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GEOMETRY OF HIGHER ORDER RELATIVE SPECTRA AND PROJECTION METHODS

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Abstract. Let H be a densely defined linear operator acting on a Hilbert space $\mathcal{H}$, let P be the orthogonal projection onto a closed linear subspace $\mathcal{L}$ and let $n \in \mathbb{N}$. The n-th order spectrum $\mathrm{Spec}_n(H, \mathcal{L})$ of H relative to $\mathcal{L}$ is the set of $z \in \mathbb{C}$ such that the restriction to $\mathcal{L}$ of the operator $P(H - zI)^n P$ is not invertible within the subspace $\mathcal{L}$. We study restrictions which may be placed on this set under given assumptions on $\mathrm{Spec}(H)$ and the behaviour of $\mathrm{Spec}_n(H, \mathcal{L})$ as $\mathcal{L}$ increases towards $\mathcal{H}$.

Keywords: *Higher order relative spectra, orthogonal projections, projection methods.*

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1. INTRODUCTION

Let H be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. In order to determine the spectrum $\mathrm{Spec}(H)$ of H, one can use the following projection method. Let $(\mathcal{L}_k)$ be a sequence of subspaces of $\mathcal{H}$ and suppose that the corresponding orthogonal projections $P_k : \mathcal{H} \to \mathcal{L}_k$ converge strongly to the identity operator. Is it true that

$$\lim_{k\to\infty} \mathrm{Spec}_1(H, \mathcal{L}_k) = \mathrm{Spec}(H), \tag{1.1}$$

where $\mathrm{Spec}_1(H, \mathcal{L}_k)$ is the spectrum of the restriction to $\mathcal{L}_k$ of the operator $P_k H P_k$ and "lim" is defined in an appropriate sense?

One can show that the inclusion

$$\lim_{k\to\infty} \mathrm{Spec}_1(H, \mathcal{L}_k) \supset \mathrm{Spec}(H),$$

is always true (see, e.g., [1], Theorem 2.3 or Lemma 7.2 below). Whether the equality holds depends on the essential spectrum $\mathrm{Spec}_\mathrm{e}(H)$ of H. If $\mathrm{Spec}_\mathrm{e}(H)$

is connected, then (1.1) is valid (see Corollary 7.3), but if $\mathrm{Spec_e}(H)$ is not connected, there exist sequences $(\mathcal{L}_k)$ such that $\lim_{k\to\infty} \mathrm{Spec}_1(H,\mathcal{L}_k)$ is strictly larger than $\mathrm{Spec}(H)$ (see Theorem 6.1 and (2.10)). The projection methods of approximating $\mathrm{Spec}(H)$ by $\mathrm{Spec}_1(H,\mathcal{L}_k)$ may lead to spurious points laying in gaps of $\mathrm{Spec_e}(H)$. In this paper we show that projection methods using higher order relative spectra are free from this defect (see Theorem 7.1 and Corollaries 3.4, 4.2).

DEFINITION 1.1. Let H be a densely defined linear operator acting on a Hilbert space $\mathcal{H}$ and let P be the orthogonal projection onto a closed linear subspace $\mathcal{L}$. We assume either that H is bounded or that $\mathcal{L}$ is finite-dimensional and contained in the domain of H^n for a given $n \in \mathbb{N}$. Let $M_n(z)$ denote the restriction to $\mathcal{L}$ of the operator $P(H - zI)^n P$, $z \in \mathbb{C}$. We define the n-th order spectrum $\mathrm{Spec}_n(H,\mathcal{L})$ of H relative to $\mathcal{L}$ to be the set of z such that the operator $M_n(z)$ is not invertible within the subspace $\mathcal{L}$.

This definition is due to E.B. Davies, who suggested that second order relative spectra might be useful for approximate computation of spectra of self-adjoint operators and started the study of their connections with complex resonances (see [2], Section 9).

$\mathrm{Spec}_n(H,\mathcal{L})$ is a non-empty compact set (see [2], Theorem 17) which may contain complex numbers even if H is self-adjoint. In Sections 6–7 we investigate the behaviour of $\mathrm{Spec}_n(H,\mathcal{L})$ as $\mathcal{L}$ increases towards $\mathcal{H}$. In particular, we prove that for any bounded self-adjoint operator H

$$\bigcup \lim_{k\to+\infty} \mathrm{Spec}_n(H,\mathcal{L}_k) \cap \mathbb{R} = \mathrm{Spec}(H) \qquad \text{if } n \text{ is even,}$$

where the union is taken over the set of all sequences $(\mathcal{L}_k)$ such that the orthogonal projections onto $\mathcal{L}_k$ converge strongly to the identity operator (see Theorem 7.1).

