J. OPERATOR THEORY
44(2000), 25–41

THE CLOSURE OF THE UNITARY ORBIT OF THE SET OF STRONGLY IRREDUCIBLE OPERATORS IN NON-WELL ORDERED NEST ALGEBRA

YOU QING JI, CHUN LAN JIANG and ZONG YAO WANG

Communicated by Norberto Salinas

Abstract. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is strongly irreducible if T does not commute with any non-trivial idempotent. A nest $\mathcal{N}$ is a chain of subspaces of H contain $\{0\}$ and $\mathcal{H}$, which is closed under intersection and closed span. The nest algebra $\mathrm{alg}\,\mathcal{N}$ associated with $\mathcal{N}$ is the set of all operators which leave each subspace in $\mathcal{N}$ invariant. This paper proves that the norm closure of the unitary orbit of the strongly irreducible operators in a nest algebra is the set of operators whose spectrum is connected if and only if $\mathcal{N}$ or $\mathcal{N}^{\perp}$ are not well-ordered.

Keywords: *Strongly irreducible operator, nest, nest algebra, unitary orbit, spectrum.*

MSC (2000): 47A, 47B, 47C.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space. $\mathcal{L}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator T on $\mathcal{H}$ is called *strongly irreducible*, or briefly, $T \in$ (SI), if T does not commute with any nontrivial idempotent. A *nest* is a chain $\mathcal{N}$ of subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$, which is closed under intersection and closed span. It is well known that for a nest $\mathcal{N}$ there is a spectral measure $E(t)$ on $[0,1]$, such that $\mathcal{N} = \{E([0,t])\mathcal{H}; t \in [0,1]\}$ and the compact subset $\mathrm{supp}E$ of $[0,1]$ is order-isomorphic to and topologically homeomorphic to $\mathcal{N}$ when $\mathcal{N}$ is given the order topology and $\mathrm{supp}E$ has the order and the related topology induced on it by the usual topology of the real line. In what follows we will denote $M_{[c,d]} = E([c,d])\mathcal{H}$ when $[c,d] \subset [0,1]$ and $M_t = M_{[0,t]}$. For each $M \in \mathcal{N}$, let $M_- = \bigcup\{M' \in \mathcal{N} : M' \not\subseteq M\}$. If $M_- \neq M$, $M \ominus M'$ is called an *atom* of $\mathcal{N}$ and the cardinal number $\dim M \ominus M_-$ is called the dimension of the atom. A nest is called continuous if it has no atoms. The nest algebra $\mathrm{alg}\,\mathcal{N}$

associated with $\mathcal{N}$ is the family of operators defined by $\operatorname{alg}\mathcal{N} = \{T \in \mathcal{L}(\mathcal{H}) : TM \subset M \text{ for all } M \in \mathcal{N}\}$.

D.A. Herrero proved the following theorem ([7]):

THEOREM H. (i) *If $\mathcal{N}$ is well ordered with finite dimensional atoms, then* $\mathcal{U}(\operatorname{alg}\mathcal{N})^- = (\mathrm{QT})$.

(ii) *If $\mathcal{N}^\perp$ is well ordered with finite dimensional atoms, then* $\mathcal{U}(\operatorname{alg}\mathcal{N})^- = (\mathrm{QT})^*$.

(iii) *If neither* (i) *nor* (ii) *holds, then*

$$\mathcal{U}(\operatorname{alg}\mathcal{N})^- = \mathcal{L}(\mathcal{H}) \quad \text{when } d = \infty, \quad \mathcal{U}(\operatorname{alg}\mathcal{N})^- = \mathcal{L}(\mathcal{H})_d \quad \text{when } d < \infty,$$

where $\mathcal{U}(\operatorname{alg}\mathcal{N})^-$ is the norm closure of the unitary orbit $\mathcal{U}(\operatorname{alg}\mathcal{N})$ of $\operatorname{alg}\mathcal{N}$, (QT) *is the set of quasitriangular operators on $\mathcal{H}$,* $(\mathrm{QT})^* := \{T \in \mathcal{L}(\mathcal{H}) : T^* \in (\mathrm{QT})\}$, $d = \sum\limits_{A\in\Lambda} \dim A$, *$\Lambda$ denotes the set of atoms of $\mathcal{N}$,*

$$\mathcal{L}(\mathcal{H})_d = \Big\{T \in \mathcal{L}(\mathcal{H}) : \sum_{\lambda\in\sigma_0(T)\setminus\sigma_{\mathrm{e}}(T)^\wedge} \dim \mathcal{H}(\lambda, T) \leqslant d\Big\},$$

$\sigma_0(T)$ is the set of normal eigenvalues of T, $\sigma_{\mathrm{e}}(T)^\wedge$ is the polynormally convex hull of the essential spectrum $\sigma_{\mathrm{e}}(T)$ of T and $\mathcal{H}(\lambda, T)$ is the Riesz spectral subspace of T associated with λ.

In [12], the authors of this paper proved that each nest algebra contains strongly irreducible operators, i.e., $\operatorname{alg}\mathcal{N} \cap (\mathrm{SI}) \neq \emptyset$. Furthermore, the authors proved that $\mathcal{U}(\operatorname{alg}\mathcal{N} \cap (\mathrm{SI}))^- = (\mathrm{QT})_{\mathrm{C}}$ if $\mathcal{N}$ is a well ordered nest, where

$$(\mathrm{QT})_{\mathrm{C}} := \{T \in (\mathrm{QT}) : \sigma(T) \text{ and the Weyl spectrum, } \sigma_{\mathrm{W}}(T) \text{ of } T \text{ are connected}\}$$

(see [13]) and $\mathcal{U}(\operatorname{alg}\mathcal{N} \cap (\mathrm{SI}))^- = \{T \in \mathcal{L}(\mathcal{H}) : \sigma(T) \text{ is connected}\}$ if $\mathcal{N}$ is a continuous nest [14]. The following is the main result of this paper.

THEOREM 1.1. *Let $\mathcal{N}$ be a maximal nest. Then $\mathcal{U}(\operatorname{alg}\mathcal{N} \cap (\mathrm{SI}))^- = \{T \in \mathcal{L}(\mathcal{H}) : \sigma(T)$ is connected$\}$ if and only if $\mathcal{N}$ and $\mathcal{N}^\perp$ are not well-ordered.*

2. PREPARATION

LEMMA 2.1. ([11], Lemma 2) *Let $A, B \in \mathcal{L}(\mathcal{H})$. Assume that*

$$\mathcal{H} = \bigvee\{\ker(\lambda - B)^k : \lambda \in \Gamma, k \geqslant 1\}$$

for a certain subset Γ of the point spectrum $\sigma_{\mathrm{p}}(B)$ of B, and $\sigma_{\mathrm{p}}(A) \cap \Gamma = \emptyset$; then τ_{AB} is injective.

LEMMA 2.2. *Let σ be the closure of a connected Cauchy domain and Ω is an open disc in σ. Then there exists an operator $A \in \mathcal{L}(\mathcal{H}) \cap (\mathrm{SI})$ such that:*

(i) *$\sigma(A) = \sigma_{\mathrm{lre}}(A) = \sigma$;*

(ii) *$\sigma_{\mathrm{p}}(A) = \Omega$, $\operatorname{nul}(A - \lambda) = 1 (\lambda \in \Omega)$, and $\sigma_{\mathrm{p}}(A^*) = \emptyset$;*

(iii) *If $\{\lambda_k\}_{k=1}^\infty \subset \Omega$, pairwise distinct and $\lim\limits_{k\to\infty} \lambda_k = \lambda_0 \in \Omega$, then $\bigvee\{\ker(A - \lambda_k) : k \geqslant 1\} = \mathcal{H}$;*

(iv) $\|(A-\lambda)^{-1}\| \leqslant 2/\mathrm{dist}\,(\lambda,\sigma)$ *for* $\lambda \notin \sigma$.

Proof. Without loss of generality we may assume that Ω is the unit disc. Let S be the backward lateral shift, i.e., $S^* = T_z^* \in \mathcal{L}(\mathcal{H}_1)$, where $\mathcal{H}_1$ is the Hardy space H^2. Let M be a diagonal operator on $\mathcal{H}_1$ with $\sigma(M) = \sigma_{\mathrm{lre}}(M) = \sigma$. Set $T = S^* \oplus M$. By a result of J. Agler, E. Franks and D.A. Herrero ([1]), for each $\varepsilon > 0$, there is a compact operator K, $\|K\| < \varepsilon$, such that $A = T + K$ is quasisimilar to $T_z^* \in \mathcal{B}_1(\Omega)$. By a result of C.L. Jiang ([15]), $A \in$ (SI). Choose ε small enough, then A satisfies (i)–(iv). ∎

THEOREM 2.3. ([9], Theorem 3.53) *Let* $A, B \in \mathcal{L}(\mathcal{H})$, *then the following are equivalent for* τ_{AB}:

(i) τ_{AB} *is surjective;*
(ii) $\sigma_{\mathrm{r}}(A) \cap \sigma_{\mathrm{l}}(B) = \emptyset$;
(iii) $\mathrm{ran}\,\tau_{AB}$ *contains the set of finite rank operators;*
(iv) $\tau_{AB}|J$ *is surjective for every norm ideal* J;

where $\tau_{AB} \in \mathcal{L}(\mathcal{L}(\mathcal{H}))$ *is given by* $\tau_{AB}(X) = AX - XB$ *for* $X \in \mathcal{L}(\mathcal{H})$.

LEMMA 2.4. *Let* σ *be the closure of a connected Cauchy domain and* Ω *be a connected open subset of* σ. *Then there exists an operator* $W \in \mathcal{L}(\mathcal{H}) \cap$ (SI) *satisfying:*

(i) $\sigma(W) = \sigma_{\mathrm{lre}}(W) = \sigma$;
(ii) $\sigma_{\mathrm{p}}(W) \subset \Omega$, $\sigma_{\mathrm{p}}(W^*) = \emptyset$;
(iii) *There exists* $\{\lambda_k\}_{k=1}^{\infty} \subset \Omega$ *such that* $\lim\limits_{k\to\infty} \lambda_k = \lambda_0 \in \Omega$, $\mathrm{nul}\,(W-\lambda_k) = \infty$ $(k \geqslant 1)$ *and* $\bigvee\{\ker(W-\lambda_k) : k \geqslant 1\} = \mathcal{H}$.

Proof. Choose a sequence $\{D_n\}_{n=0}^{\infty}$ of open discs in Ω satisfying $D_n \backslash \overline{D}_m \neq \emptyset$ $(n \neq m,\ n \neq 0)$ and $D_0 \subset \bigcap\limits_{n=1}^{\infty} D_n$.

