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A CONDITIONAL EXPECTATION FOR THE FULL FOCK SPACE

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Abstract. We define a family of conditional expectations on the algebra generated by the creation, annihilation and gauge operators on the Full Fock space over $L^2(\mathbb{R}_+)$.

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1. INTRODUCTION

There is now a well developed non-commutative stochastic calculus which deals with non-commutative analogues and generalisations of classical stochastic processes. Recently attention has turned to the example provided by the full Fock space, $\mathcal{F}$, over $L^2(\mathbb{R}_+)$. The basic processes are provided by the annihilation, creation and gauge operators $l(h), l^*(f), p(\mathcal{T})$ for $h, f \in L^2(\mathbb{R}_+)$, $\mathcal{T} \in \mathcal{B}(L^2(\mathbb{R}_+))$ (the bounded operators on $L^2(\mathbb{R}_+)$). One of the features of this situation, in contrast to [1], [2], is the absence (thus far) of a (formal) conditional expectation acting on the processes with which one can define the (formal) notion of martingale and associated processes. Without this one cannot introduce the projections associated with random times and exploit their relationship with stochastic integration ([3]). Since we shall have some more to say on this and other matters in this context ([4]), we demonstrate here that it is possible to construct a family of conditional expectations in a straightforward manner. The construction proceeds as one might expect; for example it is clear that $l(\chi_t h)$ should be the time t conditional expectation of $l(h)$. From these easy beginings the expectation is extended to the whole of $B(\mathcal{F})$. Many of the proofs are obvious, we omit these. Others demand a fuller explanation, we include some details.

2. PRELIMINARIES AND NOTATION

We define the full Fock space $\mathcal{F}$ over $L^2(\mathbb{R}_+)$ as follows:

$$\mathcal{F} \equiv \mathbb{C} \oplus \Big(\bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+)^{\otimes n} \Big); \tag{a}$$

here $\mathbb{C}$ denotes the complex numbers and $\mathcal{F}$ has the usual scalar product. Note that all scalar products are linear in the left argument. Ω will denote the vector $(1,0,0,\ldots)$. We define the annihilation operator $l(f)$ and creation operator $l^*(f)$ for $f \in L^2(\mathbb{R}_+)$, as follows

$$l(f) f_1 \otimes \cdots \otimes f_n = \langle f_1, f \rangle f_2 \otimes \cdots \otimes f_n \tag{b}$$

$$l^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n \tag{c}$$

$$l(f)\Omega = 0 \tag{d}$$

$$l^*(f)\Omega = f \tag{e}$$

for $n \geqslant 1$ and $f_1, \ldots, f_n$ in $L^2(\mathbb{R}_+)$. The operators $l(f)$ and $l^*(f)$ are bounded and mutually adjoint. Furthermore,

$$\|l(f)\| = \|l^*(f)\| = \|f\|_2.$$

Given any $\mathcal{T} \in \mathcal{B}(L^2(\mathbb{R}_+))$ we define the operator $p(\mathcal{T})$ by:

$$p(\mathcal{T}) f_1 \otimes \cdots \otimes f_n = \mathcal{T} f_1 \otimes \cdots \otimes f_n, \qquad p(\mathcal{T})\Omega = 0$$

for $f_i \in L^2(\mathbb{R}_+)$, $1 \leqslant i \leqslant n$. The operator $p(\mathcal{T})$ is bounded and $\|p(\mathcal{T})\| = \|\mathcal{T}\|$, and $p(\mathcal{T})^* = p(\mathcal{T}^*)$. For $g \in L^\infty(\mathbb{R}_+)$, g will be considered to be the element of $\mathcal{B}[L^2(\mathbb{R}_+)]$ obtained by letting g act by multiplication on $L^2(\mathbb{R}_+)$. This makes the meaning of $p(g)$ clear. Moreover the following identities hold:

$$l(g) \cdot l^*(f) = \langle f \cdot g \rangle \mathcal{I} \tag{f}$$

$$p(\mathcal{T}_1) \cdot p(\mathcal{T}_2) = p(\mathcal{T}_1 \cdot \mathcal{T}_2) \tag{g}$$

$$p(\mathcal{T}) l^*(f) = l^*(\mathcal{T} f) \tag{h}$$

$$l(g) p(\mathcal{T}) = l(\mathcal{T}^* g). \tag{i}$$

Let $D^0 \subseteq \mathcal{F}$ be the set consisting of $\lambda\Omega$ with $\lambda \in \mathbb{C}$ and $|\lambda| \leqslant 1$ and vectors of the form $u_1 \otimes \cdots \otimes u_k$ with $k \in \mathbb{N}$, the natural numbers, $u_j \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, $\|u_j\|_2 \leqslant 1$, $\|u_j\|_\infty \leqslant 1$ for $1 \leqslant j \leqslant k$. For $k = 0$, $u_1 \otimes \cdots \otimes u_k = \Omega$. D will denote the linear span of D^0. We denote the bounded operators on $\mathcal{F}$ by $\mathcal{B}(\mathcal{F})$ and by τ_{s} the strong operator topology on $\mathcal{B}(\mathcal{F})$. We collect together some elementary facts and definitions needed for the sequel.

2.1. D is dense in $\mathcal{F}$.

2.2. $\mathcal{F}$ is separable.

2.3. We note a useful lemma about the strong topology τ_{s} on bounded sets of $\mathcal{B}(\mathcal{F})$.

Lemma. *The strong operator topology on S [the unit ball of $\mathcal{B}(\mathcal{F})$] is metrisable. This metric is given by a norm on $\mathcal{B}(\mathcal{F})$.*

For $x \in \mathcal{B}(\mathcal{F})$ the norm is given by

$$\|x\|_{\mathrm{s}} = \left\{ \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\varsigma_n\|^2 \right\}^{\frac{1}{2}}$$

where $(\varsigma_n)_{n=1}^{\infty}$ is a countable base for $\mathcal{F}$.

2.4. Definition. We define $\mathcal{A}$ to be the $*$-algebra generated by the annihilation and gauge operators $l(f), p(g)$ respectively and $\mathcal{I}$, where $f \in L^2(\mathbb{R}_+)$, $g \in L^\infty(\mathbb{R}_+)$. We shall denote by $\mathcal{V}$ the Von Neumann algebra $\overline{\mathcal{A}}^{\tau_{\mathrm{s}}}$ in $\mathcal{B}(\mathcal{F})$.

2.5. Definition. $\mathcal{A}_t$ is defined to be the $*$-algebra of $\mathcal{A}$ which is generated by $\mathcal{I}$ and the operators $l(f), p(g)$ with $g \in L^\infty([0,t])$ and $f \in L^2([0,t])$, for any $t \in \mathbb{R}_+$. By $\mathcal{V}_t$ mean the strong-operator closure of $\mathcal{A}_t$. By (f), (g), (h) and (i) we note that any element of $\mathcal{A}$ can be written as a sum of *basic elements* of the form $\lambda\mathcal{I}$ or

$$l^*(f_1)\cdots l^*(f_r)p(g)l(h_1)\cdots l(h_s)$$

or

$$l^*(f_1)\cdots l^*(f_r)l(h_1)\cdots l(h_s)$$

with the convention that $r = 0$ (respectively $s = 0$) denotes an element with no creation (respectively no annihilation) operators. Here $r, s \in \mathbb{N} \cup \{0\}$ and $f_i, h_j \in L^2(\mathbb{R}_+)$ and $g \in L^\infty(\mathbb{R}_+)$, $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant s$. Furthermore if f_i, h_j, and g have support in $[0,t]$ then we get basic elements for $\mathcal{A}_t$.

2.6. Definition. We define a *process* $F(t)$ to be a function

$$F : \mathbb{R}^+ \to \{\text{operators with domain containing } \mathcal{D}\}.$$

A $\mathcal{V}$ *adapted process* is a process such that $F(t) \in \mathcal{V}_t$ and similarly for $\mathcal{A}$ adapted processes. We shall call a process *simple* if it can be written in the form

$$\sum_{j=1}^{n} F(t_j)\chi_{[t_j, t_{j+1})}$$

with $0 = t_1 \leqslant \cdots t_j \leqslant t_{j+1} \leqslant \cdots \leqslant t_{n+1} = \infty$, $1 \leqslant j \leqslant n$, $F(t_j) \in \mathcal{A}_{t_j}$.

2.7. For a subset $\mathcal{K}$ of $\mathcal{F}$ we will use the term "τ_{s}-on $\mathcal{K}$" to refer to pointwise convergence on $\mathcal{K}$.

3. CONSTRUCTION OF THE EXPECTATION

For each $t \in \mathbb{R}^+$ we will construct an expectation $E_t : \mathcal{V} \to \mathcal{V}_t$, $t \geqslant 0$ with all the standard properties. We show further that E_t is strong-operator continuous on bounded sets.