Sections 3–5 are devoted to the geometry of $\mathrm{Spec}_n(H,\mathcal{L})$ for a fixed $\mathcal{L}$. More precisely, we study restrictions which may be placed on this set under given assumptions on $\mathrm{Spec}(H)$ and in particular give an answer to a question posed in [2], Section 9. The main conclusion of this part of the analysis is Theorem 3.1, providing sharp restrictions on the possible location of $\mathrm{Spec}_n(H,\mathcal{L})$ when H is a bounded normal operator. Theorem 3.1 implies the following simple result in the case when H is self-adjoint and $n = 2$ (see also Theorem 2.6 and Remarks 3.2, 3.3).

Let $\Sigma \subset \mathbb{R}$ be an arbitrary compact set,

$$a := \min\Sigma, \quad b := \max\Sigma, \quad [a,b]\backslash\Sigma = \bigcup_l]a_l,b_l[\,, \quad]a_l,b_l[\,\cap\,]a_k,b_k[\, = \emptyset \quad \textit{if } l \neq k,$$

and let $B(c_1,c_2)$ denote the closed disk with the diameter $[c_1,c_2]$. If H is a self-adjoint operator and $\mathrm{Spec}(H) \subset \Sigma$, *then*

$$\mathrm{Spec}_2(H,\mathcal{L}) \subset B(a,b) \setminus \bigcup_l \operatorname{Int} B(a_l,b_l).$$

For any point z belonging to the right-hand side there exists a self-adjoint operator H acting on $\mathcal{H} = \mathbb{C}^2$ and a one-dimensional subspace $\mathcal{L} \subset \mathbb{C}^2$ such that

$$\operatorname{Spec}(H) \subset \Sigma \quad \textit{and} \quad z \in \operatorname{Spec}_2(H, \mathcal{L}).$$

In order to formulate the restrictions on $\operatorname{Spec}_n(H, \mathcal{L})$ when $\mathcal{L}$ is fixed and characterize its behaviour as $\mathcal{L}$ increases towards $\mathcal{H}$ in the case of normal operators (see Theorems 3.1 and 6.1), we have to define for an arbitrary compact set Σ the set $\mathcal{Q}_n(\Sigma)$ (see (2.6) and (2.7)), whose geometric structure is analysed in Section 2. The simplest results of this section are Theorems 2.6 and 2.8.

Throughout the paper, except Section 4, we deal only with bounded linear operators.

2. AUXILIARY GEOMETRIC RESULTS

Let $K \subset \mathbb{C} \setminus \{0\}$ be an arbitrary compact set,

$$\sigma(K) := \left\{ \frac{w}{|w|} : w \in K \right\}.$$

It is clear that $\sigma(K)$ lies in the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. The set $\sigma(K)$ is compact as a continuous image of a compact set. Then $\Delta(K) := \mathbb{T} \setminus \sigma(K)$ is an open set in $\mathbb{T}$. Consequently, $\Delta(K)$ is a union of an at most countable family of open arcs. Let us denote the length of the longest of these arcs by $\alpha(K)$.

For an arbitrary compact set $\Sigma \subset \mathbb{C}$ we introduce the set

$$\begin{cases} \mathcal{R}_\rho(\Sigma) := \Sigma \cup \left\{ z \in \mathbb{C} \setminus \Sigma : \alpha(\Sigma - z) \leqslant (2 - \frac{1}{\rho})\pi \right\} & \text{if } \rho \geqslant 1, \\ \mathcal{R}_\rho(\Sigma) := \emptyset & \text{if } \rho < 1. \end{cases} \tag{2.1}$$

