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REFLEXIVITY OF C_0-OPERATORS OVER A MULTIPLY CONNECTED REGION

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ABSTRACT. In this paper we show that an operator T of class C_0 over a multiply connected region is reflexive if and only if its Jordan model is reflexive. Besides, the reflexivity of T depends only on the reflexivity of a single Jordan block that can be easily calculated from the model of T.

KEYWORDS: *C_0-operator, reflexivity, Jordan model, multiply connected region.*

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1. INTRODUCTION AND NOTATION

Consider a bounded region Ω in the complex plane $\mathbb{C}$ whose boundary Γ consists of a finite number of disjoint, closed, analytic Jordan curves. A holomorphic function f on Ω is in $H^p(\Omega)$ for $1 \leqslant p < \infty$, if the subharmonic function $|f|^p$ has a harmonic majorant on Ω. For every fixed $z_0 \in \Omega$, it is possible to define a norm on $H^p(\Omega)$ by

$$\|f\| = \inf\{u(z_0)^{1/p} : u \text{ is a harmonic majorant of } |f|^p\}.$$

Denoting the harmonic measure on Γ for the point z_0 by ω, it is well-known that each $f \in H^p(\Omega)$ has nontangential boundary values f^* almost everywhere $\mathrm{d}\omega$, and f^* is in $L^p(\Gamma, \omega)$. Moreover the mapping $f \to f^*$ is an isometry from $H^p(\Omega)$ onto a closed subspace of $L^p(\Gamma, \omega)$. We will employ the same symbol f to stand both for the function and for its boundary values. A function f defined on Ω is in $H^\infty(\Omega)$ if it is holomorphic and bounded. The space $H^\infty(\Omega)$ is a closed subspace of $L^\infty(\Gamma, \omega)$ and it is a Banach algebra when endowed with the supremum norm. Finally, the mapping $f \to f^*$ is an isometry of $H^\infty(\Omega)$ onto a weak*-closed subalgebra of $L^\infty(\Gamma, \omega)$. The theory of Hardy spaces over multiply connected regions has been first studied by Rudin ([8], see also [6]).

We recall from [7] that a function $\theta \in H^\infty(\Omega)$ is said to be *inner* if $|\theta|$ is essentially constant on each component of Γ. If θ and θ' are two inner functions, we say that θ' divides θ (and we write $\theta'|\theta$) if θ can be written as $\theta = \theta'\varphi$ for some φ in $H^\infty(\Omega)$. We will denote somewhat informally such φ by θ/θ'. Moreover, if $\theta'|\theta$ and $\theta|\theta'$ we say that θ and θ' are *equivalent* and we write $\theta \equiv \theta'$. We denote by $\theta \wedge \theta'$ the *greatest common inner divisor* of θ and θ', i.e., the unique (up to equivalence) inner function which divides θ and θ' and is divisible by any other inner function dividing θ and θ' (cf. [12], Proposition 2.3.4). Clearly, this definition can be extended to a family of functions. Let $R(\Omega)$ be the space of rational functions with poles off $\overline{\Omega}$. A closed linear subspace $\mathcal{M}$ of $H^p(\Omega)$ (weak*-closed if $p = \infty$) is said to be *fully invariant* if $rf \in \mathcal{M}$ for all $f \in \mathcal{M}$ and for all $r \in R(\Omega)$. It is well-known that any fully invariant subspace of $H^p(\Omega)$ has the form $\theta H^p(\Omega)$ for some inner function θ. Two inner functions θ_1 and θ_2 generate the same subspace if and only if $\theta_1 \equiv \theta_2$.

Let H be a Hilbert space. Given a subset $\mathcal{M} \subset H$ we denote by $[\mathcal{M}]^-$ the norm-closure of $\mathcal{M}$. Given a family $\{\mathcal{M}_i\}_{i\in I} \subset H$, we denote by $\bigvee_{i\in I} \mathcal{M}_i$ the closed linear span generated by $\bigcup_{i\in I} \mathcal{M}_i$. Let $\mathcal{L}(H)$ be the algebra of bounded linear operators on H, and $\mathcal{L}(H, H')$ the algebra of bounded linear operators on H with values in a Hilbert space H'. An operator $X \in \mathcal{L}(H, H')$ is a *quasiaffinity* if it is one-to-one with dense range. An operator $T \in \mathcal{L}(H)$ is called a *quasiaffine* transform of an operator $T' \in \mathcal{L}(H')$ $(T \prec T')$ if there exists a quasiaffinity $X \in \mathcal{L}(H, H')$ such that $T'X = XT$. The operators T and T' are *quasisimilar* $(T \sim T')$ if $T \prec T'$ and $T' \prec T$. We denote by $\mathcal{F}(T', T)$ the set of all operators in $\mathcal{L}(H, H')$ intertwining T' and T, i.e., $\mathcal{F}(T', T) = \{X \in \mathcal{L}(H, H') : T'X = XT\}$.

If $K \subset \mathbb{C}$ is compact, $T \in \mathcal{L}(H)$ and $\sigma(T) \subset K$, we say that K is a *spectral set* for the operator T if $\|r(T)\| \leqslant \max\{|r(z)| : z \in K\}$, whenever r is a rational function with poles off K.

Definition 1.1. An operator $T \in \mathcal{L}(H)$ with $\overline{\Omega}$ as spectral set and with no normal summand with spectrum in Γ is said to satisfy *hypothesis* (h).

The above is the extension to more general regions of the notion of completely nonunitary operator. For each operator satisfying (h) it is possible to define a unique continuous functional calculus representation $\Phi : H^\infty(\Omega) \to \mathcal{L}(H)$, which is also continuous when both $H^\infty(\Omega)$ and $\mathcal{L}(H)$ are given the weak*-topology (cf. [12], Theorem 3.1.4).

Definition 1.2. An operator T satisfying (h) is said to be of *class* C_0 (or, equivalently, a C_0-operator) if the associated functional calculus has a non trivial kernel.

The subspace $\{u \in H^\infty(\Omega) : u(T) = 0\}$ is a fully invariant subspaces of $H^\infty(\Omega)$; hence it has the form $\theta H^\infty(\Omega)$ for some inner function θ. If T is of class C_0, the inner function θ such that $\theta H^\infty(\Omega) = \{u \in H^\infty(\Omega) : u(T) = 0\}$, is called the *minimal function* of T and is denoted by m_T (notice that the minimal function is defined to be an equivalence class of functions). If $T \in \mathcal{L}(H)$ and $T' \in \mathcal{L}(H')$ are two quasisimilar operators satisfying (h), then one is a C_0-operator if and only if so is the other, and their minimal functions coincide. The minimal function plays a role analogous in many respects to the well-known role of minimal polynomials of finite matrices in linear algebra. It is convenient to allow the operator $T = 0$ on the trivial space $\{0\}$ to belong to the class C_0; its minimal function is the function identically equal to one. The operators C_0-operators with spectrum in the unit disk were introduced by Sz.-Nagy and Foiaş ([9]) in their work on canonical models for contractions. The class C_0 is quite possibly the best understood class of non-normal operators. For a detailed presentation, the reader should refer to the monograph [3]. The operators C_0-operators over a multiply connected region have been introduced and studied in [12].