In order to construct E_t we need to define a map $\widetilde{E}_t : \mathcal{A} \to \mathcal{A}_t$ with the appropriate properties, and we shall then extend $\widetilde{E}_t$ in the following stages:

(a) For a sequence $(a_n)_{n\geqslant 1}$ in $\mathcal{A}$, for which $(a_n(\xi))$ is Cauchy in $\mathcal{F}$ for each $\xi \in D^0$ we shall demonstrate that $(\widetilde{E}_t a_n(\xi))_{n\geqslant 1}$ is also Cauchy in $\mathcal{F}$ for each $\xi \in D^0$. Furthermore, if $(a_n)_{n\geqslant 1}$ is a sequence in $\mathcal{A}$ such that $a_n \to 0$, τ_s-on D^0, then we shall show that $(\widetilde{E}_t a_n)_{n\geqslant 1}$ is a sequence in $\mathcal{A}_t$ with $\widetilde{E}_t a_n \to 0$, τ_s-on D^0.

(b) Here we will show that for any element $a \in \mathcal{A}$, $\|\widetilde{E}_t a\| \leqslant \|a\|$. In the process of proving this result we shall obtain most of the properties related to conditional expectations.

(c) We will use the previously obtained results to extend $\widetilde{E}_t : \mathcal{A} \to \mathcal{A}_t$ to $E_t : \mathcal{V} \to \mathcal{V}_t$ and we shall show that E_t satisfies the properties of a conditional expectation between two von Neumann algebras.

3.1. Notes on notation. We shall use the letters f and h to denote elements of $L^2(\mathbb{R}_+)$ which will be arguments of $l^*(\cdot)$ and $l(\cdot)$ respectively, we will use g to denote arguments of $p(\cdot)$, $g \in L^\infty(\mathbb{R}_+)$.

Furthermore, the letter a will denote elements of the algebras $\mathcal{A}$ and the letter x will denote elements of $\mathcal{V}$. In addition, for any basic elements of $\mathcal{A}$ the letter r will represent the numbers of creation operators and s the number of annihilation operators. Finally, χ_t will denote the indicator function of $[0, t]$.

4. THE MAP $\widetilde{E}_t$

We shall start this section with an important property of elements $a \in \mathcal{A}$, which will enable us to define the map $\widetilde{E}_t : \mathcal{A} \to \mathcal{A}_t$.

An element of $\mathcal{A}$ is a sum of basic elements. Recall that basic elements have the form

(a) $\lambda\mathcal{I}$;
(b) $l^*(f_1)\cdots l^*(f_r)p(g)l(h_1)\cdots l(h_s)$;
(c) $l^*(f_1)\cdots l^*(f_r)l(h_1)\cdots l(h_s)$.

We shall write an element a of $\mathcal{A}$ in a particular way that reflects how the basic elements which comprise a act on $\mathcal{F}$. So if $a = \sum_i a_i$, we shall denote by $a_{\delta,q}$ the sum of those basic elements a_i, for which the difference between the number of creation operators and the number of annihilation operators is δ; and the sum of the number of annihilation and gauge operators is q. Then we can write

$$a = \sum_\delta \sum_q a_{\delta,q}. \tag{4.1}$$

For example if

$$a = l^*(f_1) + l^*(f_2)p(g_1) + l^*(f_3)l^*(f_4)l(h_1) + l^*(f_5)l^*(f_6)p(g_2)l(h_2)$$

then

$$a = a_{1,0} + a_{1,1} + a_{1,2}$$

where

$$\begin{aligned} a_{1,0} &= l^*(f_1) \\ a_{1,1} &= l^*(f_2)p(g_1) + l^*(f_3)l^*(f_4)l(h_1) \\ a_{1,2} &= l^*(f_5)l^*(f_6)p(g_2)l(h_2). \end{aligned}$$

4.1. Definition. Given $a = \sum_{i=1}^{n} a_i$ in $\mathcal{A}$, with a_i basic elements of $\mathcal{A}$ we define:

$$a^t = \sum_{i=1}^{n} a_i^t \quad \text{with } (\lambda\mathcal{I})^t = \lambda\mathcal{I}$$

$$\{l^*(f_1)\cdots l^*(f_r)p(g)l(h_1)\cdots l(h_s)\}^t = l^*(\chi_t f_1)\cdots l^*(\chi_t f_r)p(\chi_t g)l(\chi_t h_1)\cdots l(\chi_t h_s)$$

$$\{l^*(f_1)\cdots l^*(f_r)l(h_1)\cdots l(h_s)\}^t = l^*(\chi_t f_1)\cdots l^*(\chi_t f_r)l(\chi_t h_1)\cdots l(\chi_t h_s).$$

Note that all a_i^t lie in $\mathcal{A}_t$.

4.2. Note. $a^t = \sum_\delta \sum_q a_{\delta,q}^t$ just by regrouping the basic elements of a_i. Given a vector $u_1 \otimes \cdots \otimes u_k$ in D^0 and $0 \leqslant j \leqslant k$, define,

$$\underline{v}_j = \chi_t u_1 \otimes \cdots \otimes \chi_t u_j.$$

Again we shall use the convention: $\underline{v}_0 = \Omega$ and for $k = 0$, $u_1 \otimes \cdots \otimes u_k \equiv \Omega$. We can write

$$a = \sum_{\delta=-\Delta}^{\Delta} \sum_{q=\max(0,-\delta)}^{Q(\delta)} a_{\delta,q}, \qquad a^t = \sum_{\delta=-\Delta}^{\Delta} \sum_{q=\max(0,-\delta)}^{Q(\delta)} a_{\delta,q}^t$$

with Δ the maximum $|\delta|$ of those in appearing in equation (4.1) and with $a_{\delta,q} = 0$ for all (δ, q) not appearing in equation (4.1). For each δ, $Q(\delta)$ will denote the maximum q appearing in $a_{\delta,q}$ of equation. Since the number of creation operators has to be non-negative, we need $q \geqslant -\delta$ and when $q = -\delta$ the basic elements cannot contain any gauge operators. The following result underpins our construction.

4.3. Theorem. *For a in $\mathcal{A}$ with $a = \sum_{\delta=-\Delta}^{\Delta} \sum_{q=0}^{Q(\delta)} a_{\delta,q}$ and $u_1 \otimes \cdots \otimes u_k$ in D^0 we have that:*

$$\|a^t u_1 \otimes \cdots \otimes u_k\|^2 \leqslant (4k+2) \sum_{q=0}^{k} \|a\underline{v}_q\|^2$$

for $k \in \mathbb{N} \cup \{0\}$.

Proof. Fix $k \in \mathbb{N} \cup \{0\}$. We first observe that:

$$\begin{aligned} a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k &\in L^2(\mathbb{R}_+)^{\otimes(k+\delta)} \quad \text{for } k \geqslant q, \\ &= 0 \quad \text{for } k < q; \\ \sum_{q=0}^{Q(\delta)} a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k &\in L^2(\mathbb{R}_+)^{\otimes(k+\delta)} \quad \text{for } k \geqslant q, \\ &= 0 \quad \text{for } k < q. \end{aligned}$$

Note that $k \geqslant q \Rightarrow k - q \geqslant 0 \Rightarrow k + \delta \geqslant 0$, and $L^2(\mathbb{R}_+)^{\otimes 0}$ will represent $\mathbb{C}$. So for $\delta < \delta'$ the vectors, $a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k$, $a^t_{\delta',q} u_1 \otimes \cdots \otimes u_k$ are orthogonal. Hence:

$$\begin{aligned} \|a^t u_1 \otimes \cdots \otimes u_k\|^2 &= \left\| \sum_{\delta=-\Delta}^{\Delta} \sum_{q=\max(0,-\delta)}^{Q(\delta)} a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k \right\|^2 \\ &= \sum_{\delta=-\Delta}^{\Delta} \left\| \sum_{q=\max(0,-\delta)}^{Q(\delta)} a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k \right\|^2 \end{aligned}$$

by orthogonality of the vectors, established above, and

$$\|a^t u_1 \otimes \cdots \otimes u_k\|^2 \leqslant \sum_{\delta=-\Delta}^{\Delta} \left\{ \sum_{q=\max(0,-\delta)}^{Q(\delta)} \|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\| \right\}^2 \tag{4.2}$$

by the triangle inequality.

Before going further with our proof, let us make the following remarks.

4.4. Note. $a_{\delta,q} u_1 \otimes \cdots \otimes u_k = 0$, $a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k = 0$ for $q > k$ since the first k annihilation acting on $u_1 \otimes \cdots \otimes u_k$ give the vector Ω and the next operator (gauge or annihilation) acting on Ω gives 0. Hence, without loss of generality, we can take $Q(\delta) \leqslant k$. The general term for $a^t_{\delta,q}$ is

$$\begin{aligned} a^t_{\delta,q} &= \sum_{i=1}^{n} l^*(\chi_t f^i_1) \cdots l^*(\chi_t f^i_{\delta+q-1}) p(\chi_t g^i) l(\chi_t h^i_1) \cdots l(\chi_t h^i_{q-1}) \\ &\quad + \sum_{j=1}^{m} l^*(\chi_t \widetilde{f}^j_1) \cdots l^*(\chi_t \widetilde{f}^j_{\delta+q}) \cdot l(\chi_t \widetilde{h}^j_1) \cdots l(\chi_t \widetilde{h}^j_q) \end{aligned}$$

with $m, n \in \mathbb{N} \cup \{0\}$ and when m or n is 0, there is no basic element of that corresponding form. In the particular case $\delta = 0$, $q = 0$, we have $a_{0,0} = a^t_{0,0} = \lambda \mathcal{I}$ with $\lambda \in \mathbb{C}$. So

$$\begin{aligned} &\|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2 \\ &= \Bigg\| p(\chi_t) \Bigg\{ \sum_{i=1}^{n} l^*(f^i_1) l^*(\chi_t f^i_2) \cdots l^*(\chi_t f^i_{\delta+q-1}) p(\chi_t g^i) l(\chi_t h^i_1 \cdots l(\chi_t h^i_{q-1}) \\ &\quad + \sum_{j=1}^{m} l^*(\widetilde{f}^j_1) l^*(\chi_t \widetilde{f}^j_2) \cdots l^*(\chi_t \widetilde{f}^j_{\delta+q}) l(\chi_t \widetilde{h}^j_1) \cdots l(\chi_t \widetilde{h}^j_q) \Bigg\} u_1 \otimes \cdots \otimes u_k \Bigg\|^2. \end{aligned}$$

Since we need to refer back to this last equation let us write it as

$$\|a^t_{\delta,q}u_1\otimes\cdots\otimes u_k\|^2 = \|p(\chi_t)b^t_{\delta,q}u_1\otimes\cdots\otimes u_k\|^2 \tag{4.3}$$

where $b^t_{\delta,q}$ is the term between braces above. Observe that the last creation operator to act in each of the terms of $b^t_{\delta,q}$ does *not* involve χ_t.

