)\eta - \pi_n(a)\eta\| + \|\pi_n(ba)\eta - \pi(ba)\eta\| \to 0.
\end{aligned}$$

Since π is non-degenerate, the lemma is proved. ∎

The next lemma is essential for the proof of Proposition 3.4. Some of the ideas used for the proof are taken from the proof of [3], Propositon 5.2.5.

LEMMA 3.3. *Let (A, G, δ) be a separable pointwise unitary cosystem such that $A = C_0(X, \mathcal{K}(\mathcal{H}))$ for some separable locally compact Hausdorff space X and some separable Hilbert space $\mathcal{H}$. For $x \in X$, let $\rho_x \in \mathrm{Irr}(A, \mathcal{H})$ be the evaluation map at x. Then the following holds: If $([\pi_n])_{n\in\mathbb{N}} \subset \widehat{A}$ is a sequence which converges to $[\pi] \in \widehat{A}$, and if we choose $y \in X$ and $\mu \in \mathrm{Rep}(C_0(G), \mathcal{H})$ such that $[\rho_y] = [\pi]$ and (ρ_y, μ) is a covariant representation, then there are a subsequence $([\pi_{n_m}])_{m\in\mathbb{N}}$ of $([\pi_n])_{n\in\mathbb{N}}$ and sequences $(y_m)_{m\in\mathbb{N}} \subset X$ and $(\mu_m)_{m\in\mathbb{N}} \subset \mathrm{Rep}(C_0(G), \mathcal{H})$ such that $[\rho_{y_m}] = [\pi_{n_m}]$, $\mu_m \to \mu$ in $\mathrm{Rep}(C_0(G), \mathcal{H})$, and (ρ_{y_m}, μ_m) are covariant representations for all $m \in \mathbb{N}$.*

Proof. First we show that the canonical map

$$\mathrm{Irr}(A \times_\delta G, \mathcal{H}) \to (A \times_\delta G)^\wedge$$

is surjective. Let $[\rho \times \nu] \in (A \times_\delta G)^\wedge$. By Proposition 2.7, ρ is irreducible. Since the map $X \to \widehat{A}$, $y \mapsto [\rho_y]$ is a homeomorphism, we may suppose that ρ acts on $\mathcal{H}$. But then $\rho \times \nu$ acts on $\mathcal{H}$, and the canonical map is surjective. By [2], 3.5.8, this map is also open. Furthermore, the map

$$[\rho \times \mu] \mapsto [\rho], \quad (A \times_\delta G)^\wedge \to \widehat{A}$$

is open and surjective by Proposition 2.7. It follows that the map

$$\theta : \mathrm{Irr}(A \times_\delta G, \mathcal{H}) \to \widehat{A}, \quad \rho \times \mu \mapsto [\rho]$$

is open and surjective, too.

Now let $([\pi_n])_{n\in\mathbb{N}} \subset \widehat{A}$ and $[\pi]$ such that $[\pi_n] \to [\pi]$. Let $y \in X$ such that $[\rho_y] = [\pi]$. Since δ is pointwise unitary, we can choose $\mu \in \mathrm{Rep}(C_0(G), \mathcal{H})$ such that (ρ_y, μ) is a covariant representation. Since θ is open and surjective, we obtain a subsequence $([\pi_{n_m}])_{m\in\mathbb{N}}$ of $([\pi_n])_{n\in\mathbb{N}}$ and sequences $(\rho_m)_{m\in\mathbb{N}} \subset \mathrm{Irr}(A, \mathcal{H})$ and $(\nu_m)_{m\in\mathbb{N}} \subset \mathrm{Rep}(C_0(G), \mathcal{H})$ such that $\rho_m \times \nu_m \to \rho_y \times \mu$ in $\mathrm{Irr}(A \times_\delta G, \mathcal{H})$ and $[\rho_m] = [\pi_{n_m}]$ for all $m \in \mathbb{N}$.

Since $X \to \widehat{A}$, $x \mapsto [\rho_x]$ is a homeomorphism, there is a sequence $(y_m)_{m\in\mathbb{N}} \subset X$ such that $y_m \to y$ and $[\rho_{y_m}] = [\rho_m] = [\pi_{n_m}]$ for all $m \in \mathbb{N}$. This yields a sequence $(u_m)_{m\in\mathbb{N}}$ of unitaries on $\mathcal{H}$ such that $\mathrm{Ad}\, u_m \circ \rho_{y_m} = \rho_m$ for all $m \in \mathbb{N}$. Let $\mu_m := \mathrm{Ad}\, u_m^* \circ \nu_m$. Then (ρ_{y_m}, μ_m) is a covariant representation and

$\rho_m \times \nu_m = \operatorname{Ad} u_m \circ (\rho_{y_m} \times \mu_m)$. It remains to show that $\mu_m \to \mu$. We have $\operatorname{Ad} u_m \circ (\rho_{y_m} \times \mu_m) \to \rho_y \times \mu$ in $\operatorname{Irr}(A \times_\delta G, \mathcal{H})$. By Lemma 3.2,

$$(\operatorname{Ad} u_m \circ (\rho_{y_m} \times \mu_m))^\sim \to (\rho_y \times \mu)^\sim$$

in $\operatorname{Rep}(M(A \times_\delta G), \mathcal{H})$. Let $j_A : A \to M(A \times_\delta G)$ be the canonical map. Then, for all $a \in A$, we have

$$\begin{aligned}\operatorname{Ad} u_m(\rho_{y_m}(a)) &= (\operatorname{Ad} u_m \circ (\rho_{y_m} \times \mu_m))^\sim(j_A(a)) \\ &\to (\rho_y \times \mu)^\sim(j_A(a)) = \rho_y(a),\end{aligned}$$

where the limit is taken with respect to the strong operator topology. Hence, $\operatorname{Ad} u_m \circ \rho_{y_m} \to \rho_y$ in $\operatorname{Irr}(A, \mathcal{H})$ and similarly $\operatorname{Ad} u_m \circ \mu_m \to \mu$ in $\operatorname{Rep}(C_0(G), \mathcal{H})$.

Now let $k \in \mathcal{K}(\mathcal{H})$ and $f \in C_0(X)$ such that $f(y_m) = f(y) = 1$ for all $m \in \mathbb{N}$. Then

$$u_m k u_m^* = u_m f(y_m) k u_m^* = u_m \rho_{y_m}(f \otimes k) u_m^* \to \rho_y(f \otimes k) = k$$

strongly. Since this is true for all $k \in \mathcal{K}(\mathcal{H})$, we conclude from [3], Lemma 5.2.8, that $u_m k u_m^* \to k$ in norm for all $k \in \mathcal{K}(\mathcal{H})$. Lemma 3.1 tells us that there is a sequence $(\lambda_m)_{m \in \mathbb{N}} \subset \mathbb{T}$ such that $\lambda_m u_m, \overline{\lambda}_m u_m^* \to 1_{\mathcal{H}}$ strongly. But then $\operatorname{Ad} u_m \circ \mu_m \to \mu$ implies that $\mu_m \to \mu$ in $\operatorname{Rep}(C_0(G), \mathcal{H})$. ∎

We now come to the key proposition of this chapter (compare [3], Proposition 5.2.5).

PROPOSITION 3.4. *Let (A, G, δ) be a separable pointwise unitary coaction such that A has continuous trace. Then, for any convergent sequence $([\pi_n])_{n \in \mathbb{N}} \subset \widehat{A}$, there is a subsequence $([\pi_{n_m}])_{m \in \mathbb{N}}$ such that $I := \bigcap\{\ker[\pi_{n_m}] : m \in \mathbb{N}\}$ is δ-invariant and the coaction δ^I on A/I is unitary.*

Proof. First, let us suppose that A is stable. Let $([\pi_n])_{n \in \mathbb{N}} \subset \widehat{A}$ be a sequence such that $[\pi_n] \to [\pi] \in \widehat{A}$. If $([\pi_n])_{n \in \mathbb{N}}$ has a constant subsequence, that is $[\pi_n] = [\pi]$ for infinitely many $n \in \mathbb{N}$, set $I = \ker \pi$. Then δ^I is implemented by a representation $\mu : C_0(G) \to M(\mathcal{K}(\mathcal{H}_\pi)) = M(A/I)$ of $C_0(G)$ such that (π, μ) is covariant.