Without loss of generality we may assume that D_0 is the unit disc and $D_1 = \alpha_1 + rD_0$. Let $S^* = T_z^* \in \mathcal{L}(\mathcal{H}_1)$, where $\mathcal{H}_1 = H^2$. Set $A_1 = \alpha_1 + rS^*$. Let $\mathcal{H} = \bigoplus\limits_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n = \mathcal{H}_1$ $(n \geqslant 2)$. For each $n \geqslant 2$, by Lemma 2.2, we can construct $A_n \in \mathcal{L}(\mathcal{H}_n) \cap$ (SI) satisfying:

(a) $\sigma(A_n) = \sigma_{\mathrm{lre}}(A_n) = \sigma$, $\sigma_{\mathrm{p}}(A_n) = D_n$, $\sigma_{\mathrm{p}}(A_n^*) = \emptyset$ and $\mathrm{nul}\,(A_n - \lambda) = 1$ for $\lambda \in D_n$;

(b) If $\{\mu_k\}_{k=1}^{\infty} \subset D_n$, pairwise distinct and $\lim\limits_{k\to\infty} \mu_k = \mu_0 \in D_n$, then $\bigvee\{\ker(A_n - \mu_k) : k \geqslant 1\} = \mathcal{H}_n$;

(c) $\|(A_n - \lambda)^{-1}\| \leqslant \frac{2}{\mathrm{dist}\,(\lambda,\sigma)}$ for $\lambda \notin \sigma$.

It follows from $D_n \setminus \overline{D}_m \neq \emptyset$, (b) and Lemma 2.1 that $\ker \tau_{A_n A_m} = \{0\}$ $(n \neq m)$. Since $\sigma_{\mathrm{r}}(A_1) \cap \sigma_{\mathrm{l}}(A_n) \neq \emptyset$, by Theorem 2.3, we can find a compact operator $W_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_1)$, $\|W_n\| < 2^{-n}$, such that $W_n \notin \mathrm{ran}\,\tau_{A_1 A_n}$ $(n \geqslant 2)$.

Define

$$W = \begin{bmatrix} A_1 & W_2 & W_3 & \dots \\ & A_2 & & 0 \\ & & A_3 & \\ 0 & & & \ddots \end{bmatrix} \in \mathcal{L}(\mathcal{H}).$$

Let $P \in \mathcal{A}'(W)$ be an idempotent and consider the representation

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \dots \\ P_{21} & P_{22} & P_{23} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

Since $PW = WP$, then $A_2P_{21} = P_{21}A_1$. Moreover, $\ker \tau_{A_2A_1} = \{0\}$ implies that $P_{21} = 0$. Similarly, $P_{lk} = 0$ $(l > k)$. Thus $P_{ll}A_l = A_lP_{ll}$ and $P_{ll}^2 = P_{ll}$ $(l = 1, 2, \dots)$. Since $A_l \in$ (SI), $P_{ll} = 0$ or 1 $(l = 1, 2, \dots)$. Assume that $P_{11} = 0$ (otherwise, consider $1 - P$). If $P_{22} = 1$, $W_2 \in \operatorname{ran} \tau_{A_1A_2}$, a contradiction. Thus $P_{22} = 0$ and therefore $P_{12} = 0$. By the same argument, $P_{ll} = 0$ $(l = 3, 4, \dots)$ and $P = 0$, i.e., $W \in (\mathrm{SI})(\mathcal{H})$. Let $\{\lambda_k\}_{k=1}^{\infty} \subset D_0$ be an arbitrary sequence such that $\lim\limits_{k\to\infty} \lambda_k = \lambda_0 \in D_0$, pairwise distinct, then $\bigvee\Big\{\ker\big(\bigoplus\limits_{n=2}^{\infty} A_n - \lambda_k\big) : k \geqslant 1\Big\} = \bigoplus\limits_{n=2}^{\infty} \mathcal{H}_n$ and $\bigvee\{\ker(A_1 - \lambda_k) : k \geqslant 1\} = \mathcal{H}_1$. Note that $\{\lambda_k\}_{k=1}^{\infty} \subset \rho_r(A_1)$, thus $\bigvee\{\ker(W - \lambda_n) : n \geqslant 1\} = \mathcal{H}$ and $\operatorname{nul}(W - \lambda_n) = \infty$ $(n = 0, 1, 2, \dots)$. Since $\sigma_{\mathrm{p}}(A_k) \subset D_k$ and $\sigma_{\mathrm{p}}(A_k^*) = \emptyset$ $(k = 1, 2, \dots)$, computation indicates that $\sigma_{\mathrm{p}}(W) \subset \Omega$ and $\sigma_{\mathrm{p}}(W^*) = \emptyset$. Observe that $W = \bigoplus\limits_{n=1}^{\infty} A_n + K$, where K is a compact operator and $\|(A_n - \lambda)^{-1}\| < \frac{2}{\operatorname{dist}(\lambda,\sigma)}$ for $\lambda \notin \sigma$ and $n \geqslant 1$, we have $\sigma\Big(\bigoplus\limits_{n=1}^{\infty} A_n\Big) = \sigma_{\mathrm{lre}}\Big(\bigoplus\limits_{n=1}^{\infty} A_n\Big) = \sigma$. Since $\sigma(W)$ is connected and $\sigma_{\mathrm{p}}(W^*) = \emptyset$, $\sigma(W) = \sigma_{\mathrm{lre}}(W) = \sigma$. ∎

EXAMPLE 2.5. ([10]) Define $\gamma_1 = 1$, $\gamma_2 = \frac{1}{4}$, $\gamma_3 = (\gamma_1\gamma_2)^3, \dots, \gamma_n = (\gamma_1 \cdots \gamma_{n-1})^n, \dots$, and let $\{\alpha_n\}$ be the sequence

$$\gamma_1, \gamma_2, \dots, \gamma_9, \gamma_1, \gamma_2, \dots, \gamma_{90}, \gamma_1, \gamma_2, \dots, \gamma_{900}, \gamma_1, \gamma_2, \dots, \gamma_{9000}, \gamma_1, \dots.$$

Let V be the unilateral weighted shift defined by $Ve_n = \alpha_n e_{n+1}$ $(n \geqslant 1)$ with respect to an ONB$\{e_n\}_{n=1}^{\infty}$ of the Hilbert space $\mathcal{H}$. Then V is a quasinilpotent unicellular operator and V^k is not compact for all $k = 1, 2, \dots$.

THEOREM 2.6. ([8]) *Let $R \in \mathcal{L}(\mathcal{H})$ satisfy:*
(i) *$\sigma(R)$ and $\sigma_{\mathrm{W}}(R)$ are connected and contain a connected open set Ω;*
(ii) *$\operatorname{ind}(\lambda - R) \geqslant 0$ for all $\lambda \in \rho_{\text{s-F}}(R)$ (i.e., R is a quasitriangular operator);*
(iii) *$\rho_{\text{s-F}}(R) \supset \Omega$ and $\operatorname{ind}(\lambda - R) = n$ for all $\lambda \in \Omega$.*
Then for $\varepsilon > 0$, there exists a compact operator K_ε, $\|K_\varepsilon\| < \varepsilon$, such that $R - K_\varepsilon \in \mathcal{B}_n(\Omega)$ (see the next definition).

DEFINITION 2.7. Let Ω be a bounded connected open set in $\mathbb{C}$, n is a positive integer or ∞. The set $\mathcal{B}_n(\Omega)$ of *Cowen-Douglas operators of index n* is the set of operators B in $\mathcal{L}(\mathcal{H})$ satisfying:
(i) $\sigma(B) \supset \Omega$;
(ii) $\operatorname{ran}(\lambda - B) = \mathcal{H}$ for all $\lambda \in \Omega$;
(iii) $\operatorname{nul}(\lambda - B) = n$ for all $\lambda \in \Omega$;
(iv) $\bigvee\big\{\ker(\lambda - B) : \lambda \in \Omega\big\} = \mathcal{H}$.

Note that (iv) can be replaced by (iv)$'$ or (iv)$''$ ([3]):
(iv)$'$ $\bigvee\{\ker(\lambda_0 - B)^k : k \geqslant 1\} = \mathcal{H}$ for each $\lambda_0 \in \Omega$.

(iv)$''$ $\bigvee\{\ker(\lambda_n - B) : n \geqslant 1\} = \mathcal{H}$ for all sequences $\{\lambda_n\}_{n=0}^{\infty} \subset \Omega$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda_0$.

Consider $B_1, B_2 \in \mathcal{B}_1(\Omega)$, $(0 \in \Omega)$. By Lemma 2.2 of [17], B_1 and B_2 admit the following matrix representations

$$B_1 = \begin{bmatrix} 0 & b^1_{12} & & & * \\ & 0 & b^1_{23} & & \\ & & 0 & b^1_{34} & \\ & & & 0 & \ddots \\ 0 & & & & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \vdots \end{matrix}, \quad B_2 = \begin{bmatrix} 0 & b^2_{12} & & & * \\ & 0 & b^2_{23} & & \\ & & 0 & b^2_{34} & \\ & & & 0 & \ddots \\ 0 & & & & \ddots \end{bmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{matrix},$$

where $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are ONB's of $\mathcal{H}$, and $|b^i_{nn+1}| > r > 0$ $(i = 1, 2;\ n = 1, 2, \ldots)$ for some r.

Define $r(B_1, B_2) = \overline{\lim} \Big[\prod\limits_{k=1}^{n} \Big| \frac{b^1_{kk+1}}{b^2_{kk+1}} \Big| \Big]^{\frac{1}{n}}$.

PROPOSITION 2.8. (i) *If* $r(B_1, B_2) > 1$, *then* $\ker \tau_{B_2 B_1} = \{0\}$.

(ii) *If* $r(B_1, B_2) = 1$, *then given* $\varepsilon > 0$ $(\varepsilon < r)$, *there exists a compact operator* K *satisfying:*

(a) $\|K\| < \varepsilon$;

(b) $\ker \tau_{B_1, B_2 + K} = \ker \tau_{B_2 + K, B_1} = \{0\}$;

(c) $B_2 + K \in \mathcal{B}_1(\Omega)$ *and* $r(B_1, B_2 + K) = 1$.