It is clear that $z \notin \mathcal{R}_\rho(\Sigma)$, $\rho \geqslant 1$, iff Σ is seen from z at the angle less than π/ρ. In particular, $z \notin \mathcal{R}_1(\Sigma)$ iff Σ lies in some closed half-plane not containing z, i.e. iff z does not belong to the convex hull of Σ (see, e.g., [7], Theorem V.4 (c)). Thus

$$\mathcal{R}_1(\Sigma) = \operatorname{conv}(\Sigma). \tag{2.2}$$

In the case $\Sigma = [a, b] \subset \mathbb{R}$ the set

$$\mathcal{R}_\rho([a, b]) = \mathcal{R}_\rho(\{a, b\}) = [a, b] \cup \left\{ z \in \mathbb{C} \setminus [a, b] : \left| \arg \frac{b - z}{a - z} \right| \geqslant \frac{\pi}{\rho} \right\}, \tag{2.3}$$

$\rho \geqslant 1$, where

$$-\pi < \arg \cdots \leqslant \pi, \tag{2.4}$$

is bounded by two mutually symmetric circular arcs.

It is easily seen that

$$\mathcal{R}_\rho(\Sigma_0) \subset \mathcal{R}_\rho(\Sigma) \quad \text{if} \quad \Sigma_0 \subset \Sigma. \tag{2.5}$$

The set $\mathcal{R}_\rho(\Sigma)$ has a very simple structure.

Theorem 2.1. *Let $\Sigma \subset \mathbb{C}$ be an arbitrary compact set, $\rho \geqslant 1$. Then:*
(i) $\mathcal{R}_\rho(\Sigma) = \mathcal{R}_\rho(\mathrm{conv}(\Sigma))$*;*
(ii) $\mathcal{R}_\rho(\Sigma)$ *is arcwise connected;*
(iii) $\mathbb{C} \setminus \mathcal{R}_\rho(\Sigma)$ *is arcwise connected.*

Proof. (i) It follows from (2.5) that $\mathcal{R}_\rho(\Sigma) \subset \mathcal{R}_\rho(\mathrm{conv}(\Sigma))$.

Let us take an arbitrary $z \notin \mathcal{R}_\rho(\Sigma)$. The compact set Σ lies in an angle with vertex z and size less than π/ρ. This angle is a convex set. So, $\mathrm{conv}(\Sigma)$ lies in this angle too. Consequently, $z \notin \mathcal{R}_\rho(\mathrm{conv}(\Sigma))$.

(ii) According to (i) we may suppose that Σ is convex. In this case $\sigma(\Sigma - z)$ is connected for an arbitrary $z \notin \Sigma$ as a continuous image of a connected set. Thus $\sigma(\Sigma - z)$ is a closed arc and it easily follows from (2.1) that

$$\forall\, z_1, z_2 \in \mathcal{R}_\rho(\Sigma) \quad \exists\, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Sigma : z_1 \in \mathcal{R}_\rho(\{\lambda_1, \lambda_2\}),\ z_2 \in \mathcal{R}_\rho(\{\lambda_3, \lambda_4\}).$$

Let us connect z_1 with λ_2 by a curve lying in $\mathcal{R}_\rho(\{\lambda_1, \lambda_2\}) \subset \mathcal{R}_\rho(\Sigma)$ (see (2.3) and (2.5)). Then let us connect λ_2 with λ_3 by a curve lying in $\mathcal{R}_\rho(\{\lambda_2, \lambda_3\}) \subset \mathcal{R}_\rho(\Sigma)$ (we may take the segment $[\lambda_2, \lambda_3]$). Finally, let us connect λ_3 with z_2 by a curve lying in $\mathcal{R}_\rho(\{\lambda_3, \lambda_4\}) \subset \mathcal{R}_\rho(\Sigma)$. So, for arbitrary z_1 and z_2 from $\mathcal{R}_\rho(\Sigma)$ we have constructed a curve which connects this two points and lies in $\mathcal{R}_\rho(\Sigma)$.

(iii) The compact set Σ is seen at small angles from points lying sufficiently far from it. Thus, the complement of a sufficiently large disk is an arcwise connected subset of $\mathbb{C} \setminus \mathcal{R}_\rho(\Sigma)$ and the statement will be proved if we construct a curve connecting an arbitrary point $z \in \mathbb{C} \setminus \mathcal{R}_\rho(\Sigma)$ with a point of this subset and lying in $\mathbb{C} \setminus \mathcal{R}_\rho(\Sigma)$. The set Σ lies in some angle with vertex at z and size less than π/ρ and it is clear that the ray opposite to the bisector of this angle lies in $\mathbb{C} \setminus \mathcal{R}_\rho(\Sigma)$. ∎