If there is no constant subsequence, we may suppose that $[\pi_n] \neq [\pi_m]$ for $n \neq m$ and $[\pi_n] \neq [\pi]$ for $n \in \mathbb{N}$. Since A is stable and has continuous trace, it follows from [22], Proposition 1.12, that there is a (compact) neighbourhood U of $[\pi]$ such that, for $J := \bigcap\{\ker \rho : \rho \in U\}$, we have $A/J \cong C_0(X, \mathcal{K}(\mathcal{H}))$ for some separable locally compact space X and some separable Hilbert space $\mathcal{H}$. Thus, by Remark 2.2 (iv), we may suppose that $A = C_0(X, \mathcal{K}(\mathcal{H}))$. (Note that, by [12], Theorem 5.3 (2), every ideal of A is δ-invariant.) Choose $x \in X$ and

$\mu \in \mathrm{Rep}(C_0(G), \mathcal{H})$ such that $[\rho_x] = [\pi]$ and (ρ_x, μ) is covariant. Since $[\pi_n] \to [\pi]$, Lemma 3.3 gives us a subsequence $([\pi_{n_m}])_{m\in\mathbb{N}}$ and sequences $(x_m)_{m\in\mathbb{N}} \subset X$ and $(\mu_m)_{m\in\mathbb{N}} \subset \mathrm{Rep}(C_0(G), \mathcal{H})$ such that $[\rho_{x_m}] = [\pi_{n_m}]$, (ρ_{x_m}, μ_m) is covariant for all $m \in \mathbb{N}$, and $\mu_m \to \mu$ in $\mathrm{Rep}(C_0(G), \mathcal{H})$.

Let $V = \overline{\{x_m : m \in \mathbb{N}\}}$ and $I = \bigcap\{\ker \pi_{n_m} : m \in \mathbb{N}\}$. We are now able to define the desired homomorphism $\phi : C_0(G) \to M(A/I)$ which implements the coaction δ^I. For $f \in C_0(G)$, define a map $\varphi(f) : V \to \mathcal{L}(\mathcal{H})$ by $\varphi(f)(x_m) := \mu_m(f)$ and $\varphi(f)(x) := \mu(f)$. Since all the x_m and x are mutually distinct, $\varphi(f)$ is well defined. We show that $\varphi(f) \in M(C(V, \mathcal{K}(\mathcal{H}))) = C^{\mathrm{b}}_{\mathrm{s}}(V, M(\mathcal{K}(\mathcal{H})))$. First, $\varphi(f)$ is bounded by $\|f\|_\infty$. So it remains to show that $\varphi(f) : V \to M(\mathcal{K}(\mathcal{H}))$ is strictly continuous. Since x is the only cluster point of V and since every sequence in V which converges to x and whose members are pairwise distinct is a subsequence of $(x_m)_{m\in\mathbb{N}}$, it suffices to concentrate on this sequence. But by construction,

$$\varphi(f)(x_m) = \mu_m(f) \to \mu(f) = \varphi(f)(x)$$

$*$-strongly. Since the $*$-strong operator topology and the strict topology in $M(\mathcal{K}(\mathcal{H}))$ coincide on bounded sets, it follows that $\varphi(f)$ is strictly continuous. We obtain a map $\varphi : C_0(G) \to M(C(V, \mathcal{K}(\mathcal{H})))$ which is a homomorphism since the μ_m and μ are homomorphisms. Now consider the isomorphism

$$\psi : A/I \to C(V, \mathcal{K}(\mathcal{H})), \quad \psi(a + I)(v) = a(v).$$

Then $\phi := \psi^{-1} \circ \varphi : C_0(G) \to M(A/I)$ is the desired homomorphism. We have to show that

(i) ϕ is non-degenerate;

(ii) ϕ implements δ^I.

(i) In order to show that ϕ is non-degenerate, it suffices to show that φ is. We show that $\varphi(C_0(G))C(V, \mathcal{K}(\mathcal{H})) = C(V, \mathcal{K}(\mathcal{H}))$. Let $a \in C(V, \mathcal{K}(\mathcal{H}))$ and $\varepsilon > 0$. We shall find an $h \in C_0(G)$ such that $\|\varphi(h)a - a\| \leqslant \varepsilon$. Since $\mathcal{K}(\mathcal{H})$ is the closed linear span of projections of rank one, we may suppose that $a = f \otimes P_\xi$ with $f \in C(V)$ and $\xi \in \mathcal{H}$ such that $\|f\|_\infty = \|\xi\| = 1$ and P_ξ is the projection onto $\mathbb{C}\,\xi$.

Let $M = \{\mu_m : m \in \mathbb{N}\} \cup \{\mu\}$ and $h \in C_0(G)$. Then

$$\begin{aligned}
\|\varphi(h)a - a\| &= \sup_{x\in V} \|\varphi(h)(x)a(x) - a(x)\| \\
&= \sup_{x\in V} \|f(x)(\varphi(h)(x)P_\xi - P_\xi)\| \\
&\leqslant \sup_{x\in V} \|\varphi(h)(x)P_\xi - P_\xi\| \qquad (\text{since } \|f\|_\infty = 1) \\
&= \sup_{\nu\in M} \|\nu(h)P_\xi - P_\xi\| \qquad (\text{by definition of } \varphi) \\
&= \sup_{\nu\in M} \sup_{\eta\in\mathcal{H},\, \|\eta\|=1} \|\nu(h)P_\xi\eta - P_\xi\eta\| \\
&\leqslant \sup_{\nu\in M} \|\nu(h)\xi - \xi\| \qquad (\text{since } \|\xi\| = 1).
\end{aligned}$$

Thus, it suffices to find an $h \in C_0(G)$ with $\sup_{\nu\in M} \|\nu(h)\xi - \xi\| \leqslant \varepsilon$. Since μ is non-degenerate, there are $g \in C_0(G)$ and $\eta \in \mathcal{H}$ with $\|\eta\| = 1$ and $\xi = \mu(g)\eta$. By construction, we have $\mu_m(g)\eta \to \mu(g)\eta = \xi$. So there is an $m_0 \in \mathbb{N}$ such that $\|\xi - \mu_m(g)\eta\| \leqslant \varepsilon/3$ for all $m \geqslant m_0$. Since the representations μ_m and μ are non-degenerate, we find (by using an approximate identity in $C_0(G)$) an $h \in C_0(G)$ such that $\|h\|_\infty \leqslant 1$, $\|hg - g\| \leqslant \varepsilon/3$, $\|\mu(h)\xi - \xi\| \leqslant \varepsilon$ and $\|\mu_m(h)\xi - \xi\| \leqslant \varepsilon$ for all $m < m_0$. Now, if $m \geqslant m_0$, then

$$\begin{aligned}
&\|\mu_m(h)\xi - \xi\| \\
&\leqslant \|\mu_m(h)\xi - \mu_m(h)\mu_m(g)\eta\| + \|\mu_m(h)\mu_m(g)\eta - \mu_m(g)\eta\| + \|\mu_m(g)\eta - \xi\| \\
&\leqslant \|\xi - \mu_m(g)\eta\| + \|hg - g\| + \|\mu_m(g)\eta - \xi\| \leqslant \varepsilon.
\end{aligned}$$

This implies that $\sup_{\nu\in M} \|\nu(h)\xi - \xi\| \leqslant \varepsilon$, and we have shown that φ is non-degenerate.

(ii) Remember that, for $a \in A = C_0(X, \mathcal{K}(\mathcal{H}))$ and $z \in C^*_{\mathrm{r}}(G)$, we have

$$\delta(a)(1 \otimes z) \in C_0(X, \mathcal{K}(\mathcal{H})) \otimes C^*_{\mathrm{r}}(G) = C_0(X, \mathcal{K}(\mathcal{H}) \otimes C^*_{\mathrm{r}}(G)).$$

So we can approximate $\delta(a)(1\otimes z)$ by a finite sum $\sum a_i \otimes z_i$ with $a_i \in C_0(X, \mathcal{K}(\mathcal{H}))$ and $z_i \in C^*_{\mathrm{r}}(G)$. Using the fact that $(a_i \otimes z_i)(v) = a_i(v) \otimes z_i$ for all $v \in X$ we see that

$$\begin{aligned}
(((\psi \circ q) \otimes \mathrm{id})(\delta(a)(1 \otimes z)))(v) &\approx \sum (((\psi \circ q) \otimes \mathrm{id})(a_i \otimes z_i))(v) \\
&= \sum (\psi(q(a_i)) \otimes z_i)(v) = \sum (\psi(q(a_i))(v) \otimes z_i \\
&= \sum a_i(v) \otimes z_i = \sum \rho_v(a_i) \otimes z_i \\
&= (\rho_v \otimes \mathrm{id})\Big(\sum a_i \otimes z_i\Big) \approx (\rho_v \otimes \mathrm{id})(\delta(a)(1 \otimes z))
\end{aligned}$$

for all $v \in X$. Hence, $(((\psi \circ q) \otimes \mathrm{id})(\delta(a)))(v) = (\rho_v \otimes \mathrm{id})(\delta(a))$ for all $v \in X$. It follows from the definition of φ that $\mu_m \otimes \mathrm{id}(W_G) = \varphi \otimes \mathrm{id}(W_G)(x_m)$ for all $m \in \mathbb{N}$. Now we obtain

$$\begin{aligned}(\psi \otimes \mathrm{id})(\delta^I(q(a)))(x_m) &= ((\psi \circ q) \otimes \mathrm{id})(\delta(a))(x_m) = (\rho_{x_m} \otimes \mathrm{id})(\delta(a)) \\ &= (\mu_m \otimes \mathrm{id})(W_G) \cdot (\rho_{x_m}(a) \otimes 1) \cdot (\mu_m \otimes \mathrm{id})(W_G^*) \\ &= (\varphi \otimes \mathrm{id}(W_G))(x_m) \cdot (a \otimes 1)(x_m) \cdot (\varphi \otimes \mathrm{id}(W_G^*))(x_m) \\ &= [\varphi \otimes \mathrm{id}(W_G) \cdot (\psi(q(a)) \otimes 1) \cdot \varphi \otimes \mathrm{id}(W_G^*)](x_m)\end{aligned}$$

and the same is true for x. Hence,

$$(\psi \otimes \mathrm{id})(\delta^I(q(a))) = (\varphi \otimes \mathrm{id})(W_G) \cdot (\psi(q(a)) \otimes 1) \cdot (\varphi \otimes \mathrm{id})(W_G^*)$$

in $M(C_0(V, \mathcal{K}(\mathcal{H})) \otimes C_r^*(G))$, and, if we apply $(\psi \otimes \mathrm{id})^{-1}$ to this equation, we see that $\psi^{-1} \circ \varphi$ implements δ^I. It follows (ii). So the result is shown provided that A is stable.