Proof. (ii) Denote $d_i = 1 - \varepsilon/2^i$ $(i = 1, 2, \ldots)$. Since

$$\overline{\lim_{n\to\infty}} \Big[\prod_{k=1}^{n} \frac{b^1_{kk+1}}{b^2_{kk+1} d_1} \Big]^{\frac{1}{n}} = d_1 > 1,$$

there exists n_1 such that

$$\prod_{k=1}^{n_1} \frac{b^1_{kk+1}}{b^2_{kk+1} d_1} > 2.$$

Set $\beta_k = 1 - d_1$ $(1 \leqslant k \leqslant n_1)$. Since

$$\overline{\lim_{n\to\infty}} \Big[\Big(\prod_{k=1}^{n_1} \frac{b^1_{kk+1}}{b^2_{kk+1}(1 - \beta_k)} \Big) \Big(\prod_{k=n_1+1}^{n} \frac{b^1_{kk+1} d_2}{b^2_{kk+1}} \Big) \Big]^{\frac{1}{n}} = d_2 < 1,$$

we can find $n_2 > n_1$ such that

$$\prod_{k=1}^{n_1} \frac{b^1_{kk+1}}{b^2_{kk+1}(1 - \beta_k)} \cdot \prod_{k=n_1+1}^{n_2} \frac{b^1_{kk+1} d_2}{b^2_{kk+1}} < \frac{1}{2}.$$

Set $\beta_k = 1 - 1/d_2$ $(n_1 + 1 \leqslant k \leqslant n_2)$. Inductively, we can define

$$\beta_k = \begin{cases} 1 - d_{2l-1}, & n_{2l-2} + 1 \leqslant k \leqslant n_{2l-1}, \\ 1 - \frac{1}{d_{2l}}, & n_{2l-1} + 1 < k \leqslant n_{2l}, \end{cases}$$

such that

$$\prod_{k=1}^{n_{2l-1}} \frac{b^1_{kk+1}}{b^2_{kk+1}(1 - \beta_k)} > 2^l, \quad \prod_{k=1}^{n_{2l}} \frac{b^1_{kk+1}}{b^2_{kk+1}(1 - \beta_k)} < 2^{-l}, \qquad l = 1, 2, \ldots, \tag{2.1}$$

and $\lim\limits_{k\to\infty}\beta_k = 0$ and $\sup\limits_k |\beta_k| < \frac{\varepsilon}{2}$.

Define $K'e_k = -b^2_{kk+1}\beta_k e_{k-1}$ $(k = 2, 3, \ldots)$ and $K'e_1 = 0$. Then K' is compact and $\|K'\| < \varepsilon/2$. It is easily seen that $B'_2 + K' \in \mathcal{B}_1(\Omega)$. If $B'_1 X = X(B'_2 + K')$ for some $X \in \mathcal{L}(\mathcal{H})$, we can prove that

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ & x_{22} & \cdots \\ 0 & & \ddots \end{bmatrix}$$

with respect to $\{e_n\}$ and

$$x_{nn} = \prod_{k=1}^{n-1} \frac{b^2_{kk+1}(1-\beta_k)}{b^1_{kk+1}} x_{11}, \quad n = 1, 2, \ldots.$$

By (2.1), $x_{nn} = 0$ $(n = 1, 2, \ldots)$. Similarly, a computation indicates that

$$x_{nn+1} = \frac{b^1_{nn+1}}{b^2_{12}(1-\beta_1)} \prod_{k=1}^{n} \frac{b^2_{kk+1}(1-\beta_k)}{b^1_{kk+1}} x_{12}, \quad k = 2, 3, \ldots.$$

By (2.1), $x_{nn+1} = 0$ $(n = 1, 2, \ldots)$. Generally, we can prove that $x_{ij} = 0$ $(i < j)$ and therefore, $\ker \tau_{B'_1 B'_2 + K'} = \{0\}$. By the same argument, $\ker \tau_{B'_2 + K' B'_1} = \{0\}$. From the definition of $\{\beta_k\}$, it is easy to see that $r(B'_1, B'_2 + K') = 1$. Since $B_1 \simeq B'_1$ and $B_2 \simeq B'_2$, we can find a compact operator K satisfies all requirements of (ii).

(i) If $r(B_1, B_2) > 1$, then there is a subsequence $\{n_i\}_{i=1}^{\infty}$ of natural numbers such that $n_1 < n_2 < \cdots$ and

$$\prod_{k=1}^{n_k} \frac{b^1_{kk+1}}{b^2_{kk+1}} > k, \quad k = 1, 2, \ldots.$$

By the same argument of (ii), $\ker \tau_{B_2 B_1} = \{0\}$. ∎

Let Ω be a non-empty bounded open subset of $\mathcal{C}$ with $(\overline{\Omega})^\circ = \Omega$. Let $N(\Omega)$ be the "multiplication by λ" operator acting on $L^2(\Omega, \mathrm{d}m)$. The subspace $A^2(\Omega)$ spanned by the rational functions with poles outside $\overline{\Omega}$ is invariant under $N(\Omega)$. By $N_+(\Omega)$ and $N_-(\Omega)$ we shall denote the restriction of $N(\Omega)$ to $A^2(\Omega)$ and its compression to $L^2(\Omega, \mathrm{d}m) \ominus A^2(\Omega)$, respectively, i.e.,

$$N(\Omega) = \begin{bmatrix} N_+(\Omega) & G \\ 0 & N_-(\Omega) \end{bmatrix} \begin{matrix} A^2(\Omega) \\ L^2(\Omega, \mathrm{d}A) \ominus A^2(\Omega) \end{matrix},$$

where $N_+(\Omega)$ is called *Bergmann operator*.

Lemma 2.9. *Consider a connected compact subset σ of $\mathbb{C}$ and pairwise disjoint connected open subsets Ω_k $(0 \leqslant k \leqslant l,\ 0 \leqslant l \leqslant \infty)$ of σ and given a sequence $\{n_k\}_{k=1}^{l}$ of numbers such that $\{n_k\}_{k=0}^{l} \subset \mathbb{N} \cup \{\infty\}$, $n_0 = \infty$ and $1 \leqslant n_k \leqslant \infty$ $(k \geqslant 1)$. Then there exists an operator A in $\mathcal{B}_\infty(\Omega_0) \cap (\mathrm{SI})$ satisfying:*

(i) *$\sigma(A) = \sigma$, $\sigma_{\mathrm{lre}}(A) = \sigma \setminus \bigcup\limits_{k=0}^{l} \Omega_k$;*

(ii) *$\mathrm{ind}\,(A - \lambda) = \mathrm{nul}\,(A - \lambda) = n_k$ for all $\lambda \in \Omega_k$ $(k = 0, 1, \ldots, l)$.*

Proof. Denote $\Phi_k = (\overline{\Omega}_k)^\circ$, let $N_+(\Phi_k^*)$ be the Bergmann operator on $A^2(\Phi_k^*)$ and denote $A_0 = N_+(\Phi_0^*)^*$ and $A_k = N_+(\Phi_k^*)^{*(n_k)}$ $(k = 1, 2, \ldots, l)$. Thus $\sigma(A_0) = \overline{\Omega}_0$, $A_0 \in \mathcal{B}_1(\Phi_0) \cap (\mathrm{SI})$, $\sigma(A_k) = \overline{\Omega}_k$ and $A_k \in \mathcal{B}_{n_k}(\Phi_k)$ $(k = 1, 2, \ldots, l)$.

Let $\{\lambda_k\}_{k=1}^\infty$ be a dense subset of $\sigma \setminus \bigcup_{k=0}^{l} \Omega_k$. Set $T_k = \lambda_k + V^*$, where V is given in Example 2.5, and define

$$G = A_0 \oplus \Big(\bigoplus_{k=1}^{l} A_k\Big) \oplus \Big(\bigoplus_{k=1}^{\infty} T_k\Big).$$

Then G satisfies:

(a) $\sigma(G) = \sigma_{\mathrm{W}}(G) = \sigma$, $\sigma_{\mathrm{lre}}(G) = \sigma \setminus \bigcup_{k=0}^{l} \Omega_k$;

(b) $\mathrm{ind}\,(G - \lambda) = \mathrm{nul}\,(G - \lambda) = 1$ for $\lambda \in \Omega_0$;

(c) $\mathrm{ind}\,(G - \lambda) = \mathrm{nul}\,(G - \lambda) = n_k$ for $\lambda \in \Omega_k$ $(k = 1, 2, \ldots, l)$.

By Theorem 2.6, for each $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $G + K \in \mathcal{B}_1(\Omega_0)$. It is completely apparent that $G + K$ satisfies (a), (b) and (c).

Without loss of generality, we may assume that $0 \in \Omega_0$.

Note that $\mathcal{B}_1(\Phi_0) \subset \mathcal{B}_1(\Omega_0)$. By Proposition 2.8 and Theorem 2.3, there exists a compact operator K_1 with $\|K_1\| < \varepsilon$ such that if $r(G + K, A_0) \geqslant 1$,

$$(G + K) \oplus A_0^{(\infty)} + K_1 = \begin{bmatrix} G + K & D_1 & D_2 & \cdots \\ & B_1 & & \\ & & B_2 & \\ 0 & & & \ddots \end{bmatrix},$$

where $B_i \in \mathcal{B}_1(\Omega_0)$, $D_i \notin \mathrm{ran}\,\tau_{G+K,B_i}$, $\ker \tau_{B_i,G+K} = \{0\}$ $(i \geqslant 1)$ and $\ker \tau_{B_i B_j} = \{0\}$ $(i \neq j)$. If $r(G + K, A_0) < 1$,

$$(G + K) \oplus A_0^{(\infty)} + K_1 = \begin{bmatrix} B_1 & & & D_1 \\ & B_1 & & D_2 \\ & & \ddots & \vdots \\ 0 & & & G + K \end{bmatrix},$$

where $B_i \in \mathcal{B}_1(\Omega_0)$, $D_i \in \mathrm{ran}\,\tau_{B_i,G+K}$, $\ker \tau_{G+K,B_i} = \{0\}$ $(i \geqslant 1)$ and $\ker \tau_{B_i B_j} = \{0\}$ $(i \neq j)$. By the same argument of Lemma 2.4, $A := (G + K) \oplus A_0^{(\infty)} + K_1 \in \mathcal{B}_\infty(\Omega_0) \cap (\mathrm{SI})$. Thus A satisfies the requirements of the lemma. ∎

The *spectral picture* $\Lambda(T)$ of the operator T is the compact set $\sigma_{\mathrm{lre}}(T)$, plus the data corresponding to the indices of $\lambda - T$ for λ in the bounded components of $\rho_{\text{s-F}}(T)$.