Let $\Sigma \subset \mathbb{C}$ be a compact set, $z \in \mathbb{C}$, $n \in \mathbb{N}$,

$$\Sigma_z^n := \{(\lambda - z)^n : \lambda \in \Sigma\}\,, \tag{2.6}$$

$$\mathcal{Q}_n(\Sigma) := \{z \in \mathbb{C} : 0 \in \mathrm{conv}(\Sigma_z^n)\}\,. \tag{2.7}$$

It follows from the definition that $\Sigma \subset \mathcal{Q}_n(\Sigma)$ and

$$\mathcal{Q}_n(\Sigma_0) \subset \mathcal{Q}_n(\Sigma) \quad \text{if} \quad \Sigma_0 \subset \Sigma. \tag{2.8}$$

Theorem 2.2. $\mathcal{Q}_n(\Sigma)$ *is arcwise connected.*

Proof. Let us first prove that the set $\mathcal{Q}_n(\{\lambda_1, \lambda_2, \lambda_3\})$ is arcwise connected for arbitrary $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. It is clear that the circular arcs

$$\left\{ w \in \mathbb{C} : \arg \frac{\lambda_j - w}{\lambda_k - w} = \frac{2l-1}{n}\pi \right\}, \quad j,k = 1,2,3,\ l = 1, \dots, \left[\frac{n+1}{2}\right], \tag{2.9}$$

lie in $\mathcal{Q}_n(\{\lambda_1, \lambda_2, \lambda_3\})$. So, it is sufficient to prove that for any $z \in \mathcal{Q}_n(\{\lambda_1, \lambda_2, \lambda_3\})$ there exists a curve connecting z with one of the points $\lambda_1, \lambda_2, \lambda_3$ and lying in $\mathcal{Q}_n(\{\lambda_1, \lambda_2, \lambda_3\})$.

If 0 belongs to the boundary of the triangle with the vertices $(\lambda_j - z)^n$, $j = 1,2,3$, then z belongs to the arc (2.9) for some j, k and l. In this case, we may take a part of it to play the role of the connecting curve. Let now 0 belong to the interior of the triangle $\mathrm{conv}(\{(\lambda_j - z)^n\}_{j=1}^3)$. Then there exists $x_0 > 0$ such that 0 belongs to the interior of the triangle $\mathrm{conv}(\{(\lambda_j - z - x)^n\}_{j=1}^3)$ if $0 \leqslant x < x_0$ and to its boundary if $x = x_0$. It is clear that the curve consisting of the segment

$[z, z+x_0]$ and of a part of the corresponding arc (2.9) has all necessary properties. Thus we have proved that $\mathcal{Q}_n(\{\lambda_1, \lambda_2, \lambda_3\})$ is an arcwise connected set.

Let us take an arbitrary $z_1 \in \mathcal{Q}_n(\Sigma)$. According to the definition (2.7), $0 \in \operatorname{conv}(\Sigma^n_{z_1})$. Then 0 lies in the convex hull of some subset of $\Sigma^n_{z_1}$ that contains at most 3 points (see, e.g., [8], 3.25, Lemma), i.e. there exist points $\lambda_1, \lambda_2, \lambda_3 \in \Sigma$ such that $0 \in \operatorname{conv}(\{(\lambda_j - z_1)^n\}^3_{j=1})$, i.e. $z_1 \in \mathcal{Q}_n(\{\lambda_1, \lambda_2, \lambda_3\})$. Analogously, we have $\forall\, z_2 \in \mathcal{Q}_n(\Sigma)$, $\exists\, \lambda_4, \lambda_5, \lambda_6 \in \Sigma:\ z_2 \in \mathcal{Q}_n(\{\lambda_4, \lambda_5, \lambda_6\})$. Now acting as in the proof of Theorem 2.1 (ii), we can construct a curve lying in $\mathcal{Q}_n(\Sigma)$ and connecting z_1 with z_2 in three steps. ∎

It is clear that

$$\mathcal{Q}_1(\Sigma) = \operatorname{conv}(\Sigma) = \mathcal{R}_1(\Sigma) \tag{2.10}$$

(see (2.2)). For arbitrary $n \in \mathbb{N}$ we have the following result.