If A is not stable, we may stabilize the cosystem (A, G, δ). By Lemma 2.8, the stabilized coaction δ^{s} is still pointwise unitary. Let $([\pi_n])_{n \in \mathbb{N}} \subset \widehat{A}$ be a sequence which converges to $[\pi] \in \widehat{A}$. Then $[\pi_n \otimes \mathrm{id}] \to [\pi \otimes \mathrm{id}]$ in $(A \otimes \mathcal{K}(\mathcal{H}))^\wedge$. By the above, there is a subsequence $([\pi_{n_m} \otimes \mathrm{id}])_{m \in \mathbb{N}}$ such that for $J = \bigcap\{\ker(\pi_{n_m} \otimes \mathrm{id}) : m \in \mathbb{N}\}$ the coaction $(\delta^{\mathrm{s}})^J$ is unitary. Now $J = I \otimes \mathcal{K}(\mathcal{H})$ where $I = \bigcap\{\ker \pi_{n_m} : m \in \mathbb{N}\}$. It follows from Lemma 2.8 that $(\delta^{\mathrm{s}})^J = (\delta^I)^{\mathrm{s}}$, and therefore δ^I is unitary. ∎

We now come to the applications of Proposition 3.4. We begin with some notations.

Let X and T be locally compact Hausdorff spaces and suppose that G acts freely on X. We say that a continuous, open and surjective map $p : X \to T$ is a *G-bundle* if p factors through a homeomorphism $X/G \to T$. Two G-bundles are said to be isomorphic, if there is a G-homeomorphism $h : X \to Y$ such that $q \circ h = p$. A G-bundle $p : X \to T$ is called *proper* if the map

$$G \times X \to X \times X, \quad (s, x) \to (sx, x)$$

is proper in the sense that the preimage of a compact set is compact (cf. [16]). In this case, we also say that the action of G on X is free and proper. If the proper G-bundle $p : X \to T$ has a continuous section $\mathcal{S} : T \to X$, then we call $p : X \to T$ a *trivial* G-bundle. Note that this is equivalent to saying that $p : X \to T$ is isomorphic to the G-bundle $q : T \times G \to T$, where q is the projection onto T, and G acts by left translation on the second factor ([27], Proposition 4.3). Finally, a proper G-bundle $p : X \to T$ is called *locally trivial* if there are local sections, that is, every $x \in X$ has a neighbourhood U of $p(x)$ in T such that there exists a continuous section $\mathcal{S}_U : U \to p^{-1}(U)$.

The following lemma is a characterization of C^*-algebras with continuous trace, which is due to Echterhoff ([3], Proposition 5.1.4).

LEMMA 3.5. *Let A be a separable C^*-algebra. Then the following statements are equivalent:*

(i) *A has continuous trace;*

(ii) *for any convergent sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\widehat{A}$, there exists a subsequence $(\rho_{n_m})_{m\in\mathbb{N}}$ such that $A/\bigcap\{\ker\rho_{n_m} : m\in\mathbb{N}\}$ has continuous trace.*

THEOREM 3.6. *Let (A, G, δ) be a separable pointwise unitary cosystem such that A has continuous trace. Then the crossed product $A\times_\delta G$ has continuous trace, and the restriction map $\mathrm{Res} : (A\times_\delta G)^\wedge \to \widehat{A}$ is a proper G-bundle via the dual action of G on $(A\times_\delta G)^\wedge$.*

Proof. Let $B = A\times_\delta G$. To show that B has continuous trace, we use Lemma 3.5. Let $(\pi_n\times\mu_n)_{n\in\mathbb{N}}\subset\widehat{B}$ be a sequence with $\pi_n\times\mu_n\to\pi\times\mu$. We have to show that there is a subsequence $(\pi_{n_m}\times\mu_{n_m})_{m\in\mathbb{N}}$ such that for $J = \bigcap\{\pi_{n_m}\times\mu_{n_m} : m\in\mathbb{N}\}$ the quotient B/J has continuous trace. By Proposition 2.7, the map $\rho\times\nu\to\rho$ is continuous, so we have $\pi_n\to\pi$. By Proposition 3.4, there is a subsequence $(\pi_{n_m})_{m\in\mathbb{N}}$ such that, for $I = \bigcap\{\pi_{n_m} : m\in\mathbb{N}\}$, the coaction δ^I of G on A/I is unitary. Proposition 2.6 implies that $(A/I)\times_{\delta^I} G$ has continuous trace. But by [12], Chapter 4, $(A/I)\times_{\delta^I} G\cong(A\times_\delta G)/(I\times_{\delta_I} G)$, and $(I\times_{\delta_I} G)$ is generated by the set $\{j_A(a)j_{C_0(G)}(f) : a\in I,\ f\in C_0(G)\}$. Thus, $J\supset I\times_{\delta_I} G$, and B/J is a quotient of the continuous trace algebra $B/(I\times_{\delta_I} G)$, which implies that B/J has continuous trace.

Now let us prove the second part of the theorem. By [12], Theorem 5.5 (2), G acts freely on $(A\times_\delta G)^\wedge$. We show that $\mathrm{Res} : (A\times_\delta G)^\wedge\to\widehat{A}$ factors through a homeomorphism between $(A\times_\delta G)^\wedge/G$ and $\widehat{A}$. Let $q : (A\times_\delta G)^\wedge\to(A\times_\delta G)^\wedge/G$ be the quotient map. We claim that $\mathrm{Res}(\pi\times\mu) = \mathrm{Res}(\rho\times\nu)$ if and only if $q(\pi\times\mu) = q(\rho\times\nu)$ for $\pi\times\mu, \rho\times\nu\in(A\times_\delta G)^\wedge$. Since Res is open, continuous and surjective by Proposition 2.7, we then obtain a homeomorphism $h : (A\times_\delta G)^\wedge/G\to\widehat{A}$ satisfying $\mathrm{Res} = h\circ q$. So suppose that $q(\pi\times\mu) = q(\rho\times\nu)$. There is an $s\in G$ such that $s\cdot(\pi\times\mu) = \rho\times\nu$. That is, $\pi\times(\mu\circ\sigma_{s^{-1}}) = \rho\times\nu$. Thus,

$$\pi = (\pi\times(\mu\circ\sigma_{s^{-1}}))\circ j_A = (\rho\times\nu)\circ j_A = \rho.$$

Conversely, suppose that $\mathrm{Res}(\pi\times\mu) = \mathrm{Res}(\rho\times\nu)$. Then $\pi = \rho$. Thus, both (π,μ) and (π,ν) are covariant pairs. By [12], Theorem 5.5 (2), there is an $s\in G$ such that

$$((\nu\circ\sigma_{s^{-1}})\otimes\mathrm{id})(W_G) = (\nu\otimes\mathrm{id})(W_G)(1\otimes\lambda_G(s^{-1})) = (\mu\otimes\mathrm{id})(W_G).$$

Slicing yields $\mu = \nu\circ\sigma_{s^{-1}}$. Thus, $\pi\times\mu = \rho\times(\nu\circ\sigma_{s^{-1}}) = s\cdot(\rho\times\nu)$ and therefore $q(\pi\times\mu) = q(\rho\times\nu)$.

