

## THE COMPLETE SEPARABLE EXTENSION PROPERTY

HASKELL ROSENTHAL

*Communicated by William B. Arveson*

ABSTRACT. This work introduces operator space analogues of the Separable Extension Property (SEP) for Banach spaces, the Complete Separable Extension Property (CSEP) and the Complete Separable Complementation Property (CSCP). The results use the technique of a new proof of Sobczyk’s Theorem, which also yields new results for the SEP in the non-separable situation, e.g.,  $(\bigoplus_{n=1}^{\infty} Z_n)_{c_0}$  has the  $(2 + \varepsilon)$ -SEP for all  $\varepsilon > 0$  if  $Z_1, Z_2, \dots$  have the 1-SEP; in particular,  $c_0(\ell^\infty)$  has the SEP. It is proved that e.g.,  $c_0(\mathcal{R} \oplus \mathcal{C})$  has the CSEP (where  $\mathcal{R}, \mathcal{C}$  denote Row, Column space respectively) as a consequence of the general principle: if  $Z_1, Z_2, \dots$  is a uniformly exact sequence of injective operator spaces, then  $(\bigoplus_{n=1}^{\infty} Z_n)_{c_0}$  has the CSEP. Similarly, e.g.,  $\mathbf{K}_0 \stackrel{\text{def}}{=} (\bigoplus_{n=1}^{\infty} M_n)_{c_0}$  has the CSCP, due to the general principle:  $(\bigoplus_{n=1}^{\infty} Z_n)_{c_0}$  has the CSCP if  $Z_1, Z_2, \dots$  are injective separable operator spaces. Further structural results are obtained for these properties, and several open problems and conjectures are discussed.

KEYWORDS: *Operator space, local reflexivity, space of compact operators, completely complemented subspace, Sobczyk’s theorem.*

MSC (2000): 46B., 46B15, 47D25.

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## 0. INTRODUCTION

We study here “quantized” or “operator space” versions of the following well known extension property for Banach spaces.

DEFINITION 0.1. A Banach space  $Z$  is said to have the *Separable Extension Property* (SEP) provided for all separable Banach spaces  $Y$ , closed linear subspaces  $X$ , and bounded linear operators  $T : X \rightarrow Z$ , there exists a bounded linear operator  $\tilde{T} : Y \rightarrow Z$  extending  $T$ .

That is, we have the diagram

$$(0.1) \quad \begin{array}{ccc} Y & & \\ \cup & \searrow^{\tilde{T}} & \\ X & \xrightarrow{T} & Z. \end{array}$$

If  $\lambda \geq 1$  is such that  $\tilde{T}$  can always be chosen with  $\|\tilde{T}\| \leq \lambda\|T\|$ , we say  $Z$  has the  $\lambda$ -SEP.

In 1941, A. Sobczyk proved that  $c_0$  has the SEP; in fact, he showed that  $c_0$  has the 2-SEP, and “2” is best possible ([29]). In 1978, M. Zippin established the deep converse to this result: *If  $Z$  is a separable infinite-dimensional Banach space with the SEP, then  $Z$  is isomorphic to  $c_0$*  ([35]). These results in a sense “end” the study of separable Banach spaces with the SEP. To the contrary, we show below that “quantized” versions of the SEP yield a rich “open-ended” theory. These quantized versions are founded on a new proof for Sobczyk’s theorem, given in Section 1, which actually yields new information for non-separable spaces  $Z$  with the SEP. For example, we obtain in Corollary 1.7 that  $c_0(\ell^\infty)$  has the SEP; in fact the  $(2 + \varepsilon)$ -SEP for all  $\varepsilon > 0$ . This follows immediately from the new result, Corollary 1.5: *if  $Z_1, Z_2, \dots$  have the 1-SEP, then  $(Z_1 \oplus Z_2 \oplus \dots)_{c_0}$  has the  $(2 + \varepsilon)$ -SEP, for all  $\varepsilon > 0$* . A modification of our argument, due to T. Oikhberg, yields the satisfying permanence property: *if  $\lambda \geq 1$  and  $Z_1, Z_2, \dots$  have the  $\lambda$ -SEP, then  $(Z_1 \oplus Z_2 \oplus \dots)_{c_0}$  has the  $(\lambda^2 + \lambda + \varepsilon)$ -SEP for all  $\varepsilon > 0$* . (After circulating the first draft of this paper, I learned that previously known results yield that  $c_0(\ell^\infty)$  has the 2-SEP — see Remark 1.15 (iii) after Corollary 1.14 below.)

Other proofs of Sobczyk’s theorem have been given by A. Pełczyński ([23]) and W. Veech ([32]). We show in Corollary 1.23 below that Veech’s argument yields the isometric result: *Let  $c_0 \subsetneq Y \subset \ell^\infty$  with  $Y$  separable, and set  $Z = Y/c_0$ . Then the short exact sequence*

$$0 \longrightarrow c_0 \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

admits a norm-one lift.

That is, letting  $\pi : Y \rightarrow Z$  be the quotient map, there is a norm-one operator  $L : Z \rightarrow Y$  with

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & Z \\ L \swarrow & & \uparrow I \\ & & Z \end{array} .$$

Equivalently,  $c_0$  is contractively cocomplemented in  $Y$ ; that is, there is a linear projection  $P$  from  $Y$  onto  $c_0$  with  $\|I - P\| = 1$ .

(This result immediately yields that  $c_0$  has the 2-SEP, in virtue of the injectivity of  $\ell^\infty$ .) In Theorem 1.19, we use Veech’s argument to obtain another generalization of Sobczyk’s theorem: Suppose  $X, Y, Z$  are Banach spaces with  $Y$  separable and  $X \subset Y$ , and suppose  $(T_n)$  is a sequence of operators from  $X$  to  $Z$  with  $T_n \rightarrow 0$  in the Strong Operator Topology (SOT). If  $(T_n)$  admits an extension  $(T'_n)$  to  $Y$  with  $(T'_n)$  relatively compact in the SOT, then  $(T_n)$  admits an extension  $(\tilde{T}_n)$  to  $Y$  with  $\tilde{T}_n \rightarrow 0$  in the SOT.

That is, setting  $T = (T_n), T' = (T'_n), \tilde{T} = (\tilde{T}_n)$ , we have:

Hypotheses:

$$\begin{array}{ccc} Y & \xrightarrow{T'} & \ell^\infty(Z) \\ \cup & & \cup + (T'_n) \text{ SOT-relatively compact.} \\ X & \xrightarrow{T} & c_0(Z) \end{array}$$

Conclusion:

$$\begin{array}{ccc} Y & & \\ \cup & \searrow \tilde{T} & \\ X & \xrightarrow{T} & c_0(Z). \end{array}$$

The first quantized version of the SEP that we study is the Complete Separable Extension Property (CSEP) for operator spaces. The definition is obtained by simply inserting “Completely” before “Separable” in the definition of the SEP. Again, if in the diagram (0.1),  $\tilde{T}$  may always be chosen with  $\|\tilde{T}\|_{cb} \leq \lambda \|T\|_{cb}$ , we say  $Z$  has the  $\lambda$ -CSEP.

We now briefly recall the following basic concept. (For fundamental background and references, see [22] and [25].)

By an operator space  $X$ , we mean a Banach space  $X$  which is a closed linear subspace of  $\mathcal{L}(H)$ , the bounded linear operators on some Hilbert space  $H$ , endowed with its natural tensor product structure with  $\mathbf{K}$ , the space of compact operators on separable infinite dimensional Hilbert space (which we take as  $\ell^2$  for definiteness). Thus  $\mathbf{K} \otimes_{op} X$  denotes the closed linear span in  $\mathcal{L}(\ell^2 \otimes_2 H)$  of the operators  $K \otimes T$  where  $K \in \mathbf{K}$  and  $T \in X$  (and  $\ell^2 \otimes_2 H$  is the Hilbert-space

tensor product of  $\ell^2$  and  $H$ ). Given operator spaces  $X$  and  $Y$ , a linear operator  $T : X \rightarrow Y$  is called *completely bounded* if

$$I \otimes T : \mathbf{K} \otimes_{\text{op}} X \rightarrow \mathbf{K} \otimes_{\text{op}} Y$$

is bounded, where  $I$  denotes the identity operator on  $\mathbf{K}$ ; then we set  $\|T\|_{\text{cb}} = \|I \otimes T\|$ . It then follows easily that if  $X_i, Y_i$  are operator spaces and  $T_i : X_i \rightarrow Y_i$  are completely bounded, then  $T_1 \otimes T_2$  is completely bounded, with  $\|T_1 \otimes T_2\|_{\text{cb}} \leq \|T_1\|_{\text{cb}} \|T_2\|_{\text{cb}}$ . Now many natural Banach space concepts have their operator space versions. Thus, operator spaces  $X$  and  $Y$  are called *completely isomorphic* if there exists an invertible  $T : X \rightarrow Y$  with  $T$  and  $T^{-1}$  completely bounded. If  $\|T\|_{\text{cb}} \|T^{-1}\|_{\text{cb}} \leq \lambda$ , we say  $X$  and  $Y$  are  $\lambda$ -*completely isomorphic*. We then define  $d_{\text{cb}}(X, Y)$ , the completely bounded distance between  $X$  and  $Y$ , by  $d_{\text{cb}}(X, Y) = \inf\{\lambda \geq 1 : X \text{ is } \lambda\text{-completely isomorphic to } Y\}$ . If  $X \subset Y$ , with  $Y$  an operator space and  $X$  a closed linear subspace, then  $X$  is regarded as an operator subspace of  $Y$ , via its natural structure  $\mathbf{K} \otimes_{\text{op}} X \subset \mathbf{K} \otimes_{\text{op}} Y$ .  $X$  is called *completely complemented* if there is a completely bounded projection from  $Y$  onto  $X$ . We may then loosely say: *A separable operator space  $X$  has the CSEP provided it is completely complemented in every separable operator superspace.* (After the first draft of this paper was completed, it was discovered that this ‘‘loose’’ statement is actually a theorem, see [21].)

Of course  $\mathbf{K}$  may be identified with a certain Banach space of infinite matrices, namely those representing compact operators on  $\ell^2$  (with respect to its natural basis). For an operator space  $X$ ,  $\mathbf{K} \otimes_{\text{op}} X$  may also be visualized as a Banach space of infinite matrices, all of whose elements come from  $X$ . We let  $M_n$  denote all  $n \times n$  matrices of complex scalars, regarded as  $\mathcal{L}(\ell_n^2)$ ; we also let  $M_{00}$  denote all infinite matrices of scalars, with only finitely many non-zero entries. Thus we may regard  $M_n \subset M_{n+1} \subset \cdots \subset M_{00} \subset \mathbf{K}$ . Now it follows easily that if  $P_n : \mathbf{K} \rightarrow M_n$  is the canonical projection, then

$$(0.2) \quad P_n \otimes I \rightarrow I \otimes I \quad \text{in the SOT, on } \mathbf{K} \otimes_{\text{op}} X.$$

For  $T : X \rightarrow Y$  a bounded linear operator,  $n \geq 1$ , we define  $\|T\|_n$  by

$$(0.3) \quad \|T\|_n = \|P_n \otimes T\|.$$

(Equivalently, if  $I_n = \text{Identity in } \mathcal{L}(\ell_n^2)$ ,  $\|T\|_n = \|I_n \otimes T\|$ .) It then follows easily from (0.2) that  $T$  is completely bounded iff  $(\|T\|_n)$  is bounded, and then

$$(0.4) \quad \|T\|_{\text{cb}} = \sup_n \|T\|_n.$$

(This easy fact is sometimes taken as the definition of complete-boundedness.) Visualizing  $\mathbf{K} \otimes_{\text{op}} X$  as infinite matrices, we easily then have (by the closed graph theorem) that a bounded linear operator  $T : X \rightarrow Y$  is completely bounded exactly when  $(Tx_{ij})$  belongs to  $\mathbf{K} \otimes_{\text{op}} Y$  for each  $(x_{ij})$  in  $\mathbf{K} \otimes_{\text{op}} X$ ; of course then  $(I \otimes T)(x_{ij}) = (Tx_{ij})$ .

Evidently the concept of an operator space is completely captured by the Banach space  $\mathbf{K} \otimes_{\text{op}} X$ . Remarkable axioms of Z.J. Ruan (cf. [10], [25]) abstractly characterize this tensor product, without reference to the ambient Hilbert space. Finally, we note that any Banach space  $X$  can be regarded as an operator space via the so-called MIN structure (where  $\|(x_{ij})\|_{\text{MIN}} = \sup\{\|(x^*(x_{ij}))\| : x^* \in X^*, \|x^*\| = 1\}$  (with  $\|x^*(x_{ij})\|$  the norm in  $\mathcal{L}(H)$ ). Thus formally, Banach space theory is “subsumed” by operator space theory. However this observation is useless for a Banach space  $X$  unless it is closely related to  $\mathcal{L}(H)$  and its natural subspaces, preduals of such, etc. In fact, we can alternatively say that operator space theory is simply a special (but very deep!) case of the general theory of tensor products of Banach spaces.

What are some examples of operator spaces with the CSEP? Of course  $c_0$  has this property; we may “visualize”  $c_0$  as an operator space, by simply identifying it with the space of *diagonal* matrices in  $\mathbf{K}$ . Similarly, we define  $\mathcal{R}$ , the operator Row Space, to be the space of all matrices in  $\mathbf{K}$  with entries only in the first row; of course we then define  $\mathcal{C}$ , the Column Space, as all matrices with entries only in the first column.

It is easily seen that  $\mathcal{R}$  and  $\mathcal{C}$  have the 1-CSEP. We prove (see Corollary 2.15) that  $c_0(\mathcal{R} \oplus \mathcal{C})$  has the  $(2 + \varepsilon)$ -CSEP for all  $\varepsilon > 0$ . (Throughout, direct sums of operator spaces are taken in the  $\ell^\infty$ -sense.) A deep open problem: *Let  $X$  be separable with the CSEP. Is  $X$  completely isomorphic to a subspace of  $c_0(\mathcal{R} \oplus \mathcal{C})$ ?* Of course an affirmative answer would be the direct analogue of Zippin’s theorem for the CSEP. The CSEP structure problem even for subspaces of  $c_0(\mathcal{R} \oplus \mathcal{C})$  is somewhat involved, however. In Section 4, we distinguish 21 (apparently) different infinite-dimensional operator subspaces of  $c_0(\mathcal{R} \oplus \mathcal{C})$  with the CSEP, representing six isomorphically distinct Banach spaces; it is conceivable that this is the full list (up to complete isomorphism) of all infinite-dimensional separable spaces with the CSEP.

The proof that  $c_0(\mathcal{R} \oplus \mathcal{C})$  has the CSEP uses the concept of uniformly exact families of operator spaces (see Definition 2.5 below). Using Oikhberg’s modification of our argument in Section 1 mentioned above, we obtain in Corollary 2.10 that *if  $Z_1, Z_2, \dots$  are operator spaces so that  $\{Z_1, Z_2, \dots\}$  is uniformly exact and*

the  $Z_j$ 's all have the  $\lambda$ -CSEP for some  $\lambda \geq 1$ , then  $(Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$  also has the CSEP.

We show in Proposition 2.11 that for all  $n$ ,  $M_{\infty,n}$  and  $M_{n,\infty}$  are 1-uniformly exact (where, e.g.,  $M_{\infty,n}$  denotes the  $\infty \times n$ -matrices in  $\mathbf{K}$ ). Since  $M_{\infty,n}$ ,  $M_{n,\infty}$  both have the 1-CSEP, we in fact obtain the following ‘‘almost isometric’’ version of the isometric lifting property for  $c_0$  mentioned above (via Corollary 2.9): *Fix  $n$ , and suppose*

$$X = c_0(M_{\infty,n} \oplus M_{n,\infty}) \subsetneq Y \subset \ell^\infty(M_{\infty,n} \oplus M_{n,\infty})$$

with  $Y$  separable; set  $Z = Y/X$ . Then the short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

admits an almost completely contractive lift. That is, letting  $\pi : Y \rightarrow Z$  be the quotient map, then given  $\varepsilon > 0$ , there exists an  $L : Z \rightarrow Y$  with  $\|L\|_{cb} < 1 + \varepsilon$  and

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & Z \\ L \nearrow & & \uparrow I \\ & & Z \end{array} .$$

(This immediately yields that  $c_0(M_{\infty,n} \oplus M_{n,\infty})$  has the  $(2+\varepsilon)$ -CSEP for all  $\varepsilon > 0$ , in view of the fact that  $\mathcal{L}(H)$  is operator-isometrically injective (see [25]).)

*Added in proof.* Recent joint work of A. Arias and myself yields that this short exact sequence admits a completely contractive lift, and hence  $c_0(M_{\infty,n} \oplus M_{n,\infty})$  has the 2-CSEP. See A. Arias and H.P. Rosenthal, M-complete approximate identities in operator spaces, *Studia Math.*, to appear.

We also show that our argument for Theorem 1.19 (via Veech’s technique ([32])) immediately extends to the operator space version (Theorem 2.37). In particular, we obtain for fixed  $n$ , that  $c_0(M_n)$  is completely contractively cocomplemented in  $Y$  for any separable  $Y$  with  $c_0(M_n) \subsetneq Y \subset \ell^\infty(M_n)$ . Thus  $c_0(M_n)$  has the 2-CSEP.

Although the CSEP thus has its isometric aspects, the fundamental constant that enters in the CSEP is 2, unless we are already dealing with *injective* operator spaces (i.e., no restrictions on separability in the fundamental diagram (0.1) for completely bounded maps). Indeed, we show in Proposition 2.40 that if  $X$  is separable with the  $\lambda$ -CSEP and  $\lambda < 2$ , then  $X$  is reflexive, whence (via Proposition 2.36)  $X$  is  $\lambda$ -injective.