The simplest case of an operator of class C_0 is the *Jordan block* $S(\theta)$ defined as follows. Let S denote the operator of multiplication by z in $\mathcal{L}(H^2(\Omega))$, and let $\theta \in H^\infty(\Omega)$ be an inner function. We set $\mathcal{H}(\theta) = H^2(\Omega) \ominus \theta H^2(\Omega)$ and denote by $S(\theta)$ the compression of S to $\mathcal{H}(\theta)$, i.e., $S(\theta) = P_{\mathcal{H}(\theta)} S_{|\mathcal{H}(\theta)}$, where $P_{\mathcal{H}(\theta)}$ denotes the orthogonal projection onto $\mathcal{H}(\theta)$.

Using the Jordan blocks we can define more general C_0-operators. Assume that for each ordinal number α we are given an inner function $\theta_\alpha \in H^\infty(\Omega)$, such that $\theta_\alpha | \theta_\beta$ whenever $\operatorname{card}(\beta) \leqslant \operatorname{card}(\alpha)$ and $\theta_\alpha \equiv 1$ for some α (and hence $\theta_\beta \equiv 1$ for $\beta \geqslant \alpha$). The operator

$$S(\Theta) = \bigoplus_{\alpha<\gamma} S(\theta_\alpha), \quad \gamma = \min\{\beta : \theta_\beta \equiv \mathbf{1}\}$$

is called the *Jordan operator* determined by the model function $\Theta = \{\theta_\alpha : \alpha < \gamma\}$. The operator $S(\Theta)$ is of class C_0, and $m_{S(\Theta)} \equiv \theta_0$. We will denote by $\mathcal{H}(\Theta)$ the direct sum Hilbert space on which $S(\Theta)$ acts. Separably acting Jordan operators are of the form $\bigoplus\limits_{j=0}^{\infty} S(\theta_j)$, where $\{\theta_j : j \geqslant 0\}$ is a sequence of inner functions such that $\theta_{j+1} | \theta_j$.

The following theorem (cf. [12], Theorem 4.3.21) shows why Jordan operators are important in the study of the class C_0.

Theorem 1.3. *Every C_0-operator T is quasisimilar to a unique Jordan operator, called the Jordan model of T.*

Operators of class C_0 exhibit remarkable properties, which make them easier to study than general functional model operators. Here we are concerned with those properties that a C_0-operator may have in common with its Jordan model.

Before going any further, we introduce some other notions about C_0-operators. We only state the most important results we are going to deal with. The interested reader may refer to [12]. Let $\mathcal{M}$ be a closed subspace of H and $T \in \mathcal{L}(H)$ with $\sigma(T) \subset \overline{\Omega}$; $\mathcal{M}$ is said to be $R(\Omega)$-*invariant* for T if it is invariant for $r(T)$ for all $r \in R(\Omega)$. Since $R(\Omega)$ is sequentially weak*-dense in $H^\infty(\Omega)$, if $\mathcal{M}$ is an $R(\Omega)$-invariant subspace, then $u(T)\mathcal{M} \subset \mathcal{M}$ for all $u \in H^\infty(\Omega)$. Notice that if $H = H^p(\Omega)$, then $R(\Omega)$-invariant subspaces for the operators of multiplication by z are fully invariant subspaces. Any invariant subspace of a Jordan block $S(\theta)$ is also $R(\Omega)$-invariant (cf. [12], Theorem 4.1.18).

An operator T satisfying (h) is said to be *locally of class* C_0 if for every $x \in H$ there exists $u_x \in H^\infty(\Omega) - \{0\}$ such that $u_x(T)x = 0$. If T is locally of class C_0 and $x \in H$, we denote by m_x the inner function defined by $m_x H^\infty(\Omega) = \{u \in H^\infty(\Omega) : u(T)x = 0\}$. A vector $x \in H$ is said to be T-*maximal* if for every $y \in H$ we have $m_y | m_x$, and the set of T-maximal vectors is a dense G_δ in H. In particular, T is of class C_0 and $m_T \equiv m_x$ for every T-maximal vector x.

Let $T \in \mathcal{L}(H)$ be an operator with spectrum in $\overline{\Omega}$. A subset $\mathcal{M} \subset H$ with the property that $\bigvee_{r \in R(\Omega),\, m \in \mathcal{M}} r(T)m = H$ is called an $R(\Omega)$-*generating set* for T. The *multiplicity* μ_T of T is the smallest cardinality of an $R(\Omega)$-generating set for T, and it is a quasisimilarity invariant. The operator T is said to be *multiplicity-free* if $\mu_T = 1$. A multiplicity-free operator T is quasisimilar to $S(m_T)$. If $\mu_T = 1$, any vector $x \in H$ such that $\bigvee_{r \in R(\Omega)} r(T)x = H$ is said to be $R(\Omega)$-*cyclic* for T. A vector $x \in H$ is $R(\Omega)$-cyclic for T if and only if x is T-maximal. Finally, we recall that if T is an operator satisfying (h), then $\mathcal{F}_T$ denotes the set of all operators $X \in \mathcal{L}(H)$ such that $X = v(T)^{-1}u(T)$ for some $v \in \mathcal{K}_T^\infty(\Omega)$ and $u \in H^\infty(\Omega)$, where $\mathcal{K}_T^\infty(\Omega)$ is defined to be the set of $v \in H^\infty(\Omega)$ such that $v(T)$ is a quasiaffinity.