Before starting a lemma we introduce some notation. We write

$$L^*(f^i) = l^*(f^i_1)l^*(f^i_2)\cdots l^*(f^i_{\delta+q-1})$$

and

$$L(h^i) = l(h^i_1)\cdots l(h^i_{q-1}).$$

We also write $\underline{u} = u_1\otimes\cdots\otimes u_k$ and $\underline{u}_{q+1} = u_{q+1}\otimes\cdots\otimes u_k$. For a permutation π of the indicies $1,2,\ldots,\delta+q-1$, we write $\pi(f^i)$ for the vector $f^i_{\pi(1)}\otimes\cdots\otimes f^i_{\pi(\delta+q-1)}$.

4.5. LEMMA. *Let π be a permutation of the indicies $1,2,\ldots,\delta+q-1$, then*

$$\begin{aligned}&\langle L^*(f^i)p(g^i)L(h^i)\underline{u}\,,\, L^*(f^{i'})p(g^{i'})L(h^{i'})\underline{u}\rangle\\ &\quad= \langle L^*(\pi(f^i))p(g^i)L(h^i)\underline{u}, L^*(\pi(f^{i'}))p(g^{i'})L(h^{i'})\underline{u}\rangle.\end{aligned}$$

Similarly

$$\langle L^*(f^i)p(g^i)L(h^i)\underline{u}\,,\, L^*(\widetilde{f}^j)L(\widetilde{h}^j)\underline{u}\rangle = \langle L^*(\pi(f^i))p(g^i)L(h^i)\underline{u}, L^*(\pi(\widetilde{f}^j))L(\widetilde{h}^j)\underline{u}\rangle$$

and

$$\langle L^*(\widetilde{f}^j)L(\widetilde{h}^j)\underline{u}\,,\, L^*(\widetilde{f}^{j'})L(\widetilde{h}^{j'})\underline{u}\rangle = \langle L^*(\pi(\widetilde{f}^j))L(\widetilde{h}^j)\underline{u}, L^*(\pi(\widetilde{f'}^j))L(\widetilde{h}^{j'})\underline{u}\rangle.$$

Proof. We look at one case only; the others are similar.

$$L^*(f^i)p(g^i)L(h^i)\underline{u} = \left(\prod_{r=1}^{q-1}\overline{\langle\chi_t h^i_{q-r}, u_r\rangle}\right)\bigotimes_{s=1}^{\delta+q-1} f^i_s\otimes g^i\otimes\underline{u}_{q+1}.$$

So

$$\langle L^*(f^i)p(g^i)L(h^i)\underline{u},\, L^*(f^{i'})p(g^{i'})L(h^{i'})\underline{u}\rangle$$

is equal to

$$\left(\prod_{r=1}^{q-1}\overline{\langle\chi_t h^i_{q-r}, u_r\rangle}\right)\left(\prod_{r=1}^{q-1}\langle\chi_t h^{i'}_{q-r}, u_r\rangle\right)\left\langle\bigotimes_{s=1}^{\delta+q-1} f^i_s\otimes g^i\otimes\underline{u}_{q+1},\ \bigotimes_{s=1}^{\delta+q-1} f'^i_s\otimes g^{i'}\otimes\underline{u}_{q+1}\right\rangle;$$

this in turn is equal to

$$\left(\prod_{r=1}^{q-1}\overline{\langle\chi_t h^i_{q-r}, u_r\rangle}\right)\left(\prod_{r=1}^{q-1}\langle\chi_t h^{i'}_{q-r}\,,\,u_r\rangle\right)\left(\prod_{r=1}^{\delta+q-1}\langle f^i_s, f^{i'}_s\rangle\langle g^i\, g^{i'}\rangle\langle\underline{u}_{q+1},\underline{u}_{q+1}\rangle\right)$$

which is

$$\left(\prod_{r=1}^{q-1}\overline{\langle\chi_t h^i_{q-r}, u_r\rangle}\right)\left(\prod_{r=1}^{q-1}\langle\chi_t h^{i'}_{q-r},\,u_r\rangle\right)\left(\prod_{r=1}^{\delta+q-1}\langle f^i_{\pi(s)}, f^{\pi(i')}_s\rangle\langle g^i\, g^{i'}\rangle\langle\underline{u}_{q+1},\underline{u}_{q+1}\rangle\right)$$

which is equal to

$$\langle L^*(\pi(f^i))p(g^i)L(h^i)\underline{u}\,,\, L^*(\pi(f^{i'}))p(g^{i'})L(h^{i'})\underline{u}\rangle. \quad \blacksquare$$

4.6. Corollary. *Let v be a vector in $\mathcal{F}$ which is a finite sum of tensor product vectors $e_1, e_2, \ldots, e_r$ each of which being a k-fold tensor product of elements of $L^2(\mathbb{R}_+)$. For a vector of the form $u = u_1 \otimes \cdots \otimes u_k$ and a permutation ρ of the first k integers, write $\rho(u) = u_{\rho(1)} \otimes \cdots \otimes u_{\rho(k)}$ and $\rho(v)$ for $\sum_i \rho(e_i)$. Then*

$$\|v\| = \|\rho(v)\|.$$

Moreover, if $w = w_1 \otimes w_2 \otimes \cdots \otimes w_n$ and

$$v \otimes w = \sum_i e_i \otimes w$$

then

$$\|v \otimes w\| = \|v\| \cdot \|w\|.$$

Here $e_i \otimes w$ means the tensor product of the elements of e_i and w taken in the order indicated.

Proof. The proof of the lemma is easily adapted to this case. $\blacksquare$

Proof of Theorem 4.3 *continued.* Return now to the equation (4.3); since $p(\chi_t)$ is an operator of norm less than one, we have

$$\|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2 \leqslant \|b^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2.$$

Now we can express $\|b^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2$ as a sum of (products of) inner products of the form encountered in Lemma 4.5. We can apply the permutation which interchanges the first two terms in each inner product involving f's, that is f^i_1 with f^i_2, $f^{i'}_1$ with f^i_2 with similar interchanges for the f^j and $f^{j'}$. This amounts interchanging $l^*(f^i_1)$ and $l^*(\chi_t f^i_2)$ and the corresponding creations involving the other f terms in the expression for $b^t_{\delta,q}$. This done it leaves us with an operator which we can write as $p(\chi_t)c^t_{\delta,q}$ and we have

$$\|b^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2 = \|p(\chi_t)c^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2 \leqslant \|c^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\|^2.$$

Obviously we can iterate this procedure enough times to remove the χ_t's from the first $\delta + q - 1$ creation operators in every term of $a^t_{\delta,q}$. At the same time observe that

$$\langle \chi_t h^i, u\rangle = \langle h^i, \chi_t u\rangle, \qquad \langle \chi_t g u, \chi_t f\rangle = \langle g \chi_t u, f\rangle.$$