By the above, $\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$ is a G-bundle. We show that it is also proper. Let $(\pi_n)_{n\in\mathbb{N}} \subset \widehat{A}$ be a convergent sequence. By Proposition 3.4, there is a subsequence $(\pi_{n_m})_{m\in\mathbb{N}}$ such that for $I = \bigcap\{\ker \pi_{n_m} : m \in \mathbb{N}\}$, the coaction δ^I on A/I is unitary. It follows from Proposition 2.6 that

$$\mathrm{Res}_{\delta^I} : ((A/I) \times_{\delta^I} G)^\wedge \to (A/I)^\wedge$$

is a trivial G-bundle. Further, by [12], Theorem 4.8, there is a homeomorphism h between $((A/I) \times_{\delta^I} G)^\wedge$ and the set $\{\pi \times \mu \in (A \times_\delta G)^\wedge : \ker \pi \supset I\} = \mathrm{Res}^{-1}((A/I)^\wedge)$ which makes the G-bundles $\mathrm{Res}_{\delta^I} : ((A/I) \times_{\delta^I} G)^\wedge \to (A/I)^\wedge$ and $\mathrm{Res} : \mathrm{Res}^{-1}((A/I)^\wedge) \to (A/I)^\wedge$ isomorphic. Thus, $\mathrm{Res} : \mathrm{Res}^{-1}((A/I)^\wedge) \to (A/I)^\wedge$ is a trivial G-bundle. Now it follows from [3], Proposition 5.1.3, that

$$\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$$

is a proper G-bundle. ∎

For the next theorem, we recall the definition of pull-back C^*-algebras from [26]. Let A be a C^*-algebra with spectrum X, Y a locally compact Hausdorff space, and let $f : Y \to X$ be a continuous map. Then both $C_0(Y)$ and A are $C^{\mathrm{b}}(X)$-Modules in a natural way. The pull-back of A along f is the "balanced" tensor product

$$C_0(Y) \otimes_{C(X)} A$$

which is the quotient of $C_0(Y) \otimes A$ by the ideal generated by the set

$$\{gh \otimes a - h \otimes ga : g \in C^{\mathrm{b}}(X),\ h \in C_0(Y)\, a \in A\}.$$

We write $f^*A := C_0(Y) \otimes_{C(X)} A$.

Suppose that X is Hausdorff and A is represented as the algebra of sections $\Gamma_0(E)$ of a C^*-bundle $p : E \to X$. Then the pull-back bundle $q : f^*E \to Y$ is the C^*-bundle over Y consisting of all pairs $(y, e) \in Y \times E$ satisfying $f(y) = p(e)$, and q is the obvious projection onto Y. It turns out that the continuous sections of f^*E may be identified with the continuous functions $\phi : Y \to E$ such that $p(\phi(y)) = f(y)$ for all $y \in Y$. By [26], Proposition 1.3, we have $f^*(\Gamma_0(E)) \cong \Gamma_0(f^*E)$.

In the proof of Theorem 3.7, we define a map Ψ of $A \times_\delta G = \Gamma_0(E) \times_\delta G$ into $\Gamma_0(\mathrm{Res}^*E)$ quite in the same way as in [15], Theorem 1.10. However, to show the continuity of the map $\Psi(z) : (A \times_\delta G)^\wedge \to E$ we use Proposition 3.4, and we proceed similarly to the proof of [3], Theorem 5.2.9 (2).

Theorem 3.7. *Let (A,G,δ) be a separable pointwise unitary cosystem with a continuous trace algebra A. Let* $\mathrm{Res} : (A\times_\delta G)^\wedge \to \widehat{A}$ *be the restriction map. Then* $A\times_\delta G = \mathrm{Res}^* A$.

Proof. Let $p : E \to \widehat{A}$ be the C^*-bundle with fibers $A_\rho = A/\ker\rho$ such that $A = \Gamma_0(E)$ and the cross-sections are given by $\rho \mapsto a(\rho) = a + \ker\rho$. By [26], Proposition 1.3, we have $\mathrm{Res}^* A = \mathrm{Res}^*\Gamma_0(E) = \Gamma_0(\mathrm{Res}^* E)$ where $\mathrm{Res}^* E$ is the pull-back bundle over $(A\times_\delta G)^\wedge$ with respect to the restriction map $\mathrm{Res} : (A\times_\delta G)^\wedge \to \widehat{A}$. We may identify $\Gamma_0(\mathrm{Res}^* E)$ with the set of continuous functions $f : (A\times_\delta G)^\wedge \to E$ such that the map $\pi\times\mu \to \|f(\pi\times\mu)\|$ vanishes at infinity and $p(f(\pi\times\mu)) = \mathrm{Res}(\pi\times\mu) = \pi$ for all $\pi\times\mu \in (A\times_\delta G)^\wedge$.

For an irreducible representation ρ of A, let φ_ρ be the isomorphism

$$\mathcal{K}(\mathcal{H}_\rho) \to A/\ker\rho, \quad \rho(a) \to a(\rho) = a + \ker\rho.$$

Now define

$$\Psi : A\times_\delta G \to \Gamma_0(\mathrm{Res}^* E), \quad \Psi(z)(\rho\times\mu) = \varphi_\rho((\rho\times\mu)(z)).$$

To make sure that Ψ is well defined, we have to verify that the map

$$\rho\times\mu \to \varphi_\rho(\rho\times\mu(z))$$

is well defined for all $z \in A\times_\delta G$ (that is, $\varphi_\rho((\rho\times\mu)(z)) = \varphi_{\rho'}((\rho'\times\mu')(z))$ if $\rho\times\mu \cong \rho'\times\mu'$) and that Ψ maps $A\times_\delta G$ into $\Gamma_0(\mathrm{Res}^* E)$.

Fix $z\in A\times_\delta G$. Let $\rho\times\mu$ and $\rho'\times\mu'$ be two equivalent irreducible representations and u a unitary which intertwines $\rho'\times\mu'$ and $\rho\times\mu$. Then u intertwines ρ' and ρ. Because A and $A\times_\delta G$ both have continuous trace (Theorem 3.6) and therefore are liminal, there are $a,b\in A$ with $\rho(a) = \rho\times\mu(z)$ and $\rho'(b) = \rho'\times\mu'(z)$. It follows that

$$\rho(a)-\rho(b) = \rho\times\mu(z) - u\rho'(b)u^* = \rho\times\mu(z) - u(\rho'\times\mu')(z)u^* = \rho\times\mu(z) - \rho\times\mu(z) = 0.$$

Hence, $a-b \in \ker\rho = \ker\rho'$ (since $\rho\cong\rho'$), and this implies

$$\varphi_\rho((\rho\times\mu)(z)) = \varphi_\rho((\rho(a)) = a+\ker\rho = b+\ker\rho' = \varphi_{\rho'}((\rho'(b)) = \varphi_{\rho'}((\rho'\times\mu')(z)).$$

In order to show that Ψ maps z into $\Gamma_0(\mathrm{Res}^* E)$, we must check that

(i) $p(\Psi(z)(\rho\times\mu)) = \mathrm{Res}(\rho\times\mu)$ for all $\rho\times\mu \in (A\times_\delta G)^\wedge$,
(ii) the map $\rho\times\mu \to \|\Psi(z)(\rho\times\mu)\|$ vanishes at infinity and
(iii) the map $(A\times_\delta G)^\wedge \to E$, $\rho\times\mu \to \Psi(z)(\rho\times\mu)$ is continuous.

(i) Is clear since $\Psi(z)(\rho\times\mu)=\varphi_\rho(\rho\times\mu(z))\in A/\ker\rho=p^{-1}(\rho)$ and $\mathrm{Res}(\rho\times\mu)=\rho$, and (ii) holds since φ_ρ is a $*$-isomorphism and the map $\rho\times\mu\to\|\rho\times\mu\|$ vanishes at infinity by [2], 3.3.7.

It remains to verify (iii). Let $(\rho_n\times\mu_n)_{n\in\mathbb{N}}\subset(A\times_\delta G)^\wedge$ be a sequence with $\rho_n\times\mu_n\to\rho_0\times\mu_0\in(A\times_\delta G)^\wedge$. We show that $\varphi_{\rho_n}(\rho_n\times\mu_n(z))\to\varphi_{\rho_0}(\rho_0\times\mu_0(z))$ in E. Let $V\subset E$ be a neighbourhood of $\varphi_{\rho_0}(\rho_0\times\mu_0(z))$. We may suppose that

$$V=W(h,U,\varepsilon):=\{b\in E: p(b)\in U \text{ and } \|b-h(p(b))\|<\varepsilon\}$$

for some $h\in A=\Gamma_0(E)$, $U\subset\widehat{A}$ open and $\varepsilon>0$ (see [5], II 13.18). Because $\varphi_{\rho_0}(\rho_0\times\mu_0(z))\in V$, we have $\rho_0=p(\varphi_{\rho_0}(\rho_0\times\mu_0(z)))\in U$ and also $\|\varphi_{\rho_0}(\rho_0\times\mu_0(z))-h(\rho_0)\|<\varepsilon$. We have to show that:

(a) $\rho_n\in U$ and
(b) $\|\varphi_{\rho_n}(\rho_n\times\mu_n(z))-h(\rho_n)\|<\varepsilon$

for all n greater than some $n_0\in\mathbb{N}$. Since $\{j_A(a)j_{C_0(G)}(f): a\in A, f\in C_{\mathrm{c}}(G)\}$ spans a dense subset of $A\times_\delta G$, we may suppose that $z=j_A(a)j_{C_0(G)}(f)$ for some $a\in A$ and $f\in C_{\mathrm{c}}(G)$. The restriction map $\mathrm{Res}:\rho\times\mu\to\rho$ is continuous by Proposition 2.7. Hence, $\rho_n\to\rho_0$, which proves (a). Let $I=\bigcap\{\ker\rho_n : n\in\mathbb{N}\}$. By Proposition 3.4, we may suppose (by passing to a subsequence if necessary) that the coaction δ^I on A/I is unitary and implemented by a homomorphism $\phi: C_0(G)\to M(A/I)$. Let $q: A\to A/I$ be the quotient map. Since $\ker\rho_n\supset I$ for all $n\in\mathbb{N}_0$ and $\ker\rho_0\supset I$, it follows from [12], Theorem 4.8, that there are $\pi_n\times\mu_n,\pi_0\times\mu_0\in((A/I)\times_{\delta^I}G)^\wedge$ such that $\pi_n\circ q=\rho_n$, $\pi_0\circ q=\rho_0$ and $\pi_n\times\mu_n\to\pi_0\times\mu_0$. By Proposition 2.6, there is a sequence $(s_n)_{n\in\mathbb{N}}\subset G$ and $s_0\in G$ such that $\mu_n=\pi_n\circ\phi\circ\sigma_{s_n}$, $\mu_0=\pi_0\circ\phi\circ\sigma_{s_0}$ and $s_n\to s_0$.