2. PRELIMINARY RESULTS

For an arbitrary operator $T \in L(H)$ with $\sigma(T) \subset \overline{\Omega}$ we denote by $\mathcal{A}_T$ (respective, by $\mathcal{W}_T$) the weak*-closed (respective, weakly closed) subalgebra of $\mathcal{L}(H)$ generated by all operators of the form $r(T)$ with $r \in R(\Omega)$. Note that $r(T)$ is well defined as the quotient of polynomials. It is well-known that this definition of $r(T)$ concides with the definition given by the Riesz-Dunford functional calculus. If the operator T satisfies (h), then the rational functional calculus $r \to r(T)$ has a unique continuous extension to $H^\infty(\Omega)$. Since the commutant $\{T\}'$ is always a weakly closed algebra, we clearly have $\mathcal{A}_T \subset \mathcal{W}_T \subset \{T\}'$. To every operator T we associate other algebras as follows. If $\mathcal{A}$ is an arbitrary subalgebra of $\mathcal{L}(H)$, then $\mathrm{Lat}(\mathcal{A})$ denotes the collection of all closed invariant subspaces for $\mathcal{A}$, i.e. $\mathcal{M} \in \mathrm{Lat}(\mathcal{A})$ if $X\mathcal{M} \subset \mathcal{M}$ for every $X \in \mathcal{A}$. If $\mathcal{B}$ is a collection of closed subspaces of H we denote by $\mathrm{Alg}(\mathcal{B})$ the set of those $X \in \mathcal{L}(H)$ such that $X(\mathcal{M}) \subset \mathcal{M}$, for every $\mathcal{M} \in \mathcal{B}$. The subalgebra $\mathrm{Alg}(\mathcal{B})$ is always a weakly closed subalgebra of $\mathcal{L}(H)$, hence $\mathcal{A} \subset \mathrm{Alg}\,\mathrm{Lat}(\mathcal{A})$.

Definition 2.1. An algebra $\mathcal{A} \in \mathcal{L}(H)$ is said to be *reflexive* if $\mathcal{A} = \mathrm{Alg}\,\mathrm{Lat}(\mathcal{A})$. An operator T with $\sigma(T) \subset \overline{\Omega}$ is said to be *reflexive* (respective, *hyperreflexive*) if $\mathcal{W}_T$ (respective, $\{T\}'$) is reflexive.

If Ω is simply connected, then clearly $\mathrm{Lat}(T) = \mathrm{Lat}(\mathcal{W}_T) = \mathrm{Lat}(\mathcal{A}_T)$ so that T is reflexive if and only if $\mathrm{Alg}\,\mathrm{Lat}(T) = \mathcal{W}_T$. In the general case of multiply connected regions we only have $\mathrm{Lat}(T) \supset \mathrm{Lat}(\mathcal{W}_T)$. Note also that $\mathrm{Lat}(\mathcal{W}_T)$ consists of all $R(\Omega)$-invariant subspaces, and thus for Jordan blocks $S(\theta)$ we have $\mathrm{Lat}(\mathcal{W}_{S(\theta)}) = \mathrm{Lat}(S(\theta))$.

The main result of this paper is the following.

Theorem 2.2. *Let T be a C_0-operator and $S(\Theta)$, $\Theta = \{\theta_\alpha\}$, its Jordan model. Then*

(i) *T is reflexive if and only if $S(\theta_0/\theta_1)$ is reflexive;*

(ii) *T is hyperreflexive if and only if $S(m_T)$ is reflexive.*

Thus the reflexivity and the hyperreflexivity of T depends only on the reflexivity of single Jordan blocks, which can be easily calculated from the Jordan model of T. This result is known for the case in which the region Ω is the unit disk, and it is due to Bercovici, Foiaş and Sz.-Nagy ([4]; see also [2] and [10] for the case of finite defect indices).

Theorem 2.3. *For every C_0-operator T we have*

$$\{T\}'' = \{T\}' \cap \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T) = \mathcal{A}_T = \mathcal{W}_T = \mathcal{F}_T.$$

Proof. It is enough to verify the following six inclusions:

$$\mathcal{W}_T \subset \{T\}'' \subset \{T\}' \cap \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T) \subset \mathcal{F}_T \subset \{T\}'' \subset \mathcal{A}_T \subset \mathcal{W}_T.$$

The first and the last of these inclusions and the inclusion $\{T\}'' \subset \{T\}'$ are true for arbitrary operators. Let now $X \in \{T\}''$ and $\mathcal{M} \in \operatorname{Lat}(\mathcal{W}_T)$. Then by Proposition 4.3.24 in [12], $\mathcal{M} = \ker(Y)$ for some $Y \in \{T\}'$. Hence $X(\mathcal{M}) \subset \mathcal{M}$ because X and Y commute. We conclude that $\{T\}'' \subset \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T)$ and the second inclusion is proved. To prove the third inclusion we use the splitting principle (cf. [12], Theorem 4.3.1). Assume now that $X \in \{T\}' \cap \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T)$, $x \in H$ and $K = \bigvee_{r \in R(\Omega)} r(T)x$. Then $X(K) \subset K$ and $X_{|K} \in \{T_{|K}\}'$. Since $T_{|K}$ is multiplicity-free and $m_{T_{|K}} \equiv m_x$, it follows from Theorem 4.3.2 in [12] that there exist functions $u_x, v_x \in H^\infty(\Omega)$ such that $u_x \wedge v_x \equiv 1$ and $v_x(T_{|K})(X_{|K}) = u_x(T_{|K})$; in particular

$$v_x(T)Xx = u_x(T)x. \tag{2.1}$$

Let h be a T-maximal vector, and $K_0 = \bigvee_{r \in R(\Omega)} r(T)h$. By the splitting principle there exists $\mathcal{M}_0 \in \operatorname{Lat}(\mathcal{W}_T)$ such that $K_0 \cap \mathcal{M}_0 = \{0\}$ and $K_0 \vee \mathcal{M}_0 = H$. We claim that for every $g \in \mathcal{M}_0$, the vector $h + g$ is also T-maximal. Indeed, the relation $u(T)(h+g) = 0$ implies that

$$u(T)h = -u(T)g \in K_0 \cap \mathcal{M}_0,$$

and therefore $u(T)h = 0$. Thus $m_T | u$ because h is T-maximal, and therefore $h+g$ is T-maximal. Hence we have

$$v_h \wedge m_T \equiv v_{h+g} \wedge m_T \equiv 1$$

for every $g \in \mathcal{M}_0$. Next, we want to show that $v_h(T)X = u_h(T)$ so that $X = (u_h/v_h)(T)$. Applying (2.1) we get

$$(v_{h+g}(T)X - u_{h+g}(T))h = -(v_{h+g}(T)X - u_{h+g}(T))g \in K_0 \cap \mathcal{M}_0 = \{0\},$$

which yields

$$v_{h+g}(T)Xh = u_{h+g}(T)h.$$

A further application of (2.1) gives

$$v_h(T)u_{h+g}(T)h - v_{h+g}(T)u_h(T)h = v_h(T)v_{h+g}(T)Xh - v_{h+g}(T)v_h(T)Xh = 0,$$