Lemma 2.10. *Let $T \in \mathcal{L}(\mathcal{H})$ with connected spectrum $\sigma(T)$ and let $\sigma_{\mathrm{lre}}(T)$ be the closure of an analytic Cauchy domain. Then there exists an operator $A \in (\mathrm{SI})$ satisfying:*

(i) $\Lambda(A) = \Lambda(T)$;

(ii) $\min\mathrm{ind}\,(A - \lambda) = \begin{cases} 0, & \mathrm{ind}\,(T - \lambda) \neq 0, \\ 1, & \lambda \in \rho_{\text{s-F}}^\circ(T) \cap \sigma(T); \end{cases}$

(iii) *A admits a representation* $A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{matrix}$ *and there is a subset* $\{\lambda_k : k = 0, \pm 1, \pm 2, \dots\}$ *of complex numbers such that* $\operatorname{nul}(A_1 - \lambda_k) = \infty$ $(k \geqslant 0)$, $\operatorname{nul}(A_2 - \lambda_k)^* = \infty$ $(k < 0)$, $\bigvee\{\ker(A_1 - \lambda_k) : k \geqslant 0\} = \mathcal{K}_1$ *and* $\bigvee\{\ker(A_2 - \lambda_k)^* : k < 0\} = \mathcal{K}_2$, *where* $\mathcal{K}_1, \mathcal{K}_2$ *are infinite dimensional Hilbert spaces;*

(iv) *There is an open disc* $G \subset \sigma_{\rm lre}(A)$ *such that* $G \cap \sigma_{\rm p}(A_1) = G^* \cap \sigma_{\rm p}(A_2^*) = \emptyset$.

Proof. Choose an open disc G_1 such that $\overline{G}_1 \subset \sigma_{\rm lre}(T)^\circ$. Denote $\sigma = \sigma(T) \setminus G_1$, then σ is connected and $\sigma \cap \sigma_{\rm lre}(T)$ is still the closure of an analytic Cauchy domain. Let $\{\sigma_k\}_{k=0}^{l_1}$ and $\{\sigma_{-k}\}_{k=1}^{l_2}$ be the components of $\sigma \setminus \rho_{\text{s-F}}^-(T)$ and, respectively, $\sigma \setminus \rho_{\text{s-F}}^+(T)$. For each k $(-l_2 \leqslant k \leqslant l_1)$ choose an open disc Ω_k such that $\overline{\Omega}_k \subset [\sigma_k \cap \sigma_{\rm lre}(T)]^\circ$ (if for more than one k, $(\sigma_k \cap \sigma_{\rm lre}(T)) \cap (\sigma_{-j} \cap \sigma_{\rm lre}(T)) \neq \emptyset$, let Ω_{-j} equal one of the Ω_k's.) By Lemma 2.9 there is a B_k $(-l_2 \leqslant k \leqslant l_1)$ such that:

(i) if $k \geqslant 0$, $B_k \in \mathcal{B}_\infty(\Omega_k) \cap (\mathrm{SI})(\mathcal{H}_k)$, $\sigma(B_k) = \sigma_k$, $\sigma_{\rm lre}(B_k) = \sigma_k \cap [\sigma_{\rm lre}(T) \setminus \Omega_k]$, $\operatorname{ind}(B_k - \lambda) = \operatorname{nul}(B_k - \lambda) = \operatorname{ind}(T - \lambda)$ for $\lambda \in \sigma_k \cap \rho_{\text{s-F}}^+(T)$, $\operatorname{ind}(B_k - \lambda) = \operatorname{nul}(B_k - \lambda) = 1$ for $\lambda \in \sigma_k \cap \rho_{\text{s-F}}^\circ(T)$;

(ii) if $k < 0$, $B_k^* \in \mathcal{B}_\infty(\Omega_k^*) \cap (\mathrm{SI})(\mathcal{H}_k)$, $\sigma(B_k) = \sigma_k$, $\sigma_{\rm lre}(B_k) = \sigma_k \cap [\sigma_{\rm lre}(T) \setminus \Omega_k]$, $\operatorname{ind}(B_k - \lambda) = -\operatorname{nul}(B_k - \lambda)^* = \operatorname{ind}(T - \lambda)$ for $\lambda \in \sigma_k \cap \rho_{\text{s-F}}^-(T)$, $\operatorname{ind}(B_k - \lambda) = -\operatorname{nul}(B_k - 1)^* = -1$ for $\lambda \in \sigma_k \cap \rho_{\text{s-F}}^\circ(T)$.

Choose open discs G and G_2 such that $\overline{G} \cup \overline{G}_2 \subset G_1$ and $\overline{G} \cap \overline{G}_2 = \emptyset$. By Lemma 2.4, we can construct an operator $W \in (\mathrm{SI})(\mathcal{K})$ satisfying:

(i) $\sigma(W) = \sigma_{\rm lre}(W) = \overline{G}_1$;

(ii) $\sigma_{\rm p}(W) \subset G_2$, $\sigma_{\rm p}(W^*) = \emptyset$;

(iii) There exists a sequence $\{\mu_k\}_{k=0}^\infty \subset G_2$ of distinct numbers such that $\lim_{k\to\infty} \mu_k = \mu_0$, $\operatorname{nul}(W - \mu_k) = \infty$ $(k \geqslant 1)$ and $\bigvee\{\ker(W - \mu_k) : k \geqslant 1\} = \mathcal{K}$.

For each k $(0 \leqslant k \leqslant l_1)$, choose $R_k \in \mathcal{L}(\mathcal{H}_k, \mathcal{K})$ by

$$R_k \begin{cases} = 0, & \text{if } \sigma(B_k) \cap \sigma(W) = \emptyset, \\ \notin \operatorname{ran} \tau_{WB_k} \text{ and } R_k \text{ is compact,} & \text{otherwise (Theorem 2.3).} \end{cases}$$

Set $R = (R_0, R_1, \dots, R_{l_1})$.

For each pair (i, j) $(0 \leqslant i \leqslant l_1; 1 \leqslant j \leqslant l_2)$ choose $Q_{ij} \in \mathcal{L}(\mathcal{H}_{-j}, \mathcal{H}_i)$ by

$$Q_{ij} \begin{cases} = 0, & \text{if } \sigma_i \cap \sigma_{-j} = \emptyset, \\ \notin \operatorname{ran} \tau_{B_i B_{-j}}, Q_{ij} \text{ is compact,} & \text{if } \sigma_i \cap \sigma_{-j} \neq \emptyset. \end{cases}$$

Set

$$Q = \begin{bmatrix} Q_{01} & Q_{02} & \dots & Q_{0l_2} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{l_1 1} & Q_{l_1 2} & \dots & Q_{l_1 l_2} \end{bmatrix} \in \mathcal{L}\Big(\bigoplus_{k=1}^{l_2} \mathcal{H}_{-k}, \bigoplus_{k=0}^{l_1} \mathcal{H}_k\Big).$$

Define

$$A = \begin{bmatrix} W & R & 0 \\ 0 & \bigoplus_{k=0}^{l_1} B_k & Q \\ 0 & 0 & \bigoplus_{k=1}^{l_2} B_{-k} \end{bmatrix} = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{matrix},$$

where $\mathcal{K}_1 = \mathcal{K} \oplus \Big(\bigoplus_{k=0}^{l_1} \mathcal{H}_k\Big)$, $\mathcal{K}_2 = \bigoplus_{k=1}^{l_2} \mathcal{H}_{-k}$, $A_1 = \begin{bmatrix} W & R \\ 0 & \bigoplus_{k=0}^{l_1} B_k \end{bmatrix}$ and $A_2 = \bigoplus_{k=1}^{l_2} B_{-k}$. It follows from the properties of W, B_k ($-l_2 \leqslant k \leqslant l_1$) and Lemma 2.1 that $\ker \tau_{B_k B_{k'}} = \ker \tau_{B_{-k} B_{-k'}} = 0$ ($k \neq k'$), $\ker \tau_{\bigoplus_{k=1}^{l_2} B_{-k} \bigoplus_{k=0}^{l_1} B_k} = \ker \tau_{\bigoplus_{k=1}^{l_2} B_{-k} W} = \ker \tau_{\bigoplus_{k=0}^{l_1} B_k W} = \{0\}$. Since W and each B_k ($-l_2 \leqslant k \leqslant l_1$) are strongly irreducible, by Lemma 3.1 of [16] $A \in (\mathrm{SI})$. From the construction of A, we can get (i) and (ii). Note that $\sigma\Big(\bigoplus_{k=0}^{l_1} B_k\Big) \cap \overline{G} \subset \sigma\Big(\bigoplus_{k=0}^{l_1} B_k\Big) \cap G_1 \subset \sigma \cap G_1 = \emptyset$ and $\sigma\Big(\bigoplus_{k=1}^{l_2} B_{-k}\Big) \cap \overline{G} \subset \sigma\Big(\bigoplus_{k=1}^{l_2} B_{-k}\Big) \cap G_1 \subset \sigma \cap G_1 = \emptyset$. Since $\sigma_{\mathrm{p}}(W) \subset G_2$ and $\sigma_{\mathrm{p}}(W^*) = \emptyset$, $\sigma_{\mathrm{p}}(A_1) \cap G = \sigma_{\mathrm{p}}(A_2^*) \cap G^* = \emptyset$. Since $\Omega_k \cap G_1 = \emptyset$ ($-l_2 \leqslant k \leqslant l_1$), there are $\{\lambda_k\}_{k=1}^{\infty} \subset \sigma_{\mathrm{p}}(A_1)$ and $\{\lambda_{-k}^*\}_{k=1}^{\infty} \subset \sigma_{\mathrm{p}}(A_2^*)$ satisfying (iii). ∎

LEMMA 2.11. *Let σ be the closure of a connected Cauchy domain and let $\{\sigma_k\}_{k=0}^{\infty}$ and $\{\Omega_k\}_{k=1}^{\infty}$ be two classes of subsets of σ° satisfying:*

(i) *each σ_k is a connected Cauchy domain;*

(ii) *$\sigma_k \subset \sigma_{k+1}$ and $\sigma_{k+1} \setminus \overline{\sigma}_k$ is a connected Cauchy domain ($k = 0, 1, \ldots$);*

(iii) *$\sigma = \Big[\bigcup_{k=0}^{\infty} \sigma_k\Big]^-$;*

(iv) *each Ω_k is an open disc and $\Omega_k \subset \sigma_{k+1} \setminus \overline{\sigma}_k$ ($k = 1, 2, \ldots$).*

Then there exists an operator $T \in (\mathrm{SI})(\mathcal{H})$ satisfying:

(a) *$\sigma(T) = \sigma_{\mathrm{lre}}(T) = \sigma$, $\sigma_{\mathrm{p}}(T) \subset \bigcup_{k=1}^{\infty} \Omega_k$ and $\sigma_{\mathrm{p}}(T^*) = \emptyset$;*