$\mathbf{K}$  is often regarded as the quantized version of  $c_0$ , although  $\mathbf{K}_0 \stackrel{\text{def}}{=} (M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus \cdots)_{c_0}$  is another possible candidate. What has become of  $\mathbf{K}$  in our

quantization of Sobczyk’s theorem? E. Kirchberg establishes in [17] that  $\mathbf{K}$  fails the CSEP. In fact, Kirchberg obtains a separable  $C^*$ -algebra  $\mathcal{A}$  and a (two-sided) ideal  $J \subset \mathcal{A}$  with  $J$   $*$ -isomorphic to  $\mathbf{K}$  and  $J$  completely uncomplemented in  $\mathcal{A}$  (and moreover  $\mathcal{A}/J$  is an exact  $C^*$ -algebra). Now we note that our positive results hold under a formally weaker hypothesis, that of families of operator spaces of finite matrix type (see Definition 2.29). T. Oikhberg has established that if conversely  $Z_1, Z_2, \dots$  are separable operator spaces and  $\{Z_1, Z_2, \dots\}$  is *not* of finite matrix type, then  $(Z_1 \oplus Z_2 \oplus \dots)_{c_0}$  fails the CSEP (see [21]).

In particular,  $\mathbf{K}_0$  fails the CSEP (which seems quite surprising since  $c_0(M_n)$  has the 2-CSEP for all  $n$ ). (Actually, it follows from Kirchberg’s work ([17]) and the complete isomorphic invariance of exactness for  $C^*$ -algebras (cf. [25]), that  $\mathbf{K}_0$  fails the CSEP, see Remarks 4.4 Section 4, [33].) It turns out that in these counterexamples, the “culprit” is the lack of local reflexivity of the containing operator space.

In Section 3, we study a different quantized version of the SEP, the Complete Separable Complementation Property (CSCP), which goes as follows: *A separable locally reflexive operator space  $Z$  has the CSCP provided every complete isomorph of  $Z$  is completely complemented in every separable locally reflexive operator superspace.* Equivalently, there exists a completely bounded  $\tilde{T}$  so that the diagram (0.1) holds, provided  $Y$  is separable locally reflexive and  $T$  is a complete surjective isomorphism. We now indeed obtain that  $\mathbf{K}_0$  has the CSCP. It follows from the proof that if e.g.,  $\mathcal{A}$  is a separable nuclear  $C^*$ -algebra, and  $J$  is a  $*$ -subalgebra  $*$ -isomorphic to  $\mathbf{K}_0$ , then  $J$  is  $(4 + \varepsilon)$ -completely complemented in  $\mathcal{A}$ , for all  $\varepsilon > 0$  (Corollary 3.7).

The main result of Section 3, Theorem 3.3, (again via the Oikhberg modification mentioned above) goes as follows: *Let  $Z_1, Z_2, \dots$  be separable operator spaces so that for some  $\lambda \geq 1$ ,  $Z_j$  is  $\lambda$ -injective for all  $j$ . Then  $(Z_1 \oplus Z_2 \oplus \dots)_{c_0}$  has the CSCP.*

After the first draft of this paper was completed, it was discovered that  $\mathbf{K}$  itself has the CSCP (see [21]). It follows directly from Theorem 3.3 that  $\left(\bigoplus_{n=1}^{\infty} (M_{n,\infty} \oplus M_{\infty,n})\right)_{c_0}$  has the CSCP. Looking at the natural completely complemented subspaces of this space, as well as  $\mathbf{K}$  itself, it follows that there are at least 11 different Banach isomorphism types among the separable infinite-dimensional operator spaces with the CSCP, and apparently at least 11 different primary such spaces (see Section 4 for the relevant definition, and Proposition 4.4 and Conjecture 4.5 for the various examples). Taking finite direct sums of these, we obtain a finite but apparently astronomically large list of separable spaces with the CSCP. The deep open question here is thus: *Does every separable space with the CSCP,*

*completely embed in  $\mathbf{K}$ ?* Of course an affirmative answer would yield the “true” quantized version of Zippin’s Theorem ([35]), and also imply it! A more accessible problem: *Classify the infinite-dimensional completely complemented subspaces of  $\mathbf{K}$  up to complete isomorphism.*

## 1. THE SEPARABLE EXTENSION PROPERTY

This section is devoted to the pure Banach space category.

DEFINITION 1.1. A Banach space  $Z$  is said to have the *Separable Extension Property* (the SEP) if for all separable Banach spaces  $Y$ , (closed linear) subspaces  $X$ , and (bounded linear) operators  $T : X \rightarrow Z$ , there exists an operator  $\tilde{T} : Y \rightarrow Z$  extending  $T$ .  $Z$  is said to have the  $\lambda$ -SEP provided for all such  $X$  and  $Y$ ,  $\tilde{T}$  may be chosen with  $\|\tilde{T}\| \leq \lambda\|T\|$ .

We recall also

DEFINITION 1.2.  $Z$  is said to be  $\lambda$ -*injective* provided for arbitrary Banach spaces  $X$  and  $Y$  with  $X \subset Y$ , every operator  $T$  from  $X$  to  $Z$  has an extension  $\tilde{T}$  from  $Y$  to  $Z$  with  $\|\tilde{T}\| \leq \lambda\|T\|$ .

The results in this section yield new properties of certain non-reflexive Banach spaces. In the separable space setting, the techniques yield a new proof of Sobczyk’s theorem that  $c_0$  has the SEP ([31]), and also yield an intuition base for the operator-space discoveries given in the following sections. The main “external” motivation for the Banach category itself is in the non-separable setting, however, because of the profound discovery of M. Zippin ([35]): *Every infinite dimensional separable Banach space with the SEP is isomorphic to  $c_0$ .* We note also that if  $Z$  is a finite-dimensional Banach space, then  $Z$  has the  $\lambda$ -SEP iff  $Z$  is  $\lambda$ -injective iff  $Z$  has the  $\lambda$ -SEP just with respect to finite-dimensional spaces  $Y$  and subspaces  $X$ . The quantitative characterization of the finite-dimensional  $\lambda$ -injectives remains as one of the profound open questions in Banach space theory.

We also note the classical theorem that *a Banach space  $Z$  is 1-injective iff  $Z$  is isomorphic to  $C(\Omega)$  for some Stonian compact Hausdorff space  $\Omega$*  (real or complex scalars). In particular,  $\ell^\infty$  is 1-injective and an  $n$ -dimensional Banach space is 1-injective iff it is isometric to  $\ell_n^\infty$ . It also remains a deep open problem whether every infinite-dimensional  $\lambda$ -injective is isomorphic to a 1-injective. We note in passing the author’s result that  $\ell^\infty$  is the smallest  $\lambda$ -injective; i.e., every infinite-dimensional  $\lambda$ -injective contains a subspace isomorphic to  $\ell^\infty$  ([27]).

It can easily be seen that a Banach space  $Z$  has the SEP iff it has the  $\lambda$ -SEP for some  $\lambda \geq 1$ . I am indebted to M. Zippin for pointing out that this holds for

general  $Z$  and not just the special case of separable  $Z$  (see the Remark following the proof of Proposition 2.3 in the next section).

Before stating the main result of this section, we give the following notation: Given  $Z_1, Z_2, \dots$  Banach spaces,  $(Z_j)_{c_0}$  denotes the space  $(Z_1 \oplus Z_2 \oplus \dots)_{c_0}$ ; i.e., the space of all sequences  $(z_j), z_j \in Z_j$  for all  $j$ , with  $\|z_j\| \rightarrow 0$ ; under the natural norm  $\|(z_j)\| = \sup_j \|z_j\|$ . Similarly  $(Z_j)_{\ell^\infty}$  denotes the space of all bounded sequences  $(z_j)$  with  $z_j \in Z_j$  for all  $j$ , under the same norm as above. Thus  $(Z_j)_{c_0}$  is a subspace of  $(Z_j)_{\ell^\infty}$ . Now for a fixed space  $Z$ ,  $c_0(Z)$  denotes the space  $(Z \oplus Z \oplus \dots)_{c_0}$ , and similarly  $\ell^\infty(Z) = (Z \oplus Z \oplus \dots)_{\ell^\infty}$ . The reader is thus cautioned that for example,  $(\ell_n^2)_{c_0}$  denotes the space  $(\ell_1^2 \oplus \ell_2^2 \oplus \dots)_{c_0}$ , while  $c_0(\ell_n^2)$  denotes the space  $(\ell_n^2 \oplus \ell_n^2 \oplus \dots)_{c_0}$ , for fixed  $n$ .

The main result of this section is as follows.

**THEOREM 1.3.** *Let  $Z_1, Z_2, \dots$  be 1-injective Banach spaces,  $X, Y$  be Banach spaces with  $X \subset Y$  and  $Y/X$  separable, and set  $Z = (Z_j)_{c_0}$ . Then for every non-zero operator  $T : X \rightarrow Z$  and every  $\varepsilon > 0$ , there exists a  $\tilde{T} : Y \rightarrow Z$  extending  $T$  with  $\|\tilde{T}\| < (2 + \varepsilon)\|T\|$ .*

Before giving the proof, we give several consequences.

**COROLLARY 1.4.** *Let  $Z$  be as in Theorem 1.3. Then  $Z$  is  $2 + \varepsilon$ -complemented in every superspace  $Y$  with  $Y/Z$  separable.*

Recall that if  $X \subset Y$ ,  $X, Y$  Banach spaces,  $X$  is called  $\lambda$ -complemented in  $Y$  if there is a (bounded linear) projection  $P$  mapping  $Y$  onto  $X$  with  $\|P\| \leq \lambda$ ;  $Y$  is called a superspace of  $Z$  if  $Z \subset Y$ . Of course 1.2 follows immediately from Theorem 1.3.

We shall see that the *proof* of Theorem 1.3 yields

**COROLLARY 1.5.** *Let  $Z_1, Z_2, \dots$  be Banach spaces with the 1-SEP. Then  $(Z_j)_{c_0}$  has the  $2 + \varepsilon$ -SEP for all  $\varepsilon > 0$ .*

**REMARK 1.6.** T. Oikhberg has observed that a modification of our argument actually yields that if the  $Z_j$ 's have the  $\lambda$ -SEP, then  $(Z_j)_{c_0}$  has the  $(\lambda(1 + \lambda) + \varepsilon)$ -SEP for all  $\varepsilon > 0$ . See Remark 1.13 below.

We note in passing that Corollary 1.5 covers certain situations not handled by Theorem 1.3. For example, let  $\Gamma$  be an uncountable set, and let  $\ell_c^\infty(\Gamma)$  denote the space of all scalar-valued functions on  $\Gamma$  with countable support. It is easily seen that  $\ell_c^\infty(\Gamma)$  has the 1-SEP; however it is known that  $\ell_c^\infty(\Gamma)$  is not injective; i.e., not  $\lambda$ -injective for any  $\lambda$ . Corollary 1.5 thus yields that  $c_0(\ell_c^\infty(\Gamma))$  has the  $(2 + \varepsilon)$ -SEP for all  $\varepsilon > 0$ .

We note finally the following result, which is perhaps the main “external” motivation for this section.

**COROLLARY 1.7.** *Let  $\varepsilon > 0$ . Then  $c_0(\ell^\infty)$  has the  $(2 + \varepsilon)$ -SEP. Moreover  $c_0(\ell^\infty)$  is  $(2 + \varepsilon)$ -complemented in every superspace in which it is of separable codimension.*

**REMARK 1.8.** After circulating the first draft of this paper, I learned that this corollary follows from a known result concerning  $M$ -ideals in Banach spaces; moreover, one obtains that “ $\varepsilon$ ” may be deleted in the statement. See Remark 1.15 (iii) below.

We now pass to the proof of Theorem 1.3. We first state a reformulation. We abuse terminology slightly and say that given Banach spaces  $X, Z_1, Z_2, \dots$ , and operators  $T_j : X \rightarrow Z_j$  for all  $j$ , that  $(T_n)$  tends to zero in the SOT (Strong Operator Topology) provided  $\|T_n x\| \rightarrow 0$  for all  $x \in X$ . (Of course if  $Z_1 = Z_n$  for all  $n$ , this is what it means to say  $T_n \rightarrow 0$  in the SOT.)

**THEOREM 1.9.** *Let  $Z_1, Z_2, \dots, X$ , and  $Y$  be Banach spaces satisfying the hypotheses of Theorem 1.3. Let  $(T_j)$  be a sequence of operators with  $T_j : Y \rightarrow Z_j$  for all  $j$ , so that  $T_j|_X \rightarrow 0$  in the SOT and  $\sup_j \|T_j\| = 1$ . For every  $\varepsilon > 0$ , there exists a sequence  $(S_j)$  of operators so that for all  $j$ ,*

- (i)  $S_j : Y \rightarrow Z_j$ ;
- (ii)  $X \subset \ker S_j$ ;
- (iii)  $\|S_j\| \leq 1 + \frac{\varepsilon}{2}$ ;

so that  $(T_j - S_j) \rightarrow 0$  in the SOT.

We first show that Theorem 1.9  $\Rightarrow$  Theorem 1.3. (The equally easy converse is not needed, and left to the reader. Of course the converse motivated the formulation of Theorem 1.9.)

Let  $T : X \rightarrow Z$  be as in the statement of Theorem 1.3; without loss of generality, suppose  $\|T\| = 1$ . Let  $(T_j)$  be the corresponding sequence so that  $Tx = (T_j x)$  for all  $x \in X$ . Now for each  $j$ , since  $Z_j$  is 1-injective, choose  $T'_j : Y \rightarrow Z_j$  with  $\|T'_j\| = \|T_j\|$  and  $T'_j|_X = T_j$ . Of course then  $\sup_j \|T'_j\| = \sup_j \|T_j\| = 1$ .

Let us abuse notation and let  $T_j$  denote also the extended operator  $T'_j$  for all  $j$ . Now the hypotheses of Theorem 1.9 hold; choose  $(S_j)$  satisfying its conclusion. Now define  $\tilde{T} : Y \rightarrow (Z_j)_{\ell^\infty}$  by

$$(1.1) \quad \tilde{T}(y) = (T_j - S_j)(y) \quad \text{for all } y \in Y.$$

It follows from the conclusion of Theorem 1.9 that  $\tilde{T}$  actually has its range in  $Z = (Z_j)_{c_0}$ , and (ii) of Theorem 1.9 insures that  $\tilde{T}$  extends  $T$ ; of course  $\|\tilde{T}\| < 2 + \varepsilon$ , proving Theorem 1.9. ■

We next require the following rather surprising

LEMMA 1.10. *Let  $Z_1, Z_2, \dots, X, Y$ , and  $(T_j)$  be as in Theorem 1.9. Assume further that  $Y/X$  is finite-dimensional and let  $F$  be a finite-dimensional subspace of  $Y$  with  $X \oplus F = Y$ ; let  $P$  be the projection of  $Y$  onto  $F$  with kernel  $X$ . Then*

$$(1.2) \quad \overline{\lim}_{n \rightarrow \infty} \|T_n P\| \leq 1.$$

*Proof.* Suppose not. By passing to a subsequence, we can suppose without loss of generality that there is a  $C > 1$  so that

$$(1.3) \quad \|T_n P\| > C \quad \text{for all } n.$$

So for each  $n$ , choose  $x_n \in X$  and  $f_n \in F$  with

$$(1.4) \quad \|x_n \oplus f_n\| = 1 \quad \text{and} \quad \|T_n f_n\| > C.$$

Of course then  $\|f_n\| \leq \|P\|$  for all  $n$ ; since  $F$  is finite-dimensional, we can suppose by passing to a further subsequence, that

$$(1.5) \quad f_n \rightarrow f \quad \text{in norm,}$$

for a certain  $f$  in  $F$ . But then  $\|T_n(f_n - f)\| \rightarrow 0$  as  $n \rightarrow \infty$ , whence by (1.4) and (1.5),

$$(1.6) \quad \|T_n f\| > C \quad \text{for all } n \text{ sufficiently large.}$$

But also since  $x_n \oplus f_n - x_n \oplus f \rightarrow 0$  in norm by (1.5),

$$(1.7) \quad \lim_{n \rightarrow \infty} \|x_n \oplus f\| = 1 \quad \text{by (1.4).}$$

Now let  $\varepsilon > 0$  with  $1 + \varepsilon < C$ , and by (1.7), choose  $k$  with

$$(1.8) \quad \|x_k \oplus f\| \leq 1 + \varepsilon.$$

Then since  $\|T_n\| \leq 1$  for all  $n$ ,

$$(1.9) \quad \|T_n(x_k \oplus f)\| = \|T_n(x_k) + T_n(f)\| \leq 1 + \varepsilon.$$

Hence since  $\lim_{n \rightarrow \infty} \|T_n(x_k)\| = 0$  by hypothesis,

$$(1.10) \quad \overline{\lim}_{n \rightarrow \infty} \|T_n f\| \leq 1 + \varepsilon.$$

This of course contradicts (1.6). ■

REMARK 1.11. Lemma 1.10 does not require the hypothesis that the  $Z_j$ 's be 1-injective, and furthermore it immediately yields the conclusion of Theorem 1.9 in case  $Y/X$  is finite-dimensional. Indeed, choose  $m$  so that  $\|T_n P\| < 1 + \varepsilon$  for  $n \geq m$ . Let  $S_n = 0$  for  $n < m$  and  $S_n = T_n P$  for  $n \geq m$ . Then  $T_n - S_n = T_n(I - P)$  for  $n \geq m$ , so for  $y \in Y$ , since  $(I - P)y \in X$ ,  $\lim_{n \rightarrow \infty} \|(T_n - S_n)(y)\| = \|T_n(I - P)(y)\| = 0$ .