so that $m_T \equiv m_h|(v_h u_{h+g} - v_{h+g}u_h)$. Therefore

$$v_h(T)u_{h+g}(T) = v_{h+g}(T)u_h(T),$$

which entails

$$v_{h+g}(T)v_h(T)X(h+g) = v_h(T)u_{h+g}(T)(h+g) = v_{h+g}(T)u_h(T)(h+g).$$

Since $v_{h+g} \wedge m_T \equiv 1$, the operator $v_{h+g}(T)$ is a quasiaffinity, and the last equality above implies

$$v_h(T)X(h+g) = u_h(T)(h+g),$$

and in virtue of (2.1) we conclude that $v_h(T)Xg = u_h(T)g$. Thus $v_h(T)X_{|\mathcal{M}_0} = u_h(T)_{|\mathcal{M}_0}$ and, since $v_h(T)X_{|K_0} = u_h(T)_{|K_0}$ by definition of u_h and v_h, we have

$$v_h(T)X = u_h(T)X_{|K_0 \vee \mathcal{M}_0} = u_h(T).$$

Hence $X \in \mathcal{F}_T$ and the third inclusion is proved. The inclusion $\mathcal{F}_T \subset \{T\}''$ is true for every operator T satisfying (h). Indeed, if $X = (u/v)(T) \in \mathcal{F}_T$ and $Y \in \{T\}'$, we must have

$$v(T)XY = u(T)Y = Yu(T) = Yv(T)X = v(T)YX,$$

which implies $XY = YX$ ($v(T)$ is one-to-one since $v \in \mathcal{K}_T^\infty(\Omega)$). The proof of the inclusion $\{T\}'' \subset \mathcal{A}_T$ is based on a classical argument, essentially due to von Neumann. Let $X \in \{T\}''$ and denote by T' and X' the direct sum of infinitely many copies of T and X, respectively. Then T' is an operator of class C_0 with $m_T \equiv m_{T'}$ and $X' \in \{T'\}''$. From the second inclusion, which has already been proved for all C_0 operators, we have $X' \in \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_{T'})$. Let

$$V = \left\{ Y : \sum_{j=0}^{\infty} \|Yh_j - Xh_j\|^2 < \varepsilon^2 \right\}$$

be an arbitrary ultrastrong neighborhood of X, and set $h = \bigoplus_{j=0}^{\infty} h_j$. The $R(\Omega)$-cyclic subspace $K = \bigvee_{r \in R(\Omega)} r(T')h$ is then invariant for X' so that there exists $r \in R(\Omega)$ satisfying the inequality $\|X'h - r(T')h\| < \varepsilon$. But this means that $r(T) \in V$ and we conclude that $X \in \mathcal{A}_T$. ∎

Note that the function v_h in the preceding proof can be chosen independently of X (see the remark after Proposition 4.2.7 in [12]). So we have proved the following result.

Corollary 2.4. *For every C_0-operator T there exists a function $v \in H^\infty(\Omega)$ such that $v \wedge m_T \equiv 1$, and every operator $X \in \mathcal{A}_T$ can be written as $X = (u/v)(T)$ for some $u \in H^\infty(\Omega)$.*

There are some immediate consequences of Theorem 2.3 for the reflexivity of C_0-operators, whose proofs are left to the interested reader.

Corollary 2.5. *A C_0-operator T is reflexive if and only if* $\operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T) \subset \{T\}'$.

Corollary 2.6. *Let T be a C_0-operator, and let $\{\mathcal{M}_j : j \in J\} \subset \operatorname{Lat}(\mathcal{W}_T)$ be such that $\bigvee_{j\in J} \mathcal{M}_j = H$. If $T_{|\mathcal{M}_j}$ is reflexive for every $j \in J$, then T is reflexive.*

Corollary 2.7. *Assume that T is a reflexive C_0-operator, and let $X \in \mathcal{W}_T$. Then $T_{|[\operatorname{range} X]^-}$ is also reflexive.*

In order to characterize reflexive operators in terms of their Jordan models, we need to prove that reflexivity of C_0-operators is a quasisimilarity invariant. To this aim we introduce an auxiliary property.

Definition 2.8. An operator T satisfying (h) is said to have *property* $(*)$ if for any quasiaffinity $X \in \{T\}'$ there exist a quasiaffinity $Y \in \{T\}'$ and $u \in H^\infty(\Omega)$ such that $XY = YX = u(T)$.

Of course XY is a quasiaffinity so that $u \in \mathcal{K}_T^\infty(\Omega)$. The proof of the following lemma is the same as in the case of the disk with suitable modifications (cf. [3], Lemma 4.1.11 and Lemma 4.1.12).

Lemma 2.9. *Let T and T' be two quasisimilar operators satisfying* (h). *Then:*

(i) *T has property $(*)$ if and only if T' has property $(*)$;*

(ii) *if T has property $(*)$ then we can find $u \in H^\infty(\Omega)$ and quasiaffinities $A \in \mathcal{F}(T', T)$ and $B \in \mathcal{F}(T, T')$ such that $AB = u(T)$ and $BA = u(T')$;*

(iii) *if T is of class C_0 and has property $(*)$, then T is reflexive if and only if T' is reflexive.*

It is not true that every operator of class C_0 has property $(*)$. We can, however, produce a large family of operators with property $(*)$ that will suffice for our purposes.

PROPOSITION 2.10. *Let θ_0 and θ_1 be two inner functions such that $\theta_1|\theta_0$. Then the operator $T = S(\theta_0) \oplus S(\theta_1)$ has property $(*)$.*

The proof of this proposition is based on the following lemmas.

LEMMA 2.11. *Let $T \in \mathcal{L}(H)$ be a C_0-operator, K a Banach space and $X : K \to H$ a continuous linear map such that $\bigvee\limits_{r\in R(\Omega)} r(T)XK = H$. Then the set*

$$\{k \in K : m_{Xk} \equiv m_T\}$$

is a dense G_δ in K.