Theorem 2.3. *Let $\Sigma \subset \mathbb{C}$ be an arbitrary compact set, $n \in \mathbb{N}$. Then:*
(i) $\mathcal{Q}_n(\Sigma) \subset \mathcal{R}_n(\Sigma)$;
(ii) $\mathcal{Q}_n(\Sigma) = \mathcal{R}_n(\Sigma)$ *if Σ is connected;*
(iii) $\mathcal{R}_n(\Sigma) = \mathcal{Q}_n(\operatorname{conv}(\Sigma))$.

Proof. (i) Let us take an arbitrary $z \notin \mathcal{R}_n(\Sigma)$. Then the set $\Sigma - z$ lies in an open angle with vertex at 0 and size less than π/n. Consequently Σ^n_z lies in an open half-plane whose boundary contains 0. Thus $0 \notin \operatorname{conv}(\Sigma^n_z)$, i.e. $z \notin \mathcal{Q}_n(\Sigma)$.

(ii) Let Σ be connected and $z \in \mathcal{R}_n(\Sigma)$. If $z \in \Sigma$, then clearly $z \in \mathcal{Q}_n(\Sigma)$. Let now z belong to $\mathcal{R}_n(\Sigma) \setminus \Sigma$. Then

$$\alpha(\Sigma - z) \leqslant \left(2 - \frac{1}{n}\right)\pi. \tag{2.11}$$

The compact set $\sigma(\Sigma - z)$ is connected as a continuous image of a connected set. Therefore $\sigma(\Sigma - z)$ is a closed arc. It follows from (2.11) that the length of this arc is not less than π/n. Consequently, there exist $\lambda_1, \lambda_2 \in \Sigma$ such that

$$\arg\frac{\lambda_1 - z}{\lambda_2 - z} = \frac{\pi}{n}, \quad \text{i.e., } \arg\frac{(\lambda_1 - z)^n}{(\lambda_2 - z)^n} = \pi.$$

The last equality means that $0 \in [(\lambda_1 - z)^n, (\lambda_2 - z)^n] \subset \operatorname{conv}(\Sigma^n_z)$, i.e. $z \in \mathcal{Q}_n(\Sigma)$.

(iii) The equality follows from (ii), Theorem 2.1 (i) and the fact that $\operatorname{conv}(\Sigma)$ is connected. ∎

The statement of Theorem 2.3 (ii) is not generally true if Σ is not connected. The set $\mathcal{Q}_n(\Sigma)$ may have rather complicated structure, unlike the set $\mathcal{R}_n(\Sigma)$. For example, if Σ consists of two points a and b, the set $\mathcal{Q}_n(\Sigma) = \mathcal{Q}_n(\{a, b\})$ is the union of the circular arcs

$$\left\{w \in \mathbb{C} : \left|\arg\frac{b - w}{a - w}\right| = \frac{2l - 1}{n}\pi\right\}, \quad l = 1, \ldots, \left[\frac{n+1}{2}\right],$$

(cf. (2.3), (2.4)). Gaps in Σ may cause gaps in $\mathcal{Q}_n(\Sigma)$. We will demonstrate this effect in the case $\Sigma \subset \mathbb{R}$.

Suppose $-\infty < a \leqslant a_0 < b_0 \leqslant b < +\infty$. Let

$$\begin{aligned} S^m_n(a, a_0, b_0, b) :=& \operatorname{Int} \mathcal{R}_{\frac{n}{2m-1}}([a_0, b_0]) \setminus (\mathcal{R}_{\frac{n}{2m+1}}([a, b]) \\ &\cup \mathcal{R}_n([a, a_0]) \cup \mathcal{R}_n([b_0, b])), \end{aligned} \tag{2.12}$$

$m = 1, \ldots, \left[\frac{n}{2}\right]$, where "Int" denotes the interior of the corresponding set.