Note also that in $u_1 \otimes \cdots \otimes u_k$ the vectors $u_{q+1}, u_{q+2}, \ldots, u_k$ are unaffected by the action of the elements of $a^t_{\delta,q}$. Putting all of this together we arrive at

$$
\begin{aligned}
&\left\|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\right\|^2 \\
&\leqslant \Bigg\| \Bigg\{ \sum_{i=1}^{n} l^*(f_1^i) \cdots l^*(f^i_{\delta+q-1}) p(g^i) l(h_1^i) \cdots l(h^i_{q-1}) \\
&\quad + \sum_{j=1}^{m} l^*(\widetilde{f}_1^j) \cdots l^*(\widetilde{f}^j_{\delta+q-1}) l^*(\chi_t \widetilde{f}^j_{\delta+q}) l(\widetilde{h}_1^j) \cdots l(\widetilde{h}_q^j) \Bigg\} \chi_t u_1 \otimes \cdots \otimes \chi_t u_q \Bigg\|^2 \\
&= \Bigg\| \sum_{i=1}^{n} f_1^i \otimes \cdots \otimes f^i_{\delta+q-1} \otimes \chi_t g^i u_q \overline{\langle h_1^i, \chi_t u_{q-1}\rangle \cdots \langle h^i_{q-1}, \chi_t u_1\rangle} \\
&\quad + \sum_{j=1}^{m} \widetilde{f}_1^j \otimes \cdots \otimes \widetilde{f}^j_{\delta+q-1} \otimes \chi_t \widetilde{f}^j_{\delta+q} \overline{\langle \widetilde{h}_1^j, \chi_t u_q\rangle \cdots \langle \widetilde{h}_q^j, \chi_t u_1\rangle} \Bigg\|^2 \\
&= \Bigg\| \sum_{i=1}^{n} \chi_t g^i u_q \otimes f_2^i \otimes \cdots \otimes f^i_{\delta+q-1} \otimes f_1^i \overline{\langle h_1^i, \chi_t u_{q-1}\rangle \cdots \langle h^i_{q-1}, \chi_t u_1\rangle} \\
&\quad + \sum_{j=1}^{m} \chi_t \widetilde{f}^j_{\delta+q} \otimes \widetilde{f}_2^j \otimes \cdots \otimes \widetilde{f}^j_{\delta+q-1} \otimes \widetilde{f}_1^j \overline{\langle \widetilde{h}_1^j, \chi_t u_q\rangle \cdots \langle \widetilde{h}_q^j, \chi_t u_1\rangle} \Bigg\|^2
\end{aligned}
$$

we have interchanged the first and last terms of the tensor products as above

$$
\begin{aligned}
&\left\|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\right\|^2 \\
&= \Bigg\| p(\chi_t) \Bigg\{ \sum_{i=1}^{n} g^i \chi_t u_q \otimes f_2^i \otimes \cdots \otimes f^i_{\delta+q-1} \otimes f_1^i \overline{\langle h_1^i, \chi_t u_{q-1}\rangle \cdots \langle h^i_{q-1}, \chi_t u_1\rangle} \\
&\quad + \sum_{j=1}^{m} \widetilde{f}^j_{\delta+q} \otimes \widetilde{f}_2^j \otimes \cdots \otimes \widetilde{f}^j_{\delta+q-1} \otimes \widetilde{f}_1^j \overline{\langle \widetilde{h}_1^j, \chi_t u_q\rangle \cdots \langle \widetilde{h}_q^j, \chi_t u_1\rangle} \Bigg\} \Bigg\|^2 \\
&\leqslant \Bigg\| \sum_{i=1}^{n} f_1^i \otimes \cdots \otimes f^i_{\delta+q-1} \otimes g^i \chi_t u_q \overline{\langle h_1^i, \chi_t u_{q-1}\rangle \cdots \langle h^i_{q-1}, \chi_t u_1\rangle} \\
&\quad + \sum_{j=1}^{m} \widetilde{f}_1^j \otimes \cdots \otimes \widetilde{f}^j_{\delta+q} \overline{\langle \widetilde{h}_1^j, \chi_t u_q\rangle \cdots \langle \widetilde{h}_q^j, \chi_t u_1\rangle} \Bigg\|^2
\end{aligned}
$$

by using $\|p(\chi_t)\| = 1$ and interchanging the first and last terms of the products as above

$$
\begin{aligned}
&\left\|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\right\|^2 \\
&= \Bigg\| \Bigg\{ \sum_{i=1}^{n} l^*(f_1^i) \cdots l^*(f^i_{\delta+q-1}) p(g^i) l(h_1^i) \cdots l(h^i_{q-1}) \\
&\quad + \sum_{j=1}^{m} l^*(\widetilde{f}_1^j) \cdots l^*(\widetilde{f}^j_{\delta+q}) l(\widetilde{h}_1^j) \cdots l(\widetilde{h}_q^j) \Bigg\} \chi_t u_1 \otimes \cdots \otimes \chi_t u_q \Bigg\|^2
\end{aligned}
$$

or in other words

$$\|a^t_{\delta,q} u_1 \otimes \cdots \otimes u_k\| \leqslant \|a_{\delta,q}\chi_t u_1 \otimes \cdots \otimes \chi_t u_q\|.$$

Recalling the notation of Corollary 4.6,

$$a_{\delta,q}\chi_t u_1 \otimes \cdots \otimes \chi_t u_q \otimes \chi_t u_{q+1} \otimes \cdots \otimes \chi_t u_{q+r}$$

is equal to

$$\left\{a_{\delta,q}\chi_t u_1 \otimes \cdots \otimes \chi_t u_q\right\} \otimes \chi_t u_{q+1} \otimes \cdots \otimes \chi_t u_{q+r}.$$

When $q = 0$, this amounts to:

$$a_{\delta,0}\chi_t u_1 \otimes \cdots \otimes \chi_t u_r = (a_{\delta,0}\Omega) \otimes \chi_t u_1 \otimes \cdots \otimes \chi_t u_r.$$

For convenience, we define

$$\sum_{q=\max(0,-\delta)}^{Q(\delta)} a_{\delta,q} = a_\delta.$$

For $1 \leqslant q' \leqslant Q(\delta)$, we can write

$$a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'} = \sum_{q=\max(0,-\delta)}^{Q(\delta)} a_{\delta,q}\chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'}$$

and the right hand side of the last equation is equal to

$$\sum_{q=\max(0,-\delta)}^{q'} \left(a_{\delta,q}\chi_t u_1 \otimes \cdots \otimes \chi_t u_q\right) \otimes \chi_t u_{q+1} \otimes \cdots \otimes \chi_t u_{q'};$$

also

$$\begin{aligned}&\left\{a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'-1}\right\} \otimes \chi_t u_{q'}\\ &\quad = \sum_{q=\max(0,-\delta)}^{q'-1} \left(a_{\delta,q}\chi_t u_1 \otimes \cdots \otimes \chi_t u_q\right) \otimes \chi_t u_{q+1} \otimes \cdots \chi_t u_{q'-1} \otimes \chi_t u_{q'}.\end{aligned}$$

Substracting

$$a_{\delta,q'}\chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'} = a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'} - \left\{a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'-1}\right\} \otimes \chi_t u_{q'}.$$

By taking norms and using the triangle inequality:

$$\|a_{\delta,q'}\chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'}\| \leqslant \|a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'}\| + \|a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'-1}\|$$

since $\|\chi_t u_{q'}\|_2 \leqslant 1$ while for $q' = 0$ we have $\|a_{\delta,0}\Omega\| \leqslant \|a_\delta \Omega\|$. We combine these inequalities in the following

$$\|a^t_{\delta,q'} u_1 \otimes \cdots \otimes u_k\| \leqslant \|a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'}\| + \|a_\delta \chi_t u_1 \otimes \cdots \otimes \chi_t u_{q'-1}\|;$$

substituting in equation (4.2) we get

$$\|a^t u_1 \otimes \cdots \otimes u_k\|^2$$
$$\leqslant \sum_{\delta=-\Delta}^{\Delta} \left\{ \sum_{q=\max(1,-\delta)}^{Q(\delta)} \big(\|a_\delta \underline{v}_q\| + \|a_\delta \underline{v}_{q-1}\|\big) + \|a_\delta \Omega\| \right\}^2$$
$$\leqslant \sum_{\delta=-\Delta}^{\Delta} (2Q(\delta)+1) \cdot \left\{ \sum_{q=\max(1,-\delta)}^{Q(\delta)} (\|a_\delta \underline{v}_q\|^2 + \|a_\delta \underline{v}_{q-1}\|^2) + \|a_\delta \Omega\|^2 \right\}$$

since there are at most $2Q(\delta)+1$ terms in the expression in $\{\cdot\}$

$$\|a^t u_1 \otimes \cdots \otimes u_k\|^2$$
$$\leqslant \sum_{\delta=-\Delta}^{\Delta} (2Q(\delta)+1) \cdot \left\{ \sum_{q=\max(0,-(\delta+1))}^{Q(\delta)} \|a_\delta \underline{v}_q\|^2 + \sum_{q=\max(0,-(\delta+1))}^{Q(\delta)-1} \|a_\delta \underline{v}_q\|^2 \right\}$$
$$\leqslant 2 \sum_{\delta=-\Delta}^{\Delta} (2Q(\delta)+1) \cdot \sum_{q=\max(0,-(\delta+1))}^{Q(\delta)} \|a_\delta \underline{v}_q\|^2$$
$$\leqslant (4k+2) \sum_{\delta=-\Delta}^{\Delta} \sum_{q=\max(0,-(\delta+1))}^{k} \|a_\delta \underline{v}_q\|^2 \quad \text{since } Q(\delta) \leqslant k$$
$$\leqslant (4k+2) \sum_{\delta=-\Delta}^{\Delta} \sum_{q=0}^{k} \|a_\delta \underline{v}_q\|^2 = (4k+2) \sum_{q=0}^{k} \sum_{\delta=-\Delta}^{\Delta} \|a_\delta \underline{v}_q\|^2 = (4k+2) \sum_{q=0}^{k} \|a \underline{v}_q\|^2$$

because $a_\delta \underline{v}_q$ are orthogonal for different value of δ, and $a = \sum_{\delta=-\Delta}^{\Delta} a_\delta$. ∎