Proof. The proof closely imitates that of Theorem 3.3.5 in [12]. We provide the relevant details. First we recall that to any inner function m_x we can associate a subharmonic function u_x by:

$$u_x(z) = -\sum_{z\in\Omega} \mu(\zeta) g(z,\zeta) + \int\limits_{\Gamma} \frac{\partial g}{\partial n}(\zeta, z)\mathrm{d}\nu(\zeta),$$

where $m_x \equiv B_\mu S_\nu$ is the factorization provided by Theorem 2.2.11 in [12]. For a fixed $z_0 \in \Omega$, denote $a = \inf\limits_{k\in K}\{\exp u_{Xk}(z_0)\}$. Then the set

$$\sigma_j = \{k \in K : \exp u_{Xk}(z_0) \geqslant a + 1/j\} = X^{-1}\{h \in H : \exp u_h(z_0) \geqslant a + 1/j\}$$

is closed for $j \geqslant 1$, and it has empty interior. It follows that the set

$$\{k \in K : \exp u_{Xk}(z_0) = a\}$$

is a dense G_δ in K. Then the set

$$\mathcal{M} = \Big\{k \in K : \exp u_{Xk}(z) = \inf_{h\in K}\{\exp u_{Xh}(z)\}, z \in \Omega\Big\}$$

is a dense G_δ in K. If $k \in \mathcal{M}$ it follows that $m_{Xh}|m_{Xk}$ for every $h \in K$, and hence

$$m_{Xk}(T)(XK) = \{0\}.$$

The last relation clearly implies

$$m_{Xk}(T)\bigg(\bigvee_{r\in R(\Omega)} r(T)XK\bigg) = \{0\}$$

and hence $m_{Xk}(T) = 0$, from which we deduce $m_{Xk} \equiv m_T$. ∎

If θ is an inner function and $f \in H^2(\Omega)$, we say that $\theta|f$ if $f = \theta g$ for some $g \in H^2(\Omega)$. Given a family $\{f_j\}_{j\in J}$ of functions in $H^2(\Omega)$, the greatest common inner divisor $\bigwedge_{j\in J} f_j$ is defined to be the unique (up to equivalence) inner function dividing each f_j and multiple of any common inner divisor of the family. Its existence can be easily proved using the fully invariant subspace of $H^2(\Omega)$ given by $\bigvee_{j\in J} f_j H^\infty(\Omega)$.

Lemma 2.12. *Let $\{f_j\}_{j\geqslant 0}$ be a bounded sequence of functions in $H^2(\Omega)$ and let θ be an inner function. The set of $\{a_j\} \subset \ell^1$ satisfying the relation*

$$\Big(\sum_{j=0}^{\infty} a_j f_j\Big) \wedge \theta \equiv \Big(\bigwedge_{j=0}^{\infty} f_j\Big) \wedge \theta$$

is a dense G_δ in ℓ^1.

Proof. We may assume without loss of generality that $\Big(\bigwedge_{j=0}^{\infty} f_j\Big) \wedge \theta \equiv 1$. Indeed, we may replace θ by θ/φ and each f_j by f_j/φ, where $\varphi \equiv \Big(\bigwedge_{j=0}^{\infty} f_j\Big) \wedge \theta$. Under this additional assumption, the invariant subspace for $S(\theta)$ generated by the vectors $\{P_{\mathcal{H}(\theta)} f_j : j \geqslant 0\}$ is $\mathcal{H}(\theta)$. Indeed, if the invariant subspace for $S(\theta)$ generated by the vectors $\{P_{\mathcal{H}(\theta)} f_j : j \geqslant 0\}$ is $\varphi H^2(\Omega) \ominus \theta H^2(\Omega)$, then $\varphi \,\Big|\, \Big(\sum_{j=0}^{\infty} a_j f_j\Big) \wedge \theta$, and thus $\varphi \equiv 1$. We can therefore apply Lemma 2.11 with $H = \mathcal{H}(\theta)$, $K = \ell^1$ and $X : K \to H$ defined by

$$X(\{a_j\}) = P_{\mathcal{H}(\theta)}\Big(\sum_{j=0}^{\infty} a_j f_j\Big),$$

with $\{a_j\} \in \ell^1$. Hence the set of sequences $a \in \ell^1$ such that $m_{Xa} \equiv \theta$ is a dense G_δ in ℓ^1. Finally, the condition $m_{Xa} \equiv \theta$ is equivalent to $Xa \wedge \theta \equiv 1$, which in turn is equivalent to $\Big(\sum_{j=0}^{\infty} a_j f_j\Big) \wedge \theta \equiv 1$. ∎

Proof of Proposition 2.10. Let P_0 and P_1 denote the projections of $H = \mathcal{H}(\theta_0) \oplus \mathcal{H}(\theta_1)$ onto $\mathcal{H}(\theta_0)$ and $\mathcal{H}(\theta_1)$, respectively. If $X \in \{T\}'$, then $P_i^* X P_j \in \mathcal{F}(S(\theta_j), S(\theta_i))$ for $0 \leqslant i,\, j \leqslant 1$, and in virtue of Theorem 4.1.2 in [12] we can find functions $a_{ij} \in H^\infty(\Omega)$ such that

$$\theta_i | a_{ij}\theta_j \tag{2.2}$$

and

$$P_i^* X P_j h = P_{\mathcal{H}(\theta_i)}(a_{ij} h) \tag{2.3}$$

for $h \in \mathcal{H}(\theta_j)$ and $0 \leqslant i, j \leqslant 1$. Conversely, if

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

is a matrix of functions in $H^\infty(\Omega)$ for which (2.2) holds, then there exists an operator $X \in \{T\}'$ satisfying (2.3). Of course, the matrix A is not uniquely determined by X. We can always change a_{ij} into $a_{ij} + u_{ij}\theta_j$, where u_{ij} are arbitrary functions in $H^\infty(\Omega)$. Assume for the moment that $\theta_0 \wedge \det(A) \equiv 1$, where $\det(A) \equiv a_{00}a_{11} - a_{01}a_{10}$. Then the matrix

$$B = \begin{pmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{pmatrix}$$

determines an operator $Y \in \{T\}'$, and the immediate relations $AB = BA = uI$, $u = \det(A)$, imply that $XY = YX = u(T)$. Moreover, since $m_T \equiv \theta_0$, the fact that $\theta_0 \wedge u \equiv 1$ implies that $u \in \mathcal{K}_T^\infty(\Omega)$, and therefore $u(T)$ is a quasiaffinity. The considerations above indicate that, in order to show that T has property $(*)$, it suffices to prove that for every quasiaffinity $X \in \{T\}'$ we can find a matrix A satisfying (2.2) and (2.3) and such that $\theta_0 \wedge \det(A) \equiv 1$. Assume therefore that X is a quasiaffinity, and the matrix A satisfies (2.3) and (2.3). We first note that

$$a_{00} \wedge a_{01} \wedge \theta_0 \equiv 1. \tag{2.4}$$

Indeed, if $q \equiv a_{00} \wedge a_{01} \wedge \theta_0$ then we see from (2.3) that $P_{\mathcal{H}(\theta_0)}XH \subset qH^2(\Omega) \ominus \theta_0 H^2(\Omega)$, and hence $q \equiv 1$ because X has dense range. Moreover, we have

$$\theta_1 \wedge \det(A) \equiv 1. \tag{2.5}$$

Indeed, if $p \equiv \theta_1 \wedge \det(A)$ and we define

$$h = P_{\mathcal{H}(\theta_0)}(-a_{01}\theta_1/p) \oplus P_{\mathcal{H}(\theta_1)}(a_{00}\theta_1/p)$$