Since φ_{ρ_n} is an isomorphism, we have that

$$\begin{aligned}\|\pi_n(q(a)\phi(\sigma_{s_n}(f))-q(h))\| &= \|\varphi_{\rho_n}(\pi_n(q(a)\phi(\sigma_{s_n}(f))))-\varphi_{\rho_n}(\pi_n(q(h)))\|\\ &= \|\varphi_{\rho_n}(\rho_n\times\mu_n(z))-h(\rho_n)\|\end{aligned}$$

for all $n\in\mathbb{N}$, and similarly

$$\|\pi_0(q(a)\phi(\sigma_{s_0}(f))-q(h))\|=\|\varphi_{\rho_0}(\rho_0\times\mu_0(z))-h(\rho_0)\|.$$

Especially, $\|\pi_0(q(a)\phi(\sigma_{s_0}(f))-q(h))\|<\varepsilon$, and there exists a $\gamma>0$ such that $\|\pi_0(q(a)\phi(\sigma_{s_0}(f))-q(h))\|<\varepsilon-\gamma$. The element $q(a)\phi_0(\sigma_{s_0}(f))-q(h)$ is in A/I. By [2], 3.3.9 and since $\pi_n\to\pi_0$, there is an $n_0\in\mathbb{N}$ such that

$$\|\pi_n(q(a)\phi(\sigma_{s_0}(f))-q(h))\|<\varepsilon-\gamma$$

for all $n \geqslant n_0$. Further, we may choose this n_0 such that $\|\sigma_{s_n}(f) - \sigma_{s_0}(f)\| < \gamma/\|a\|$. It follows

$$\begin{aligned}\|\varphi_{\rho_n}((\rho_n \times \mu_n)(z)) - h(\rho_n)\| &= \|\pi_n(q(a)\phi(\sigma_{s_n}(f)) - q(h))\| \\ &\leqslant \|\pi_n(q(a)\phi(\sigma_{s_n}(f)) - q(a)\phi(\sigma_{s_0}(f)))\| + \|\pi_n(q(a)\phi(\sigma_{s_0}(f)) - q(h))\| \\ &\leqslant \|a\|\,\|\sigma_{s_n}(f) - \sigma_{s_0}(f)\| + \|\pi_n(q(a)\phi(\sigma_{s_0}(f)) - q(h))\| \\ &\leqslant \gamma + \varepsilon - \gamma = \varepsilon\end{aligned}$$

for all $n \geqslant n_0$. This proves (b), and Ψ is well defined.

We now verify that Ψ is an isomorphism. First, Ψ is isometric, because

$$\begin{aligned}\|\Psi(z)\| &= \sup\{\|\Psi(z)(\rho \times \mu)\| : \rho \times \mu \in (A \times_\delta G)^\wedge\} \\ &= \sup\{\|\varphi_\rho((\rho \times \mu)(z))\| : \rho \times \mu \in (A \times_\delta G)^\wedge\} \\ &= \sup\{\|(\rho \times \mu)(z)\| : \rho \times \mu \in (A \times_\delta G)^\wedge\} \\ &= \|z\|.\end{aligned}$$

For $f \in C_0((A \times_\delta G)^\wedge)$ and $h \in \Gamma_0(\mathrm{Res}^* E)$, let $f \cdot h$ be defined by $(f \cdot h)(\rho \times \mu) = f(\rho \times \mu)h(\rho \times \mu)$ for all $\rho \times \mu \in (A \times_\delta G)^\wedge$. It follows from the definition of Ψ that

$$\{\Psi(z)(\rho \times \mu) : z \in A \times_\delta G\} = A_\rho$$

for all $\rho \times \mu \in (A \times_\delta G)^\wedge$. Thus, by [2], 10.2.5, the set

$$\{f \cdot \Psi(z) : f \in C_0((A \times_\delta G)^\wedge),\ z \in A \times_\delta G\}$$

spans a dense subspace in $\Gamma_0(\mathrm{Res}^* E)$. But now

$$f(\rho \times \mu)\Psi(z)(\rho \times \mu) = \varphi_\rho(f(\rho \times \mu) \cdot (\rho \times \mu)(z)) = \varphi_\rho((\rho \times \mu)(z_f z)) = \Psi(z_f z)(\rho \times \mu)$$

for all $f \in C_0((A \times_\delta G)^\wedge)$ and $z \in A \times_\delta G$, where z_f denotes the image of f under the Dauns-Hofmann isomorphism. Thus, $f \cdot \Psi(z) = \Psi(z_f z)$ for all $f \in C_0((A \times_\delta G)^\wedge)$ and $z \in A \times_\delta G$. It follows from this that $\Psi(A \times_\delta G)$ is dense in $\Gamma_0(\mathrm{Res}^* E)$, and the surjectivity follows since Ψ is isometric. ∎

To each separable continuous trace algebra A we can associate an element $\varepsilon(A)$ of the third Čeck cohomology group $H^3(\widehat{A}, \mathbb{Z})$, the so-called *Dixmier-Douady Class* of A (see [2], Chapter 10). Here we use the letter ε instead of the more common letter δ to avoid confusion with the image $\delta(A)$ of A when δ is a coaction on A. Any continuous map $f : X \to Y$ between two locally compact Hausdorff spaces X and Y induces a homomorphism $f^* : H^3(Y, \mathbb{Z}) \to H^3(X, \mathbb{Z})$. If A is a continuous trace algebra with spectrum Y, then, by [26], Proposition 1.4 (1), $f^* A$ is a continuous trace algebra with spectrum X and $\varepsilon(f^* A) = f^*(\varepsilon(A)) \in H^3(X, \mathbb{Z})$. Now Theorem 3.7 yields the following

COROLLARY 3.8. *Let (A, G, δ) be as in Theorem 3.7, and let $\varepsilon(A)$ be the Dixmier-Douady-Class of A. Then $\varepsilon(A \times_\delta G) = \mathrm{Res}^*(\varepsilon(A))$.*

4. APPLICATIONS

In this section, we give the applications of Theorem 3.6 and Theorem 3.7 as stated in the introduction. We start with the definition of exterior equivalence. Let $\Sigma : C^*_{\mathrm{r}}(G)\otimes C^*_{\mathrm{r}}(G) \to C^*_{\mathrm{r}}(G)\otimes C^*_{\mathrm{r}}(G)$ be the flip map. Recall that, for an element $W \in M(A\otimes C^*_{\mathrm{r}}(G))$, we define $W_{12} = W\otimes 1$ and $W_{13} = \mathrm{id}\otimes\Sigma(W\otimes 1)$.

DEFINITION 4.1. Let (A, G, δ) be a cosystem. A unitary $U \in M(A\otimes C^*_{\mathrm{r}}(G))$ is called a *δ-cocycle* if

$$(\mathrm{id}\otimes\delta_G)(U) = U_{12}\cdot(\delta\otimes\mathrm{id})(U); \tag{4.1}$$

$$(\mathrm{Ad}\, U\circ\delta)(A)(1\otimes C^*_{\mathrm{r}}(G)) \subset A\otimes C^*_{\mathrm{r}}(G). \tag{4.2}$$

If δ and ε are coactions on A, then we say that ε is *exterior equivalent* to δ if there is a δ-cocycle U such that $\varepsilon = \mathrm{Ad}\, U\circ\delta$.

REMARK 4.2. Let (A, G, δ) be a cosystem and $U \in M(A\otimes C^*_{\mathrm{r}}(G))$ a δ-cocycle. In [12], Chapter 2, Landstad et al. mentioned that $\mathrm{Ad}\, U\circ\delta$ is a coaction on A. Further, exterior equivalence is an equivalence relation, and the unitary coactions are precisely those which are exterior equivalent to the trivial coaction (see [21], Chapter 2).