Each $a \in \mathcal{A}$ can be written as a sum of basic elements in a non unique way. We shall denote these different representations by $\pi, \rho, \sigma, \ldots$. So for a in $\mathcal{A}$ with a representation π:

$$a = \sum_i a_i \equiv a_{(\pi)}$$

and a representation ρ:

$$a = \sum_j a'_j \equiv a_{(\rho)}.$$

Note that the definition of a^t depended on a given representation of a and so we now have:

$$a^t_{(\pi)} = \sum_i a^t_i, \qquad a^t_{(\rho)} = \sum_j {a'}^t_j$$

for two different representations π, ρ of a. To define $\widetilde{E}_t$ we need to show that in fact $a^t_{(\pi)} = a^t_{(\rho)}$ for different π and ρ. Given any

$$a_{(\pi)} = \sum_{i=1}^{n} a_i, \qquad b_{(\sigma)} = \sum_{j=1}^{m} b_j$$

define

$$(a-b)_{(\pi-\sigma)} = \sum_{i=1}^{n} a_i - \sum_{j=1}^{m} b_j \tag{4.4}$$

which is a representation of $a-b$ in $\mathcal{A}$. Hence,

$$(a-b)^t_{(\pi-\sigma)} = \sum_{i=1}^{n} a_i^t - \sum_{j=1}^{m} b_j^t = a^t_{(\pi)} - b^t_{(\sigma)}.$$

Thus:

$$a^t_{(\pi)} - a^t_{(\rho)} = (a-a)^t_{(\pi-\rho)}$$

and

$$\|(a-a)^t_{(\pi-\rho)} u_1 \otimes \cdots \otimes u_k\|^2 \leqslant (4k+2) \sum_{q=0}^{k} \|(a-a)\underline{v}_q\|^2 = 0$$

for all $u_1 \otimes \cdots \otimes u_k \in D$, hence $(a-a)^t_{(\pi-\rho)} = 0$. So $a^t_{(\pi)} = a^t_{(\rho)}$. Now we can make

4.7. Definition. We can now define the function $\widetilde{E}_t : \mathcal{A} \to \mathcal{A}_t$ by

$$\widetilde{E}_t(a) = a^t_{(\pi)}$$

for any representation π of a.

Before we discuss the properties of $\widetilde{E}_t$ we prove a theorem used in the extension to $\mathcal{V}$.

4.8. Theorem. (i) *If $(a^{(n)})_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ which is Cauchy τ_{s}-on D^0 then so is $(\widetilde{E}_t a^{(n)})_{n=1}^{\infty}$.*

(ii) *If $(a^{(n)})_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ with $a^{(n)} \to 0$, τ_{s}-on D^0 then $\widetilde{E}_t a^{(n)} \to 0$ likewise.*

Proof. (i) Let $u_1 \otimes \cdots \otimes u_k \in D$ for $k \in \mathbb{N} \cup \{0\}$. With $\underline{v}_q = \chi_t u_1 \otimes \chi_t u_2 \otimes \chi_t \otimes \cdots \otimes \chi_t u_q$ we have that $a^{(n)}\underline{v}_q$ is Cauchy in $\mathcal{F}$ for $0 \leqslant q \leqslant k$. Hence $\forall \varepsilon > 0$, $\exists\, N(\varepsilon)$ such that $\forall n, m \geqslant N(\varepsilon)$

$$\|[a^{(n)} - a^{(m)}]\underline{v}_q\| < \varepsilon, \quad 0 \leqslant q \leqslant k.$$

By Theorem 4.3

$$\|\widetilde{E}_t(a^{(n)} - a^{(m)}) u_1 \otimes \cdots \otimes u_k\|^2 \leqslant (4k+2) \sum_{q=0}^{k} \varepsilon^2 = (k+1)(4k+2)\varepsilon^2$$

hence $(\widetilde{E}_t a^{(n)} u_1 \otimes \cdots \otimes u_k)_{n=1}^{\infty}$ is Cauchy in $\mathcal{F}$. So $(\widetilde{E}_t a^{(n)})_{n=1}^{\infty}$ is τ_{s}-on D^0 Cauchy.

(ii) Suppose now that $a^{(n)} \to 0$, τ_{s}-on D^0. Let $u_1 \otimes \cdots \otimes u_k \in D$; $\forall \varepsilon > 0$, $\exists\, N(\varepsilon)$ such that $\forall n \geqslant N(\varepsilon)$:

$$\|a^{(n)}\underline{v}_q\| < \varepsilon, \quad 0 \leqslant q \leqslant k.$$

Again by Theorem 4.3

$$\|\widetilde{E}_t a^{(n)} u_1 \otimes \cdots \otimes u_k\|^2 \leqslant (4k+2)(k+1)\varepsilon^2.$$

Hence $\widetilde{E}_t a^{(n)} u_1 \otimes \cdots \otimes u_k \to 0$ in $\mathcal{F}$, and so $\widetilde{E}_t a^{(n)} \to 0$, τ_{s}-on D^0. ∎

4.9. Remark. The above results can be extended to "τ_{s}-on D" by linearity.

5. PROPERTIES OF $\widetilde{E}_t$

5.1. THEOREM. (i) $\widetilde{E}_t$ *is a surjective linear map onto* $\mathcal{A}_t$ *with* $\widetilde{E}_t(I) = I$*;*
(ii) $\widetilde{E}_t^2 = \widetilde{E}_t$*;*
(iii) $\forall\, a \in \mathcal{A}$*,* $\big(\widetilde{E}_t a\big)^* = \widetilde{E}_t(a^*)$*;*
(iv) $\widetilde{E}_t\big[(\widetilde{E}_t a) \cdot b\big] = \widetilde{E}_t a \cdot \widetilde{E}_t b = \widetilde{E}_t\big[a \cdot (\widetilde{E}_t b)\big]$*.*

Proof. It is enough to consider basic elements and then to extend the result to $\mathcal{A}$. The proofs are straightforward, the details may be found in [7]. ∎

We shall now prove the following fundamental property of $\widetilde{E}_t$: $\widetilde{E}_t(aa^*) \geqslant 0$, $\forall\, a \in \mathcal{A}$ (with positivity in the operator sense).

First we introduce the notation we will use in the proof of this property. In particular:

(i) $|a|^2 = a^*a$, $\forall\, a \in \mathcal{A}$, and so $|a^*|^2 = aa^*$.

(ii) $\mathrm{R}(a) = \frac{a+a^*}{2}$, the real part of the operator a.

(iii) For any $a \in \mathcal{A}$ we write $a = \sum\limits_{i=1}^{n} a_i$ where a_i denotes the sum of those basic elements which have the same number, $s(i)$, of annihilation operators and where $0 \leqslant s(1) < s(2) < \cdots < s(n)$, $n \in \mathbb{N}$.

(iv) Further, we write $a_i = \sum\limits_{j=1}^{m(i)} a^{i,j}$ with

$$a^{i,j} = l^*(f_1^{s(i),j}) \cdots l^*(f_{r(i,j)}^{s(i),j}) p_{\nu_{i,j}}(g^{s(i),j}) l(h_{s(i)}^{s(i),j}) \cdots l(h_1^{s(i),j})$$

where $r(i,j)$ is the number of creation operators for the basic element $a^{i,j}$, $\nu_{i,j}$ ($= 0$ or 1) is the number of gauge operators of $a^{i,j}$, and $p_0(g^{s(i),j}) = I$; while $p_1(g^{s(i),j}) = p_(g^{s(i),j})$. Let us make some contractions of notation. So, much as before, we will write

$$\begin{aligned} L^*(f_{r(i,j)}^{s(i),j}) &= l^*(f_1^{s(i),j}) \cdots l^*(f_{r(i,j)}^{s(i),j}) \\ p_{\nu_{i,j}} &= p_{\nu_{i,j}}(g^{s(i),j}) \\ L(h_{s(i)}^{s(i),j}) &= l(h_{s(i)}^{s(i),j}) \cdots l(h_1^{s(i),j}). \end{aligned}$$

Notice that in the notation for L^* it is implicit that the index that counts the f's runs from 1 up to $r(i,j)$ while in L the index runs from $s(i)$ down to 1. We will need to consider different ranges for the counting indicies, so we will indicate these with the sufficies of the arguments of L^* and L. So, for example $L(h_{s(1),k}^{s(i),j})$ is the product of the annihilation operators $l(h_r^{s(i),j})$ with r running from $s(1)$ to k as we read left to right. It is worth noting that taking the adjoint of an L or L^* reverses the order of its counting index.

A preparatory lemma follows.

5.2. Lemma. *For $d \in \mathcal{A}$ with*

$$d = \sum_{i=1}^{N} \sum_{j=1}^{m(i)} l^*(f_1^{s(i),j}) \cdots l^*(f_{r(i,j)}^{s(i),j}) p_{\nu_{i,j}} (g^{s(i),j}) l(h_{s(i)}^{s(i),j}) \cdots l(h_1^{s(i),j}),$$

we have for any $0 \leqslant k \leqslant s(1)$,

$$dd^* \geqslant \left| \left\{ \sum_{i=1}^{N} \sum_{j=1}^{m(i)} L^*(f_{r(i,j)}^{s(i),j}) p_{\nu_{i,j}} L(h_{s(i),s(1)+1}^{s(i),j}) L(\chi_t h_{s(1),k+1}^{s(i),j} L(h_k^{s(i),j}) \right\}^* \right|^2.$$

Note that for $k = 0$ the term $L(h_k^{s(i),j})$ does not appear in the above expression.