an easy calculation (using (2.2) and the fact that $P_{\mathcal{H}(\theta)}(aP_{\mathcal{H}(\theta)}f) = P_{\mathcal{H}(\theta)}(af)$, if $a \in H^\infty(\Omega)$, $f \in H^2(\Omega)$ and θ is inner) shows that $P_0Xh = 0$ and

$$P_1Xh = P_{\mathcal{H}(\theta_1)}(\theta_1 \det(A)/p) = 0.$$

By the injectivity of X we must have $h = 0$ and therefore $\theta_0 | (-a_{01}\theta_1/p)$ and $\theta_1 | (a_{00}\theta_1/p)$. We deduce that $p | (a_{01}\theta_1/\theta_0)$ and $p | a_{00}$. Since $(a_{01}\theta_1/\theta_0) | a_{01}$ and $p | \theta_1$ by definition of p, we easily have $p | (\theta_1 \wedge a_{01} \wedge a_{00})$ and thus $p \equiv 1$ by (2.4). Now (2.4) and (2.5) imply

$$(\theta_1 a_{00} \wedge \theta_1 a_{01} \wedge \det(A)) \wedge \theta_0 \equiv 1. \tag{2.6}$$

Indeed, if r denotes the left-hand-side of (2.6), then $r|\det(A)$, and so by (2.5) $r\wedge\theta_1\equiv 1$. Then we see that the relation $r|\theta_1 a_{00}$ (respective, $r|\theta_1 a_{01}$) implies $r|a_{00}$ (resp., $r|a_{01}$) and hence $r|a_{00}\wedge a_{01}\wedge\theta_0$. Using (2.4), we conclude that $r\equiv 1$. An easy application of Lemma 2.11 implies the existence of scalars λ,μ such that

$$(\det(A)+\lambda\theta_1 a_{00}+\mu\theta_1 a_{01})\wedge\theta_0\equiv 1.$$

We now define

$$A'=\begin{pmatrix} a_{00} & a_{01}\\ a_{10}-\mu\theta_1 & a_{11}+\lambda\theta_1\end{pmatrix}$$

and note that, by the remarks above, A' also determines X. Finally we have

$$\det(A')\equiv\det(A)+\lambda\theta_1 a_{00}+\mu\theta_1 a_{01}$$

and hence $\theta_0\wedge\det(A')\equiv 1$. The proposition is proved. ∎

Proposition 2.10 certainly applies to $T=S(\theta_0)$ since it is allowed to take $\theta_1\equiv 1$. The proposition and Lemma 2.9 already show that reflexivity is a quasisimilarity invariant for operators of class C_0 with multiplicity $\leqslant 2$. It would be therefore interesting to know which Jordan operators with multiplicity $\leqslant 2$ are reflexive.

3. PROOF OF THE MAIN THEOREM

The following lemma is contained in Proposition 4.1.14 in [12]. We recall that any invariant subspace of a Jordan block $S(\theta)$ is also $R(\Omega)$-invariant.

LEMMA 3.1. *Let θ be a non-invertible inner function.*

(i) *Every invariant subspace $\mathcal{M}$ of $S(\theta)$ has the form $\varphi H^2(\Omega)\ominus\theta H^2(\Omega)$ for some inner divisor φ of θ. We have $\varphi H^2(\Omega)\ominus\theta H^2(\Omega)=\ker\big((\theta/\varphi)(S(\theta))\big)=\mathrm{range}\big(\varphi(S(\theta))\big)$.*

(ii) *If $\mathcal{M}=\varphi H^2(\Omega)\ominus\theta H^2(\Omega)$ is an invariant subspace for $S(\theta)$, then there exists an invertible operator $Z\in\mathcal{L}(\mathcal{H}(\theta/\varphi),\varphi H^2(\Omega)\ominus\theta H^2(\Omega))$ such that $S(\theta)_{|\mathcal{M}}Z=ZS(\theta/\varphi)$.*

The proof of the following result is based on very explicit knowledge of the invariant subspaces of a Jordan block.

PROPOSITION 3.2. *Let θ_0 and θ_1 be two inner functions such that $\theta_1|\theta_0$. The operator $T = S(\theta_0) \oplus S(\theta_1)$ is reflexive if and only if the Jordan block $S(\theta_0/\theta_1)$ is reflexive.*

Proof. An easy application of Corollary 4.1.16 in [12] shows that

$$\text{range}(\theta_1(T)) = (\theta_1 H^2(\Omega) \ominus \theta_0 H^2(\Omega)) \oplus \{0\}$$

and thus from Lemma 3.1, $T_{|\text{range}(\theta_1(T))}$ is similar to $S(\theta_0/\theta_1)$. If T is reflexive, then $S(\theta_0/\theta_1)$ is reflexive by Corollary 2.7. Assume that $X \in \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T)$. The subspaces $\mathcal{H}(\theta_0) \oplus \{0\}$ and $\{0\} \oplus \mathcal{H}(\theta_1)$ belong to $\operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T)$, hence they are invariant for X and therefore X can be written as $X = X_0 \oplus X_1$ with $X_j \in \operatorname{Alg}\operatorname{Lat}(S(\theta_j))$ for $j = 0, 1$. Let $Z : \mathcal{H}(\theta_1) \to (\theta_0/\theta_1)H^2(\Omega) \ominus \theta_0 H^2(\Omega)$ be defined as in the preceding lemma with $\theta = \theta_0$ and $\varphi = \theta_0/\theta_1$, and consider the subspaces $\mathcal{M}_0, \mathcal{M}_1 \in \operatorname{Lat}(\mathcal{W}_T)$ described by:

$$\begin{aligned}\mathcal{M}_0 &= \{(Zh \oplus h) : h \in \mathcal{H}(\theta_1)\}\\ \mathcal{M}_1 &= \{(ZS(\theta_1)h \oplus h) : h \in \mathcal{H}(\theta_1)\}.\end{aligned}$$

The inclusion $X\mathcal{M}_0 \subset \mathcal{M}_0$ yields

$$X_0 Z h = Z X_1 h,$$

and the inclusion $X\mathcal{M}_1 \subset \mathcal{M}_1$ yields

$$X_0 Z S(\theta_1) h = Z S(\theta_1) X_1 h$$

for every $h \in \mathcal{H}(\theta_1)$. We combine the second equality above with the first in which h is replaced by $S(\theta_1)h$ to obtain

$$Z S(\theta_1) X_1 h = Z X_1 S(\theta_1) h$$

for every $h \in \mathcal{H}(\theta_1)$. Since Z is invertible, this last equality shows that $X_1 \in \{S(\theta_1)\}'$ and hence there exists $u \in H^\infty(\Omega)$ such that $X_1 = u(S(\theta_1))$ by Corollary 4.1.3 in [12]. Thus we deduce the existence of an operator $Y_0 \in \operatorname{Alg}\operatorname{Lat}(S(\theta_0))$ such that

$$X - u(T) = Y_0 \oplus 0 \in \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T). \tag{3.1}$$