We thus obtain the following consequence:

COROLLARY 1.12. *Let  $X \subset Y$  with  $Y/X$  finite-dimensional,  $(Z_j)$  a given sequence of Banach spaces, and  $T : Y \rightarrow (Z_j)_{\ell^\infty}$  a bounded linear operator with  $TX \subset (Z_j)_{c_0}$ . Then given  $\varepsilon > 0$ , there exists  $\tilde{T} : Y \rightarrow (Z_j)_{c_0}$  so that  $\tilde{T}$  extends  $T|_X$  and  $\|\tilde{T}\| < (2 + \varepsilon)\|T\|$ .*

We now pass to the

*Proof of Theorem 1.9.* By the above remarks, we may assume that  $Y/X$  is infinite-dimensional. We may then choose  $y_1, y_2, \dots$  so that  $y_1, y_2, \dots$  are linearly independent over  $X$ , and  $Y$  is the closed linear span of  $X$  and of the  $y_n$ 's. For all  $k$ , let  $F_k = [y_1, \dots, y_k]$  and  $Y_k = X + F_k$ . Thus  $F_k$  is  $k$ -dimensional,  $X \cap F_k = \{0\}$ , and

$$(1.11) \quad \overline{\bigcup_{k=1}^{\infty} Y_k} = Y.$$

(Throughout, for any (finite or infinite) sequence  $(w_j)$  of elements of a Banach space,  $[(w_j)]$  denotes the closed linear span of the  $w_j$ 's.)

For each  $k$ , let  $P_k : Y_k \rightarrow F_k$  be the projection of  $Y_k$  onto  $F_k$  with kernel  $X$ . Let  $\varepsilon > 0$ . We shall construct for each  $k$ , a sequence  $(S_n^{(k)})$  of operators with the following properties for all  $n$ ;

$$(1.12) \quad S_n^{(k)} : Y_k \rightarrow Z_n,$$

$$(1.13) \quad X \subset \ker S_n^{(k)},$$

$$(1.14) \quad \|S_n^{(k)}\| < 1 + \frac{\varepsilon}{2},$$

$$(1.15) \quad S_n^{(k+1)}|_{Y_k} = S_n^{(k)},$$

and also

$$(1.16) \quad S_n^{(k)} = T_n P_k \quad \text{for all } n \text{ sufficiently large.}$$

Once  $S_n^{(k)}$  has been constructed satisfying (1.12)–(1.16) for all  $k$  and appropriate  $n$ 's, for each  $n$  let  $S_n$  be the unique bounded linear operator from  $Y$  to  $Z_n$  so that

$$(1.17) \quad S_n|_{Y_k} = S_n^{(k)}.$$

(As ordered pairs in  $Y \times Z_n$ ,  $S_n = \overline{\bigcup_{n=1}^{\infty} S_n^{(k)}}$ ). It follows from (1.14) and (1.15) that  $S_n$  is well defined and of course  $\|S_n\| \leq 1 + \frac{\varepsilon}{2} < 1 + \varepsilon$ ; (1.13) yields that  $X \subset \ker S_n$ . Of course then  $(T_n - S_n)$  is a uniformly bounded sequence of operators; for each  $k$  and  $y \in Y_k$ , we have by (1.16) and (1.17) that

$$(1.18) \quad (T_n - S_n)(y) = (T_n - T_n P_k)(y) = T_n(I - P_k)(y)$$

for all  $n$  sufficiently large, whence since  $(I - P_k)(y) \in X$ ,

$$(1.19) \quad \|(T_n - S_n)(y)\| \rightarrow 0.$$

Thus  $\|(T_n - S_n)(y)\| \rightarrow 0$  for all  $y \in Y$ , since this holds on the dense subset  $\bigcup_{k=1}^{\infty} Y_k$ .

Hence  $(S_n)$  satisfies the conclusion of Theorem 1.9.

We now construct the sequences  $(S_n^{(k)})_{n=1}^{\infty}$  by induction on  $k$ . For convenience, set  $S_n^{(0)} = 0$  for all  $n$ . Let  $k \geq 0$  and suppose  $S_n^{(k)}$  has been defined, satisfying (1.12)–(1.14) for all  $n$  and (1.16) for all  $n$  sufficiently large. By Lemma 1.10, choose  $M_{k+1}$  so that for all  $n \geq M_{k+1}$

$$(1.20) \quad \|T_n P_{k+1}\| < 1 + \frac{\varepsilon}{2}$$

and also (in case  $k \geq 1$ ) so that (1.16) holds. Now define  $S_n^{(k+1)} = T_n P_{k+1}$  for  $n \geq M_{k+1}$ . For  $n < M_{k+1}$ , since  $Z_n$  is 1-injective, simply choose  $S_n^{(k+1)}$  an extension of  $S_n^{(k)}$  from  $Y_k$  to  $Y_{k+1}$  with  $\|S_n^{(k+1)}\| = \|S_n^{(k)}\|$ .

(This procedure is also valid in the setting of Corollary 1.5; in this case we have that  $Y$  will be assumed separable; the assumption that  $Z_n$  has the 1-SEP, allows us again to choose  $S_n^{(k+1)}$  as above.)

We now have that  $S_n^{(k+1)}$  satisfies (1.12)–(1.14) and (1.16) (for “ $k$ ” =  $k + 1$ ) for all appropriate  $n$ . Finally, we check that (1.15) holds. For  $n < M_{k+1}$ , this is immediate. For  $n \geq M_{k+1}$ , we have since  $P_k P_{k+1} = P_k$ , that

$$(1.21) \quad S_n^{(k)} = T_n P_k = T_n P_{k+1} |_{Y_k} = S_n^{(k+1)} |_{Y_k}$$

as desired. ■

REMARK 1.13. (i) A modification of this argument yields that if one instead assumes the  $Z_j$ 's are  $\lambda$ -injective in Theorem 1.3, then one obtains that  $\tilde{T}$  may be chosen as in the conclusion, with  $\|\tilde{T}\| < (\lambda(1 + \lambda) + \varepsilon)\|T\|$ . It follows as above that if  $Z_1, Z_2, \dots$  have the  $\lambda$ -SEP, then  $(Z_j)_{c_0}$  has the  $(\lambda(1 + \lambda) + \varepsilon)$ -SEP for all  $\varepsilon > 0$ . The modification and these attendant consequences are due to T. Oikhberg.

Briefly, assume the  $Z_j$ 's are  $\lambda$ -injective and the  $T_j$ 's as in Theorem 1.9, we obtain the  $S_j$ 's satisfying the conclusion with  $\|S_j\| < \lambda + \frac{\varepsilon}{2}$  for all  $j$ , as follows: we construct for each  $k$ , a sequence  $S_n^{(k)}$  of operators so that there is an  $M_k$  so that for all  $n$ ,

- (a) if  $n < M_k$  then  $S_n^{(k)} : Y \rightarrow Z_n$  and  $\|S_n^{(k)}\| < \lambda + \frac{\varepsilon}{2}$ ;
- (b) if  $n \geq M_k$ ,  $S_n^{(k)} : Y_k \rightarrow Z_n$ ,  $S_n^{(k)} = T_n P_k$ , and  $\|S_n^{(k)}\| < 1 + \frac{\varepsilon}{2\lambda}$ ;
- (c) (1.13) and (1.15) hold.

(This modification holds in the complete category also; we give the full details in the next section.)

(ii) Oikhberg has recently further noted that one may *eliminate* " $\varepsilon > 0$ " in the statements of Theorems 1.3 and 1.9, by instead constructing  $S_n^{(k)}$  in the proof of Theorem 1.9 so that  $\|S_n^{(k)}\| \leq 1$  and  $S_n^{(k)} = (1 - 2^{-k})T_n P_k$  for  $n \geq M_k$  say, satisfying (1.15) for  $n < M_k$  and (1.12), (1.13) for all  $n$ . The same variation may be used to eliminate " $\varepsilon > 0$ " in the preceding remark.

We note now a further consequence of the proof of Theorem 1.3. Given  $\lambda \geq 1$  and Banach spaces  $X, Y$  with  $X \subset Y$ , we say that  $X$  is  $\lambda$ -*cocomplemented* in  $Y$  if there is a (linear) projection  $P$  from  $Y$  onto  $X$  with  $\|I - P\| \leq \lambda$ . We say that  $X$  is *contractively cocomplemented* provided  $X$  is 1-cocomplemented;  $X$  is *almost contractively cocomplemented* provided  $X$  is  $(1 + \varepsilon)$ -cocomplemented for all  $\varepsilon > 0$ .

**COROLLARY 1.14.** *Let  $Z_1, Z_2, \dots$  be 1-injective Banach spaces,  $X = (Z_j)_{c_0}$ , and let  $Y$  be a (closed linear) subspace of  $(Z_j)_{\ell^\infty}$  with  $Y/X$  separable. Then  $X$  is almost contractively cocomplemented in  $Y$ .*

*Proof.* We easily deduce this from Theorem 1.9. Define  $T_n : Y \rightarrow Z_n$  by  $y = (T_1(y), T_2(y), \dots, T_n(y), \dots)$  for all  $y \in Y$ . ( $T_n$  is just the restriction of the  $n^{\text{th}}$  coordinate projection on  $(Z_j)_{\ell^\infty}$  to  $Y$ .)

Let  $\varepsilon > 0$  and  $(S_n)$  be chosen satisfying the conclusion of Theorem 1.9. Now defining  $P(y) = (T_n(y) - S_n(y))$  for all  $y \in Y$ , it follows that  $P$  is a projection from  $Y$  onto  $X$ . Indeed, since  $T_n - S_n \rightarrow 0$  in the SOT,  $P$  has its range in  $X$ . But if  $x \in X$ ,  $S_n(x) = 0$  for all  $n$  and so  $x = (T_n(x))_{n=1}^\infty = (T_n(x) - S_n(x))_{n=1}^\infty$ . Of course then  $(I - P)(y) = (S_n(y))$  so

$$\|I - P\| = \sup_n \|S_n\| \leq 1 + \frac{\varepsilon}{2} < 1 + \varepsilon. \quad \blacksquare$$

**REMARK 1.15.** (i) Corollary 1.4 is actually "stronger" than Theorem 1.3. Indeed, let  $X, Y$  and  $T : X \rightarrow (Z_j)_{c_0}$  be as in the statement of Theorem 1.3. It follows easily from the 1-injectivity of the  $Z_j$ 's that also  $(Z_j)_{\ell^\infty}$  is 1-injective. Hence we may choose  $T' : Y \rightarrow (Z_j)_{\ell^\infty}$  extending  $T$  so that  $\|T'\| = \|T\|$ . Now let

$\tilde{Y}$  denote the closed linear span of  $(Z_j)_{c_0}$  and  $T'(Y)$ . Then  $\tilde{Y}/(Z_j)_{c_0}$  is separable, so given  $\varepsilon > 0$ , choose  $P$  a projection from  $\tilde{Y}$  onto  $(Z_j)_{c_0}$  with  $\|I - P\| < 1 + \varepsilon$ . Of course then  $\|P\| < 2 + \varepsilon$ . Now  $\tilde{T} = PT$  is the desired extension of  $T$ , with  $\|\tilde{T}\| < (2 + \varepsilon)\|T\|$ .

(ii) Using the modification of the proof of Theorem 1.9 given above, we obtain the following generalization: *Let  $Z_1, Z_2, \dots$  be  $\lambda$ -injective Banach spaces, and  $X$  and  $Y$  as in Corollary 1.14. Then for all  $\varepsilon > 0$ ,  $X$  is  $(\lambda^2 + \varepsilon)$ -cocomplemented in  $Y$ .*

(iii) After the first draft of this paper was completed, it was brought to my attention by Bill Johnson that Corollary 1.14 actually follows from a known theorem concerning  $M$ -ideals in Banach spaces, and in fact one obtains the stronger conclusion that  *$X$  is contractively cocomplemented in  $Y$*  ( $X, Y$  as in Corollary 1.14). The theorem, due to T. Ando ([2]), T. Andersen ([1]), and later refined by M. Choi and E. Effros ([8]), yields the following result (see Theorem 2.1, page 59 of [13]). *Consider a short isometric exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ . Assume that  $Z$  is separable and  $X$  is an  $M$ -ideal in  $Y$  and an  $L^1$ -predual; then the sequence admits a contractive lift. Equivalently, regarding  $X \subset Y$ , then  $X$  is contractively cocomplemented in  $Y$ .* To obtain Corollary 1.14, we use the known theorem that each  $Z_j$  in its hypotheses is isometric to  $C(\Omega_j)$  for some extremely disconnected space  $\Omega_j$ . Making this identification, it then follows that  $X$  is actually an algebraic closed ideal in  $(Z_j)_{\ell^\infty}$ , which of course may be regarded as a  $C(\Omega)$ -space. Thus  $X$  is an  $M$ -ideal and an  $L^1$ -predual, and the result follows. However we note that the generalization of Corollary 1.14 given in the previous remark does *not* follow from this  $M$ -ideal result (unless every  $\lambda$ -injective Banach space is isomorphic to a 1-injective, a famous open problem, as noted above).

Lemma 1.10 and Remark 1.11 yield an interesting consequence for  $(Z_j)_{c_0}$ , for general Banach spaces  $Z_1, Z_2, \dots$

DEFINITION 1.16. Let  $X \subset Y$  be given Banach spaces.  $X$  is said to be *locally complemented* in  $Y$  if there is a  $\lambda \geq 1$  so that

$$(1.22) \quad \begin{aligned} &X \text{ is } \lambda\text{-complemented in } Z \text{ for all } X \subset Z \subset Y \\ &\text{with } Z/X \text{ finite-dimensional.} \end{aligned}$$

When (1.22) holds, we say  $X$  is locally  $\lambda$ -complemented in  $Z$ .

COROLLARY 1.17. *Let  $Z_1, Z_2, \dots$  be arbitrary Banach spaces. Then  $(Z_j)_{c_0}$  is locally almost contractively cocomplemented in  $(Z_j)_{\ell^\infty}$ .*

The statement means that  $(Z_j)_{c_0}$  is almost contractively cocomplemented in  $Y$  for all  $Y$  with  $(Z_j)_{c_0} \subset Y \subset (Z_j)_{\ell^\infty}$  and  $Y/(Z_j)_{c_0}$  finite-dimensional. Thus

$(Z_j)_{c_0}$  is locally  $(2 + \varepsilon)$ -complemented in  $(Z_j)_{\ell^\infty}$ , for all  $\varepsilon > 0$ . The proof of Corollary 1.17 follows immediately from Remark 1.11 and the argument for Corollary 1.14.

REMARK 1.18. W. Johnson and T. Oikhberg have obtained a stronger result when the  $Z_j$ 's are separable ([15]); see Remark 1.24.

As noted above, our argument yields a new proof of Sobczyk's Theorem that  $c_0$  has the SEP, but we pay an "ε" price, for in fact  $c_0$  has the 2-SEP by [29], (and "2" is the best possible here). We recapture this result through the following extension theorem, whose proof uses a technique due to W. Veech ([32]).

THEOREM 1.19. *Let  $Z$  be an arbitrary Banach space,  $X \subset Y$  separable Banach spaces, and  $T : Y \rightarrow \ell^\infty(Z)$  a bounded linear operator so that*

(i)  $T(X) \subset c_0(Z)$ ;

(ii)  $(T_n)$  is relatively compact in the SOT, where  $T(y) = (T_n(y))$  for all  $y$ .

*Then there exists an operator  $\tilde{T} : Y \rightarrow c_0(Z)$  extending  $T|_X$  with  $\|\tilde{T}\| \leq 2\|T\|$ .*

REMARK 1.20. The hypotheses hold for any finite-dimensional  $Z$ . This easily yields the fact that  $c_0$  has the 2-SEP; we give the detailed proof in Corollary 1.23 below.

Theorem 1.19 is a consequence of the following two simple lemmas.

LEMMA 1.21. *There exists a norm  $|\cdot|$  on  $\mathcal{L}(Y, Z)$  so that letting  $\mathcal{M} = (\mathcal{L}(Y, Z), |\cdot|)$ , then the  $\mathcal{M}$ -topology coincides with the SOT topology on bounded subsets of  $\mathcal{L}(Y, Z)$ .*

*Proof.* Let  $d_1, d_2, \dots$  be a countable dense subset of the unit ball of  $Y$  and define  $|\cdot|$  by

$$(1.23) \quad |S| = \sum_{n=1}^{\infty} \frac{\|S(d_n)\|}{2^n}.$$

It is easily verified that  $\mathcal{M}$  is a normed linear space. Moreover, if  $(T_n)$  is a bounded sequence in  $\mathcal{L}(Y, Z)$  and  $T \in \mathcal{L}(Y, Z)$ , then  $T_n \rightarrow T$  in the SOT iff  $T_n(d_j) \rightarrow T(d_j)$  for all  $j$  iff  $|T_n - T| \rightarrow 0$ . ■

LEMMA 1.22. *Let  $(\mathcal{M}, \rho)$  be a metric space,  $\mathcal{S}$  be a closed subset of  $\mathcal{M}$ , and  $(T_n)$  be a sequence in  $\mathcal{M}$  so that  $\{T_1, T_2, \dots\}$  is relatively compact and all cluster points of  $(T_n)$  lie in  $\mathcal{S}$ . There exists a sequence  $(S_n)$  of points in  $\mathcal{S}$  so that*

$$(1.24) \quad \rho(T_n, S_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Proof.* Define  $d_n$  by

$$(1.25) \quad d_n = \text{dist}(T_n, \mathcal{S}) \stackrel{\text{def}}{=} \inf\{\rho(T_n, S) : S \in \mathcal{S}\}.$$

Of course (1.24) is simply the assertion that  $d_n \rightarrow 0$  as  $n \rightarrow \infty$ . Were this false, by passing to a subsequence if necessary, we can assume without loss of generality that there is a  $d > 0$  so that

$$(1.26) \quad d_n \geq d \quad \text{for all } n.$$

Choose  $n_1 < n_2 < \dots$  so that  $(T_{n_i})$  converges to  $S$ , say. By hypothesis,  $S \in \mathcal{S}$ . Now  $\rho(T_{n_i}, S) \rightarrow 0$  as  $i \rightarrow \infty$ , yet  $\rho(T_{n_i}, S) \geq \text{dist}(T_{n_i}, \mathcal{S}) \geq d_{n_i} \geq d > 0$  for all  $i$ , a contradiction. ■

*Proof of Theorem 1.19.* We may assume without loss of generality that  $\|T\| = 1$ . Let  $\mathcal{M}$  be as in Lemma 1.21, and define  $\mathcal{S}$  by

$$(1.27) \quad \mathcal{S} = \{S \in \mathcal{L}(Y, Z) : \|S\| \leq 1 \text{ and } X \subset \ker S\}.$$

Now, the hypotheses imply that all SOT-cluster points of  $(T_n)$  lie in  $\mathcal{S}$ . Indeed, if  $T_{n_i} \rightarrow T$  in the SOT, then for  $x \in X$ ,  $T_{n_i}(x) \rightarrow T(x)$ , but  $T_{n_i}(x) \rightarrow 0$  in norm; of course  $\|T\| \leq 1$  since  $\|T_j\| \leq 1$  for all  $j$ . Thus the hypotheses of Lemma 1.22 apply (where of course  $\mathcal{M}$  is endowed with the standard metric,  $\rho(x, y) = |x - y|$ ). Hence we may choose a sequence  $(S_n)$  in  $\mathcal{S}$  so that  $|T_n - S_n| \rightarrow 0$  as  $n \rightarrow \infty$ ; i.e., by Lemma 1.21,

$$(1.28) \quad T_n - S_n \rightarrow 0 \quad \text{in the SOT.}$$

Now define  $\tilde{T}$  by  $\tilde{T}(y) = (T_n - S_n)(y)$  for all  $y \in Y$ . Then  $\tilde{T}$  is the desired extension of  $T|X$ . ■

We may now deduce the following rather surprising consequence of this proof, analogous to Corollary 1.14.