Proof.

$$dd^* = \sum_{i=1}^{N} \sum_{i'=1}^{N} \sum_{j=1}^{m(i)} \sum_{j'=1}^{m(i')} \left(\prod_{r=1}^{s(1)} \langle h_r^{s(i'),j'}, h_r^{s(i),j} \rangle \right) L^*(f_{r_{(i,j)}}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i)}^{s(i),j})$$
$$\circ\, L^*(h_{s(i),s(1)+1}^{s(i'),j'}) L^*(h_{s(1)+1,s(i')}^{s(i'),j'}) p_{\nu_{i',j'}}^* L(f_{r_{(i',j')}}^{s(i'),j'}).$$

The term on the right hand side is equal to

$$\left| \left\{ \sum_{i=1}^{N} \sum_{j=1}^{m(i)} L^*(f_{r_{(i,j)}}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i),s(1)+1}^{s(i),j}) L(h_k^{s(i),j}) L(h_{s(1),k+1}^{s(i),j}) \right\}^* \right|^2$$
$$\geqslant \left| \left\{ \sum_{i=1}^{N} \sum_{j=1}^{m(i)} L^*(f_{r_{(i,j)}}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i),s(1)+1}^{s(i),j}) L(h_k^{s(i),j}) L(h_{s(1),k+1}^{s(i),j}) p(\chi_t) \right\}^* \right|^2.$$

The inequality is achieved simply by using the fact that for $a, b \in \mathcal{A}$ with $\|b\| \leqslant 1$, we have $ab^*ba^* \leqslant aa^*$. Now equation (i) of Section 2 tells us that $l(h_{k+1}^{s(i),j}) p(\chi_t) = l(\chi_t h_{k+1}^{s(i),j})$. Notice also that in the expression for dd^* and the last term, the annihilation operators $l(h_1^{s(i),j}) \cdots l(h_{s(1)}^{s(i),j})$ interact with each other to give inner product terms. As a consequence, the *order* of these terms can be varied (uniformly in every term of the double sum) without changing it. (There is a variant of Lemma 4.5 here.) So, by permuting the annihilation operators in $L(h_{s(1),k+2}^{s(i),j}) l(\chi_t h_{k+1}^{s(i),j})$ and using the simple operator inequality above to insert a $p(\chi_t)$, we can arrive at the inequality

$$dd^* \geqslant \left| \left\{ \sum_{i=1}^{N} \sum_{j=1}^{m(i)} L^*(f_{r_{(i,j)}}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i),s(1)+1}^{s(i),j}) L(h_k^{s(i),j}) L(\chi_t h_{s(1),k+1}^{s(i),j}) \right\}^* \right|^2.$$

By one last permutation of the annihilation operators we get

$$dd^* \geqslant \left| \left\{ \sum_{i=1}^{N} \sum_{j=1}^{m(i)} L^*(f_{r_{(i,j)}}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i),s(1)+1}^{s(i),j}) L(\chi_t h_{s(1),k+1}^{s(i),j}) L(h_k^{s(i),j}) \right\}^* \right|^2. \quad \blacksquare$$

5.3. Theorem. *For each $a \in \mathcal{A}$, $\widetilde{E}_t(aa^*) \geqslant 0$.*

Proof. Let $a = \sum_{i=1}^{n} \sum_{j=1}^{m(i)} a^{i,j}$ as before with

$$a^{i,j} = L^*(f_{r(i,j)}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i)}^{s(i),j}).$$

Define

$$b_k = \left| \left\{ \sum_{i=n-k+1}^{n} a_i \right\}^* \right|^2, \quad 1 \leqslant k \leqslant n,\, b_0 = 0,$$

and note that $b_n = aa^* = |a^*|^2$; furthermore, define

$${}^{(\mu)}a_i = \sum_{j=1}^{m(i)} L^*(\chi_t f_{r(i,j)}^{s(i),j)}) p_{\nu_{i,j}} L(\chi_t h_{s(i),s(n-\mu)+1}^{s(i),j}) L(h_{s(n-\mu)}^{s(i),j})$$

with $n - i < \mu \leqslant n - 1$, and

$${}^{(n-i)}a_i = \sum_{j=1}^{m(i)} L^*(\chi_t f_{r_{(i,j)}}^{s(i),j)}) p_{\nu_{i,j}} L(h_{s(i)}^{s(i),j}).$$

Finally, write $p_{\nu_{i,j}}(\chi_t)$ for $p_{\nu_{i,j}}(\chi_t g^{s(n),j})$. We now consider

$$\begin{aligned}
\widetilde{E}_t b_1 &= \widetilde{E}_t(a_n a_n^*) = \widetilde{E}_t \left\{ \sum_{j=1}^{m(n)} \sum_{j'=1}^{m(n)} a^{n,j} a^{n,j'*} \right\} \\
&= \widetilde{E}_t \left\{ \sum_{j=1}^{m(n)} \sum_{j'=1}^{m(n)} L^*(f_{r_{(n,j)}}^{s(n),j}) p_{\nu_{n,j}} L(h_{s(n)}^{s(n),j}) L^*(h_{s(n)}^{s(n),j'}) p_{\nu_{n,j'}}^* L(f_{r_{(n,j')}}^{s(n),j'}) \right\} \\
&= \sum_{j=1}^{m(n)} \sum_{j'=1}^{m(n)} \left(\prod_{r=1}^{s(n)} \langle h_r^{s(n),j'}, h_r^{s(n),j} \rangle \right) L^*(\chi_t f_{r_{(n,j)}}^{s(n),j}) p_{\nu_{n,j}}(\chi_t) p_{\nu_{n,j'}}^*(\chi_t) L(\chi_t f_{r(n,j')}^{s(n),j'}) \\
&= \sum_{j=1}^{m(n)} \sum_{j'=1}^{m(n)} L^*(\chi_t f_{r_{(n,j)}}^{s(n),j}) p_{\nu_{n,j}}(\chi_t) L(h_{s(n)}^{s(n),j}) L^*(h_{s(n)}^{s(n),j'}) p_{\nu_{n,j'}}(\chi_t) L(\chi_t f_{r_{(n,j')}}^{s(n),j'}) \\
&= \left| \left\{ \sum_{j=1}^{m(n)} L^*(\chi_t f_{r_{(n,j)}}^{s(n),j}) p_{\nu_{n,j}}(\chi_t) L(h_{s(n)}^{s(n),j}) \right\}^* \right|^2 = {}^{(0)}a_n \cdot {}^{(0)}a_n^*. \quad \blacksquare
\end{aligned}$$

We shall now show that:

$$\widetilde{E}_t(b_{\mu+1}) \geqslant \left| \left\{ {}^{(\mu)}a_n + {}^{(\mu)}a_{n-1} + \cdots + {}^{(\mu)}a_{n-\mu} \right\}^* \right|^2$$

for $0 \leqslant \mu \leqslant n - 1$. Indeed, we showed previously that:

$$\widetilde{E}_t(b_1) = {}^{(0)}a_n \cdot {}^{(0)}a_n^*$$

We shall show this by proving that

$$\begin{aligned}
&\widetilde{E}_t(b_\mu) \geqslant \left| \{ {}^{(\mu-1)}a_n + \cdots + {}^{(\mu-1)} a_{n-\mu+1} \}^* \right|^2 \\
\Rightarrow\ &\widetilde{E}_t(b_{\mu+1}) \geqslant \left| \{ {}^{(\mu)}a_n + \cdots + {}^{(\mu)} a_{n-\mu} \}^* \right|^2, \quad \text{for } 0 \leqslant \mu \leqslant n - 1.
\end{aligned}$$

To begin with

$$
\begin{aligned}
\widetilde{E}_t(b_{\mu+1}) &= \widetilde{E}_t\Big|\Big\{\sum_{i=n-\mu}^{n} a_i\Big\}^*\Big|^2 \\
&= \widetilde{E}_t\Big[\Big\{\Big(\sum_{i=n-\mu+1}^{n} a_i\Big) + a_{n-\mu}\Big\}\Big\{\Big(\sum_{i'=n-\mu+1}^{n} a_{i'}^*\Big) + a_{n-\mu}^*\Big\}\Big] \\
&= \widetilde{E}_t\Big\{b_\mu + 2\mathrm{R}\Big(a_{n-\mu}\cdot\sum_{i'=n-\mu+1}^{n} a_{i'}^*\Big) + |a_{n-\mu}^*|^2\Big\} \qquad (5.1)\\
&\geqslant \Big|\Big\{{}^{(\mu-1)}a_n + \cdots + {}^{(\mu-1)}a_{n-\mu+1}\Big\}^*\Big|^2 \\
&\quad + 2\mathrm{R}\Big\{\widetilde{E}_t\Big(a_{n-\mu}\cdot\sum_{i'=n-\mu+1}^{n} a_{i'}^*\Big)\Big\} + \widetilde{E}_t|a_{n-\mu}^*|^2
\end{aligned}
$$

using the hypothesis and because $\widetilde{E}_t\{\mathrm{R}(a)\} = \mathrm{R}(\widetilde{E}_t a)$. Consider first:

$$
\begin{aligned}
\widetilde{E}_t|a_{n-\mu}^*|^2 &= \widetilde{E}_t(a_{n-\mu}\cdot a_{n-\mu}^*) \\
&= \widetilde{E}_t\Big\{\sum_{j=1}^{m(n-\mu)}\sum_{j'=1}^{m(n-\mu)} L^*(f_{r(n-\mu,j)}^{s(n-\mu),j})p_{\nu_{n-\mu,j}}L(h_{s(n-\mu)}^{s(n-\mu),j}) \\
&\qquad \cdot L^*(h_{s(n-\mu)}^{s(n-\mu),j})p_{\nu_{n-\mu,j'}}^* L(f_{r(n-\mu,j')}^{s(n-\mu),j'})\Big\} \\
&= \sum_{j=1}^{m(n-\mu)}\sum_{j'=1}^{m(n-\mu)}\Big(\prod_{r=1}^{s(n-\mu)}\langle h_r^{s(n-\mu),j'}, h_r^{s(n-\mu),j}\rangle\Big) \\
&\qquad \cdot L^*(\chi_t f_{r_{(n-\mu,j)}}^{s(n-\mu),j})p_{\nu_{n-\mu,j}}(\chi_t)p_{\nu_{n-\mu,j'}}(\chi_t)^* L(\chi_t f_{r(n-\mu,j')}^{s(n-\mu),j'}) \\
&= \Big|\Big\{\sum_{j=1}^{m(n-\mu)} L^*(\chi_t f_{r(n-\mu,j)}^{s(n-\mu),j})p_{\nu_{n-\mu,j}}(\chi_t)L(h_{s(n-\mu)}^{s(n-\mu),j})\Big\}^*\Big|^2 \\
&= {}^{(\mu)}a_{n-\mu}\cdot{}^{(\mu)}a_{n-\mu}^* = |{}^{(\mu)}a_{n-\mu}^*|^2.
\end{aligned}
$$

Now, the following inequality follows from Lemma 5.2

$$
\big|\big\{{}^{(\mu-1)}a_n + \cdots + {}^{(\mu-1)}a_{n-\mu+1}\big\}^*\big|^2 \geqslant \big|\big\{{}^{(\mu)}a_n + \cdots + {}^{(\mu)}a_{n-\mu+1}\big\}^*\big|^2
$$

using the notation of that lemma, $d = {}^{(\mu-1)}a_n + \cdots + {}^{(\mu-1)}a_{n-\mu+1}$, $s(n-\mu+1)$ is the minimum number of annihilation operators of d, corresponding to $s(1)$ of Lemma 5.2, and $s(n-\mu)$ corresponds to k in Lemma 5.2.

Finally, we consider the term

$$
\widetilde{E}_t\Big\{a_{n-\mu}\cdot\sum_{i'=n-\mu+1}^{n} a_{i'}^*\Big\}
$$

which is equal to

$$\sum_{i'=n-\mu+1}^{n} \widetilde{E}_t\Big\{ \sum_{j=1}^{m(n-\mu)} L^*(f^{s(n-\mu),j}_{r(n-\mu,j)}) p_{\nu_{n-\mu,j}} L(h^{s(n-\mu),j}_{s(n-\mu)})$$

$$\cdot \sum_{j'=1}^{m(i')} L^*(h^{s(i'),j'}_{s(i')}) p^*_{\nu_{i',j'}} L(f^{s(i'),j'}_{r(i',j')}) \Big\}$$

$$= \sum_{i'=n-\mu+1}^{n} \sum_{j=1}^{m(n-\mu)} \sum_{j'=1}^{m(i')} \Big\{ \Big(\prod_{r=1}^{s(n-\mu)} \langle h^{s(i'),j'}_r, h^{s(n-\mu),j}_r \rangle \Big) L^*(\chi_t f^{s(n-\mu),j}_{r(n-\mu,j)}) p_{\nu_{n-\mu,j}}(\chi_t)$$

$$\cdot L^*(\chi_t h^{s(i'),j'}_{s(n-\mu)+1,s(i')}) p_{\nu_{i',j'}}(\chi_t)^* L(\chi_t f^{s(i'),j'}_{r(i',j')}) \Big\}$$

$$= \Big\{ \sum_{j=1}^{m(n-\mu)} L^*(\chi_t f^{s(n-\mu),j}_{r(n-\mu,j)} p_{\nu_{n-\mu,j}}(\chi_t) L(h^{s(n-\mu),j}_{s(n-\mu)}) \Big\}$$

$$\cdot \Big\{ \sum_{i'=n-\mu+1}^{n} \sum_{j'=1}^{m(i)} L^*(\chi_t f^{s(i'),j}_{r(i',j')}) p_{\nu_{i',j'}}(\chi_t) L(\chi_t h^{s(i'),j'}_{s(i'),s(n-\mu)}) L(h^{s(i'),j'}_{s(n-\mu)}) \Big\}^*$$

$$= {}^{(\mu)}a_{n-\mu} \cdot \sum_{i'=n-\mu+1}^{n} {}^{(\mu)}a^*_{i'}.$$

Hence inequality (5.1) becomes

$$\widetilde{E}_t(b_{\mu+1}) \geqslant \big|\big\{ {}^{(\mu)}a_n + \cdots {}^{(\mu)}a_{n-\mu+1} \big\}^*\big|^2 + 2\mathrm{R}\Big[{}^{(\mu)}a_{n-\mu} \cdot \sum_{i'=n-\mu+1}^{n} {}^{(\mu)}a^*_{i'} \Big] + \big| {}^{(\mu)}a^*_{n-\mu} \big|^2$$

$$= \big|\big\{ {}^{(\mu)}a_n + \cdots + {}^{(\mu)}a_{n-\mu} \big\}^*\big|^2$$

as required. Finally, using this result $(n-1)$-times beginning with $\widetilde{E}_t(b_1) = {}^{(0)}a_n \cdot {}^{(0)}a^*_n$ we get

$$\widetilde{E}_t(b_n) \geqslant \big|\big\{ {}^{(n-1)}a_n + \cdots + {}^{(n-1)}a_1 \big\}^*\big|^2.$$

Hence

$$\widetilde{E}_t(aa^*) = \widetilde{E}_t b_n \geqslant 0. \quad \blacksquare$$

5.4. Corollary. *If $b \in \mathcal{A}$ and $b \geqslant 0$, then $\widetilde{E}_t(b) \geqslant 0$.*

Proof. Let $\overline{\mathcal{A}}^{\tau_n}$ denote the norm closure of $\mathcal{A}$. For $b \in \mathcal{A}$, $b \geqslant 0$, there is an element $a \in \overline{\mathcal{A}}^{\tau_n}$ with $b = aa^*$, since $\overline{\mathcal{A}}^{\tau_n}$ is a C^*-algebra. Since $a \in \overline{\mathcal{A}}^{\tau_n}$ there is a sequence $(a_n)_{n=1}^{\infty}$ in $\mathcal{A}$ with $a_n \to a$ in τ_n-topology. Since multiplication and $*$-operation are τ_n-continuous $a_n \cdot a^*_n \to b$ in the τ_n-topology. Hence, $a_n a^*_n \to b$ pointwise on D^0 or $b - a_n \cdot a^*_n \to 0$ pointwise on D^0. Now by Theorem 4.8, $\widetilde{E}_t(b - a_n \cdot a^*_n) \to 0$ pointwise on D^0. By linearity, $\widetilde{E}_t(b - a_n \cdot a^*_n) \to 0$ pointwise on D. So $\forall u \in D$:

$$\langle \widetilde{E}_t bu, u \rangle = \lim_{n\to\infty} \langle \widetilde{E}_t(a_n \cdot a^*_n)u, u \rangle \geqslant 0$$

by Theorem 5.3.

Finally, for $h \in \mathcal{F}$, $\exists$ a sequence $(u_n)_{n=1}^{\infty}$ in $\operatorname{span} D$ with $u_n \to h$; then

$$\langle \widetilde{E}_t b h, h\rangle = \lim_{n\to\infty} \langle \widetilde{E}_t b u_n, u_n\rangle$$

since $\widetilde{E}_t b$ is a bounded operator and thus $\langle \widetilde{E}_t b h, h\rangle \geqslant 0$. Therefore $\widetilde{E}_t b \geqslant 0$. ∎

LEMMA 5.5. *For every $a \in \mathcal{A}$*

$$\widetilde{E}_t(a\cdot a^*) \geqslant \widetilde{E}_t a \cdot \widetilde{E}_t a^*,$$

and

$$\|\widetilde{E}_t a\| \leqslant \|a\|.$$

Proof. By Theorem 5.3, for the element $a - \widetilde{E}_t a$ we have

$$\widetilde{E}_t\big\{(a-\widetilde{E}_t a)\cdot(a-\widetilde{E}_t a)^*\big\} \geqslant 0$$

$$\Rightarrow \widetilde{E}_t\big\{a\cdot a^* - \widetilde{E}_t a\cdot a^* - a\cdot \widetilde{E}_t a^* + \widetilde{E}_t a\cdot \widetilde{E}_t a^*\big\} \geqslant 0$$

$$\Rightarrow \widetilde{E}_t(a\cdot a^*) - \widetilde{E}_t a\cdot \widetilde{E}_t a^* - \widetilde{E}_t a \widetilde{E}_t a^* + \widetilde{E}_t a\cdot \widetilde{E}_t a^* \geqslant 0$$

(by Theorem 5.1 (iii)–(iv))

$$\Rightarrow \widetilde{E}_t(a\cdot a^*) \geqslant \widetilde{E}_t a\cdot \widetilde{E}_t a^*$$

$$\Rightarrow \|\widetilde{E}_t(aa^*)\| \geqslant \|\widetilde{E}_t a\cdot \widetilde{E}_t a^*\|$$

(since for any two positive operators a, b in $\mathcal{A}$ with

$a \geqslant b$ we have $\|a\| = \sup_{\|h\|\leqslant 1}\langle ah, h\rangle \leqslant \sup_{\|h\|\leqslant 1}\langle bh, h\rangle = \|b\|$)

$$\Rightarrow \|\widetilde{E}_t(aa^*) \geqslant \|\widetilde{E}_t a\|^2$$

(since the bounded operators on $\mathcal{F}$ form a C^*-algebra).