(b) *there is a subset $\{\mu_n\}_{n=1}^{\infty}$ of $\sigma_{\mathrm{p}}(T)$ such that $\mathrm{nul}\,(T - \mu_n) = \infty$ ($n = 1, 2, \ldots$) and $\bigvee\{\ker(T - \mu_n) : n \geqslant 1\} = \mathcal{H}$;*

(c) *if $A \in \mathcal{L}(\mathcal{H})$ such that $\sigma(A) \cap \sigma^\circ = \emptyset$, then $\ker \tau_{AT} = \ker \tau_{TA} = \{0\}$.*

Proof. According to Lemma 2.4 we can construct an operator $T_k \in (\mathrm{SI})(\mathcal{H}_k)$ such that $\sigma(T_k) = \sigma_{\mathrm{lre}}(T_k) = \sigma_k$, $\sigma_{\mathrm{p}}(T_k) \subset \Omega_k$, $\sigma_{\mathrm{p}}(T_k^*) = \emptyset$ and there is a sequence $\{\lambda_n^k\}_{n=0}^{\infty} \subset \Omega_k$ satisfying $\lim\limits_{n\to\infty} \lambda_n^k = \lambda_0$, $\mathrm{nul}\,(T_k - \lambda_n^k) = \infty$ ($n = 1, 2, \ldots$) and $\bigvee\{\ker(T_k - \lambda_k^n) : n \geqslant 1\} = \mathcal{H}_k$ ($k = 1, 2, \ldots$). Since $\sigma_{\mathrm{r}}(T_1) \cap \sigma_{\mathrm{l}}(T_k) = \sigma_1 \cap \sigma_k \neq \emptyset$ ($k \geqslant 2$), there is a compact operator $D_k \notin \mathrm{ran}\, \tau_{T_1 T_k}$, $\|D_k\| < 2^{-k}$ ($k \geqslant 2$).

Set

$$T = \begin{bmatrix} T_1 & D_2 & D_3 & \cdots \\ & T_2 & & \\ & & T_3 & \\ 0 & & & \ddots \end{bmatrix} \in \mathcal{L}(\mathcal{H}),$$

where $\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$. Since $\{\Omega_k\}_{k=1}^{\infty}$ are pairwise disjoint, $\ker \tau_{T_i T_j} = \{0\}$ ($i \neq j$). By the same argument of Lemma 2.4, $T \in (\mathrm{SI})$. It follows from the construction of T that T satisfies (i) and (ii). By Lemma 2.1, $\ker \tau_{AT} = \{0\}$. If there is an operator $X \in \mathcal{L}(\mathcal{H})$ such that $TX = XA$, let $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix}$; then we have $T_2X_2 = X_2A, \dots, T_nX_n = X_nA$, $(n \geqslant 2)$. Since $\sigma(A) \cap \sigma^\circ = \emptyset$ and $\sigma(T_n) = \sigma_n \subset \sigma^\circ$, $\sigma(A) \cap \sigma(T_n) = \emptyset$. Thus $X_n = 0$ $(n \geqslant 2)$ and $T_1X_1 = X_1A$. For the same reason $X_1 = 0$ and $X = 0$, i.e., $\ker \tau_{TA} = \{0\}$. ∎

LEMMA 2.12. *Let $n \in \mathbb{N}$ or $n = \infty$, let σ be a connected compact subset of $\mathbb{C}$ and Ω be a connected open subset of σ° such that $\sigma^\circ \setminus \overline{\Omega} \neq \emptyset$. Then there exists an operator $A \in (\mathrm{SI})(\mathcal{H})$ satisfying:*

(i) *$\sigma(A) = \sigma$, $\sigma_{\mathrm{lre}}(A) = \sigma \setminus \Omega$, $\sigma_{\mathrm{p}}(A^*) = \emptyset$;*

(ii) *$\mathrm{ind}\,(A - \lambda) = n$ for $\lambda \in \Omega$;*

(iii) *there exists a subset $\{\lambda_k\}_{k=1}^\infty$ of σ such that $\mathrm{nul}\,(A - \lambda_k) = \infty$ $(k \geqslant 1)$ and $\bigvee\{\ker(A - \lambda_k) : k \geqslant 1\} = \mathcal{H}$.*

Proof. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$. Choose open discs G_1, G_2 such that $\overline{G}_2 \subset G_1 \subset \overline{G}_1 \subset \sigma^\circ \setminus \overline{\Omega}$. According to Lemma 2.9, we can construct an operator $A_1 \in \mathcal{B}_\infty(G_1) \cap (\mathrm{SI})(\mathcal{H}_1)$ satisfying $\sigma(A_1) = \sigma$, $\sigma_{\mathrm{lre}}(A_1) = \sigma \setminus (G_1 \cup \Omega)$ and $\mathrm{ind}\,(A_1 - \lambda) = n$ for $\lambda \in \Omega$. By Lemma 2.4, we can find an operator $A_2 \in (\mathrm{SI})(\mathcal{H}_2)$ satisfying $\sigma(A_2) = \sigma_{\mathrm{lre}}(A_2) = \overline{G}_1$, $\sigma_{\mathrm{p}}(A_2) \subset G_2$, $\sigma_{\mathrm{p}}(A_2^*) = \emptyset$ and there exists a sequence $\{\mu_i\}_{i=1}^\infty \subset G_2$ such that $\mathrm{nul}\,(A_2 - \mu_i) = \infty$ $(i \geqslant 1)$ and $\bigvee\{\ker(A_2 - \mu_i) : i \geqslant 1\} = \mathcal{H}_2$. By Lemma 2.1 $\ker \tau_{A_2A_1} = \{0\}$. By Theorem 2.3, there is a compact operator $K \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $K \notin \mathrm{ran}\,\tau_{A_1A_2}$.

Define $A = \begin{bmatrix} A_1 & K \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$. By the same argument of Lemma 2.4, $A \in (\mathrm{SI})(\mathcal{H})$ and satisfies (i), (ii) and (iii). ∎

LEMMA 2.13. *Let $T \in \mathcal{L}(\mathcal{H})$ with connected spectrum $\sigma(T)$ and assume that $\sigma_{\mathrm{lre}}(T)$ is the closure of an analytic Cauchy domain, then there exists an operator $W \in (\mathrm{SI})(\mathcal{H})$ satisfying:*

(i) *$\Lambda(W) = \Lambda(T)$;*

(ii) $\min \mathrm{ind}\,(W - \lambda) = \begin{cases} 0, & \text{if } \lambda \in \rho^{\pm}_{\text{s-F}}(W), \\ 1, & \text{if } \lambda \in \sigma(W) \cap \rho^{\circ}_{\text{s-F}}(W); \end{cases}$

(iii) $W = \begin{bmatrix} W_1 & * \\ 0 & W_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$, *where $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$, and there is a sequence $\{\lambda_k : k = 0, \pm 1, \pm 2, \dots\}$ of numbers such that $\bigvee\{\ker(W_1 - \lambda_k)^* : k \geqslant 0\} = \mathcal{H}_1$ and $\bigvee\{\ker(W_2 - \lambda_k) : k < 0\} = \mathcal{H}_2$;*

(iv) *there is an open disc $G \subset \sigma_{\mathrm{lre}}(W)$ such that $G \cap \sigma_{\mathrm{p}}(W_2) = G^* \cap \sigma_{\mathrm{p}}(W_1^*) = \emptyset$.*

Proof. Assume that

$$\{\Omega_{1i}\}_{i=1}^{l_1} \text{ are the components of } \rho^{-}_{\text{s-F}}(T),$$
$$\{\Omega_{2j}\}_{j=1}^{l_2} \text{ are the components of } \rho^{\circ}_{\text{s-F}}(T) \cap \sigma(T),$$
$$\{\Omega_{3k}\}_{k=1}^{l_3} \text{ are the components of } \rho^{+}_{\text{s-F}}(T).$$

Choose connected Cauchy domains Φ_{ij} in $\sigma(T)$ $(i = 1, 2, 3; j = 1, 2, \cdots, j_i)$ such that $\Phi_{ij} \supset \Omega_{ij}$, $\Phi_{ij} \setminus \overline{\Omega}_{ij}$ are connected Cauchy domains, $\{\overline{\Phi}_{ij}\}$ are pairwise disjoint and $\sigma(T) \setminus \bigcup \Phi_{ij}$ is the closure of an analytic Cauchy domain.

Choose an open disc $\sigma_0 \subset [\sigma(T) \setminus \bigcup \Phi_{ij}]^\circ$. Let $\{\sigma_k\}_{k=1}^{l_4}$ be the components of $\sigma(T) \setminus [\sigma_0^\circ \cup (\bigcup \Phi_{ij})]$. Choose an open disc G such that $\overline{G} \subset \sigma_0^\circ$. For each k $(0 \leqslant k \leqslant l_4)$, according to Lemma 2.11, we can construct an operator $E_k \in (\mathrm{SI})(\mathcal{H})$ satisfying:

(i) $\sigma(E_k) = \sigma_{\mathrm{lre}}(E_k) = \sigma_k$;

(ii) $\sigma_{\mathrm{p}}(E_0) = \emptyset$ and there is a subset $\{\mu_n : n \geqslant 1\}$ of $\sigma_0 \setminus G$ such that $\mathrm{nul}\,(E_0 - \mu_n)^* = \infty$, $\bigvee\{\ker(E_0 - \mu_n)^* : n \geqslant 1\} = \mathcal{H}$ and $G^* \cap \sigma_{\mathrm{p}}(E_0^*) = \emptyset$;

(iii) For each $k \geqslant 1$, $\sigma_{\mathrm{p}}(E_k^*) = \emptyset$ and there is a subset $\{\mu_{kn} : n \geqslant 1\}$ of σ_k such that $\mathrm{nul}\,(E_k - \mu_{kn}) = \infty$, $\bigvee\{\ker(E_k - \mu_{kn}) : n \geqslant 1\} = \mathcal{H}$;

(iv) For each k and each operator F, if $\sigma(F) \cap \sigma_k^\circ = \emptyset$, then $\ker \tau_{E_k F} = \ker \tau_{F E_k} = \{0\}$.

According to Lemma 2.12, we construct the following (SI) operators.