**COROLLARY 1.23.** *Let  $Z$  be a finite-dimensional Banach space and  $Y$  be a separable subspace of  $\ell^\infty(Z)$  containing  $c_0(Z)$ . Then  $c_0(Z)$  is contractively cocomplemented in  $Y$ .*

*Proof.* Of course we assume  $Y \neq c_0(Z)$ . Let  $T$  denote the identity injection of  $Y$  into  $\ell^\infty(Z)$ , and let  $y = (T_1(y), T_2(y), \dots, T_n(y), \dots)$  for all  $y \in Y$ . Since  $Z$  is finite-dimensional and of course  $T_n : Y \rightarrow Z$  with  $\|T_n\| \leq 1$  for all  $n$ ,  $(T_n)$  is relatively compact in the SOT on  $\mathcal{L}(Y, Z)$ . Now choose  $(S_n)$  as in the proof of Theorem 1.19. It follows that defining  $P : Y \rightarrow c_0(Z)$  by

$$P(y) = (T_n)(y) - S_n(y) \quad \text{for all } y \in Y$$

then  $P$  is a linear projection from  $Y$  onto  $c_0(Z)$ , and  $(I - P)(y) = (S_n(y))$  for all  $y$ , whence  $\|I - P\| = \sup \|S_n\| = 1$ . ■

REMARK 1.24. This result has been obtained in [15], using a different argument. The authors of [15] also study the family of separable Banach spaces  $Z$  so that  $Z$  is complemented in  $Y$  for all separable  $Y$  with  $c_0(Z) \subset Y \subset \ell^\infty(Z)$ , obtaining quite nice results, including the fact that there exists a sequence  $(E_n)$  of finite dimensional Banach spaces so that  $Z = (E_n)_{c_0}$  fails this property. It is also proved in [15] that for  $Z$  separable,  $c_0(Z)$  is locally contractively cocomplemented in  $\ell^\infty(Z)$ , thus removing the “almost” from Corollary 1.17 above, when the  $Z_j$ 's are separable.

As noted in the introduction, we immediately obtain the following isometric property for  $c_0$  itself.

COROLLARY 1.25.  $c_0$  is contractively cocomplemented in any separable superspace which lies in  $\ell^\infty$ .

In turn, this yields Sobczyk's Theorem.

COROLLARY 1.26.  $c_0$  has the 2-SEP.

*Proof.* Let  $X \subset Y$  be separable Banach spaces and let  $T : X \rightarrow c_0$  be a given bounded linear operator. Define the sequence  $(f_n)$  in  $X^*$  by  $Tx = (f_1(x), f_2(x), \dots)$  for all  $x$ . Now fixing  $n$ ,  $\|f_n\| \leq \|T\|$ ; let  $f'_n$  be a Hahn-Banach extension of  $f_n$  to  $Y$ . Let  $T' : Y \rightarrow \ell^\infty$  be defined by  $T'(y) = (f'_1(y), f'_2(y), \dots)$ . Then of course  $\|T'\| = \|T\|$ . Let  $\tilde{Y}$  denote the closed linear span of  $Y$  and  $c_0$ , and choose  $P$  a projection from  $\tilde{Y}$  onto  $c_0$  with  $\|I - P\| = 1$ . Then  $\tilde{T} = PT'$  yields the desired extension of  $T$  to  $Y$  with  $\|\tilde{T}\| \leq 2\|T\|$ . ■

REMARK 1.27. Corollary 1.25 also follows directly from the  $M$ -ideal theorem cited in Remark 1.15 (iii).

## 2. THE COMPLETE SEPARABLE EXTENSION PROPERTY

As noted in the Introduction, most of the results of the previous section follow from their operator space versions given here. However the *techniques* of proof come from the arguments in Section 1, so we have chosen to present the Banach category first, in the interest of clarity.

We first recall the definition given in the Introduction.

DEFINITION 2.1. An operator space  $Z$  is said to have the *Complete Separable Extension Property* (the CSEP) if for any separable operator space  $Y$ , subspace  $X$ , and completely bounded operator  $T : X \rightarrow Z$ , there exists a completely bounded operator  $\tilde{T} : Y \rightarrow Z$  extending  $T$ .  $Z$  is said to have the  $\lambda$ -CSEP provided for all such  $X$  and  $Y$ ,  $\tilde{T}$  may be chosen with  $\|\tilde{T}\|_{\text{cb}} \leq \lambda\|T\|_{\text{cb}}$ .

Again, we have the operator space analogue of injectivity in Banach spaces.

DEFINITION 2.2. An operator space  $Z$  is said to be *isomorphically injective* provided for arbitrary operator spaces  $X$  and  $Y$  with  $X \subset Y$ , every completely bounded map  $T : X \rightarrow Z$  admits a completely bounded extension  $\tilde{T} : Y \rightarrow Z$ .  $Z$  is called  $\lambda$ -*injective* if the extension  $\tilde{T}$  may always be chosen with  $\|\tilde{T}\|_{\text{cb}} \leq \lambda\|T\|_{\text{cb}}$ . Finally,  $Z$  is called *isometrically injective* if it is 1-injective.

It is a basic theorem in operator space theory that  $\mathcal{L}(H)$  is isometrically injective for all Hilbert spaces  $H$ . The theorem was proved for the fundamental case of completely positive maps and self-adjoint operator spaces in the domain by W.B. Arveson ([5]), and later, in general, by V. Paulsen (cf. [22]) and G. Wittstock ([34]). See also [25] for a proof from the abstract operator-space viewpoint. It follows easily that an operator space  $X$  is isomorphically injective provided it is completely complemented in some complete isometric embedding  $\tilde{X}$  into  $\mathcal{L}(H)$ ; moreover if  $P : \mathcal{L}(H) \rightarrow \tilde{X}$  is a completely bounded projection, then  $X$  is  $\lambda$ -injective, where  $\lambda = \|P\|_{\text{cb}}$ . We will mainly be concerned with isometrically injective operator spaces here. Unlike the Banach space category, there are separable infinite-dimensional examples. A complete classification of these has been given by G. Robertson ([27]) (see Section 4 below). See also work of Z.J. Ruan giving a characterization of the 1-injectives as ‘‘corners’’ of injective  $C^*$ -algebras ([28]). Finally, we note that the 1-CSEP is studied for  $C^*$ -algebras by R.R. Smith and D.P. Williams ([30]).

The 1-injectivity of  $\mathcal{L}(H)$  easily yields the following result.

PROPOSITION 2.3. *Let  $X$  be a separable operator space. The following are equivalent:*

- (i)  $X$  has the CSEP;
- (ii)  $X$  is completely complemented in every separable operator space  $Y$  with  $X \subset Y$ ;
- (iii) There is a  $\lambda \geq 1$  so that  $X$  is  $\lambda$ -completely complemented in every separable operator space  $Y$  with  $Y \supset X$ .

Moreover, if  $X$  satisfies (iii),  $X$  has the  $\lambda$ -CSEP.

*Proof.* (i)  $\Rightarrow$  (ii). Trivial.

(ii)  $\Rightarrow$  (iii). We may assume  $X \subset \mathcal{L}(H)$  (with  $H$  separable infinite-dimensional Hilbert space). We shall prove

(iii') *there is a  $\lambda \geq 1$  so that  $X$  is completely  $\lambda$ -complemented in every separable superspace of  $X$  contained in  $\mathcal{L}(H)$ .*

Were this false, we could choose  $Y_1, Y_2, \dots$  separable operator subspaces of  $\mathcal{L}(H)$ , so that for all  $n$ ,  $X \subset Y_n$  but  $X$  is not completely  $n$ -complemented in  $Y_n$ . Then letting  $Y = [Y_1, Y_2, \dots]$ ,  $Y$  is separable,  $X \subset Y \subset \mathcal{L}(H)$ , but  $X$  is not-completely complemented in  $Y$ , a contradiction.

Now suppose  $X$  satisfies (iii'). Let then  $Y \subset Z$  be separable operator spaces and  $T : Y \rightarrow X$  a completely bounded operator. Choose  $\tilde{T} : Z \rightarrow \mathcal{L}(H)$  extending  $T$ , with  $\|\tilde{T}\|_{\text{cb}} = \|T\|_{\text{cb}}$  (by the fundamental theorem cited above). Now letting  $E = \overline{X + TY}$ ,  $E$  is separable, so choose  $P : E \rightarrow X$  a completely bounded projection with  $\|P\|_{\text{cb}} \leq \lambda$ . Then  $T' \stackrel{\text{def}}{=} P\tilde{T}$  is an extension of  $T$  to  $Z$ , and  $\|T'\|_{\text{cb}} \leq \|P\|_{\text{cb}}\|\tilde{T}\|_{\text{cb}} \leq \lambda\|T\|_{\text{cb}}$ , proving that  $X$  has the  $\lambda$ -CSEP (so of course (iii)) holds. ■

REMARK 2.4. I am indebted to M. Zippin for the following simple argument, which shows that *any* Banach space (respective operator space)  $Z$  with the SEP (respective CSEP) has the  $\lambda$ -SEP (respective  $\lambda$ -CSEP) for some  $\lambda \geq 1$ . If not, we can choose for every  $n$ , separable Banach spaces (respective operator spaces)  $X_n \subset Y_n$  and bounded (respective completely bounded) operators  $T_n : X_n \rightarrow Z$  so that  $\|T_n\| < \frac{1}{n^2}$  (respective  $\|T_n\|_{\text{cb}} < \frac{1}{n^2}$ ) and for any bounded (respective completely bounded) extension  $\tilde{T}_n : Y_n \rightarrow Z$ ,  $\|\tilde{T}_n\| > n$  (respective  $\|\tilde{T}_n\|_{\text{cb}} > n$ ). Now let  $Y = (Y_1 \oplus Y_2 \oplus \dots)_{c_0}$  and  $X = (X_1 \oplus X_2 \oplus \dots)_{c_0}$  and define  $T : X \rightarrow Z$  by  $T((x_n)) = \sum T_n x_n$  for all  $(x_n) \in X$ . Then  $T$  is a bounded (respective completely bounded) operator from  $X$  into  $Z$  with no bounded (respective completely bounded) extension  $\tilde{T} : Y \rightarrow Z$ .

We next pass to a rather strong condition on operator spaces, which we will use to produce examples of spaces with the CSEP.

DEFINITION 2.5. A family  $\mathcal{Z}$  of operator spaces is called *uniformly exact* if there is a  $C \geq 1$  and a function  $\mathbf{n} : \mathbb{N} \rightarrow \mathbb{N}$  so that for all  $Z \in \mathcal{Z}$  and all  $k$  and  $k$ -dimensional subspaces  $F$  of  $Z$ , there exists a  $G \subset M_{\mathbf{n}(k)}$  with

$$(2.1) \quad d_{\text{cb}}(F, G) \leq C.$$

In case  $C$  works, we say  $\mathcal{Z}$  is *C-uniformly exact*. In case  $\mathbf{n}$  works, we say  $\mathbf{n}$  is a *uniformity function* for  $\mathcal{Z}$ . We say an operator space  $Z$  is *uniformly exact* (respective *C-uniformly exact*) in case  $\mathcal{Z} = \{Z\}$  has the corresponding property.

It follows that an operator space  $Z$  is  $C$ -exact as defined in [25] precisely when every finite dimensional subspace of  $Z$  is  $C + \varepsilon$ -uniformly exact for every  $\varepsilon > 0$ . If  $X$  is a Banach space endowed with the MIN operator space structure, then  $X$  is  $(1 + \varepsilon)$ -uniformly exact for every  $\varepsilon > 0$ . We may now state the main result of this section, (which yields Theorem 1.3 in view of the last comment above).

**THEOREM 2.6.** *Let  $Z_1, Z_2, \dots$  be  $\lambda$ -injective operator spaces so that  $\{Z_1, Z_2, \dots\}$  is  $C$ -uniformly exact for some  $C \geq 1$ , and set  $Z = (Z_j)_{c_0}$ . Let  $X \subset Y$  be operator spaces with  $X \subset Y$  and  $Y/X$  separable. Then for every non-zero completely bounded operator  $T : X \rightarrow Z$  and every  $\varepsilon > 0$ , there exists a completely bounded operator  $\tilde{T} : Y \rightarrow Z$  extending  $T$  with  $\|\tilde{T}\|_{cb} < (C\lambda^2 + \lambda + \varepsilon)\|T\|_{cb}$ .*

**REMARK 2.7.** We had originally obtained this result for  $\lambda = 1$ . This more general result follows via the proof-modification due to Oikhberg, mentioned in Section 1.

We give again several consequences before passing to the proof.

**COROLLARY 2.8.** *Let  $Z$  be as in Theorem 2.6 and let  $\varepsilon > 0$ . Then  $Z$  is completely  $(C\lambda^2 + \lambda + \varepsilon)$ -complemented in every operator superspace  $Y$  with  $Y/Z$  separable.*

Of course this corollary follows immediately from Proposition 2.3. Inserting “completely” before the “cocomplemented” definition given preceding Corollary 1.14, we again discover the following consequence of the proof of Theorem 2.6.

**COROLLARY 2.9.** *Let  $Z_1, Z_2, \dots$  satisfy the hypotheses of Proposition 2.3,  $Z = (Z_j)_{c_0}$ , and let  $Y$  be an operator space with  $(Z_j)_{c_0} \subset Y \subset (Z_j)_{\ell^\infty}$  and  $Y/Z$  separable; let  $\varepsilon > 0$ . Then  $Z$  is  $(C\lambda^2 + \varepsilon)$ -completely cocomplemented in  $Y$ .*

We may thus conclude, if  $\{Z_1, Z_2, \dots\}$  is  $(1 + \varepsilon)$ -uniformly exact for every  $\varepsilon > 0$  and the  $Z_j$ 's are 1-injective, then  $Z$  is almost completely contractively cocomplemented in  $Y$ .

Again, we have the following analogue of Corollary 1.5 (which follows immediately from Proposition 2.3 if the  $Z_j$ 's are 1-injective).

**COROLLARY 2.10.** *If  $Z_1, Z_2, \dots$  are operator-spaces with the  $\lambda$ -CSEP and  $\{Z_1, Z_2, \dots\}$  is  $C$ -uniformly exact, then  $(Z_j)_{c_0}$  has the  $(C\lambda^2 + \lambda + \varepsilon)$ -CSEP for all  $\varepsilon > 0$ .*

The next result, combined with Theorem 2.6, yields our examples of separable spaces with the CSEP.

PROPOSITION 2.11. *For all  $n$ ,  $M_{\infty,n}$  and  $M_{n,\infty}$  are 1-uniformly exact, with uniformity function  $\mathbf{n}(k) = k \cdot n$ .*

REMARK 2.12. Every isometrically injective separable operator space  $Z$  is completely isometric to a subspace of  $M_{\infty,n} \oplus M_{n,\infty}$  for some  $n$ . It thus follows from Corollary 2.9 that, for such  $Z$ 's,  $c_0(Z)$  is almost completely contractively cocomplemented in  $Y$  for all separable  $Y$  with  $c_0(Z) \subset Y \subset \ell^\infty(Z)$ .

The following corollary gives the main “separable” motivation for Theorem 2.6.

COROLLARY 2.13. *For all  $n$ ,  $c_0(M_{\infty,n} \oplus M_{n,\infty})$  has the  $(2 + \varepsilon)$ -CSEP for all  $\varepsilon > 0$ .*

REMARK 2.14. As noted in the introduction, Kirchberg’s work in [17] yields that  $(M_n)_{c_0}$  fails the CSEP. In view of this, it seems rather surprising that the above family of spaces has the CSEP with a good uniform constant.

Of course Corollary 2.13 yields the immediate

COROLLARY 2.15.  *$c_0(\mathcal{R} \oplus \mathcal{C})$  has the  $(2 + \varepsilon)$ -CSEP for all  $\varepsilon > 0$ .*

The fundamental open problem for the *characterization* of separable spaces with the CSEP goes as follows:

PROBLEM 2.16. *Let  $X$  be a separable operator space with the CSEP. Is  $X$  completely isomorphic to a completely contractively complemented subspace of  $c_0(\mathcal{R} \oplus \mathcal{C})$ ?*

A more “refined” version of this problem is given in Section 4 below.