Now $\|a\|^2\mathcal{I} \geqslant aa^*$ and $\widetilde{E}_t$ is positivity preserving so $\|a\| \geqslant \|\widetilde{E}_t a\|$. ∎

We can now extend the expectation to $\mathcal{V}$.

6. DEFINITION OF E_t

6.1. LEMMA. (i) *If $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ which is Cauchy in the τ_{s}-topology, then $(\widetilde{E}_t a_n)_{n=1}^{\infty}$ is also Cauchy in the τ_{s}-topology.*

(ii) *If $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ such that $a_n \to 0$ in the τ_{s}-topology, then $\widetilde{E}_t a_n \to 0$ in the τ_{s}-topology.*

Proof. (i) As (a_n) is τ_{s} Cauchy on $\mathcal{F}$ it is certainly Cauchy τ_{s}-on D. By Theorem 4.8, $(\widetilde{E}_t a_n)_{n=1}^{\infty}$ is Cauchy τ_{s}-on D. By Banach-Steinhaus Theorem

$$\sup_n \|a_n\| < \infty$$

and by Lemma 5.5

$$\sup_n \|\widetilde{E}_t a_n\| \leqslant \sup_n \|a_n\| < \infty.$$

So we have a uniformly bounded sequence of operators pointwise convergent on a linear set dense in $\mathcal{F}$. It follows that $(\widetilde{E}_t a_n)_{n=1}^{\infty}$ is Cauchy in the τ_{s}-topology.

(ii) The proof of this is similar to (i). ∎

6.2. LEMMA. *If $x \in \mathcal{V}$, $\exists (a_n)_{n=1}^{\infty}$ in $\mathcal{A}$ such that $a_n \to x$ in the τ_s-topology.*

Proof. Kaplansky's density theorem tells us that any operator T in $\mathcal{V}_1$ is the τ_s-limit of a net of operators from $\mathcal{A}$. Since in our case the τ_s-topology is metrisable on bounded subsets of $B(\mathcal{F})$ we can choose a subsequence from this net converging τ_s to T. ∎

6.3. DEFINITION. Let $x \in \mathcal{V}$, by Lemma 6.2 we can choose an $(a_n)_{n=1}^{\infty}$ in $\mathcal{A}$ with $a_n \to x$ in the τ_s-topology. By Theorem 4.8 (i), since (a_n) is τ_s Cauchy then so is $\big(\widetilde{E}_t(a_n)\big)$. Since $\mathcal{V}_t$ is complete for the τ_s-topology it follows that $\widetilde{E}_t a_n$ converges to an element of $\mathcal{V}_t$. Define

$$E_t x = \lim \widetilde{E}_t a_n \in \mathcal{V}_t.$$

Suppose now that (b_n) is another sequence from $\mathcal{A}$ converging τ_s to x. Then $a_n - b_n \to 0$ and $(a_n - b_n)$ is a sequence in $\mathcal{A}$. By Theorem 4.8 (ii), $\widetilde{E}_t(a_n - b_n) \to 0$ in the τ_s-topology, or $\widetilde{E}_t b_n \to E_t x$. So E_t is well defined. Finally, we note that if $x \in \mathcal{A}$, $E_t x = \widetilde{E}_t x$.

6.4. LEMMA. (Properties if E_t) (i) *E_t is a linear map from $\mathcal{V}$ onto $\mathcal{V}_t$;*
(ii) *$(E_t x)^* = E_t x^*$, $\forall x \in \mathcal{V}$;*
(iii) *$E_t^2 = E_t$;*
(iv) *$E_t[(E_t x) \cdot y] = E_t x \cdot E_t y = E_t[x \cdot (E_t y)]$, $\forall x, y \in \mathcal{V}$;*
(v) *$E_t x \geqslant 0$, $\forall x \in \mathcal{V}_+$;*
(vi) *$\|E_t x\| \leqslant \|x\|$, $\forall x \in \mathcal{V}$.*
These properties show that E_t is a conditional expectation.

Proof. Items (i), (ii), (iii) are proved in the obvious way.
(iv) Suppose $x, y \in \mathcal{V}$ with $a_n \to x$, $b_n \to y$ in the τ_s-topology and $(a_n), (b_n) \subseteq \mathcal{A}$. Then, by Banach–Steinhaus Theorem

$$\sup_n \|a_n\|_\infty < \infty, \qquad \sup_n \|b_n\|_\infty < \infty.$$

Furthermore, $\widetilde{E}_t a_n \to E_t a$ in the τ_s-topology, and $\sup_n \|\widetilde{E}_t a_n\| < \infty$. Now,

$$E_t\big[(E_t x) \cdot y\big] = E_t\big[\big\{\lim_{n\to\infty} (\widetilde{E}_t a_n)\big\} \cdot \big\{\lim_{n\to\infty} b_n\big\}\big]$$
$$= E_t\big[\lim_{n\to\infty} \big(\widetilde{E}_t a_n \cdot b_n\big)\big]$$

(since multiplication is continuous in the τ_s-topology)

$$= \lim_{n\to\infty} \widetilde{E}_t\big[(\widetilde{E}_t a_n) \cdot b_n\big]$$

(by definition of E_t and since $(\widetilde{E}_t a_n) \cdot b_n \in \mathcal{A}$, $\forall n \in \mathbb{N}$)

$$= \lim_{n\to\infty} \big[(\widetilde{E}_t a_n) \cdot (\widetilde{E}_t b_n)\big]$$

(properties of $\widetilde{E}_t$)

$$= \big(\lim_{n\to\infty} \widetilde{E}_t a_n\big) \cdot \big(\lim_{n\to\infty} \widetilde{E}_t b_n\big)$$

(continuity of multiplication in the τ_s-topology on bounded sets)

$$= E_t x \cdot E_t y.$$

Furthermore,

$$\begin{aligned}\{E_t[x \cdot (E_t y)]\}^* &= E_t\{[x \cdot E_t y]^*\} \quad \text{(by (ii))}\\ &= E_t\{E_t y^* \cdot x^*\} \quad \text{(by (ii))}\\ &= E_t y^* \cdot E_t x^* \quad \text{(by above)}\\ &= [E_t x \cdot E_t y]^* \quad \text{(by (ii))}\end{aligned}$$

$$E_t[E_t(x) \cdot y] = E_t x \cdot E_t y = E_t[x \cdot (E_t y)], \quad \forall x, y \in \mathcal{V}.$$

Item (v) uses the fact that $\forall x \in \mathcal{V}_+$, $\exists y = y^* \in \mathcal{V}$ such that $y^2 = x$. Approximating y with a sequence (a_n) of Hermitian elements from $\mathcal{A}$, we see that (a_n^2) approximates x, and each a_n^2 is a positive operator.

$$\widetilde{E}_t(a_n^2) \to E_t(y^2) = E_t x$$

in the τ_s-topology. So $E_t x$ is a strong operator limit of positive operators, so it is positive.

(vi) follows from (v) with x replased by $(x - E_t x) \cdot (x - E_t x)^*$. ∎

6.5. Theorem. *E_t is τ_s-continuous on bounded sets of $\mathcal{V}$.*

Proof. It suffices to show that:

$$E_t : (\mathcal{V}_1, \tau_s\text{-topology}) \to (\mathcal{V}_1, \tau_s\text{-topology})$$

is continuous. ($\|E_t x\| \leqslant \|x\|$ so $E_t(\mathcal{V}_1) \subset \mathcal{V}_1$.)

By Lemma 2.3, $(\mathcal{V}_1, \tau_s\text{-topology})$ is metrisable, and we shall denote this metric by ρ.

So all we need to show is that if $x_n \to x$ in ρ in $\mathcal{V}_1$ then $E_t x_n \to E_t x$ in ρ.

By definition of $E_t x_n$, $\exists a_n \in \mathcal{A}$ such that:

$$\rho(x_n, a_n) < \frac{1}{n} \quad \text{and} \quad \rho(E_t x_n, E_t a_n) < \frac{1}{n}.$$

Hence

$$\rho(a_n, x) \leqslant \rho(a_n, x_n) + \rho(x_n, x) \to 0.$$

So $a_n \to x$ in the ρ-topology.

By the definition of $E_t x$, $E_t a_n \to E_t x$ in ρ.

Hence:

$$\rho(E_t x_n, E_t x) \leqslant \rho(E_t x_n, E_t a_n) + \rho(E_t a_n, E_t x) \to 0$$

as required. ∎

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