Step 1. Construct $A_i \in (\mathrm{SI})(\mathcal{H})$ $(1 \leqslant i \leqslant l_1)$ such that $\sigma(A_i) = \overline{\Phi}_{1i}$, $\sigma_{\mathrm{p}}(A_i) = \emptyset$, $\sigma_{\mathrm{lre}}(A_i) = \overline{\Phi}_{1i} \setminus \Omega_{1i}$, $\mathrm{ind}\,(A_i - \lambda) = \mathrm{ind}\,(T - \lambda)$ for $\lambda \in \Omega_{1i}$ and there is a countable subset Λ_{1i} of $\sigma(A_i)$ such that $\mathrm{nul}\,(A_i - \lambda)^* = \infty$ $(\lambda \in \Lambda_{1i})$ and $\bigvee\{\ker(A_i - \lambda)^* : \lambda \in \Lambda_{1i}\} = \mathcal{H}$.

Step 2. Construct $B_k \in (\mathrm{SI})(\mathcal{H})$ $(1 \leqslant k \leqslant l_3)$ such that $\sigma(B_k) = \overline{\Phi}_{3k}$, $\sigma_{\mathrm{p}}(B_k^*) = \emptyset$, $\sigma_{\mathrm{lre}}(B_k) = \overline{\Phi}_{3k} \setminus \Omega_{3k}$, $\mathrm{ind}\,(B_k - \lambda) = \mathrm{ind}\,(T - \lambda)$ for $\lambda \in \Omega_{3k}$ and there is a countable subset Λ_{3k} of $\sigma(B_k)$ such that $\mathrm{nul}\,(B_k - \lambda) = \infty$ $(\lambda \in \Lambda_{3k})$ and $\bigvee\{\ker(B_k - \lambda) : \lambda \in \Lambda_{3k}\} = \mathcal{H}$.

Step 3. Construct $C_j \in (\mathrm{SI})(\mathcal{H})$ $(1 \leqslant j \leqslant l_2)$ such that $\sigma(C_j) = \overline{\Phi}_{2j}$, $\sigma_{\mathrm{p}}(C_j) = \emptyset$, $\sigma_{\mathrm{lre}}(C_j) = \overline{\Phi}_{2j} \setminus \Omega_{2j}$, $\mathrm{ind}\,(C_j - \lambda) = -1$ for $\lambda \in \Omega_{2j}$ and there is a countable subset $\Lambda_{2j} \in \sigma(C_j)$ such that $\mathrm{nul}\,(C_j - \lambda)^* = \infty$ $(\lambda \in \Lambda_{3j})$ and $\bigvee\{\ker(C_j - \lambda)^* : \lambda \in \Lambda_{3j}\} = \mathcal{H}$.

Step 4. Construct $D_h \in (\mathrm{SI})(\mathcal{H})$ $(1 \leqslant h \leqslant l_2)$ such that $\sigma(D_h) = \overline{\Phi}_{2h}$, $\sigma_{\mathrm{p}}(D_h^*) = \emptyset$, $\sigma_{\mathrm{lre}}(D_h) = \overline{\Phi}_{2h} \setminus \Omega_{2h}$, $\mathrm{ind}\,(D_h - \lambda) = 1$ for $\lambda \in \Omega_{2h}$ and there is a countable subset Λ_{4h} of $\sigma(D_h)$ such that $\mathrm{nul}\,(D_h - \lambda) = \infty$ $(\lambda \in \Lambda_{4h})$ and $\bigvee\{\ker(D_h - \lambda) : \lambda \in \Lambda_{4h}\} = \mathcal{H}$.

By the definitions, it is easily seen that

$$\ker \tau_{A_i A_j} = \ker \tau_{B_i B_j} = \ker \tau_{C_i C_j} = \ker \tau_{D_i D_j} = \ker \tau_{E_i E_j} = \{0\}, \quad i \neq j.$$

Set $A = \bigoplus_{i=1}^{l_1} A_i \in \mathcal{L}(\mathcal{H}^{(l_1)})$, $B = \bigoplus_{k=1}^{l_3} B_k \in \mathcal{L}(\mathcal{H}^{(l_3)})$, $C = \bigoplus_{j=1}^{l_2} C_j$, $D = \bigoplus_{h=1}^{l_2} D_h \in \mathcal{L}(\mathcal{H}^{(l_2)})$ and $E = \bigoplus_{k=1}^{l_4} E_k \in \mathcal{L}(\mathcal{H}^{(l_4)})$.

Define $Q_i \in \mathcal{L}(\mathcal{H})$ $(1 \leqslant i \leqslant l_4)$ as follows

$$Q_i = \begin{cases} \text{compact and } \notin \operatorname{ran} \tau_{E_0 E_i}, & \text{if } \sigma(E_i) \cap \sigma(E_0) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Set $X_0 = (Q_1, Q_2, \ldots, Q_{l_4}) \in \mathcal{L}(\mathcal{H}^{(l_4)}, \mathcal{H})$.

Define $X_1 = (Q_{ij})_{l_1 \times l_4} \in \mathcal{L}(\mathcal{H}^{(l_4)}, \mathcal{H}^{(l_1)})$ as follows

$$Q_{ij} = \begin{cases} \text{compact and } \notin \operatorname{ran} \tau_{A_i E_j}, & \text{if } \sigma(A_i) \cap \sigma(E_j) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

$X_2 \in \mathcal{L}(\mathcal{H}^{(l_4)}, \mathcal{H}^{(l_2)})$, and $X_4 = \mathcal{L}(\mathcal{H}^{(l_4)}, \mathcal{H}^{(l_3)})$ are defined similarly. $X_3 = (M_{ij})_{l_2 \times l_4} \in \mathcal{L}(\mathcal{H}^{(l_4)}, \mathcal{H}^{(l_2)})$ is defined as follows: M_{ij} is compact and $M_{ij} + K \notin \operatorname{ran} \tau_{D_i E_j}$ for all $K \in \mathcal{K}(\mathcal{H})$ if $\sigma(D_i) \cap \sigma(E_j) = \overline{\Phi}_{1i} \cap \sigma_j \neq \emptyset$ (Theorem 2.3) and $M_{ij} = 0$ if $\sigma(D_i) \cap \sigma(E_j) = \emptyset$.

Define

$$W = \begin{bmatrix} E_0 & & & & & X_0 \\ & A & & 0 & & X_1 \\ & & C & & & X_2 \\ & & & D & & X_3 \\ & 0 & & & B & X_4 \\ & & & & & E \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}^{(l_1)} \\ \mathcal{H}^{(l_2)} \\ \mathcal{H}^{(l_2)} \\ \mathcal{H}^{(l_3)} \\ \mathcal{H}^{(l_4)} \end{matrix}.$$

Assume that $P \in \mathcal{A}'(W)$ is an idempotent. It follows from Lemma 2.1 and the properties of $\{E_k\}$ that P admits the following representation

$$P = \begin{bmatrix} P_1 & & & & & P_{16} \\ & P_2 & & 0 & & P_{26} \\ & & P_3 & & & P_{36} \\ & & P_{43} & P_4 & & P_{46} \\ & & & & P_5 & P_{56} \\ & 0 & & & & P_6 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}^{(l_1)} \\ \mathcal{H}^{(l_2)} \\ \mathcal{H}^{(l_2)} \\ \mathcal{H}^{(l_3)} \\ \mathcal{H}^{(l_4)} \end{matrix}.$$

Since $E_0 \in$ (SI) and since A, B, C, D, E are direct sums of (SI) operators with disjoint spectrum respectively, $P_1 = 0$ or 1, $P_2 = \bigoplus_{i=1}^{l_1} \delta_{2i}$, $P_3 = \bigoplus_{i=1}^{l_2} \delta_{3i}$, $P_4 = \bigoplus_{i=1}^{l_2} \delta_{4i}$, $P_5 = \bigoplus_{i=1}^{l_3} \delta_{5i}$ and $P_6 = \bigoplus_{i=1}^{l_4} \delta_{6i}$, where $\delta_{ji} = 0$ or 1. Without loss of generality, we can assume that $P_1 = 0$. By the argument of Lemma 3.1 of [15], we can get $P_2 = P_3 = P_5 = P_6 = 0$. Since $PW = WP$, $P_{43}X_2 + P_4X_3 + P_{46}E = DP_{46}$. Note that X_2 is compact, thus $P_{43}X_2$ is compact. For each j $(1 \leqslant j \leqslant l_2)$, there must exists an integer k such that $\sigma_{\mathrm{re}}(D_j) \cap \sigma_{\mathrm{le}}(E_k) = \overline{\Phi}_{1j} \cap \sigma_k \neq \emptyset$. Suppose that $P_{46} = (L_{ih})_{l_2 \times l_4}$, then

$$D_j L_{jk} - L_{jk} E_k = \delta_{4j} M_{jk} + K,$$

where K is a compact operator. By the choice of M_{jk}, $\delta_{4j} = 0$. Thus $P_4 = 0$. Since $P^2 = P$, $P = 0$ and $W \in$ (SI).

Set $W_1 = \begin{bmatrix} E_0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}$, $W_2 = \begin{bmatrix} D & 0 & X_3 \\ 0 & B & X_4 \\ 0 & 0 & E \end{bmatrix}$, then $W = \begin{bmatrix} W_1 & * \\ 0 & W_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$, where $\mathcal{H}_1 = \mathcal{H}^{(l_1+l_2+1)}$, $\mathcal{H}_2 = \mathcal{H}^{(l_2+l_3+l_4)}$. By the properties of $\{A_i\}$ and $\{C_i\}$ we have $\min \operatorname{ind}(W_1 - \lambda) = 0$ for $\lambda \in \rho_{\text{s-F}}(T) \cap \sigma(T)$ and

$$\operatorname{ind}(W_1 - \lambda) = \begin{cases} \operatorname{ind}(T - \lambda), & \lambda \in \rho_{\text{s-F}}^{-}(T), \\ -1, & \lambda \in \rho_{\text{s-F}}^{\circ}(T) \cap \sigma(T). \end{cases}$$

By the properties of E_0, $\{A_i\}$ and $\{C_i\}$, we can find a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of numbers such that $\operatorname{nul}(W_1 - \lambda_k)^* = \infty$ $(k \geqslant 0)$ and $\bigvee\{\ker(W_1 - \lambda_k)^* : k \geqslant 0\} = \mathcal{H}_1$.

Similarly, by the properties of $\{E_i\}, \{B_i\}$ and $\{D_i\}$, we have $\min \operatorname{ind}(W_2 - \lambda) = 0$ for $\lambda \in \rho_{\text{s-F}}(T) \cap \sigma(T)$,

$$\operatorname{ind}(W_2 - \lambda) = \begin{cases} \operatorname{ind}(T - \lambda), & \lambda \in \rho_{\text{s-F}}^{+}(T), \\ 1, & \lambda \in \rho_{\text{s-F}}^{\circ}(T) \cap \sigma(T), \end{cases}$$

and there is a sequence $\{\lambda_k\}_{k=-1}^{-\infty}$ of numbers such that $\mathrm{nul}\,(W_2-\lambda_k)=\infty$ $(k\leqslant -1)$ and $\bigvee\{\ker(W_2-\lambda_k): k\leqslant -1\}=\mathcal{H}_2$.