It seems very likely that the fundamental problem reduces to the

EMBEDDING PROBLEM 2.17. *Let  $X$  be a separable operator space with the CSEP. Is  $X$  completely isomorphic to a subspace of  $c_0(\mathcal{R} \oplus \mathcal{C})$ ?*

In turn, Corollary 2.9 leads us to the following

QUANTITATIVE EMBEDDING PROBLEM 2.18. *Let  $\lambda \geq 1$ , and let  $X$  be separable with the  $\lambda$ -CSEP. Is there a  $\beta$  depending only on  $\lambda$ , and an  $n$  (depending on  $X$ ), so that  $d_{cb}(Y, X) \leq \beta$  for some subspace  $Y$  of  $c_0(M_{\infty,n} \oplus M_{n,\infty})$ ?*

Before dealing with the proofs of the results stated above, we give some information concerning the relationship between the CSEP and isomorphic injectivity for operator spaces. The following result follows quickly from known theorems.

PROPOSITION 2.19. *Let  $X$  be a non-reflexive operator space. If  $X$  is completely isomorphic to a completely complemented subspace of some  $C^*$ -algebra, then  $X$  contains a subspace completely isomorphic to  $c_0$ . If moreover  $X$  is completely isomorphic to a completely complemented subspace of some von Neumann algebra, then  $X$  contains a subspace completely isomorphic to  $\ell^\infty$ .*

COMMENT 2.20. This result also holds if one deletes the term “completely” from all occurrences in its statement.

*Proof.* Suppose first, without loss of generality that  $X \subset \mathcal{A}$ ,  $\mathcal{A}$  a  $C^*$ -algebra, and  $P : \mathcal{A} \rightarrow X$  is a completely bounded projection onto  $X$ . Since  $P$  is non-weakly compact, a result of H. Pfister ([24]) yields that there exists a commutative  $C^*$ -subalgebra  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$  with  $P|_{\tilde{\mathcal{A}}}$  non-weakly compact. By the uniqueness of the operator-space structure for  $C^*$ -algebras, it follows that  $\tilde{\mathcal{A}}$  has MIN as its inherited operator space structure. By a result of A. Pełczyński ([23]), there exists a subspace  $E$  of  $\tilde{\mathcal{A}}$  with  $E$  isomorphic (and hence completely isomorphic) to  $c_0$ , so that  $P|_E$  is a Banach isomorphism. Since  $P$  is completely bounded and  $E$  has the MIN structure,  $P|_E$  is in fact a complete isomorphism onto its range, proving the first assertion. Now if  $\mathcal{A}$  is a von Neumann algebra, then of course  $\mathcal{A}$  is also a dual Banach space, and in fact the canonical projection  $\Pi : \mathcal{A}^{**} \rightarrow \mathcal{A}$  is then completely bounded. Now of course  $E^{**}$  is completely isomorphic to  $\ell^\infty$ ; regarding  $E^{**} = E^{\perp\perp} \subset \mathcal{A}^{**}$ , the operator  $T$  defined by  $T = P\Pi|_{E^{**}}$  is then a non-weakly compact completely bounded operator into  $X$ . By a result of the author ([27]), it follows that there is a subspace  $Z$  of  $E^{**}$  with  $Z$  isomorphic to  $\ell^\infty$  and  $T|_Z$  a Banach isomorphism. Again,  $E^{**}$  has the MIN structure, hence so does  $Z$ , so as before,  $T|_Z$  is a complete isomorphism of  $Z$  onto its range. ■

The following result is now immediate.

COROLLARY 2.21. *Let  $X$  be a non-reflexive operator space. If  $X$  has the CSEP and is separable,  $X$  contains a subspace completely isomorphic to  $c_0$ . If  $X$  is isomorphically injective,  $X$  contains a subspace completely isomorphic to  $\ell^\infty$ .*

The next result shows that reflexive separable operator spaces are isomorphically injective provided they have the CSEP.

PROPOSITION 2.22. *Let  $X$  be a reflexive separable operator space with the  $\lambda$ -CSEP. Then  $X$  is a  $\lambda$ -injective operator space.*

*Proof.* Assume  $X \subset \mathcal{L}(H)$  (with  $H$  separable infinite-dimensional Hilbert space) and let  $\mathcal{S}$  be the family of separable subspaces  $Y$  of  $\mathcal{L}(H)$ , with  $X \subset Y$ ; direct  $\mathcal{S}$  by inclusion. For each  $\alpha \in \mathcal{S}$ , let  $P_\alpha : Y \rightarrow X$  be a complete projection of  $Y$  onto  $X$ , with  $\|P_\alpha\|_{cb} \leq \lambda$ . We may now use the reflexivity of  $Y$  and the

Tychonoff theorem to produce a completely bounded projection from  $\mathcal{L}(H)$  onto  $Y$ . For each  $\alpha \in \mathcal{S}$ , define  $\tilde{P}_\alpha : \mathcal{L}(H) \rightarrow X$  by  $\tilde{P}_\alpha(v) = 0$  if  $v \notin \alpha$ ;  $\tilde{P}_\alpha(v) = P_\alpha(v)$  if  $v \in \alpha$ .  $\tilde{P}_\alpha$  is neither continuous nor linear; nevertheless, the weak-compactness of the ball of  $X$  yields a subnet  $(P_{\alpha_\beta})_{\beta \in \mathcal{D}}$  of  $(P_\alpha)_{\alpha \in \mathcal{S}}$  so that  $Pv \stackrel{\text{def}}{=} \lim_{\beta \in \mathcal{D}} P_{\alpha_\beta}(v)$  exists weakly for all  $v \in B(H)$ . Since of course every  $v \in B(H)$  is contained in some  $\alpha \in \mathcal{S}$ , it follows that  $P$  is indeed a linear projection from  $\mathcal{L}(H)$  into  $X$ . Finally, we also have that for all  $n$ ,

$$\|P\|_n \leq \overline{\lim}_{\alpha \in \mathcal{S}} \|P_\alpha\|_n \leq \lambda$$

whence  $\|P\|_{\text{cb}} \leq \lambda$ , showing that  $X$  is indeed  $\lambda$ -injective. ■

COMMENT 2.23. This (rather outrageous) use of the Tychonoff theorem is due to J. Lindenstrauss.

REMARK 2.24. We show in Proposition 2.40 below that if  $X$  is a separable operator space with the  $\lambda$ -CSEP and  $\lambda < 2$ , then  $X$  is reflexive and (hence is  $\lambda$ -injective).

Work of G. Pisier yields immediately that every separable reflexive operator space which is isomorphically injective is Hilbertian, i.e., Banach isomorphic to Hilbert space (cf. [26]). Evidently Corollary 2.21 also yields that every isomorphically injective separable operator space is reflexive (and so Hilbertian).

Of course the natural (and far from obvious!) special problem in this setting is then as follows:

*Let  $X$  be a separable infinite-dimensional isomorphically injective operator space. Is  $X$  completely isomorphic to  $\mathcal{R}, \mathcal{C}$ , or  $\mathcal{R} \oplus \mathcal{C}$ ?*

(This problem has been solved affirmatively for  $X$  isometrically injective by A. Robertson ([26]).) A remarkable result of T. Oikhberg ([20]) yields that the answer is *affirmative* if  $X$  is completely isomorphic to a subspace of  $\mathcal{R} \oplus \mathcal{C}$ . Finally, we note the following quantitative problem, whose positive solution implies an affirmative answer to the preceding question, in virtue of Oikhberg's result.

*Let  $X$  be a separable operator space which is  $\lambda$ -injective. Is there a  $\beta$ , depending only on  $\lambda$ , and an  $n$  (depending on  $X$ ) so that  $d_{\text{cb}}(X, Y) \leq \beta$  for some subspace  $Y$  of  $M_{\infty, n} \oplus M_{n, \infty}$ ?*

Before dealing with the main result of this section, we give the

*Proof of Proposition 2.11.* We identify  $M_{\infty, n}$  with  $\mathcal{C} \otimes \mathcal{R}_n$  endowed with its natural operator space structure (where  $\mathcal{R}_n$  denotes the  $n$ -dimensional row space).

Letting  $e_1, \dots, e_n$  be the natural orthonormal basis of  $\mathcal{R}_n$ , any vector  $v \in \mathcal{C} \otimes \mathcal{R}_n$  has the form

$$(2.2) \quad v = \sum_{i=1}^n u_i \otimes e_i \quad \text{for unique } u_1, \dots, u_n \in \mathcal{C}.$$

In fact, the map  $P_i$  which sends  $v$  to  $u_i$  yields a projection from  $\mathcal{C} \otimes \mathcal{R}_n$  onto  $\mathcal{C}$ . Now, letting  $F$  be a  $k$ -dimensional subspace of  $\mathcal{C} \otimes \mathcal{R}_n$  and setting  $V_i = PF_i$ , we have that  $V_i$  is a subspace of  $\mathcal{C}$  with  $\dim V_i \leq k$  for all  $i$ , and clearly

$$(2.3) \quad F \subset \text{span}_{1 \leq i \leq n} V_i \otimes e_i \subset \mathcal{V} \otimes \mathcal{R}_n$$

where  $\mathcal{V} = V_1 + \dots + V_n$ . Evidently  $m \stackrel{\text{def}}{=} \dim \mathcal{V} \leq k \cdot n$ ; by homogeneity of  $\mathcal{C}$ ,  $\mathcal{V} \otimes \mathcal{R}_n$  is completely isometric to  $M_{m,n}$ , which in turn is completely isometric to a subspace of  $M_{k \cdot n}$ . The proof for  $M_{n,\infty}$  is of course the same. ■

We now prove Theorem 2.6, giving the full details of the modification of our original argument, due to T. Oikhberg. We shall see the argument is essentially the same as the one alluded to in section one after inserting the appropriate quantizations. We first give the reformulation analogous to Theorem 1.9.

**THEOREM 2.25.** *Let  $Z_1, Z_2, \dots, X$ , and  $Y$  be operator spaces satisfying the hypotheses of Theorem 2.6. Let  $(T_j)$  be a sequence of completely bounded operators, with  $T_j : Y \rightarrow Z_j$  for all  $j$ , so that  $T_j|_X \rightarrow 0$  in the SOT, and  $\sup_j \|T_j\|_{\text{cb}} = 1$ . For every  $\varepsilon > 0$ , there exists a sequence  $(S_j)$  of completely bounded operators so that for all  $j$ ,*

- (i)  $S_j : Y \rightarrow Z_j$ ,
- (ii)  $X \subset \ker S_j$ ,
- (iii)  $\|S_j\|_{\text{cb}} < C\varepsilon + \frac{\varepsilon}{2\lambda}$ ,

so that  $(T_j - S_j) \rightarrow 0$  in the SOT.

We first give the proof that Theorem 2.25  $\Rightarrow$  Theorem 2.6. Let  $T : X \rightarrow Z$  be as in the statement of Theorem 2.6, and let  $(\tilde{T}_j)$  be the sequence so that  $Tx = (\tilde{T}_jx)$  for all  $x \in X$ . For each  $j$ , since  $Z_j$  is  $\lambda$ -injective, we may choose  $T'_j : Y \rightarrow Z_j$  with  $\|T'_j\|_{\text{cb}} \leq \lambda\|\tilde{T}_j\|_{\text{cb}}$  and  $T'_j|_X = \tilde{T}_j$ . Of course then

$$(2.4) \quad \sup_j \|T'_j\|_{\text{cb}} \stackrel{\text{def}}{=} \beta \leq \lambda\|T\|_{\text{cb}}.$$

Let  $T_j = T'_j/\beta$  for all  $j$ . Now the hypotheses of Theorem 2.25 hold; choose  $(S_j)$  satisfying its conclusion. Now define  $\tilde{\tilde{T}} : Y \rightarrow (Z_j)_{\ell^\infty}$  by

$$(2.5) \quad \tilde{\tilde{T}}(y) = (T_j - S_j)(y) \quad \text{for all } y \in Y.$$

It follows from the conclusion of Theorem 2.25 that  $\tilde{T}$  actually has its range in  $Z = (Z_j)_{c_0}$ , and Theorem 2.25 (ii) insures that  $\tilde{T}$  extends  $\frac{1}{\beta}T$ . Thus defining  $\tilde{T} = \beta\tilde{T}$ ,  $\tilde{T}$  extends  $T$  and by (2.4) and Theorem 2.25 (iii)

$$(2.6) \quad \|\tilde{T}\|_{\text{cb}} < \beta \left(1 + C\lambda + \frac{\varepsilon}{2}\right) < (C\lambda^2 + \lambda + \varepsilon)\|T\|_{\text{cb}}. \quad \blacksquare$$

We need again the analogue for Lemma 1.10, which actually holds with no uniform exactness assumption.

LEMMA 2.26. *Let  $Z_1, Z_2, \dots$  be arbitrary operator spaces,  $X \subset Y$  operator spaces with  $Y/X$  finite-dimensional, and  $(T_j)$  a sequence of completely bounded operators so that for all  $j$ ,  $T_j : Y \rightarrow Z_j$  with  $\|T_j\|_{\text{cb}} \leq 1$ , so that  $T_j|_X \rightarrow 0$  in the SOT. Let  $F$  be a finite-dimensional subspace of  $Y$  with  $X \oplus F = Y$ , and let  $P$  be the projection of  $Y$  onto  $F$  with kernel  $X$ . Then for all positive integers  $n$ ,*

$$(2.7) \quad \overline{\lim}_{j \rightarrow \infty} \|T_j P\|_n \leq 1.$$

*Proof.* Suppose not. Then by passing to a subsequence if necessary, we can fix an  $n$  and without loss of generality, choose  $C > 1$  and  $(A_j)$  a norm-one sequence in  $M_n(Y) = \mathcal{L}(\ell_n^2) \otimes Y$  with

$$(2.8) \quad \|(I \otimes T_j) \cdot (I \otimes P)(A_j)\| > C \quad \text{for all } j.$$

Now setting

$$(2.9) \quad A_j = \begin{bmatrix} y_{11}^j & \cdots & y_{1n}^j \\ \vdots & & \vdots \\ y_{n1}^j & \cdots & y_{nn}^j \end{bmatrix}$$

for each  $j$ , choose  $U_j \in M_n(X)$  and  $V_j \in M_n(F)$  so that

$$(2.10) \quad A_j = U_j \oplus V_j$$

whence by (2.8) and the fact that  $(I \otimes P)(A_j) = V_j$

$$(2.11) \quad \|(I \otimes T_j)(V_j)\| > C.$$

Of course for each  $j$ , we may set

$$(2.12)(i) \quad U_j = \begin{bmatrix} x_{11}^j & \cdots & x_{1n}^j \\ \vdots & & \vdots \\ x_{n1}^j & \cdots & x_{nn}^j \end{bmatrix}$$

and

$$(2.12)(ii) \quad V_j = \begin{bmatrix} f_{11}^j & \cdots & f_{1n}^j \\ \vdots & & \vdots \\ f_{n1}^j & \cdots & f_{nn}^j \end{bmatrix}.$$

Thus (2.11) means that

$$(2.13) \quad \left\| \begin{bmatrix} T_j f_{11}^j & \cdots & T_j f_{1n}^j \\ \vdots & & \vdots \\ T_j f_{n1}^j & \cdots & T_j f_{nn}^j \end{bmatrix} \right\| > C \quad \text{for all } j.$$

Since  $F$  is a finite-dimensional space,  $P$  is bounded and of course completely bounded; in particular, the sequences  $(U_j)$  and  $(V_j)$  are both bounded, so by compactness of bounded subsets of  $M_n(F)$ , by passing to a further subsequence if necessary, we may assume for some  $V \in M_n(F)$  that

$$(2.14) \quad V_j \rightarrow V \quad \text{in norm.}$$

(In other words, we have

$$V = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix}$$

and for each  $i$  and  $k$ ,  $f_{ik}^j \rightarrow f_{ik}$  in norm.) But then

$$(2.15) \quad \|(I \otimes T_j)(V_j - V)\| \rightarrow 0$$

whence by (2.11)

$$(2.16) \quad \varliminf_{j \rightarrow \infty} \|(I \otimes T_j)(V)\| \geq C.$$

That is

$$\varliminf_{j \rightarrow \infty} \left\| \begin{bmatrix} T_j f_{11} & \cdots & T_j f_{1n} \\ \vdots & & \vdots \\ T_j f_{n1} & \cdots & T_j f_{nn} \end{bmatrix} \right\| \geq C.$$

Now also

$$(2.17) \quad \|U_j \oplus V\| \rightarrow 1$$

since  $\|U_j \oplus V_j\| = \|A_j\| = 1$  for all  $j$  and by (2.14),  $\|U_j \oplus V_j\| - \|U_j \oplus V\| \rightarrow 0$ . Now fix  $\varepsilon > 0$  with  $1 + \varepsilon < C$ , and choose  $k$  with

$$(2.18) \quad \|U_k \oplus V\| < 1 + \varepsilon \quad (\text{using (2.17)}).$$

Then for all  $j$ ,

$$(2.19) \quad \|(I \otimes T_j)(U_k \oplus V)\| \leq \|T_j\|_{\text{cb}} \|U_k \oplus V\| < 1 + \varepsilon.$$

Since  $T_j|X \rightarrow 0$  in the SOT,  $I \otimes T_j|M_n(X) \rightarrow 0$  in the SOT, whence

$$(2.20) \quad \lim_{j \rightarrow \infty} \|(I \otimes T_j)(U_k \oplus V)\| - \|(I \otimes T_j)(V)\| = 0.$$

Thus we obtain from (2.19) and (2.20) that

$$(2.21) \quad \overline{\lim}_{j \rightarrow \infty} \|(I \otimes T_j)(V)\| \leq 1 + \varepsilon,$$

contradicting (2.16). ■

We now apply a useful result of R. Smith to obtain the following consequence.