LEMMA 4.3. *Let (A, G, δ) be a cosystem. Let M be the representation of $C_0(G)$ as multiplication operators on $L^2(G)$ and $V = (M\otimes\mathrm{id})(W_G)$. Then the unitary $1_A\otimes V \in M((A\otimes\mathcal{K}(L^2(G)))\otimes C^*_{\mathrm{r}}(G))$ is a δ^{s}-cocycle where δ^{s} is the stabilization of δ.*

Proof. For abbreviation, let $C = C^*_{\mathrm{r}}(G)$, $\mathcal{K} = \mathcal{K}(L^2(G))$ and $B = A\otimes\mathcal{K}$. Further, let $\Sigma^{\mathcal{K}}_C : C\otimes\mathcal{K}\to\mathcal{K}\otimes C$ and $\Sigma^C_C : C\otimes C\to C\otimes C$ denote the flip maps. Since $1_A\otimes V = ((1_A\otimes M)\otimes\mathrm{id})(W_G)$ and $1_A\otimes M$ is a non-degenerate homomorphism of $C_0(G)$ into $M(B) = M(A\otimes\mathcal{K})$, it follows from Lemma 1.2 that $1_A\otimes V \in M(B\otimes C)$. Note that

$$\Sigma^{\mathcal{K}}_C\otimes\mathrm{id}_C(1_C\otimes V) = \mathrm{id}_{\mathcal{K}}\otimes\Sigma^C_C(V\otimes 1_C).$$

Hence,

$$\begin{aligned}
\delta^{\mathrm{s}}\otimes\mathrm{id}_C(1_A\otimes V) &= ((\mathrm{id}_A\otimes\Sigma^{\mathcal{K}}_C)\circ(\delta\otimes\mathrm{id}_{\mathcal{K}})\otimes\mathrm{id}_C)(1_A\otimes V)\\
&= (\mathrm{id}_A\otimes\Sigma^{\mathcal{K}}_C\otimes\mathrm{id}_C)(\delta\otimes\mathrm{id}_{\mathcal{K}}\otimes\mathrm{id}_C(1_A\otimes V))\\
&= (\mathrm{id}_A\otimes\Sigma^{\mathcal{K}}_C\otimes\mathrm{id}_C)(1_A\otimes 1_C\otimes V)\\
&= (\mathrm{id}_A\otimes\mathrm{id}_{\mathcal{K}}\otimes\Sigma^C_C)(1_A\otimes V\otimes 1_C)\\
&= (1_A\otimes V)_{13}.
\end{aligned}$$

Since $(1_A \otimes V)$ satisfies

$$\mathrm{id}_B \otimes \delta_G(1_A \otimes V) = (1_A \otimes V)_{12}(1_A \otimes V)_{13}$$

by [21], Lemma 1.2, we obtain (4.1). By [10], Theorem 8, $\mathrm{Ad}(1_A \otimes V) \circ \delta^{\mathrm{s}}$ is a coaction, so (4.2) is satisfied, too. Thus, $1_A \otimes V$ is a δ^{s}-cocycle. ∎

Let (A, G, α) be a C^*-dynamical system, and $A \times_{\alpha,\mathrm{r}} G$ be the reduced crossed product of (A, G, α). Let π be any faithful representation on some Hilbert space $\mathcal{H}$. By [17], Theorem 7.7.5, $A \times_{\alpha,\mathrm{r}} G$ acts faithfully on $\mathcal{H} \otimes L^2(G)$ via the representation $\mathrm{Ind}\,\pi = \widetilde{\pi} \times (1 \otimes \lambda_G)$, where $\widetilde{\pi}$ is the representation of A on $\mathcal{H} \otimes L^2(G)$ defined by $(\widetilde{\pi}(a)\xi)(s) := \pi(\alpha_{s^{-1}}(a))\xi(s)$ for $a \in A$, $\xi \in \mathcal{H} \otimes L^2(G)$ and $s \in G$. Let $V \in \mathcal{L}(L^2(G) \otimes L^2(G))$ be as in Lemma 4.3. Define $\widehat{\alpha} : A \times_{\alpha,\mathrm{r}} G \to M((A \times_{\alpha,\mathrm{r}} G) \otimes C^*_{\mathrm{r}}(G))$ by

$$\widehat{\alpha}(x) := (1_{\mathcal{H}} \otimes V)(x \otimes 1)(1_{\mathcal{H}} \otimes V)$$

for $x \in A \times_{\alpha,\mathrm{r}} G$. It follows from the calculations in [11], pp. 255–257, that $\widehat{\alpha}$ is a non-degenerate coaction of G on $A \times_{\alpha,\mathrm{r}} G$. We call $\widehat{\alpha}$ the *dual coaction* of α.

THEOREM 4.4. *Let (A, G, δ) be a pointwise unitary separable cosystem such that A has continuous trace and such that G is a Lie group. Then δ is locally unitary.*

Proof. By Theorem 3.6, the restriction map $\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$ is a proper G-bundle. Since G is a Lie group, it follows from [16], Section 4.1, that $\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$ is a locally trivial G-bundle. Let $\delta' := \widehat{\widehat{\delta}}$ be the double dual coaction of G on $(A \times_\delta G) \times_{\hat{\delta},\mathrm{r}} G$. By [12], Theorem 5.14, δ' is locally unitary. It follows from [10], Theorem 8, that there is an isomorphism

$$\psi : (A \times_\delta G) \times_{\hat{\delta},\mathrm{r}} G \to A \otimes \mathcal{K}(L^2(G))$$

which carries δ' to the coaction $(\mathrm{Ad}(1 \otimes V)) \circ \delta^{\mathrm{s}}$. Here δ^{s} is the stabilized coaction of δ (see Lemma 2.8), and $V = (M \otimes \mathrm{id})(W_G)$ with M being the representation of $C_0(G)$ as multiplication operators on $L^2(G)$. Since $1 \otimes V$ is a δ^{s}-cocycle by Lemma 4.3, δ^{s} is locally unitary by [12], Remark 5.12. Now Lemma 2.8 implies that δ is locally unitary, too. ∎

In [28], Corollary 2.2, Rosenberg showed that a pointwise unitary action of a compactly generated and second countable abelian group G on a separable continuous trace algebra A is automatically locally unitary. Since every compactly generated abelian group is the dual group of a Lie group, the preceding theorem is a generalization of part of Rosenberg's theorem.

The following corollary is a generalization of [15], Corollary 1.11, in the case where G is a Lie group.

COROLLARY 4.5. *Let (A, G, δ) and (A, G, ε) be two separable pointwise unitary cosystems such that A has continuous trace. If δ and ε are exterior equivalent, then the proper G-bundles $\mathrm{Res}_\delta : (A \times_\delta G)^\wedge \to \widehat{A}$ and $\mathrm{Res}_\varepsilon : (A \times_\varepsilon G)^\wedge \to \widehat{A}$ are isomorphic. If G is a Lie group, then the converse is also true.*

Proof. If δ and ε are exterior equivalent, then, by [21], Proposition 2.8, there is an isomorphism $\phi : A \times_\delta G \to A \times_\varepsilon G$ such that ϕ intertwines the actions $\widehat{\delta}$ and $\widehat{\varepsilon}$ and $\phi \circ j_A^\delta = j_A^\varepsilon$. This induces a homeomorphism $h : (A \times_\varepsilon G)^\wedge \to (A \times_\delta G)^\wedge$ which is G-equivariant and which satisfies $\mathrm{Res}_\delta \circ h = \mathrm{Res}_\varepsilon$. Thus, the G-bundles are isomorphic.

If G is a Lie group, then δ and ε are locally unitary, and therefore, in this case, the converse is true by [12], Theorem 5.11. ∎

REMARK 4.6. Let (A, G, δ) and (A, G, ε) be as in Corollary 4.5, and suppose that the corresponding G-bundles are isomorphic via the G-equivariant homeomorphism $h : (A \times_\varepsilon G)^\wedge \to (A \times_\delta G)^\wedge$. From h one can construct an isomorphism $\phi : A \times_\delta G \to A \times_\varepsilon G$ such that $\phi \circ j_A^\delta = j_A^\varepsilon$, and ϕ intertwines the actions $\widehat{\delta}$ and $\widehat{\varepsilon}$. We may regard A as a subalgebra of $M(A \times_\delta G)$. Let

$$U := ((\phi^{-1} \circ j_G^\varepsilon) \otimes \mathrm{id})(W_G) \cdot (j_G^\delta \otimes \mathrm{id})(W_G).$$

Then U is a unitary element of $M((A \times_\delta G) \otimes C_\mathrm{r}^*(G))$. If we knew that $U \in M(A \otimes C_\mathrm{r}^*(G))$, then straightforward calculations would show that U satisfies (4.1) and $\varepsilon = \mathrm{Ad}\, U \circ \delta$. In the special case when $A \subset A \times_\delta G$ and G is amenable (for example when G is compact) one can show that $aS_f(U)$ and $S_f(U)a$ satisfy Landstad's coconditions (see [20], Definition 4.1) for all $a \in A$ and $f \in B_\mathrm{r}(G)$. So $aS_f(U), S_f(U)a \in A$ for all $a \in A$ and $f \in B_\mathrm{r}(G)$, by [20], Theorem 4.3. Then it follows from [1] that $U \in M(A \otimes C_\mathrm{r}^*(G))$. Thus, in this special case, the converse of Corollary 4.5 is also true.