It follows from $G\cap\Big[\Big(\bigcup\limits_{k=1}^{l_4}\sigma_k\Big)\cup\Big(\bigcup\{\Phi_{ij}: i=1,2,3;\ j=1,2,\ldots,l_i\}\Big)\Big]$ and the properties of E_0 that we have $G\cap\sigma_{\mathrm{p}}(W_2)=\emptyset$ and $G^*\cap\sigma_{\mathrm{p}}(W_1^*)=\emptyset$. Thus W satisfies (iii) and (iv) of the lemma. It is easy to see that W satisfies (i) and (ii). Thus the proof of the lemma is now complete. ∎

3. PROOF OF THEOREM 1.1

In [13], we have proved that if $\mathcal{N}$ is well-ordered with finite dimensional atoms, then $\mathcal{U}(\mathrm{alg}\,\mathcal{N}\cap(\mathrm{SI}))^-=(\mathrm{QT})_{\mathrm{C}}$. Thus we only need to show that if $\mathcal{N}$ is maximal and $\mathcal{N}$ and $\mathcal{N}^\perp$ are not well-ordered, then

$$\mathcal{U}(\mathrm{alg}\,\mathcal{N}\cap(\mathrm{SI}))^-=\{T\in\mathcal{L}(\mathcal{H}):\sigma(T)\ \text{ is connected}\}.$$

Given an operator $T\in\mathcal{L}(\mathcal{H})$ with connected $\sigma(T)$ and given $\varepsilon>0$, by the theory of approximation of Hilbert space operators, there is an operator $T_\varepsilon\in\mathcal{L}(\mathcal{H})$ with $\sigma(T_\varepsilon)$ connected such that $\sigma_{\mathrm{lre}}(T_\varepsilon)$ is the closure of an analytic Cauchy domain and $\|T-T_\varepsilon\|<\varepsilon$. Thus for the maximal nest $\mathcal{N}$, with $\mathcal{N}$ and $\mathcal{N}^\perp$ not well-ordered, it suffices to show that for each operator T with connected $\sigma(T)$ and whose $\sigma_{\mathrm{lre}}(T)$ is the closure of an analytic Cauchy domain, we always can find an (SI) operator A in $\mathrm{alg}\,\mathcal{N}$ such that $\|UAU^*-T\|<\varepsilon$, where U is a unitary operator, i.e., it is needed to show that

$$\begin{aligned}\Delta:=\{T\in\mathcal{L}(\mathcal{H}):&\,\sigma(T)\text{ is connected and }\sigma_{\mathrm{lre}}(T)\text{ is the}\\ &\text{closure of an analytic Cauchy domain}\}\subset\mathcal{U}(\mathrm{alg}\,\mathcal{N}\cap(\mathrm{SI}))^-.\end{aligned}$$

If $\mathcal{N}$ and $\mathcal{N}^\perp$ are not well-ordered, there are three possibilities.

Case A. There are $\{t_n\}_{n=-\infty}^{\infty}\subset[0,1]$ such that

$$0=t_0<t_1<t_2<\cdots<t_n<\cdots<t_{-n}<\cdots<t_{-2}<t_{-1}=1,$$

$\lim\limits_{n\to\infty}t_n=\lim\limits_{n\to\infty}t_{-n}$ and $\dim M_{(t_{n-1},t_n]}=\infty$ $(n=\pm1,\pm2,\ldots)$, where

$$M_{(t_{n-1},t_n]}=E\big((t_{n-1},t_n]\big)\mathcal{H}$$

and E is the spectral measure associated with $\mathcal{N}$.

Case B. There are $t_0,t_1,t_2,t_3\in[0,1]$, such that $0<t_0<t_1<t_2<t_3<1$ and

$$\begin{aligned}&\mathcal{N}_0:=\{M_t:0\leqslant t\leqslant t_0\}\text{ is atomic},\\ &\mathcal{N}_1:=\{M_t\ominus M_{t_0}:t\leqslant t_1\}\text{ has the type }\omega+1,\\ &\mathcal{N}_2:=\{M_t\ominus M_{t_1}:t_1\leqslant t\leqslant t_2\}\text{ is atomic},\\ &\mathcal{N}_3:=\{M_t\ominus M_{t_2}:t_2\leqslant t\leqslant t_3\}\text{ has the type }1+\omega^*,\\ &\mathcal{N}_4:=\{M_t\ominus M_{t_3}:t_3\leqslant t\leqslant 1\}\text{ is atomic},\end{aligned}$$

where $M_t=M_{[0,t]}=E([0,t])\mathcal{H}$.

Case C. There are $t_0, t_1, t_2, t_3 \in [0,1]$ such that $0 < t_0 < t_1 < t_2 < t_3 < 1$ and

$$\mathcal{N}_0 := \{M_t : 0 \leqslant t \leqslant t_0\} \text{ is atomic,}$$
$$\mathcal{N}_1 := \{M_t \ominus M_{t_0} : t_0 \leqslant t \leqslant t_1\} \text{ has the type } 1+\omega^*,$$
$$\mathcal{N}_2 := \{M_t \ominus M_{t_1} : t_1 \leqslant t \leqslant t_2\} \text{ is atomic,}$$
$$\mathcal{N}_3 := \{M_t \ominus M_{t_2} : t_2 \leqslant t \leqslant t_3\} \text{ has the type } \omega+1,$$
$$\mathcal{N}_4 := \{M_t \ominus M_{t_3} : t_3 \leqslant t \leqslant 1\} \text{ is atomic.}$$

In Case A, according to Lemma 2.10, there exists an operator $A \in$ (SI) such that $\Lambda(A) = \Lambda(T)$, $\min\operatorname{ind}(A-\lambda) \leqslant \min\operatorname{ind}(T-\lambda)$ for $\lambda \in \rho_{\text{s-F}}(A)$ and $A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{matrix}$, where

$$A_1 = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \vdots \end{matrix}, \quad A_2 = \begin{bmatrix} \ddots & & & * \\ & \lambda_{-3} & & \\ & & \lambda_{-2} & \\ 0 & & & \lambda_{-1} \end{bmatrix} \begin{matrix} \vdots \\ \mathcal{H}_{-3} \\ \mathcal{H}_{-2} \\ \mathcal{H}_{-1} \end{matrix},$$

$\mathcal{H}_n = \bigvee\{\ker(A_1-\lambda_k) : 1 \leqslant k \leqslant n\} \ominus \mathcal{H}_{n-1}$, $\mathcal{H}_{-n} = \bigvee\{\ker(A_2-\lambda_k) : -n \leqslant k \leqslant -1\} \ominus \mathcal{H}_{-n+1}$ $(n = 1,2,\ldots)$, $\mathcal{H}_0 = \{0\}$, $\dim\mathcal{H}_n = \infty$ $(n = \pm1, \pm2, \ldots)$, $\mathcal{K}_1 = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ and $\mathcal{K}_2 = \bigoplus_{n=-1}^{-\infty} \mathcal{H}_n$, $\{\lambda_k : k = \pm1, \pm2, \ldots\}$ are given in Lemma 2.10 (iii).

By Similarity Orbit Theorem ([2]), $T \in \mathrm{S}(A)^-$, i.e., for each $\varepsilon > 0$, there exists an invertible operator X such that $\|XAX^{-1} - T\| < \varepsilon$. It is easily seen that XAX^{-1} admits a same matrix representation with respect to another decomposition of the space,

$$XAX^{-1} = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & * & & \\ & & \lambda_3 & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_{-3} & & \\ & 0 & & & & & \lambda_{-2} & \\ & & & & & & & \lambda_{-1} \end{bmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \vdots \\ \vdots \\ \mathcal{M}_{-3} \\ \mathcal{M}_{-2} \\ \mathcal{M}_{-1} \end{matrix},$$

where $\dim\mathcal{M}_n = \infty$ $(n = \pm1, \pm2, \ldots)$.

Choose a unitary operator U so that $U\mathcal{M}_n = M_{(t_{n-1}, t_n]}$ $(n = \pm1, \pm2, \ldots)$, then $UXAX^{-1}U^* \in \operatorname{alg}\mathcal{N} \cap (\mathrm{SI})$, i.e., $T \in \mathcal{U}(\operatorname{alg}\mathcal{N} \cap (\mathrm{SI}))^-$.

If B is the case, for simplicity we only prove the conclusion of the theorem when $t_0 = 0$ and $t_3 = 1$. Denote the operator A in Case A by A_1 which satisfies (i), (ii), (iii) and (iv) of Lemma 2.10. Let $\{f_\alpha\}_{\alpha\in\Lambda}$ be the unit vectors of the atoms of $\mathcal{N}_2$, $\bigvee\{f_\alpha : \alpha \in \Lambda\} = M_{t_2} \ominus M_{t_1}$. Assume that G is the open disc contained in $\sigma_{\text{lre}}(A)$ given in Lemma 2.10 (iv), then choose $c_\alpha \in G$ $(\alpha \in \Lambda)$ such that $\{c_\alpha\}$ is pairwise distinct and define $A_3 = \sum c_\alpha f_\alpha \otimes f_\alpha$. By the construction of A_1 in Lemma 2.10, $G \subset \sigma_{\text{lre}}(A_1)$. Thus for each α there is a unit vector $g_\alpha \in \mathcal{K}_1$ such that $g_\alpha \notin \operatorname{ran}(A_1 - c_\alpha)$. Let $\{d_\alpha\}_{\alpha\in\Lambda}$ be positive numbers satisfying $\sum_{\alpha\in\Lambda} d_\alpha = 1$. Set $K = \sum_{\alpha\in\Lambda} d_\alpha g_\alpha \otimes f_\alpha$ and

$$A = \begin{bmatrix} A_1 & K & A_{12} \\ 0 & A_3 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{K}_1 \\ M_{t_2} \ominus M_{t_1} \\ \mathcal{K}_2 \end{matrix} .$$

Then it is easily seen that $\Lambda(A) = \Lambda(T)$ and $\min\operatorname{ind}(A-\lambda) \leqslant \min\operatorname{ind}(T-\lambda)$ for $\lambda \in \rho_{\text{s-F}}(T)$. By Lemma 2.10 (iii), (iv) we have $\ker\tau_{A_3A_1} = \ker\tau_{A_2A_3} = \{0\}$.