**COROLLARY 2.27.** *Let  $Z_1, Z_2, \dots, X, Y$ , and  $(T_j)$ ,  $F$  and  $P$  be as in Lemma 2.26, and assume  $\{Z_1, Z_2, \dots\}$  is  $C$ -uniformly exact. Then*

$$(2.22) \quad \overline{\lim}_{j \rightarrow \infty} \|T_j P\|_{\text{cb}} \leq C.$$

*Proof.* Roger Smith's lemma ([29]) yields that for all  $n$ , operator spaces  $X$ , and linear maps  $T : X \rightarrow M_n$ ,

$$(2.23) \quad \|T\|_{\text{cb}} = \|T\|_n.$$

(See [25] for the operator space formulation and another proof.) Let  $k = \dim F$  and  $n = \mathbf{n}(k)$  where  $\mathbf{n}$  is the  $C$ -uniformity function for  $\{Z_1, Z_2, \dots\}$ . Then fixing  $j$ , the range of  $T_j P$  is a subspace of  $Z_j$  of dimension at most  $n$ , hence we obtain from (2.23) and Definition 2.5 that

$$(2.24) \quad \|T_j P\|_{\text{cb}} \leq C \|T\|_n$$

which immediately yields (2.22) in virtue of Lemma 2.26. ■

The next result follows from this corollary in the same manner as the corresponding Banach space result follows from Lemma 1.10 (see Remark 1.11).

**COROLLARY 2.28.** *Let  $X \subset Y$  be operator spaces with  $Y/X$  finite-dimensional,  $(Z_j)$  a sequence of operator spaces so that  $\{Z_1, Z_2, \dots\}$  is  $C$ -uniformly exact, and  $T : Y \rightarrow (Z_j)_{\ell^\infty}$  a non-zero completely bounded operator with  $TX \subset (Z_j)_{c_0}$ . Then given  $\varepsilon > 0$ , there exists  $\tilde{T} : Y \rightarrow (Z_j)_{c_0}$  so that  $\tilde{T}$  extends  $T|_X$  and  $\|\tilde{T}\|_{cb} \leq (C + 1 + \varepsilon)\|T\|_{cb}$ .*

*Proof of Theorem 2.25.* Assume (in virtue of Corollary 2.28) that  $Y/X$  is infinite-dimensional, and let  $y_1, y_2, \dots, F_k, Y_k$  and  $P_k$  be as in the proof of Theorem 1.9; let  $\varepsilon > 0$ . We construct for each  $k$ , a sequence  $(S_n^{(k)})$  of operators so that there is an  $M_k$  such that for all  $n$ ,

$$(2.25) \text{ if } n < M_k, \text{ then } S_n^{(k)} : Y \rightarrow Z_n \text{ and } \|S_n^{(k)}\|_{cb} < C\lambda + \frac{\varepsilon}{2\lambda};$$

$$(2.26) \text{ if } n \geq M_k, S_n^{(k)} : Y_k \rightarrow Z_n, S_n^{(k)} = T_n P_k \text{ and } \|S_n^{(k)}\|_{cb} < C + \frac{\varepsilon}{2\lambda^2};$$

$$(2.27) X \subset \ker S_n^{(k)};$$

$$(2.28) S_n^{(k+1)}|_{Y_k} = S_n^{(k)}.$$

Letting  $S_n = \overline{\bigcup_{k=1}^{\infty} S_n^{(k)}}$ , it follows that  $S_n$  is a well-defined completely bounded operator with  $\|S_n\|_{cb} \leq C\lambda + \frac{\varepsilon}{2\lambda} < C\lambda + \frac{\varepsilon}{\lambda}$  for all  $n$ . Just as before, it then follows that  $(S_n)$  satisfies the conclusion of Theorem 2.25.

Again, we construct the sequences  $(S_n^{(k)})_{n=1}^{\infty}$  by induction on  $k$ , setting  $S_n^{(0)} = 0$  for all  $n$ . Let  $k \geq 0$  and suppose  $S_n^{(k)}, M_k$  have been defined, satisfying (2.25)–(2.27). By Corollary 2.27, choose  $M_{k+1} > M_k$  so that for all  $n \geq M_{k+1}$ ,

$$(2.29) \qquad \|T_n P_{k+1}\|_{cb} < C + \frac{\varepsilon}{2\lambda^2}.$$

Now for  $n \geq M_{k+1}$ , let  $S_n^{(k+1)} = T_n P_{k+1}$ . Since also  $S_n^{(k)} = T_n P_k$  by (2.26), we have that (2.28) holds.

Now for  $M_k \leq n < M_{k+1}$ , since  $Z_n$  is a  $\lambda$ -injective operator space, choose  $S_n^{(k+1)}$  a linear extension of  $S_n^{(k)}$  from  $Y_k$  to  $Y$  with

$$(2.30) \qquad \|S_n^{(k+1)}\|_{cb} \leq \lambda \|S_n^{(k)}\|_{cb} < C\lambda + \frac{\varepsilon}{2\lambda} \quad (\text{by (2.26)}).$$

Finally, for  $n < M_k$ , let  $S_n^{(k+1)} = S_n^{(k)}$ . Evidently (2.28) holds for all  $n < M_{k+1}$ , and of course we have that (2.25)–(2.27) hold replacing “ $k$ ” by “ $k + 1$ ”. (Again the procedure is also valid in the setting of Corollary 2.10, since the separability of  $Y$  and the assumption that  $Z_n$  has the  $\lambda$ -CSEP allows us to do this.) ■

Theorem 2.25 (and hence Theorem 2.6) holds under a hypotheses weaker than that of uniform exactness. Here is the relevant concept.

DEFINITION 2.29. A family  $\mathcal{Z}$  of operator spaces is said to be of *finite matrix type* if there is a  $C \geq 1$  so that for any finite-dimensional operator space  $F$  there is an  $n = \mathbf{n}(F)$  so that

$$(2.31) \quad \|T\|_{\text{cb}} \leq C\|T\|_n \quad \text{for all linear operators } T : F \rightarrow Z \text{ and all } Z \in \mathcal{Z}.$$

If  $C$  works, we say that  $\mathcal{Z}$  is of  $C$ -finite matrix type, or briefly,  $\mathcal{Z}$  is  $C$ -finite; if the function  $\mathbf{n}$  works, we say that  $\mathcal{Z}$  is  $C$ -finite with function  $\mathbf{n}$ .

(Note that the domain of  $\mathbf{n}$  is the family of all finite-dimensional operator spaces.) An operator space  $Z$  is  $C$ -finite provided  $\{Z\}$  is  $C$ -finite.

Thanks to the result of R. Smith cited in the proof of Corollary 2.27, it follows that if  $\mathcal{Z}$  is  $C$ -uniformly exact,  $\mathcal{Z}$  is  $C$ -finite.

PROPOSITION 2.30. *Theorems 2.6 and 2.14 both hold if one replaces the assumption that  $\{Z_1, Z_2, \dots\}$  is  $C$ -uniformly exact by the assumption that  $\{Z_1, Z_2, \dots\}$  is  $C$ -finite.*

Of course it follows that also Corollaries 2.9 and 2.10 hold under this weaker assumption. T. Oikhberg has actually obtained a converse to this result, which goes as follows (see [21]): *Let  $Z_1, Z_2, \dots$  be separable operator spaces so that  $(Z_j)_{c_0}$  has the CSEP. Then  $\{Z_1, Z_2, \dots\}$  is of finite matrix type.*

Proposition 2.30 follows readily from the argument for Theorem 2.25 and the following simple, useful result about operator spaces:

FACT 2.31. *Let  $Y, Z$  be operator spaces and  $T : Y \rightarrow Z$  be a completely bounded map; let  $X$  be a (closed linear) subspace of  $Y$ , let  $\ker T \supset X$ ,  $\pi : Y \rightarrow Y/X$  be the quotient map, and  $\tilde{T} : Y/X \rightarrow Z$  be the canonical map with  $T = \tilde{T}\pi$ . Then*

$$(i) \quad \|T\|_n = \|\tilde{T}\|_n \quad \text{for all } n,$$

and hence

$$(ii) \quad \|T\|_{\text{cb}} = \|\tilde{T}\|_{\text{cb}}.$$

(This fact in turn follows from the natural, elementary result: if  $n \geq 1$  and  $(y_{ij}) \in M_n(Y)$  (regarded as contained in  $M_{00}(Y)$ ), then  $\|\pi(y_{ij})\| = \inf\{\|(y_{ij}) - (x_{ij})\| : (x_{ij}) \in M_n(X)\}$ ).

*Proof of Proposition 2.30.* The main new observation is that the conclusion of Corollary 2.27 holds if we assume instead that  $\{Z_1, Z_2, \dots\}$  is  $C$ -finite. Indeed, let  $G = Y/X$  and let  $n = \mathbf{n}(G)$ , where  $\mathbf{n}$  is as in Definition 2.29. Now fix  $j$ , let

$\pi : Y \rightarrow Y/X$  be the quotient map, and  $\tilde{T}_j : G \rightarrow Z_j$  the canonical map with  $\tilde{T}_j\pi = T_jP$ . Then

$$\begin{aligned}
 \|T_jP\|_{cb} &= \|\tilde{T}_j\|_{cb} && \text{by Fact 2.31} \\
 (2.31) \quad &\leq C\|\tilde{T}_j\|_n && \text{since } Z_j \text{ is } C\text{-exact} \\
 &= C\|T_jP\|_n && \text{by Fact 2.31.}
 \end{aligned}$$

Hence we deduce from Lemma 2.26 that

$$\overline{\lim}_{j \rightarrow \infty} \|T_jP\|_{cb} \leq C$$

as desired. The proof of the modified statement of Theorem 2.25 is now identical to the argument given above, whence Proposition 2.30 follows. ■

We finally deal with certain quantized formulations of the later results of Section 1. We first note that Corollary 2.9 follows from Theorem 2.25 in exactly the same way as Corollary 1.14 follows from Theorem 1.9. We also have the following quantized form of local complementability (Definition 1.16).

DEFINITION 2.32. Let  $X \subset Y$  be given operator spaces.  $X$  is said to be *completely locally complemented* in  $Y$  if there is a  $\lambda \geq 1$  so that

$$\begin{aligned}
 (2.32) \quad &X \text{ is a completely } \lambda\text{-complemented in } Z \text{ for all} \\
 &X \subset Z \subset Y \text{ with } Z/X \text{ finite-dimensional.}
 \end{aligned}$$

When (2.32) holds, we say  $X$  is *completely locally  $\lambda$ -complemented* in  $Z$ . We also say that  $X$  is *completely locally  $\lambda$ -cocomplemented* in  $Y$  if for all  $X \subset Z \subset Y$  with  $Z/X$  finite-dimensional there exists a projection  $P : Z \rightarrow X$  with  $\|I - P\|_{cb} \leq \lambda$ .

Proposition 2.30, Corollary 2.28 and the arguments for Corollary 1.17 now immediately yield the following result.

COROLLARY 2.33. *Let  $Z_1, Z_2, \dots$  be a sequence of operator spaces so that  $\{Z_1, Z_2, \dots\}$  is  $C$ -finite, and let  $Z = (Z_j)_{c_0}$ . Then  $Z$  is completely locally  $(C + \varepsilon)$ -cocomplemented in  $(Z_j)_{\ell^\infty}$ .*

Corollary 1.12 also immediately yields

COROLLARY 2.34. *Let  $Z_1, Z_2, \dots$  be a sequence of one-injective operator spaces and again let  $Z = (Z_j)_{c_0}$ . Then  $Z$  is Banach  $(2 + \varepsilon)$ -locally complemented in any operator superspace.*

REMARK 2.35. Corollary 2.34 has also been obtained in a different way in [15].

The concept of local complementability may be used to refine the formulation of Proposition 2.22, as follows.

PROPOSITION 2.36. *Let  $X$  be a reflexive operator space, and suppose that  $X$  is completely locally  $\lambda$ -complemented in every operator space  $Y$  with  $X \subset Y$ . Then  $X$  is  $\lambda$ -injective.*

*Proof.* The proof is really the same as the argument for Proposition 2.22. We assume that  $X \subset \mathcal{L}(H)$  for  $H$  a (not necessarily separable) Hilbert space, and then let  $\mathcal{S}$  be the family of subspaces  $Y$  of  $H$  with  $X \subset Y$  and  $\dim Y/X < \infty$ .  $\mathcal{S}$  is again directed by inclusion, and the argument that  $X$  is completely  $\lambda$ -complemented in  $\mathcal{L}(H)$  now follows just as before. ■

We finally note the quantized versions of Theorem 1.19 and its consequences.

THEOREM 2.37. *Replace “bounded” by “completely bounded” and “ $\|\tilde{T}\| \leq 2\|T\|$ ” by “ $\|\tilde{T}\|_{\text{cb}} \leq 2\|T\|_{\text{cb}}$ ” in the statement of Theorem 1.19.*

*Proof.* Assume without loss of generality that  $\|T\|_{\text{cb}} = 1$ . Let  $\mathcal{M}$  be as in Lemma 1.21, and define  $\mathcal{S}$  by

$$(2.33) \quad \mathcal{S} = \{S \in \mathcal{L}(Y, Z) \mid \|S\|_{\text{cb}} \leq 1 \text{ and } X \subset \ker S\}.$$

Again, we see that all SOT-cluster points of  $(T_n)$  lie in  $\mathcal{S}$ , for if  $T_{n_i} \rightarrow S$  in the SOT, also for each fixed  $k$ ,  $\|T_{n_i} - S\|_k \rightarrow 0$ , whence  $\|T_{n_i}\|_k \rightarrow \|S\|_k$ , so  $\|S\|_k \leq 1$ , and thus  $\|S\|_{\text{cb}} \leq 1$ . Of course also  $S(X) = 0$  for all  $x \in X$ ; so  $S \in \mathcal{S}$ . This argument also shows that  $\mathcal{S}$  is SOT-closed. The rest of the argument is the same as for Theorem 1.19. ■

The next result follows again immediately from Theorem 2.37 and the proof of Corollary 1.23.

COROLLARY 2.38. *Let  $Z$  be a finite-dimensional operator space and  $Y$  a separable subspace of  $\ell^\infty(Z)$  containing  $c_0(Z)$ . Then  $c_0(Z)$  is completely contractively cocomplemented in  $Y$ .*

COROLLARY 2.39.  *$c_0(M_n)$  has the 2-CSEP, for all  $n$ .*

This follows immediately from Corollary 2.38 and the fact that  $M_n$  is a 1-injective operator space. The next section gives a “saving property” for the space  $(M_n)_{c_0}$  in view of its failure to have the CSEP.

Our last result of this section shows that separable non-injective operator spaces cannot have the  $\lambda$ -CSEP if  $\lambda < 2$ .

PROPOSITION 2.40. *Let  $X$  be a separable operator space with the  $\lambda$ -CSEP. If  $\lambda < 2$ , then  $X$  is reflexive (and hence is  $\lambda$ -injective by Proposition 2.22).*

We first require the corresponding “pure” Banach space result. (This may be part of the subject’s folklore. The argument I give is due to W.B. Johnson, and I am most grateful to him for providing this elegant proof.)

LEMMA 2.41. *Let  $X$  be a separable Banach space containing a subspace  $Y$  isomorphic to  $c_0$ . Given  $\varepsilon > 0$ , there exists a subspace  $Z$  of  $Y$  so that  $Z$  is  $(1 + \varepsilon)$ -isomorphic to  $c_0$  and  $Z$  is  $(1 + \varepsilon)$ -complemented in  $X$ .*

*Proof.* Let  $\delta > 0$  be such that  $(1 + \delta)^2 < 1 + \varepsilon$ . By a result of R.C. James ([14]), we may choose a subspace  $E$  of  $Y$  with  $E$   $(1 + \delta)$ -isomorphic to  $c_0$ . It follows that we may choose a basis  $(e_j)$  for  $E$  so that for all  $(c_j)$  in  $c_0$ ,

$$(2.34) \quad \sup_j \sum |c_j| \leq \left\| \sum c_j e_j \right\| \leq (1 + \delta) \sup |c_j|.$$

Let  $(f_n)$  be a Hahn-Banach extension to  $X$  of the biorthogonal functionals to  $(e_n)$ . By passing to a subsequence, we may assume without losing the generality that  $(f_n)$  converges  $w^*$  in  $X^*$ . Now define  $g_n$  by

$$(2.35) \quad g_n = \frac{f_{2n} - f_{2n-1}}{2}.$$

It follows that  $g_n \rightarrow 0$   $w^*$  and moreover (by (2.34)),

$$(2.36) \quad \|g_n\| \leq 1 + \delta \quad \text{for all } n.$$

Finally, let  $z_n = e_{2n} - e_{2n-1}$  for all  $n$ ; then let  $Z = [z_n]$ . Of course  $Z$  is  $(1 + \varepsilon)$ -isomorphic to  $c_0$ , and  $(g_n)$  is biorthogonal to  $(z_n)$ . Thus we may define a projection  $P : X \rightarrow Z$  by

$$(2.37) \quad Px = \sum g_n(x) z_n \quad \text{for all } x \in X.$$

It follows that if  $x \in X$ , then

$$(2.38) \quad \begin{aligned} \|Px\| &\leq (1 + \delta) \sup_n |g_n(x)| && \text{by (2.34)} \\ &\leq (1 + \delta)^2 \|x\| && \text{by (2.36)}. \end{aligned}$$

Hence  $P$  is indeed a projection onto  $Z$  with  $\|P\| < 1 + \varepsilon$ . ■

*Proof of Proposition 2.40.* Suppose to the contrary that  $X$  is not reflexive. Then  $X$  contains a subspace isomorphic to  $c_0$  by Corollary 2.21. Now let  $\varepsilon > 0$ , to be decided later, and choose by Lemma 2.41 a subspace  $Z$  of  $X$  which is (Banach)  $(1 + \varepsilon)$ -isomorphic to  $c_0$  and  $(1 + \varepsilon)$ -complemented in  $X$ . Now let  $Y$  be a separable subspace of  $Z^{**}$  with  $Z \subset Y$ , let  $i : Z \rightarrow X$  be the identity injection, and also let  $P : X \rightarrow Z$  be a projection with  $\|P\| < 1 + \varepsilon$ . Since  $X$  has the  $\lambda$ -CSEP, letting  $Y$  have its natural operator space structure, we find a completely bounded extension  $\tilde{i} : Y \rightarrow X$  with  $\|\tilde{i}\|_{\text{cb}} \leq \lambda$ . But then letting  $Q = P\tilde{i}$ ,  $Q$  is a projection from  $Y$  onto  $Z$  and

$$(2.39) \quad \|Q\| < (1 + \varepsilon)\lambda.$$

Since  $Z$  is  $(1 + \varepsilon)$ -isomorphic to  $c_0$ , it now follows that if  $\tilde{Y}$  is separable with  $c_0 \subset \tilde{Y} \subset \ell^\infty$ , then

$$(2.40) \quad c_0 \text{ is } (1 + \varepsilon)^2\lambda\text{-complemented in } \tilde{Y}.$$

But this implies that  $c_0$  itself has the  $(1 + \varepsilon)^2\lambda$ -SEP, hence by Sobczyk's result ([29]),  $(1 + \varepsilon)^2\lambda \geq 2$ . Of course we then need only choose  $\varepsilon > 0$  with  $(1 + \varepsilon)^2\lambda < 2$ , to arrive at the desired contradiction. ■

### 3. THE COMPLETE SEPARABLE COMPLEMENTATION PROPERTY

In this section we study the following concept, more general than the CSEP.