THEOREM 2.4. *Let* $\Sigma = [a, a_0] \cup [b_0, b]$, $n \in \mathbb{N}$. *Then*

$$\mathcal{Q}_n(\Sigma) = \mathcal{R}_n([a,b]) \setminus \bigcup_{m=1}^{\left[\frac{n}{2}\right]} S_n^m(a, a_0, b_0, b). \tag{2.13}$$

Proof. We will first prove that the LHS of (2.13) is a subset of the RHS. Let us take an arbitrary z not belonging to the RHS of (2.13). Then

$$z \notin \mathcal{R}_n([a,b]) \quad \text{or} \quad z \in \bigcup_{m=1}^{\left[\frac{n}{2}\right]} S_n^m(a, a_0, b_0, b).$$

In the first case $z \notin \mathcal{Q}_n(\Sigma)$ according to Theorem 2.3 (i). So, we have to consider the second case, i.e. the case

$$z \in S_n^m(a, a_0, b_0, b) \quad \text{for some } m = 1, \ldots, \left[\frac{n}{2}\right]. \tag{2.14}$$

Suppose that

$$\alpha(\Sigma_z^n) \leqslant \pi \tag{2.15}$$

(see (2.6)). It is clear that $\Delta(\Sigma_z^n)$ consists of two arcs at most. If $\Delta(\Sigma_z^n)$ is empty or consists of one arc, then it is easy to see that (2.15) implies the existence two points $\mu_1, \mu_2 \in \sigma(\Sigma_z^n)$ such that $\mu_1 = -\mu_2$. If $\Delta(\Sigma_z^n)$ consists of two open arcs l_1, l_2, then $\sigma(\Sigma_z^n)$ consists of two closed arcs γ_1 and γ_2 (one or both of them may be degenerate, i.e. contain only one point). The set $-\gamma_1$ cannot be covered by l_1 or l_2, because in this case the covering arc would contain a point diametrically opposite to its end-point and, consequently, its length would be greater than π. This, however, contradicts (2.15). Therefore, the connected set $-\gamma_1$ cannot be covered by the union of the disjoint open arcs l_1 and l_2. Thus

$$\exists\, \mu_1 \in \gamma_1 \subset \sigma(\Sigma_z^n): \quad \mu_2 = -\mu_1 \in \sigma(\Sigma_z^n). \tag{2.16}$$

So, (2.15) implies (2.16).

We can rewrite (2.16) in the following form

$$\exists\, \lambda_1, \lambda_2 \in [a, a_0] \cup [b_0, b]: \quad \arg \frac{(\lambda_1 - z)^n}{(\lambda_2 - z)^n} = \pi. \tag{2.17}$$

It follows from (2.3), (2.12) and (2.14) that if $\lambda_1, \lambda_2 \in [a, a_0]$, then the angle between $\lambda_1 - z$ and $\lambda_2 - z$ is less than π/n. This, however, contradicts (2.17). By the same reason we can exclude the case $\lambda_1, \lambda_2 \in [b_0, b]$. Let us suppose that $\lambda_1 \in [a, a_0]$ and $\lambda_2 \in [b_0, b]$ (or $\lambda_2 \in [a, a_0]$ and $\lambda_1 \in [b_0, b]$). Using the inclusion

$$z \in \operatorname{Int} \mathcal{R}_{\frac{n}{2m-1}}([a_0, b_0]) \setminus \mathcal{R}_{\frac{n}{2m+1}}([a,b])$$

(see (2.12), (2.14)), we obtain the following inequality for the angle β between $\lambda_1 - z$ and $\lambda_2 - z$

$$\frac{2m-1}{n}\pi < \beta < \frac{2m+1}{n}\pi. \tag{2.18}$$

It is clear that (2.18) contradicts (2.17).

So, we have proved that (2.17) leads to a contradiction. Therefore (2.15) leads to a contradiction. Consequently, we have proved that $\alpha(\Sigma_z^n) > \pi$, i.e. $0 \notin \mathcal{R}_1(\Sigma_z^n) = \operatorname{conv}(\Sigma_z^n)$, i.e. $z \notin \mathcal{Q}_n(\Sigma)$ (see (2.1), (2.2), (2.7)). Thus the LHS of (2.13) is a subset of the RHS.

Now we have to prove the opposite inclusion. Let us take an arbitrary z from the RHS of (2.13). There is nothing to prove if

$$z \in \mathcal{R}_n([a,a_0]) \cup \mathcal{R}_n([b_0,b]) = \mathcal{Q}_n([a,a_0]) \cup \mathcal{Q}_n([b_0,b]) \subset \mathcal{Q}_n(\Sigma)$$