For every inner divisor q of θ_0/θ_1 we consider the subspace $\mathcal{N}_q \in \operatorname{Lat}(\mathcal{W}_T)$ defined by

$$\mathcal{N}_q = \{(Z(h) \oplus P_{\mathcal{H}(\theta_1)} h) : h \in \mathcal{H}(\theta_0/q)\},$$

where $Z : \mathcal{H}(\theta_0/q) \to qH^2(\Omega) \ominus \theta_0 H^2(\Omega)$ is as in Lemma 3.1 with $\theta = \theta_0$ and $\varphi = q$. The inclusion $(Y_0 \oplus 0)\mathcal{N}_q \subset \mathcal{N}_q$ means that for every $h \in \mathcal{H}(\theta_0/q)$ we have

$$Y_0(Z(h)) = Z(h')$$

for some $h' \in \mathcal{H}(\theta_0/q)$ such that $P_{\mathcal{H}(\theta_1)}h' = 0$. This last equality implies that $h' \in \theta_1 H^2(\Omega)$ so that $h' \in \theta_1 H^2(\Omega) \cap \mathcal{H}(\theta_0/q) = \theta_1 H^2(\Omega) \cap (H^2(\Omega) \ominus (\theta_0/q)H^2(\Omega))$. We then have that

$$Y_0(qH^2(\Omega) \ominus \theta_0 H^2(\Omega)) \subset q\theta_1 H^2(\Omega) \ominus \theta_0 H^2(\Omega), \tag{3.2}$$

for all q inner divisor of θ_0/θ_1. If $q = 1$ and $q = \theta_0/\theta_1$ we obtain the particular cases

$$\operatorname{range}(Y_0) \subset \theta_1 H^2(\Omega) \ominus \theta_0 H^2(\Omega), \quad \ker(Y_0) \supset (\theta_0/\theta_1)H^2(\Omega) \ominus \theta_0 H^2(\Omega). \tag{3.3}$$

Relations (3.3) can be used to find an operator in $\operatorname{Alg}\operatorname{Lat}(S(\theta_0/\theta_1))$. Let $Z : \mathcal{H}(\theta_0/\theta_1) \to \theta_1 H^2(\Omega) \ominus \theta_0 H^2(\Omega)$ be defined as in Lemma 3.1 with $\theta = \theta_0$ and $\varphi = \theta_1$. Then $V = Z^{-1}$ is an invertible operator such that

$$VS(\theta_0)_{|\theta_1 H^2(\Omega)\ominus\theta_0 H^2(\Omega)} = S(\theta_0/\theta_1)V.$$

Moreover, by the fact that $\theta_0|\theta_1$, we have $S(\theta_0/\theta_1)V = P_{\mathcal{H}(\theta_0/\theta_1)}S(\theta_0)V$, and thus

$$VS(\theta_0)_{|\theta_1 H^2(\Omega)\ominus\theta_0 H^2(\Omega)} = P_{\mathcal{H}(\theta_0/\theta_1)}S(\theta_0)V.$$

Let us now consider the operator $W = VY_{0|\mathcal{H}(\theta_0/\theta_1)}$. We claim that $W \in \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_{S(\theta_0/\theta_1)})$. To prove this, let us consider $\mathcal{M} \in \operatorname{Lat}(S(\theta_0/\theta_1))$; by Lemma 3.1, there exists an inner function q such that $q|(\theta_0/\theta_1)$ and $\mathcal{M} = qH^2(\Omega) \ominus (\theta_0/\theta_1)H^2(\Omega)$. Hence (3.2) implies $W(\mathcal{M}) \subset \mathcal{M}$. Assume now that $S(\theta_0/\theta_1)$ is reflexive. Then $W \in \{S(\theta_0/\theta_1)\}'$ and hence, using (3.3),

$$\begin{aligned}
&V(Y_0 S(\theta_0) - S(\theta_0)Y_0)_{|\mathcal{H}(\theta_0/\theta_1)} \\
&\quad = \big(VY_0 P_{\mathcal{H}(\theta_0/\theta_1)}S(\theta_0) - V(S(\theta_0)_{|\theta_1 H^2(\Omega)\ominus\theta_0 H^2(\Omega)})Y_0\big)_{|\mathcal{H}(\theta_0/\theta_1)} \\
&\quad = WS(\theta_0/\theta_1) - S(\theta_0/\theta_1)W = 0.
\end{aligned}$$

Thus $Y_0 S(\theta_0) = S(\theta_0)Y_0$ on $\mathcal{H}(\theta_0/\theta_1)$, and on the orthogonal complement $(\theta_0/\theta_1)H^2(\Omega) \ominus \theta_0 H^2(\Omega)$ of this space, $Y_0 S(\theta_0) - S(\theta_0)Y_0 = 0$ by (3.3), and therefore $Y_0 \in \{S(\theta_0)\}'$. Hence, if $S(\theta_0/\theta_1)$ is reflexive, (3.1) and the preceding argument entail that every $X \in \operatorname{Alg}\operatorname{Lat}(\mathcal{W}_T)$ commutes with T. The conclusion follows from Corollary 2.5. ∎

We are now ready to prove the main result of this paper.

Proof of Theorem 2.2.(i). Assume that $T \in \mathcal{L}(H)$ and $X \in \mathcal{F}(S(\Theta), T)$ is a quasiaffinity. The operators $T_{|[\mathrm{range}(\theta_1(T))]^-}$ and $S(\Theta)_{|\mathrm{range}(\theta_1(S(\Theta)))}$ are quasisimilar since $X_{|\mathrm{range}(\theta_1(S(\Theta)))}$ is a quasiaffinity intertwining them. Thus $T_{|[\mathrm{range}(\theta_1(T))]^-}$ is quasisimilar to $S(\theta_0/\theta_1)$, being $S(\Theta)_{|\mathrm{range}(\theta_1(S(\Theta)))}$ similar to $S(\theta_0/\theta_1)$ by Lemma 3.1. If T is reflexive, it follows from Corollary 2.7, Lemma 2.9 and Proposition 2.10 that $S(\theta_0/\theta_1)$ is reflexive.