**DEFINITION 3.1.** A separable locally reflexive operator space  $Z$  has the *Complete Separable Complementation Property* (the CSCP) if whenever  $Y$  is a separable locally reflexive operator space,  $X$  is a subspace of  $Y$ , and  $T : X \rightarrow Z$  is a complete surjective isomorphism,  $T$  has a completely bounded extension  $\tilde{T} : Y \rightarrow Z$ .

In other words,  $Z$  has the CSCP provided it is locally reflexive and every complete isomorph of  $Z$  is completely complemented in every separable locally reflexive operator superspace.

**REMARK 3.2.** After the first draft of this paper was completed, it was discovered that a locally reflexive separable  $Z$  has the CSCP provided it is completely complemented in every locally reflexive separable operator superspace  $Y$  with  $Z \subset Y \subset \mathcal{L}(H)$  (see [21]).

Evidently this property is invariant under complete isomorphisms. The main result of this section is as follows.

**THEOREM 3.3.** *Let  $\lambda \geq 1$  and let  $Z_1, Z_2, \dots$  be separable  $\lambda$ -injective operator spaces. Then  $(Z_j)_{c_0}$  has the CSCP.*

**COROLLARY 3.4.**  $\mathbf{K}_0$  has the CSCP.

In fact, our proof yields that if  $Y$  is a  $C$ -locally reflexive separable superspace of  $(Z_j)_{c_0}$ , then  $(Z_j)_{c_0}$  is completely  $C\lambda^3 + C\lambda^2 + \lambda + \varepsilon$ -complemented in  $Y$ , for all  $\varepsilon > 0$ . As in the preceding section, this theorem follows via the modification of T. Oikhberg of our original construction for the case  $\lambda = 1$ . Because of the known structure of the separable isometric injectives ([26]), Theorem 3.3 for the case  $\lambda = 1$  is equivalent to:  $(M_{\infty, n} \oplus M_{n, \infty})_{c_0}$  has the CSCP. After the first draft of this paper was completed, it was discovered that  $\mathbf{K}$  (the space of compact operators on  $\ell^2$ ), has the CSCP. (The proof uses Corollary 3.4 — see [21].) The main structural problem for this property is as follows:

**PROBLEM 3.5.** *Is every space with the CSCP completely isomorphic to a subspace of  $\mathbf{K}$ ?*

We discuss further aspects of this problem in Section 4. Let us also note that by T. Oikhberg's result (see [21]), Theorem 3.3 fails without the assumption of local reflexivity in the definition of the CSCP. Positive motivation for Theorem 3.3 and Definition 3.1 is given by the following result:

**COROLLARY 3.6.** *Let  $Z_1, Z_2, \dots$  be as in the statement of Theorem 3.3,  $\mathcal{A}$  be a separable nuclear  $C^*$ -algebra, and  $\tilde{Z}$  be a subspace of  $\mathcal{A}$  which is completely isomorphic to  $(Z_j)_{c_0}$ . Then  $\tilde{Z}$  is completely complemented in  $\mathcal{A}$ .*

This follows immediately from Theorem 3.3, in virtue of the fact that nuclear  $C^*$ -algebras are 1-locally reflexive ([9]). The quantitative version of Theorem 3.3 yields

**COROLLARY 3.7.** *Let  $\mathcal{A}$  be a separable nuclear  $C^*$ -algebra and  $\mathcal{K}_0$  be a subspace which is completely isometric to  $\mathbf{K}_0 = (M_n)_{c_0}$ . Then for all  $\varepsilon > 0$ ,  $\mathcal{K}_0$  is completely  $(3 + \varepsilon)$ -complemented in  $\mathcal{A}$ .*

Corollary 3.6 also suggests the following

**PROBLEM 3.8.** *Let  $Z$  be a separable operator space which completely embeds in some nuclear  $C^*$ -algebra. Suppose that every complete embedding of  $Z$  into a nuclear separable  $C^*$ -algebra  $\mathcal{A}$  is completely complemented in  $\mathcal{A}$ . Does  $Z$  have the CSCP?*

We now deal with the proof of Theorem 3.3. As has been the case in the preceding section, the arguments hold in considerable generality; *local complementability* (cf. Definition 2.32) plays a key role in the discussion.

**THEOREM 3.9.** *Let  $\lambda \geq 1$ , and let  $Z_1, Z_2, \dots$  be  $\lambda$ -injective operator spaces and  $X \subset Y$  operator spaces with  $X$  locally complemented in  $Y$  and  $Y/X$  separable. Let  $T : X \rightarrow Z$  be a completely bounded operator, where  $Z = (Z_j)_{c_0}$ . Then  $T$  admits a completely bounded extension  $\tilde{T} : Y \rightarrow Z$ .*

**REMARK 3.10.** The proof yields that if  $X$  is completely locally  $\beta$ -cocomplemented in  $Y$ , then for all  $\varepsilon > 0$ ,  $\tilde{T}$  may be chosen with  $\|\tilde{T}\|_{cb} < (\beta\lambda^2 + \lambda + \varepsilon)\|T\|_{cb}$  (if  $T \neq 0$ ).

We first note an immediate consequence.

**COROLLARY 3.11.** *Let  $Z_1, Z_2, \dots$  and  $Z$  be as in Theorem 3.9 and let  $Y$  be an operator space with  $Z \subset Y$ ,  $Y/Z$  separable, and  $Z$  locally complemented in  $Y$ . Then  $Z$  is completely complemented in  $Y$ .*

**REMARK 3.12** Again, we obtain that if  $Z$  is completely locally  $\beta$ -cocomplemented in  $Y$ ,  $Z$  is completely  $(\beta\lambda^2 + \lambda + \varepsilon)$ -complemented in  $Y$  for all  $\varepsilon > 0$ .

As before, we first reformulate Theorem 3.9.

**THEOREM 3.13.** *Let  $Z_1, Z_2, \dots, Z, X$ , and  $Y$  be operator spaces satisfying the hypotheses of Theorem 3.9, and suppose  $X$  is completely locally  $C$ -cocomplemented in  $Y$ . Let  $(T_j)$  be a sequence of completely bounded operators with  $T_j : Y \rightarrow Z_j$  for all  $j$ , so that  $T_j|_X \rightarrow 0$  in the SOT and  $\sup_j \|T_j\|_{cb} = 1$ . For every  $\varepsilon > 0$ , there exists a sequence  $(S_j)$  of completely bounded operators so that for all  $j$ :*

- (i)  $S_j : X \rightarrow Z_j$ ;
- (ii)  $X \subset \ker S_j$ ;
- (iii)  $\|S_j\|_{cb} < C\lambda + \frac{\varepsilon}{2\lambda}$ ,

so that  $(T_j - S_j) \rightarrow 0$  in the SOT.

The proof that Theorem 3.13  $\Rightarrow$  Theorem 3.9 is again the same as the one showing Theorem 2.25  $\Rightarrow$  Theorem 2.6; this proof also yields the quantitative statement in the Remarks following Theorem 3.7, as well as the following quantitative variation of Corollary 3.11 (all objects as in its statement): *If  $Z$  is completely locally  $C$ -cocomplemented in  $Y$ , then  $Z$  is  $(C\lambda + \varepsilon)$ -cocomplemented in  $Y$  for all  $\varepsilon > 0$ .*

The proof of Theorem 3.9 (i.e., of Theorem 3.13) is analogous to the proofs of Theorems 1.3 and 2.6; it requires a different (again rather surprising) lemma, replacing Lemmas 1.10 and 2.26.

LEMMA 3.14. *Let  $Z_1, Z_2, \dots, X, Y$  be arbitrary operator spaces with  $X \subset Y$  and  $Y/X$  finite-dimensional. Let  $(T_n)$  be a sequence of completely bounded operators with  $T_j : Y \rightarrow Z_j$  for all  $j$ , so that  $T_n|_X \rightarrow 0$  in the SOT. Let  $P$  and  $Q$  be linear projections on  $Y$  with  $\ker P = \ker Q = X$  and  $\dim \text{Range } P = \dim \text{Range } Q = \dim Y/X$ . Then*

$$(3.1) \quad \lim_{n \rightarrow \infty} \|T_n P - T_n Q\|_{\text{cb}} = 0.$$

Hence

$$(3.2) \quad \overline{\lim}_{n \rightarrow \infty} \|T_n P\|_{\text{cb}} = \overline{\lim}_{n \rightarrow \infty} \|T_n Q\|_{\text{cb}}.$$

*Proof.* Let  $S_n = T_n Q$  for all  $n$ . Then

$$(3.3) \quad T_n - S_n \rightarrow 0 \quad \text{in the SOT.}$$

Indeed, if  $y \in Y$ ,  $(T_n - S_n)(y) = T_n(I - Q)(y) \rightarrow 0$  in norm since  $(I - Q)y \in X$ .

Let  $F = \text{Range } P$ . Since  $F$  is finite-dimensional, (3.3) yields

$$(3.4) \quad \|(T_n - S_n)|_F\| \rightarrow 0$$

whence

$$(3.5) \quad \|(T_n - S_n)|_F\|_{\text{cb}} \leq (\dim F) \|(T_n - S_n)|_F\| \rightarrow 0.$$

Now

$$(3.6) \quad QP = Q \quad \text{since} \quad Q(I - P) = 0.$$

Hence

$$(3.7) \quad \begin{aligned} \|T_n P - T_n Q\|_{\text{cb}} &= \|T_n P - S_n P\| && \text{by (3.6)} \\ &\leq \|(T_n - S_n)|_F\|_{\text{cb}} \|P\|_{\text{cb}} \rightarrow 0. \end{aligned}$$

(Note that  $P$  is completely bounded since it is a continuous finite rank operator.) ■

REMARK 3.15. This proof could be given “more conceptually” by noting that  $\ker(P - Q) \supset X$  and hence the operator  $T_n(P - Q)$  “lives” on  $Y/X$ , a finite-dimensional space; in fact  $T_n(P - Q) = T_n((I - Q) - (I - P)) \rightarrow 0$  in the SOT on  $Y/X$ , so the cb-norms of the sequence  $(T_n(P - Q))$ , as operators on  $Y/X$ , go to zero.

The proof of Theorem 3.13 is actually identical to that for Theorem 2.25 once we draw the following consequence of Lemma 3.14.

LEMMA 3.16. *Assuming the hypotheses of Lemma 3.14, let  $X \subset Y_0 \subset Y$  with  $Y_0/X$  finite-dimensional and let  $P : Y_0 \rightarrow Y_0$  be a finite-rank projection with  $\ker P = X$  and  $\text{rank } P = \dim Y_0/X$ . Then*

$$(3.8) \quad \overline{\lim}_{j \rightarrow \infty} \|T_j P\|_{\text{cb}} \leq C.$$

*Proof.* By hypotheses, there exists a projection  $Q$  with  $\ker P = X$  and  $\text{rank } Q = \dim Y_0/X$ , so that  $\|Q\|_{\text{cb}} \leq C$ . Hence of course

$$(3.9) \quad \overline{\lim}_{n \rightarrow \infty} \|T_n Q\|_{\text{cb}} \leq C.$$

Now (3.8) follows immediately from Lemma 3.14, in virtue of (3.2). ■

COMMENT 3.17. Lemma 3.16 holds for arbitrary  $Z_j$ 's; i.e., the assumption of  $\lambda$ -injectivity is not needed here.

The proof of Theorem 3.13 is now word for word the same as that for Theorem 2.25, except that we replace Corollary 2.27 by Lemma 3.16 in the discussion. ■

We need one last ingredient for the proof of Theorem 3.3 (the main result in this section).

(If  $X \subset Y$ , we identify  $X^{**}$  with  $X^{\perp\perp} \subset Y^{**}$ .)

LEMMA 3.18. *Let  $X$  and  $Y$  be operator spaces with  $X \subset Y$ ,  $X^{**}$  isomorphically injective, and  $Y$  locally reflexive. Then  $X$  is completely locally complemented in  $Y$ .*

REMARK 3.19. The proof yields that if  $X^{**}$  is  $\lambda$ -injective and  $Y$  is  $C$ -locally reflexive, then for all  $\varepsilon > 0$ ,  $X$  is completely locally  $(C\lambda + C + \varepsilon)$ -cocomplemented in  $Y$ , hence  $X$  is completely locally  $(C\lambda + C + 1 + \varepsilon)$ -complemented in  $Y$ .

We delay the proof of this lemma, showing instead how we obtain Theorem 3.13. In fact, we have the more general

THEOREM 3.20. *Let  $Z_1, Z_2, \dots$  be reflexive  $\lambda$ -injective operator spaces,  $Z = (Z_j)_{c_0}$ , and  $X \subset Y$  be operator spaces with  $Y/X$  separable and  $Y$  locally reflexive. Let  $T : X \rightarrow Z$  be a complete surjective isomorphism. Then  $T$  admits a completely bounded extension  $\tilde{T} : Y \rightarrow Z$ .*

REMARK 3.21. (i) If  $Y$  is  $C$ -locally reflexive and  $\|T\|_{\text{cb}}\|T^{-1}\|_{\text{cb}} = \gamma$ , we obtain for  $\varepsilon > 0$  that  $\tilde{T}$  may be chosen with

$$\|\tilde{T}\|_{\text{cb}} < (C\gamma\lambda^3 + C\lambda^2 + \lambda + \varepsilon)\|T\|_{\text{cb}}.$$

(ii) As noted in Section 2, every separable isomorphically injective operator space is reflexive, so Theorem 3.20 indeed yields Theorem 3.3. Actually, more care in the proof yields that *the conclusion of Theorem 3.20 holds if one deletes the reflexivity assumption from its hypotheses*. Hence we obtain the “quantized” version of Corollary 1.4 (with a worse constant): *Let  $Z$  be as in Theorem 3.20 (but drop the assumption that the  $Z_j$ ’s are reflexive). Then  $Z$  is  $(C\lambda^3 + C\lambda^2 + \lambda + \varepsilon)$ -completely complemented in every  $C$ -locally reflexive superspace  $Y$  with  $Y/Z$  separable.*

*Proof of Theorem 3.20.* Let  $C$  and  $\gamma$  be as in Remark 3.21 (i). It follows from the hypotheses that  $X^{**}$  is completely  $\gamma$ -isomorphic to  $(Z_j)_{\ell^\infty}$ , a  $\lambda$ -injective operator space. Hence  $X^{**}$  is completely  $(\gamma\lambda + 1)$ -cocomplemented in  $Y^{**}$ , so the proof of Lemma 3.18 yields that given  $\varepsilon > 0$ ,  $X$  is completely locally  $(C(\gamma\lambda + 1) + \varepsilon)$ -cocomplemented in  $Y$ . Hence (by playing with  $\varepsilon$ ) we obtain from Theorem 3.9 that the extension  $\tilde{T}$  may be chosen with

$$(3.10) \quad \|\tilde{T}\|_{cb} \leq (C\gamma\lambda^3 + C\lambda^2 + \lambda + \varepsilon)\|T\|_{cb}. \quad \blacksquare$$

We now deal with Lemma 3.18. Let us first recall the precise concept of operator space local reflexivity (reformulated in the spirit of the original Banach space concept given by J. Lindenstrauss and H.P. Rosenthal in [18], as refined in [16]).

DEFINITION 3.22. An operator space  $X$  is called  *$C$ -locally reflexive* if for all  $\varepsilon > 0$ , and finite dimensional subspaces  $F$  and  $G$  of  $X^*$  and  $X^{**}$  respectively, there exists a linear operator  $T : G \rightarrow X$  satisfying

$$(3.11) \quad \langle Tg, f \rangle = \langle g, f \rangle \quad \text{for all } g \in G, f \in F$$

and

$$(3.12) \quad \|T\|_{cb} < C + \varepsilon.$$

As shown in [16], Banach spaces are thus 1-locally reflexive. Remarkable permanence properties given in [11] yield that if  $X$  is a  $C$ -locally reflexive operator space, then every subspace of  $X$  is  $C$ -locally reflexive; moreover as noted above, nuclear  $C^*$ -algebras are 1-locally reflexive.