(see Theorem 2.3 (ii) and (2.8)). If $z \notin \mathcal{R}_n([a,a_0]) \cup \mathcal{R}_n([b_0,b])$, then there exist

$$\lambda_1 \in [a,a_0], \quad \lambda_2 \in [b_0,b] \quad \text{and} \quad l = 1,\ldots,\left[\frac{n+1}{2}\right]$$

such that the angle between $\lambda_1 - z$ and $\lambda_2 - z$ equals $\frac{2l-1}{n}\pi$, because otherwise z would belong to $S_n^m(a,a_0,b_0,b)$ for some m. It is clear that for these λ_1 and λ_2 we have

$$\arg\frac{(\lambda_1 - z)^n}{(\lambda_2 - z)^n} = \pi,$$

i.e. $0 \in [(\lambda_1 - z)^n, (\lambda_2 - z)^n] \subset \operatorname{conv}(\Sigma_z^n)$, i.e. $z \in \mathcal{Q}_n(\Sigma)$. ∎

Remark 2.5. We have proved in fact that if z belongs to the LHS of (2.13), then there exist λ_1 and $\lambda_2 \in \Sigma$ such that

$$0 \in [(\lambda_1 - z)^n, (\lambda_2 - z)^n]. \tag{2.19}$$

A similar result holds in a more general situation. Namely, if a compact set $\Sigma \subset \mathbb{C}$ is connected or consists of two connected components, and $z \in \mathcal{Q}_n(\Sigma)$, then there exist λ_1 and $\lambda_2 \in \Sigma$ satisfying (2.19). Indeed, if $z \in \Sigma$, we may take $\lambda_1 = \lambda_2 = z$. If $z \in \mathcal{Q}_n(\Sigma) \setminus \Sigma$, the inclusion $0 \in \operatorname{conv}(\Sigma_z^n) = \mathcal{R}_1(\Sigma_z^n)$ (see (2.2)) is equivalent to $\alpha(\Sigma_z^n) \leqslant \pi$ (see (2.1)) and we can repeat a part of the proof of Theorem 2.4 (from (2.15) to (2.17)).

Above we considered the case of one gap in Σ. Let now $\Sigma \subset \mathbb{R}$ be an arbitrary compact set,

$$a = \min\Sigma, \quad b = \max\Sigma. \tag{2.20}$$

The open set $]a,b[\setminus\Sigma$ is a union of an at most countable family of mutually disjoint open intervals

$$]a,b[\setminus\Sigma = \bigcup_l\,]a_l,b_l[\,. \tag{2.21}$$

Theorem 2.4 implies

$$\mathcal{Q}_n(\Sigma) \subset \mathcal{R}_n([a,b]) \setminus \bigcup_l \bigcup_{m=1}^{\left[\frac{n}{2}\right]} S_n^m(a,a_l,b_l,b). \tag{2.22}$$

The equality in (2.22) generally does not hold if $n \geqslant 4$. Let for example $\Sigma = \{0, \pm 1\}$ and $z = \mathrm{i}\cot\frac{7\pi}{4n}$, $n \geqslant 4$. Taking into account the fact that the angle between $\pm 1 - z$ and $0 - z = -z$ equals $\frac{7\pi}{4n}$, we can easily see that

$$z \in \mathcal{R}_n([-1,0]) \cap \mathcal{R}_n([0,1]), \quad 0 \notin \operatorname{conv}\{(\pm 1 - z)^n, (-z)^n\}.$$

Consequently, z belongs to the RHS of (2.22) (see (2.12)) and does not belong to the LHS (see (2.6), (2.7)).

The reason of the described effect is that different gaps in Σ may "interact" and cause new gaps in $\mathcal{Q}_n(\Sigma)$, which cannot be predicted by Theorem 2.4. However, this does not happen if $n = 2$ or 3 (see also (2.10)).

For arbitrary $c_1, c_2 \in \mathbb{R}$, let $B(c_1, c_2)$ be the closed disk with the diameter $[c_1, c_2]$.