Assume that P is an idempotent commuting with A and

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{matrix} \mathcal{K}_1 \\ M_{t_2} \ominus M_{t_1} \\ \mathcal{K}_2 \end{matrix} ,$$

then by Lemma 2.1, $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}$. Observe that $P' = \begin{bmatrix} P_{11} & 0 & P_{13} \\ 0 & 0 & 0 \\ 0 & 0 & P_{33} \end{bmatrix}$ is an idempotent commuting with $\begin{bmatrix} A_1 & 0 & A_{12} \\ 0 & 0 & 0 \\ 0 & 0 & A_2 \end{bmatrix}$ and $A' = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \in (\text{SI})$, thus $P' = 0$ or 1. Without loss of generality we can assume that $P' = 0$, or $P = \begin{bmatrix} 0 & P_{12} & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since $PA = AP$, $P_{12}A_3 = A_1P_{12} + KP_2$. It follows from $P_{22}A_3 = A_3P_{22}$ and pairwise distinction of c_α's that $P_{22} = \bigoplus_{\alpha\in\Lambda} \delta_\alpha$, where $\delta_\alpha = 0$ or 1. Thus for each $\alpha \in \Lambda$

$$(A_1P_{12} - P_{12}A_3)f_\alpha = A_1P_{12}f_\alpha - c_\alpha P_{12}f_\alpha = -\delta_\alpha d_\alpha g_\alpha .$$

Since $g_\alpha \notin \operatorname{ran}(A_1 - c_\alpha)$, $\delta_\alpha = 0$. Therefore $P = 0$ and $A \in (\text{SI})$. By Similarity Orbit Theorem ([2]), $T \in \mathrm{S}(A)^-$, i.e., for each $\varepsilon > 0$ there exists an invertible operator X such that $\|XAX^{-1} - T\| < \varepsilon$. By Lemma 2.10 (iii), A_1 and A_2^* admit upper triangular matrix representations

$$A_1 = \begin{bmatrix} \lambda_0 & & & * \\ & \lambda_1 & & \\ & & \lambda_2 & \\ 0 & & & \ddots \end{bmatrix} \begin{matrix} e_0^1 \\ e_1^1 \\ e_2^1 \\ \vdots \end{matrix}, \quad A_2 = \begin{bmatrix} \ddots & & & * \\ & \lambda_{-3} & & \\ & & \lambda_{-2} & \\ 0 & & & \lambda_{-1} \end{bmatrix} \begin{matrix} \vdots \\ e_3^2 \\ e_2^2 \\ e_1^2 \end{matrix}$$

with respect to some ONB$\{e_n^1\}_{n=0}^\infty$ of $\mathcal{K}_1$ and, respectively, ONB$\{e_n^2\}_{n=1}^\infty$ of $\mathcal{K}_2$.

Set

$$\mathcal{M} = \left\{ \begin{matrix} \bigvee_{i=1}^{n} \{e_i^1\} (n = 0, 1, 2, \ldots); \bigvee_{i=1}^{\infty} \{e_i^1\} \oplus N (N \in \mathcal{N}_2); \\ \bigvee_{i=1}^{\infty} \{e_i^1\} \oplus (M_{t_2} \ominus M_{t_1}) \oplus \bigvee_{j=n}^{\infty} \{e_j^2\} (n = 0, 1, 2, \ldots) \end{matrix} \right\},$$

then $\mathcal{M}$ is a maximal atomic nest, and unitarily equivalent to $\mathcal{N}$. Thus, there exists a unitary operator U such that $UXAX^{-1}U^* \in \operatorname{alg}\mathcal{N}$. Therefore $T \in \mathcal{U}(\operatorname{alg}\mathcal{N} \cap (\text{SI}))^-$.

For Case C, we only prove the conclusion of the theorem when $t_1 = t_2$.

According to Lemma 2.13 we get an operator $W \in$ (SI) satisfying (i)–(iv) of Lemma 2.13. Let $W = \begin{bmatrix} W_1 & W_{12} \\ 0 & W_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$.

Let $N_{-\infty} = \bigcap\{M_{t_n} : -\infty < n < \infty\}$, $N_\infty = \bigvee\{M_{t_n} : -\infty < n < \infty\}$. Let $\mathcal{N}_- = \{M_t \in \mathcal{N} : 0 \leqslant t \leqslant t_0\}$, $\mathcal{N}_+ = \{M_t \ominus M_{t_3} : t_3 \leqslant t \leqslant 1\}$. Let $\{f_\alpha\}_{\alpha\in\Lambda_1}$ and $\{g_\beta\}_{\beta\in\Lambda_2}$ be the unit vectors of the atoms of $\mathcal{N}_-$ and, respectively, $\mathcal{N}_+$. Define $B_1 = \sum\limits_{\alpha\in\Lambda_1} c_\alpha f_\alpha \otimes f_\alpha$ and $B_2 = \sum\limits_{\beta\in\Lambda_2} d_\beta g_\beta \otimes g_\beta$, where $\{c_\alpha, \alpha \in \Lambda_1; d_\beta, \beta \in \Lambda_2\} \subset G \subset \sigma_{\rm lre}(W)$ are pairwise distinct and G is given in Lemma 2.13 (iv). By the similar way of Case B, construct operators $E_1 \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2, \bigvee\{f_\alpha : \alpha \in \Lambda_1\})$ and $E_2 \in \mathcal{L}(\bigvee\{g_\beta : \beta \in \Lambda_2\}, \mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $E_1^* f_\alpha \notin {\rm ran}\,(W_1 - c_\alpha)^*$, $E_2 g_\beta \notin {\rm ran}\,(W_2 - d_\beta)$ $(\alpha \in \Lambda_1, \beta \in \Lambda_2)$.

Set

$$A = \begin{bmatrix} B_1 & E_1 & 0 \\ 0 & W & E_2 \\ 0 & 0 & B_2 \end{bmatrix} \begin{matrix} \bigvee\{f_\alpha : \alpha \in \Lambda_1\} \\ \mathcal{H}_1 \oplus \mathcal{H}_2 \\ \bigvee\{g_\beta : \beta \in \Lambda_2\} \end{matrix}.$$

By the same argument of Case B, $A \in$ (SI) and $T \in {\rm S}(A)^-$. Thus for each $\varepsilon > 0$, $\|XAX^{-1} - T\| < \varepsilon$ for some invertible operator X. Note that by (i), (ii) and (iii) of Lemma 2.13

$$W = \begin{bmatrix} \ddots & & & & & & \\ & \lambda_{-2} & & & & * & \\ & & \lambda_{-1} & & & & \\ & & & \lambda_0 & & & \\ & & & & \lambda_1 & & \\ & 0 & & & & \lambda_2 & \\ & & & & & & \ddots \end{bmatrix} \begin{matrix} \vdots \\ e_{-2} \\ e_{-1} \\ e_0 \\ e_1 \\ e_2 \\ \vdots \end{matrix}$$

with respect to some ${\rm ONB}\{e_n\}_{n=-\infty}^{\infty}$ of $\mathcal{H}_1 \oplus \mathcal{H}_2$. Thus by the argument of Case B, there is a unitary operator U such that $UXAX^{-1}U^* \in {\rm alg}\mathcal{N}$ and therefore $T \in \mathcal{U}({\rm alg}\mathcal{N} \cap ({\rm SI}))^-$. The proof of the theorem is now complete. ∎

The second and the third author were partially supported by NNSFC.

REFERENCES

1. J. AGLER, E. FRANKS, D.A. HERRERO, Spectral pictures of operators quasisimilar to the unilateral shift, *J. Reine Angew. Math.* **422**(1991), 1–20.
2. C. APOSTOL, L.A. FIALKOW, D.A. HERRERO, D. VOICULESCU, *Approximation of Hilbert Space Operator.* II, Res. Notes Math., vol. 102, Longman, Harlow–Essex 1984.
3. M.J. COWEN, R.G. DOUGLAS, Complex geometry and operator theory, *Bull. Amer. Math. Soc.* **83**(1977), 131–133.
4. K.R. DAVIDSON, *Nest Algebra*, Res. Notes Math., vol. 191, Longman, Harlow–Essex 1988.
5. R.G. DOUGLAS, *Banach Algebras Techniques in Operator Theory*, Academic Press, New York–London 1972.

6. L.A. FIALKOW, A note on the range of the operator $X \mapsto AX - XB$, *Illinois J. Math.* **25**(1981), 112–124.
7. D.A. HERRERO, Compact perturbations of nest algebras, index obstructions and a problem of Arveson, *J. Funct. Anal.* **55**(1984), 78–109.
8. D.A. HERRERO, Spectral pictures of operators in the Cowen-Douglas class $\mathcal{B}_n(\Omega)$ and its closure, *J. Operator Theory* **10**(1987), 213–222.
9. D.A. HERRERO, *Approximation of Hilbert space operators.* I, 2ed ed., Res. Notes Math., vol. 224, Longman, Harlow–Essex 1990.
10. D.A. HERRERO, A unicellular universal quasinilpotent operator, *Proc. Amer. Math. Soc.* **110**(1990), 649–652.
11. D.A. HERRERO, C.L. JIANG, Limits of strongly irreducible operators and the Riesz decomposition theorem, *Michigan Math. J.* **37**(1990), 283–291.
12. Y.Q. JI, C.L. JIANG, Z.Y. WANG, Strongly irreducible operators in nest algebras, *Integral Equations Operator Theory* **28**(1997), 28–44.
13. Y.Q. JI, C.L. JIANG, Z.Y. WANG, Strongly irreducible operators in nest algebras with well-ordered nest, *Michigan Math. J.* **44**(1997), 85–98.
14. Y.Q. JI, C.L. JIANG, Z.Y. WANG, Strongly irreducible operators in continuous nest, to appear.
15. C.L. JIANG, Strongly irreducible operator and Cowen-Douglas operators, *Northeast. Math. J.* **1**(1991), 1–3.
16. C.L. JIANG, Z.Y. WANG, The spectral picture and the closure of the similarity orbit of strongly irreducible operators, *Integral Equations Operator Theory* **24**(1996), 81–105.
17. C.L. JIANG, S.H. SUN, Z.Y. WANG, Essentially normal operator+compact operator = strongly irreducible operator, *Trans. Amer. Math. Soc.* **349**(1997), 217–233.

YOUQING JI
Department of Mathematics
Jilin University
Changchun, 130023
P.R. CHINA

CHUNLAN JIANG
Dept. of Applied Math. and Physics
Hebei University of Technology
Tianjin, 300103
P.R. CHINA

ZONGYAO WANG
Department of Mathematics
East China University
of Science and Technology
Shanghai, 200237
P.R. CHINA

Received January 16, 1998.