Lemma 3.16 is an immediate consequence of the following technical result (whose proof is the operator space analogue of an argument in [12]).

**SUBLEMMA 3.23.** *Let  $X \subset Y$  be operator spaces with  $\dim Y/X < \infty$  so that  $Y$  is  $C$ -locally reflexive and  $X^{**}$  is completely  $\beta$ -cocomplemented in  $Y^{**}$ . Then for all  $\varepsilon > 0$ ,  $X$  is completely  $(C\beta + \varepsilon)$ -cocomplemented in  $Y$ .*

Let us first deduce Lemma 3.18. Let  $\varepsilon > 0$ . Assuming that  $X^{**}$  is  $\lambda$ -injective,  $X^{**}$  is completely  $\lambda$ -complemented in  $Y^{**}$ . Now assuming  $Y$  is  $C$ -locally reflexive, if  $Y_0$  is a subspace of  $Y$  with  $X \subset Y_0$  and  $Y_0/X$  finite-dimensional,  $Y_0$  is also  $C$ -locally reflexive and of course  $X^{**}$  is also completely  $\lambda$ -complemented in  $Y_0^{**}$ , hence  $X^{**}$  is completely  $(\lambda + 1)$ -cocomplemented in  $Y_0^{**}$ . Thus by Sublemma 3.23,  $X$  is completely  $(C(\lambda + 1) + \varepsilon)$ -cocomplemented in  $Y_0$ . ■

*Proof of Sublemma 3.23.* Of course we identify  $X^{**}$  with  $X^{\perp\perp}$ . Let  $F = X^\perp$ . The hypotheses actually imply that there exists a projection  $P$  from  $Y^*$  onto  $X^\perp$  satisfying

$$(3.13) \quad \|P\|_{\text{cb}} \leq \beta.$$

Indeed, if  $Q$  is a projection on  $Y^{**}$  with  $\ker Q = X^{**}$  and  $\|Q\|_{\text{cb}} \leq \beta$ , then  $P = Q^*|_{Y^*}$  has the desired property, where we regard  $Y^* \subset Y^{***}$ . (In fact, the range of  $Q^*$  equals  $X^{\perp\perp\perp} = X^\perp$ , because  $Y/X$  is finite-dimensional.)

Now define  $G$  by

$$(3.14) \quad G = P^*(Y^{**}).$$

Of course  $G$  is finite-dimensional; hence since  $Y$  is  $C$ -locally reflexive, given  $\varepsilon > 0$ , choose  $T : G \rightarrow Y$  a linear operator with

$$(3.15) \quad \|T\|_{\text{cb}} < C + \frac{\varepsilon}{\beta}$$

and

$$(3.16) \quad \langle Tg, f \rangle = \langle g, f \rangle \quad \text{for all } g \in G, f \in F.$$

Finally, define  $H$  by

$$(3.17) \quad H = T(G).$$

We now claim that  $H$  yields the desired decomposition of  $Y$ . Now it follows immediately from (3.14) that

$$(3.18) \quad Y^{**} = F^\perp \oplus G.$$

This and (3.16) imply that  $T$  is one-to-one and  $H \cap X = \{0\}$ . Indeed, suppose  $g \in G$  and  $Tg = 0$ . Then  $\langle Tg, f \rangle = \langle g, f \rangle = 0$  for all  $f \in F$ , whence by (3.18),  $g = 0$ . But if  $Tg \in X$ , then since  $X^\perp = F$ ,  $\langle Tg, f \rangle = 0 = \langle g, f \rangle$  for all  $f \in F$ , so of course  $g = Tg = 0$ .

Since  $\dim Y/X = \dim Y^{**}/X^{**} = \dim G$ , we have now deduced

$$(3.19) \quad Y = X \oplus H.$$

Now let  $R$  be the projection from  $Y$  onto  $H$  with  $\ker R = X$ . We claim

$$(3.20) \quad \|R\|_{\text{cb}} < C\beta + \varepsilon.$$

We need the fundamental duality pairing for operator spaces. Fix  $K_1, \dots, K_m$  in  $\mathbf{K}$ . Then given  $y_1, \dots, y_m$  in  $Y$ ,  $y_1^*, \dots, y_\ell^*$  in  $Y^*$ , and  $L_1, \dots, L_\ell$  in  $\mathbf{K}$ , we define

$$(3.21) \quad \left\langle \sum_{i=1}^m K_i \otimes y_i, \sum_{j=1}^{\ell} L_j \otimes y_j^* \right\rangle = \sum_{i,j} y_j^*(y_i) K_i \otimes L_j.$$

(Here, the last term is an operator on  $\ell^2 \otimes \ell^2$ .) Then we have (cf. [25])

$$(3.22) \quad \left\| \sum K_i \otimes y_i \right\| = \sup \left\{ \left\| \left\langle \sum K_i \otimes y_i, \sum L_j \otimes y_j^* \right\rangle \right\| : \left\| \sum L_j \otimes y_j^* \right\| = 1 \right\}.$$

Now applying this duality statement to  $Y^*$  rather than  $Y$ , it follows by our definition of  $P$  and  $G$ , that given  $g_1, \dots, g_m$  in  $G$  (and  $K_1, \dots, K_m$  as above), then

$$(3.23) \quad \left\| \sum K_i \otimes g_i \right\| \leq \beta \sup \left\{ \left\| \left\langle \sum K_i \otimes g_i, \sum L_j \otimes f_j \right\rangle \right\| : f_1, \dots, f_\ell \in F, L_1, \dots, L_\ell \text{ in } \mathbf{K}, \text{ and } \left\| \sum L_j \otimes f_j \right\| = 1 \right\}.$$

Finally, let  $h_1, \dots, h_n$  in  $H$ ,  $x_1, \dots, x_m$  in  $X$  and  $K_1, \dots, K_m$  as above. We must prove:

$$(3.24) \quad \left\| \sum K_i \otimes R(x_i + h_i) \right\| \stackrel{(\text{trivial})}{=} \left\| \sum K_i \otimes h_i \right\| \leq (C\beta + \varepsilon) \left\| \sum K_i \otimes (x_i + h_i) \right\|.$$

Now choose unique  $g_1, \dots, g_m$  in  $G$  with  $h_i = Tg_i$  for all  $i$ . Then

$$(3.25) \quad \begin{aligned} & \left\| \sum K_i \otimes Tg_i \right\| \leq (C + \varepsilon/\beta) \left\| \sum K_i \otimes g_i \right\| && \text{by (3.15)} \\ & \leq (C + \varepsilon/\beta)\beta \sup \left\{ \left\| \left\langle \sum K_i \otimes g_i, \sum L_j \otimes f_j \right\rangle \right\| : \right. \\ & \quad \left. f_i \text{'s in } F, L_j \text{'s in } \mathbf{K}, \text{ and } \left\| \sum L_j \otimes f_j \right\| = 1 \right\} && \text{by (3.23)} \\ & = (C\beta + \varepsilon) \sup \left\{ \left\| \left\langle \sum K_i \otimes (x_i + Tg_i), \sum L_j \otimes f_j \right\rangle \right\| : \right. \\ & \quad \left. f_i \text{'s in } F, L_j \text{'s in } \mathbf{K} \text{ and } \left\| \sum L_j \otimes f_j \right\| = 1 \right\} \end{aligned}$$

by (3.18) and the fact that  $F = X^\perp$

$$\leq (C\beta + \varepsilon) \left\| \sum K_i \otimes (x_i + Tg_i) \right\| \quad \text{by (3.21).}$$

This proves (3.24), completing the proof. ■

## 4. EXAMPLES OF SPACES WITH THE CSEP AND THE CSCP

Our preceding results yield certain lists of separable infinite-dimensional operator spaces with the CSEP and CSCP. It is conceivable that these lists are complete (up to complete isomorphism).

The results stated in this section are direct consequences of the work in the preceding sections and previously known facts. The conjectures we formulate here are strongly believed to be true, and should be “accessible.” On the other hand, the problems we formulate are (probably) at a considerably deeper level.

We first give a basic definition; the operator space analogue of a well known Banach space concept.

**DEFINITION 4.1.** An operator space  $X$  is called *primary* if whenever  $Y$  and  $Z$  are operator spaces with  $X$  completely isomorphic to  $Y \oplus Z$ , then  $X$  is completely isomorphic to  $Y$  or  $Z$ .

All of our examples of spaces with the CSEP (respective CSCP) are direct sums of primary spaces with the CSEP (respective CSCP).

We first treat the CSEP. Recall that  $\mathcal{R}, \mathcal{C}$  denote infinite-dimensional row and column space, respectively, and  $\mathcal{R}_n, \mathcal{C}_n$   $n$ -dimensional row and column space, respectively.

**PROPOSITION 4.2.** *There are at least six isomorphically different Banach spaces among the separable infinite-dimensional operator spaces with the CSEP, namely*

$$(4.1) \quad c_0, \quad (\ell_n^2)_{c_0}, \quad c_0(\ell^2), \quad \ell^2, \quad c_0 \oplus \ell^2, \quad \text{and} \quad (\ell_n^2)_{c_0} \oplus \ell^2.$$

*Proof.* Standard Banach space results easily yield these spaces are isomorphically distinct (cf. [7]). Of course Sobczyk’s theorem yields that  $c_0$  has the 2-CSEP; as a Banach space,  $R$  is just isometric to  $\ell^2$ , and  $R$  has the 1-CSEP. Corollary 2.13 yields immediately that  $(R_n)_{c_0}$  and  $c_0(R)$  have the  $(2+\varepsilon)$ -CSEP for all  $\varepsilon > 0$ , and  $(R_n)_{c_0}$  is isometric to  $(\ell_n^2)_{c_0}$  and  $c_0(R)$  is isometric to  $c_0(\ell^2)$ . Finally,  $c_0 \oplus R$  has the 2-CSEP, and this is just  $c_0 \oplus \ell^2$  in the Banach space category. ■

**PROBLEM 4.3.** *Let  $X$  be a separable infinite-dimensional operator space with the CSEP. Is  $X$  Banach isomorphic to one of the six spaces in (4.1)?*

By the results in [7], the first four spaces in (4.1) are primary, and moreover, every infinite dimensional complemented subspace of  $c_0(\ell^2)$  (the largest one), is isomorphic to one of these six. Thus Problem 4.3 has an affirmative answer if every separable space with the CSEP is completely isomorphic to a subspace of  $c_0(\mathcal{R}) \oplus c_0(\mathcal{C})$ .

CONJECTURE 4.4. *There are at least seven completely isomorphically distinct separable infinite-dimensional primary operator spaces with the CSEP, namely:*

$$(4.2) \quad c_0, \quad (\mathcal{R}_n)_{c_0}, \quad (\mathcal{C}_n)_{c_0}, \quad \mathcal{R}, \quad \mathcal{C}, \quad c_0(\mathcal{R}), \quad c_0(\mathcal{C}).$$

As before, it follows immediately from Corollary 2.13 that all these spaces have the CSEP (indeed all are completely isometric to completely contractively complemented subspaces of  $c_0(\mathcal{R}) \oplus c_0(\mathcal{C})$ ). It is also easily seen that all these spaces are isomorphically distinct as operator spaces, and it is essentially trivial that  $c_0, \mathcal{R}$ , and  $\mathcal{C}$  are all primary (in fact they are prime). The content of the conjecture thus becomes: the remaining spaces in (4.2) are all primary.

PROBLEM 4.5. *Let  $X$  be a separable infinite-dimensional primary operator space with the CSEP. Is  $X$  completely isomorphic to one of the seven spaces listed in (4.2)?*

CONJECTURE 4.6. *There are at least 21 completely isomorphically distinct separable operator spaces with the CSEP, namely:*

- (i) *the seven spaces listed in (4.2);*
- (b) *the nine spaces  $c_0 \oplus \mathcal{R}$ ,  $c_0 \oplus \mathcal{C}$ ,  $(\mathcal{R}_n)_{c_0} \oplus \mathcal{R}$ ,  $(\mathcal{R}_n)_{c_0} \oplus (\mathcal{C}_n)_{c_0}$ ,  $(\mathcal{R}_n)_{c_0} \oplus \mathcal{C}$ ,  $(\mathcal{C}_n)_{c_0} \oplus \mathcal{R}$ ,  $\mathcal{R} \oplus \mathcal{C}$ ,  $(\mathcal{C}_n)_{c_0} \oplus \mathcal{C}$ ,  $c_0(\mathcal{R}) \oplus c_0(\mathcal{C})$ ;*
- (iii) *the five spaces  $c_0 \oplus \mathcal{R} \oplus \mathcal{C}$ ,  $(\mathcal{R}_n)_{c_0} \oplus (\mathcal{C}_n)_{c_0} \oplus \mathcal{R}$ ,  $(\mathcal{R}_n)_{c_0} \oplus (\mathcal{C}_n)_{c_0} \oplus \mathcal{C}$ ,  $(\mathcal{R}_n)_{c_0} \oplus \mathcal{R} \oplus \mathcal{C}$ ,  $(\mathcal{R}_n)_{c_0} \oplus (\mathcal{C}_n)_{c_0} \oplus \mathcal{R} \oplus \mathcal{C}$ .*

*Moreover, any finite direct sum of any of these spaces is again completely isomorphic to one of them.*

As above, it follows immediately from the results of Section 2 that all these spaces have the CSEP. We leave the remaining assertions of this conjecture to the ambitious reader.

PROBLEM 4.7. *Is every separable infinite-dimensional operator space with the CSEP completely isomorphic to one of the 21 spaces in Conjecture 4.6?*

We now deal with the CSCP. It is conceivable that the separable infinite-dimensional operator spaces with the CSCP are precisely those which are completely isomorphic to completely complemented subspaces of  $\mathbf{K}$ . Accordingly, we discuss the evident spaces with this property; recall that  $\mathbf{K}_0$  denotes the space  $(M_n)_{c_0}$ . The following result is due to J. Arazy and J. Lindenstrauss (see Theorem 5 and Remark (i), p. 107, of [4]).

PROPOSITION 4.8. *There are at least 11 isomorphically distinct Banach spaces, isomorphic to an infinite-dimensional complemented subspace of  $\mathbf{K}$ , namely*

- (i) *the seven spaces  $c_0$ ,  $\ell^2$ ,  $(\ell_n^2)_{c_0}$ ,  $c_0(\ell^2)$ ,  $\mathbf{K}_0$ ,  $(M_{\infty,n})_{c_0}$ , and  $\mathbf{K}$ ;*
- (ii) *the four spaces  $c_0 \oplus \ell^2$ ,  $\ell^2 \oplus (\ell_n^2)_{c_0}$ ,  $\ell^2 \oplus \mathbf{K}_0$ , and  $c_0(\ell^2) \oplus \mathbf{K}_0$ .*

It is known that all the spaces in (a), except possibly  $(M_{\infty,n})_{c_0}$ , are primary. The primariness of the first four is noted above ([7]). The result that  $\mathbf{K}$  and  $\mathbf{K}_0$  are primary, is due to J. Arazy ([3]). I conjecture that also  $(M_{\infty,n})_{c_0}$  is primary, but this remains an open question.

PROBLEM 4.9. *Is every infinite-dimensional completely complemented subspace of  $\mathbf{K}$  Banach-isomorphic to one of the 11 spaces listed in (i) and (ii) of Proposition 4.8?*

It is conceivable that every infinite dimensional complemented subspace of  $\mathbf{K}$  is isomorphic to one of these 11 spaces; this problem is raised in [4]. Problem 4.9 might be somewhat more accessible. Of course our motivation here is that by the results of Section 3 (respective [21] for  $\mathbf{K}$  itself), all of the spaces listed in Problem 4.9 are Banach-isomorphic to operator spaces with the CSCP.

CONJECTURE 4.10. *There are at least 11 completely isomorphically distinct primary operator spaces, each completely isometric to a completely contractively complemented subspace of  $\mathbf{K}$ , namely:*

- (i) *the seven spaces listed in (4.2);*
- (b) *the four spaces  $\mathbf{K}$ ,  $\mathbf{K}_0$ ,  $(M_{\infty,n})_{c_0}$  and  $(M_{n,\infty})_{c_0}$ .*

Using the known Banach space result, Proposition 4.8, it is not hard to see that all the listed spaces are completely isomorphically distinct, and all are completely contractively complemented in  $\mathbf{K}$ . The content of the conjecture thus becomes: all these spaces are primary. (It seems likely the work in [3] should yield that  $\mathbf{K}$  and  $\mathbf{K}_0$  are primary operator spaces, but we have not verified this.) Again, by the results of Section 3, (and [21] for the case of  $\mathbf{K}$  itself) all these spaces have the CSCP.

PROBLEM 4.11. *Let  $X$  be a separable infinite-dimensional primary operator space with the CSCP. Is  $X$  completely isomorphic to one of the spaces listed in (i) and (ii) of Conjecture 4.10?*

Of course a motivation to classify the (apparently finite but rather immense number of) finite-direct sums of these 11 spaces would be provided by an affirmative answer to the following (obviously deep) problem:

PROBLEM 4.12. *Is every operator space with the CSCP completely isomorphic to a finite direct sum of primary operator spaces?*

*Acknowledgements.* Some of these results were discovered during the August 1997 Workshop in Operator Spaces at Texas A&M University. I wish to thank the many participants with whom I had most useful, stimulating discussions. Special thanks to Gilles Pisier and Timur Oikhberg for their invaluable feedback on the operator-space constructs so crucial in this work.

The research for this paper was partially supported by NSF Grant DMS-9500874 and TARP Grant ARP-275.

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HASKELL ROSENTHAL  
Department of Mathematics  
The University of Texas at Austin  
Austin, TX 78712  
USA  
E-mail: rosenthl@math.utexas.edu

Received February 20, 1998; revised January 15, 1999.