Conversely, assume that $S(\theta_0/\theta_1)$ is reflexive and for each ordinal α consider the subspaces $H_\alpha, K_\alpha \in \mathrm{Lat}(\mathcal{W}_T)$ defined by

$$H_\alpha = \Big[\Big\{X\Big(\bigoplus f_\beta\Big) : f_\beta = 0 \text{ for } \beta \neq \alpha\Big\}\Big]^-$$

$$K_\alpha = \Big[\Big\{X\Big(\bigoplus f_\beta\Big) : f_0 \in (\theta_0/\theta_\alpha)H^2(\Omega) \ominus \theta_0 H^2(\Omega),\ f_\alpha = 0 \text{ for } \alpha \neq 0\Big\}\Big]^- .$$

The restriction $T_{|H_0 \vee H_1}$ is quasisimilar to $S(\theta_0) \oplus S(\theta_1)$, while $T_{|H_\alpha \vee K_\alpha}$ is quasisimilar to $S(\theta_\alpha) \oplus S(\theta_\alpha)$ for $\alpha > 0$. This is a consequence of Proposition 4.4.22 in [11], since a suitable restriction of X provides the needed intertwining operators. All these restrictions are then reflexive by Lemma 2.9, Proposition 2.10 and Proposition 3.2. Finally, we note that

$$(H_0 \vee H_1) \vee \Big(\bigvee_{\alpha \geqslant 1} H_\alpha \vee K_\alpha\Big) = \bigvee_{\alpha \geqslant 0} H_\alpha = H$$

and the reflexivity follows from Corollary 2.6. ∎

In order to complete the proof of Theorem 2.2, we need the following result about quasisimilarity invariance (cf. [3], Proposition 4.1.24).

Proposition 3.3. *If the operators T and T' are quasisimilar, and one of them is hyperreflexive, then so is the other.*

Proof of Theorem 2.2.(ii). The preceding proposition shows that we can restrict ourselves to operators T of the form $S(\Theta)$, where Θ is a model function. Assume first that $S(\Theta)$ is hyperreflexive and $X \in \mathrm{Alg\,Lat}(S(\theta_0))$. We claim that the operator $Y = \bigoplus_\alpha Y_\alpha$, where $Y_0 = X$ and $Y_\alpha = 0$ for $\alpha \neq 0$, belongs to $\mathrm{Alg\,Lat}(\{S(\Theta)\}')$. Indeed, a subspace $\mathcal{M} \in \mathrm{Lat}(\{S(\Theta)\}')$ is of the form $\mathcal{M} = \bigoplus_\alpha \mathcal{M}_\alpha$, with $\mathcal{M}_\alpha \in \mathrm{Lat}(S(\theta_\alpha))$, and this clearly implies that $Y\mathcal{M} \subset \mathcal{M}$. Thus $Y \in \{S(\Theta)\}'$ by the assumption that $S(\Theta)$ is hyperreflexive, and hence $X \in \{S(\theta_0)\}'$. The reflexivity of $S(\theta_0)$ follows from Corollary 2.5.

Conversely, assume that $S(\theta_0)$ is reflexive. By Lemma 3.1 we have that $S(\theta_\alpha)$ is similar to $S(\theta_0)_{|\mathrm{range}(\theta_0/\theta_\alpha)(S(\theta_0))}$, and therefore, by Lemma 2.9, Proposition 2.10 and Corollary 2.7, $S(\theta_\alpha)$ is reflexive for every ordinal α. For $\alpha \leqslant \beta$, let $Z_{\alpha\beta} : \mathcal{H}(\theta_\beta) \to (\theta_\alpha/\theta_\beta)H^2(\Omega) \ominus \theta_\alpha H^2(\Omega)$ be as in Lemma 3.1, with $\theta = \theta_\alpha$ and $\varphi = \theta_\alpha/\theta_\beta$. Let us define operators $R_{\alpha\beta} \in \{S(\Theta)\}'$ as follows: $R_{\alpha\beta}\Big(\bigoplus_\gamma h_\gamma\Big) = \bigoplus_\gamma k_\gamma$, where

$$k_\gamma = \begin{cases} 0 & \text{for } \gamma \neq \alpha, \\ P_{\mathcal{H}(\theta_\alpha)}h_\beta & \text{for } \gamma = \alpha > \beta, \\ Z_{\alpha\beta}h_\beta & \text{for } \gamma = \alpha \leqslant \beta. \end{cases}$$

Clearly $Z_{\alpha\alpha} = I$, and thus $P_\alpha = R_{\alpha\alpha}$ coincides with the orthogonal projection of $\mathcal{H}(\Theta)$ onto its α-component subspace. For every A in $\operatorname{Alg}\operatorname{Lat}(\{S(\Theta)\}')$ we have $P_\alpha A P_\beta \in \operatorname{Alg}\operatorname{Lat}(\{S(\Theta)\}')$ and $A = \sum_{\alpha,\beta} P_\alpha A P_\beta$ unconditionally in the strong operator topology. To conclude the proof, it will suffice to show that each $P_\alpha A P_\beta$ commutes with $S(\Theta)$. Now, the operators $R_{\beta\alpha}P_\alpha A P_\beta$ and $P_\alpha A P_\beta R_{\beta\alpha}$ also belong to $\operatorname{Alg}\operatorname{Lat}(\{S(\Theta)\}')$ and have the form $\bigoplus_\gamma T_\gamma$ with $T_\gamma = 0$ for $\gamma \neq \beta$ and $\gamma \neq \alpha$, respectively. Considering hyperinvariant subspaces of the form $\ker(\theta(S(\Theta)))$ such that $\theta|\theta_0$, it is easy to see that $T_\gamma \in \operatorname{Alg}\operatorname{Lat}(S(\theta_\gamma))$ for each γ, so that T_γ commutes with $S(\theta_\gamma)$ by the reflexivity of $S(\theta_\gamma)$. Thus $R_{\beta\alpha}P_\alpha A P_\beta$ and $P_\alpha A P_\beta R_{\beta\alpha}$ commute with $S(\Theta)$, hence

$$R_{\beta\alpha}(P_\alpha A P_\beta S(\Theta) - S(\Theta)P_\alpha A P_\beta) = (P_\alpha A P_\beta S(\Theta) - S(\Theta)P_\alpha A P_\beta)R_{\beta\alpha} = 0.$$

If the range of $R_{\beta\alpha}$ does not contain the range of P_α, it follows that $\beta < \alpha$ and therefore $R_{\beta\alpha}$ is one-to-one on the range of P_α. In either case the last equality shows that $P_\alpha A P_\beta \in \{S(\Theta)\}'$, and the theorem is proved. ∎

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