The arguments in the proof of our final theorem are almost the same as in the proof of [15], Theorem 3.1 (while the (iii) $\Rightarrow$ (i) direction was already shown by Raeburn and Rosenberg ([25])). The difference here is that our group G is not necessarily abelian. So we have to work with a dual coaction rather than a dual action. We would like to mention that we have been informed that Igor Fulman, Paul Muhly and Dana Williams independently found an alternative proof of this fact without using coactions.

Recall that, for each representation π of A on $\mathcal{H}$, $\mathrm{Ind}\, \pi$ denotes the induced representation of $A \times_{\alpha,\mathrm{r}} G$ on $\mathcal{H} \otimes L^2(G)$ (see the discussion before Theorem 4.4). Let X be a locally compact Hausdorff space. We say that A is a $C_0(X)$-algebra if there exists a non-degenerate injection ι of $C_0(X)$ into the center of $M(A)$.

Suppose further that G acts on X. Then A is called a G-$C_0(X)$-algebra if ι is G-equivariant. If A has Hausdorff spectrum, then A is always G-$C_0(\widehat{A})$-algebra via the Dauns-Hofmann theorem.

THEOREM 4.7. *Let (A, G, α) be a separable C^*-dynamical system with a continuous trace algebra A such that G acts freely on $\widehat{A}$. Then the following statements are equivalent:*

(i) *$A \times_\alpha G$ has continuous trace;*

(ii) *$A \times_{\alpha,\mathrm{r}} G$ has continuous trace;*

(iii) *G acts properly on $\widehat{A}$.*

Moreover, if one of these conditions is satisfied, then $A \times_\alpha G = A \times_{\alpha,\mathrm{r}} G$.

Proof. Since the reduced crossed product is the quotient of the full crossed product, it is clear that (i) implies (ii). So suppose that (ii) holds. First, we show that the dual coaction of α is pointwise unitary. Let $\rho \in (A \times_{\alpha,\mathrm{r}} G)^\wedge$. By the Gootman-Rosenberg Theorem ([7]), there is a $\pi \in \widehat{A}$ such that the induced primitive ideal $\mathrm{Ind}(\ker \pi) = \ker(\mathrm{Ind}\, \pi)$ contains $\ker \rho$. Then [15], Lemma 3.2 implies that $\mathrm{Ind}\, \pi$ is irreducible ([15], Lemma 3.2 is stated for G abelian, but the proof remains valid if G is an arbitrary locally compact group). Since $(A \times_{\alpha,\mathrm{r}} G)^\wedge$ is Hausdorff, this implies that $\rho \cong \mathrm{Ind}\, \pi$.

As usual, let M be the representation of $C_0(G)$ as multiplication operators on $L^2(G)$. By [6], Proposition 2.6, $(\mathrm{Ind}\, \pi, 1 \otimes M)$ is a covariant representation of $(A \times_{\alpha,\mathrm{r}} G, G, \widehat{\alpha})$. Thus, $\widehat{\alpha}$ is pointwise unitary. Since $A \times_{\alpha,\mathrm{r}} G$ has continuous trace, it follows from Theorem 3.6 that $((A \times_{\alpha,\mathrm{r}} G) \times_{\widehat{\alpha}} G)^\wedge$ is Hausdorff, and G acts properly on $((A \times_{\alpha,\mathrm{r}} G) \times_{\widehat{\alpha}} G)^\wedge$ via the double dual action $\widehat{\widehat{\alpha}}$. By the Imai-Takai duality theorem ([8]), there is an isomorphism of $(A \times_{\alpha,\mathrm{r}} G) \times_{\widehat{\alpha}} G$ onto $A \otimes \mathcal{K}(L^2(G))$ which carries the second dual action $\widehat{\widehat{\alpha}}$ into $\alpha \otimes \mathrm{Ad}\rho_G$ (ρ_G being the right regular representation). So G acts properly on $(A \otimes \mathcal{K}(L^2(G)))^\wedge$ via $\alpha \otimes \mathrm{Ad}\rho_G$. Since the homeomorphism

$$\widehat{A} \to (A \otimes \mathcal{K}(L^2(G)))^\wedge, \quad \pi \to \pi \otimes \mathrm{id}$$

intertwines the actions induced by α and $\alpha \otimes \mathrm{Ad}\rho_G$, the action of G on $\widehat{A}$ induced by α must be proper, and (iii) follows.

If G acts properly on $\widehat{A}$, then the full crossed product $A \times_\alpha G$ has continuous trace by [25], Theorem 1.1. Thus, (iii) implies (i). Since A has continuous trace, it is a G-$C_0(\widehat{A})$-algebra. Therefore, (iii) and [9], Theorem 3.13 imply that $A \times_\alpha G = A \times_{\alpha,\mathrm{r}} G$. ∎

APPENDIX A. FULL COACTIONS

In this paper, we prefer to work with reduced coactions. One reason for this is that Katayama's duality ([10]), which is used in Theorem 4.4, is only secured for reduced coactions and may fail for full coactions ([18]). However, in this appendix we show that the main results in this paper also hold for full coactions. We retrieve these results by "reducing" our full coactions. This reduction process behaves nicely in the sense that a coaction and its reduction have the same crossed products, the same covariant representations and the same invariant ideals (see Theorem A.2 and Lemma A.3). Moreover, the property of being (pointwise, respectively locally) unitary is not affected by reduction (Lemma A.4). Full coactions are defined in a similar way as reduced coactions. We replace $C^*_{\mathrm{r}}(G)$ by the full group C^*-algebra $C^*(G)$, and we work with the canonical embedding $u_G : G \to M(C^*(G))$ instead of working with the left regular representation. Further, we replace $W_G \in M(C_0(G) \otimes C^*_{\mathrm{r}}(G))$ by $w_G \in M(C_0(G) \otimes C^*(G))$, which is now given by $s \mapsto u_G(s)$. The comultiplication $\delta^f_G : C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the integrated form of the map $s \mapsto u_G(s) \otimes u_G(s)$. Note that $W_G = (\mathrm{id} \otimes \lambda_G)(w_G)$ and $\delta_G \circ \lambda_G = (\lambda_G \otimes \lambda_G) \circ \delta^f_G$. Now a full coaction δ of G on a C^*-algebra A is defined to be a non-degenerate injective $*$-homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying

$$\delta(A)(1 \otimes C^*(G)) \subset A \otimes C^*(G), \tag{A.1}$$

and

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta^f_G) \circ \delta \text{ as maps of } A \text{ into } M(A \otimes C^*(G) \otimes C^*(G)). \tag{A.2}$$

As for reduced cosystems, we say that δ is non-degenerate if equality holds in (A.1). Now, for full coactions, we define covariant representations, crossed products, invariant ideals, (locally, respectively pointwise) unitary coactions and exterior equivalence as in the reduced case by replacing $C^*_{\mathrm{r}}(G)$ by $C^*(G)$, W_G by w_G and δ_G by δ^f_G.

We say that a full cosystem (A, G, δ) is *normal* if the map $j_A : A \to M(A \times_\delta G)$ is injective. It may happen that a full cosystem (A, G, δ) is not normal (consider for example the cosystem $(C^*(G), \delta^f_G, G)$ for non-amenable G as in [18], Corollary 2.6). However, we have the following:

PROPOSITION A.1. *Every pointwise unitary full cosystem (A, G, δ) is normal.*

Proof. Since δ is pointwise unitary, it follows from the fact that the direct sum of covariant representations is again a covariant representation (see [24], before Lemma 2.10) that there exists a covariant representation (π, μ) with π faithful. Thus, δ is normal by [19], Lemma 2.2. ∎

The following theorem is a summary of some results by John Quigg ([19]).

THEOREM A.2. *Let A be a C^*-algebra. For any normal non-degenerate full coaction δ of G on A define $\delta^{\mathrm{r}} := (\mathrm{id}\otimes\lambda_G)\circ\delta$. Then $(A, G, \delta^{\mathrm{r}})$ is a non-degenerate reduced cosystem called the reduction of (A, G, δ). This reduction process yields a one-to-one correspondence between the normal non-degenerate full coactions and the non-degenerate reduced coactions of G on A. Moreover, the cosystems (A, G, δ) and $(A, G, \delta^{\mathrm{r}})$ have the same covariant representations and the same crossed product.*

LEMMA A.3. *Let (A, G, δ) be a normal non-degenerate full coaction, and let $(A, G, \delta^{\mathrm{r}})$ be its reduction. Then an ideal $I \subset A$ is δ-invariant if and only if it is δ^{r}-invariant.*

Proof. Let I be δ-invariant. Then

$$\delta^{\mathrm{r}}(I)(1\otimes C^*_{\mathrm{r}}(G)) = (\mathrm{id}\otimes\lambda_G)(\delta(I)(1\otimes C^*(G))) = (\mathrm{id}\otimes\lambda_G)(I\otimes C^*(G)) = I\otimes C^*_{\mathrm{r}}(G),$$

and I is δ^{r}-invariant. For the converse, suppose that I is δ^{r}-invariant. Then $\overline{\delta^{\mathrm{r}}(I)(1 \otimes C^*_{\mathrm{r}}(G))} = I \otimes C^*_{\mathrm{r}}(G)$, and it follows from [20], Lemma 2.2 (2), that $\overline{\delta^{\mathrm{r}}_{A(G)}(I)} = I$. We have $\delta_{A(G)}(I) = \delta^{\mathrm{r}}_{A(G)}(I)$, and this implies that $\overline{\delta_{A(G)}(I)} = I$. By [19], Corollary 1.6, $\delta(I)(1 \otimes C^*(G)) = I \otimes C^*(G)$, and I is δ-invariant. ∎

LEMMA A.4. *Let A be a liminal C^*-algebra with Hausdorff spectrum. Let δ be a normal full coaction on A, and let δ^{r} be its reduction. Then δ is (pointwise, respectively locally) unitary if and only if δ^{r} is (pointwise, respectively locally) unitary.*

Proof. The coaction δ is unitary if and only if there is a non-degenerate homomorphism $\phi : C_0(G) \to M(A)$ such that (id, ϕ) is a covariant representation of (A, G, δ). Thus, by Theorem A.2, the lemma is true for unitary coactions and also for pointwise unitary coactions. Suppose that δ is locally unitary. Since locally unitary implies pointwise unitary, it follows from [12], Theorem 5.3 (2) and Lemma A.3 that every ideal I of A is invariant for δ and δ^{r}. We have $(\delta_I)^{\mathrm{r}} = (\delta^{\mathrm{r}})_I$, and the first part of the proof implies that δ^{r} is locally unitary. The converse direction is proven in the same way. ∎

The following lemma is used in Proposition 2.7. Let X be a locally compact Hausdorff space, and let (A, G, δ) be a (full or reduced) cosystem. Recall that A is called a $C_0(X)$-algebra if there exists a non-degenerate injection $\iota : C_0(X) \to ZM(A)$. Further, as in [14], Chapter 3, we say that δ is a $C_0(X)$-coaction if $\delta(\iota(f)) = \iota(f) \otimes 1$ for all $f \in C_0(X)$.

LEMMA A.5. *Let A be a liminal C^*-algebra with Hausdorff spectrum, and let δ be a (full or reduced) pointwise unitary coaction on A. Then there is a C^*-bundle E over $\widehat{A}$ such that $A \times_\delta G = \Gamma_0(E)$, and, for each $\rho \in \widehat{A}$, the fiber B_ρ is a crossed product by a unitary (full, respectively reduced) coaction δ_ρ on the elementary algebra $A/\ker\rho$.*

Proof. By the foregoing results, we may suppose that δ is a full cosystem. Let $\iota : C_0(\widehat{A}) \to ZM(A)$ be the Dauns-Hofmann isomorphism. As δ is pointwise unitary and $\widehat{A}$ is Hausdorff, the arguments used in the proof of [12], Proposition 5.3 (1), show that $\delta(\iota(f)) = \iota(f) \otimes 1$ for all $f \in C_0(\widehat{A})$. Thus, δ is a $C_0(\widehat{A})$-coaction on the $C_0(\widehat{A})$-algebra A. Now fix $\rho \in \widehat{A}$. Since δ is pointwise unitary, the coaction $\delta_\rho := \delta^{\ker\rho}$ of G on A_ρ is unitary. The rest of the proof now follows from [14], Theorem 4.3. ∎

The results above allow us to transmit the results of Chapter 3 and Chapter 4 to full coactions. First note that Proposition 2.7 also holds for pointwise unitary full coactions since the covariant representations and the crossed products of a normal full coaction and its reduction coincide by Theorem A.2.

Let (A, G, δ) be a non-degenerate normal full cosystem, and let $(A \times_\delta G, j_A, j_{C_0(G)})$ be its crossed product. As for reduced coactions, we define a dual action of G on $A \times_\delta G$ by

$$\widehat{\delta}_s(j_A(a)j_{C_0(G)}(f)) = j_A(a)j_{C_0(G)}(\sigma_s(f))$$

for all $a \in A$, $f \in C_0(G)$ and $s \in G$, where σ_s is the right translation by $s \in G$ ([24], Corollary 2.14). By Theorem A.2, the dual actions of δ and δ^{r} agree. Since δ^{r} is pointwise unitary if and only if δ is, we obtain a result analogous to Theorem 3.6 and Theorem 3.7.

THEOREM A.6. *Let (A, G, δ) be a separable pointwise unitary full cosystem such that A has continuous trace. Then the crossed product $A \times_\delta G$ has continuous trace, and the restriction map $\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$ is a proper G-bundle. Moreover, $A \times_\delta G$ is isomorphic to the pull back $\mathrm{Res}^* A$.*

We are now going to verify that Theorem 4.4 and Theorem 4.5 hold also for full coactions.

THEOREM A.7. *Let (A, G, δ) be a separable pointwise unitary full cosystem such that A has continuous trace and such that G is a Lie group. Then δ is locally unitary.*

Proof. This is an immediate consequence of Lemma A.4 and Theorem 4.4. ∎

THEOREM A.8. *Let (A, G, δ) and (A, G, ε) be two separable pointwise unitary full cosystems such that A has continuous trace. If δ and ε are exterior equivalent, then the proper G-bundles $\mathrm{Res}_\delta : (A \times_\delta G)^\wedge \to \widehat{A}$ and $\mathrm{Res}_\varepsilon : (A \times_\varepsilon G)^\wedge \to \widehat{A}$ are isomorphic. If G is a Lie group, then the converse is also true.*

Proof. Let U be a δ-cocycle such that $\varepsilon = \mathrm{Ad}\, U \circ \delta$. Then

$$(\mathrm{id} \otimes \delta_G^f)(U) = U_{12} \cdot (\delta \otimes \mathrm{id})(U).$$

If we apply $(\mathrm{id} \otimes \lambda_G \otimes \lambda_G)$, we see that

$$(\mathrm{id} \otimes (\delta_G \circ \lambda_G))(U) = ((\mathrm{id} \otimes \lambda_G)(U))_{12} \cdot (\delta^{\mathrm{r}} \otimes \mathrm{id})((\mathrm{id} \otimes \lambda_G)(U)).$$

Thus, $(\mathrm{id} \otimes \lambda_G)(U)$ is a δ^{r}-cocycle, and $\varepsilon^{\mathrm{r}} = \mathrm{Ad}((\mathrm{id} \otimes \lambda_G)(U)) \circ \delta^{\mathrm{r}}$. Therefore, δ^{r} and ε^{r} are exterior equivalent. Since δ^{r} and ε^{r} are pointwise unitary, Theorem 4.5 yields that the G-bundles $\mathrm{Res}_\delta : (A \times_\delta G)^\wedge \to \widehat{A}$ and $\mathrm{Res}_\varepsilon : (A \times_\varepsilon G)^\wedge \to \widehat{A}$ are isomorphic.

If G is a Lie group, then δ and ε are locally unitary by Theorem A.7. If we replace λ_G by u_G and $C^*_{\mathrm{r}}(G)$ by $C^*(G)$ in the proof of [12], Theorem 5.11, we see that [12], Theorem 5.11, also holds for full locally unitary coactions. Thus, δ and ε are exterior equivalent. ∎

Note added in proof. The result in Theorem 3.6 concerning the fact that the crossed product has continuous trace is much easier verified (without using Proposition 3.4) as follows: For a C^*-algebra B, let $T^+(B)$ (respectively $T^+_M(B)$) be the cone of all positive $b \in B$ (respectively $b \in M(B)$) such that the map $\pi \mapsto \mathrm{tr}\pi(b)$ is finite and continuous on $\widehat{B}$ (see [2], 4.5.2). Let (A, G, δ) be a pointwise unitary cosystem with A having continuous trace. Since the restriction map $\mathrm{Res} : (A \times_\delta G)^\wedge \to \widehat{A}$ is well defined and continuous, the image of $T^+(A)$ under j_A is contained in $T^+_M(A \times_\delta G)$. By assumption, the linear span of $T^+(A)$ is a dense ideal in A. Hence, it follows that the linear span of $T^+(A \times_\delta G)$ is a dense ideal in $A \times_\delta G$. Thus, $A \times_\delta G$ has continuous trace.

Note that these arguments do not carry over to the second (much more important) assertion of Theorem 3.6, namely that the dual action is proper, which is the key result for the applications in Chapter 4.

Acknowledgements. I would like to thank my advisor Siegfried Echterhoff for his suggestions and guidance.

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Received February 8, 1998; revised May 25, 1998.