THEOREM 2.6. *Let $\Sigma \subset \mathbb{R}$ be an arbitrary compact set. Then* $\mathcal{Q}_2(\Sigma) = B(a,b) \setminus \bigcup_l \operatorname{Int} B(a_l, b_l)$ *(see* (2.20), (2.21)*).*

Proof. Note that $\mathcal{R}_2([a,b]) = B(a,b)$ and $S_2^1(a, a_l, b_l, b) = \operatorname{Int} B(a_l, b_l)$ (see (2.12)). According to (2.22) it is sufficient to prove that an arbitrary z from the RHS belongs to $\mathcal{Q}_2(\Sigma)$. It is easily seen that $\alpha(\Sigma_z^2) \leqslant \pi$. Consequently, $0 \in \mathcal{R}_1(\Sigma_z^2) = \operatorname{conv}(\Sigma_z^2)$ (see (2.1), (2.2)), i.e. $z \in \mathcal{Q}_2(\Sigma)$. ∎

THEOREM 2.7. *Let $\Sigma \subset \mathbb{R}$ be an arbitrary compact set. Then* $\mathcal{Q}_3(\Sigma) = \mathcal{R}_3([a,b]) \setminus \bigcup_l S_3^1(a, a_l, b_l, b)$ *(see* (2.20), (2.21)*).*

Proof. According to (2.22) it is sufficient to prove that an arbitrary z from the RHS belongs to $\mathcal{Q}_3(\Sigma)$. If $z \in [a,b]$, then $z \in \mathcal{Q}_3(\Sigma)$ (see Theorem 2.8 (ii) below). Let now z belong to $\mathcal{R}_3([a,b]) \setminus [a,b]$.

If each interval $]a_l, b_l[$ is seen from z at an angle less or equal to $\pi/3$, then $\alpha(\Sigma_z^3) \leqslant \pi$. Consequently, $0 \in \operatorname{conv}(\Sigma_z^3)$, i.e. $z \in \mathcal{Q}_3(\Sigma)$.

If there exists l such that the interval $]a_l, b_l[$ is seen from z at an angle $(\pi/3) + \beta$, $\beta > 0$, then either $[a, a_l]$ or $[b_l, b]$ is seen from z at an angle greater or equal to $\pi/3$, because otherwise z would belong to $S_3^1(a, a_l, b_l, b)$ (see (2.12)). Let us suppose for definiteness that $[a, a_l]$ is seen from z at angle $(\pi/3) + \gamma$, $\gamma \geqslant 0$. Note that $(\pi/3) + \beta + (\pi/3) + \gamma < \pi$. Consequently, $\beta + \gamma < \pi/3$ and $\beta, \gamma < \pi/3$. It is easy to see that

$$\alpha(\{a, a_l, b_l\}_z^3) = \max\{\pi - 3\beta, \pi - 3\gamma, 3(\beta + \gamma)\} \leqslant \pi.$$

Therefore, $0 \in \operatorname{conv}(\{a, a_l, b_l\}_z^3) \subset \operatorname{conv}(\Sigma_z^3)$, i.e. $z \in \mathcal{Q}_3(\Sigma)$. ∎

THEOREM 2.8. *Let $\Sigma \subset \mathbb{R}$ be an arbitrary compact set. Then:*
(i) $\mathcal{Q}_n(\Sigma) \cap \mathbb{R} = \Sigma$ *if n is even;*
(ii) $\mathcal{Q}_n(\Sigma) \cap \mathbb{R} = \operatorname{conv}(\Sigma)$ *if n is odd.*

Proof. (i) Since $\Sigma \subset \mathcal{Q}_n(\Sigma)$, we have to prove only that the LHS of (i) is a subset of Σ. Let us take an arbitrary $z \in \mathbb{R} \setminus \Sigma$. Then

$$(\lambda - z)^n \geqslant \text{const} > 0, \quad \forall\, \lambda \in \Sigma.$$

Consequently, $0 \notin \operatorname{conv}(\Sigma_z^n)$, i.e. $z \notin \mathcal{Q}_n(\Sigma)$.

(ii) Let us take an arbitrary $z \in [a,b] := \operatorname{conv}(\Sigma)$. It is clear that

$$(a - z)^n \leqslant 0 \leqslant (b - z)^n.$$

Therefore, $0 \in [(a-z)^n, (b-z)^n] \subset \operatorname{conv}(\Sigma_z^n)$, i.e. $z \in \mathcal{Q}_n(\Sigma)$. So, the RHS of (ii) is a subset of the LHS.

Now let us take an arbitrary $z \in \mathbb{R} \setminus [a, b]$. If $z > b$, then

$$-(\lambda - z)^n \geqslant \text{const} > 0, \quad \forall \lambda \in \Sigma.$$

If $z < a$, then

$$(\lambda - z)^n \geqslant \text{const} > 0, \quad \forall \lambda \in \Sigma.$$

In both cases $0 \notin \operatorname{conv}(\Sigma_z^n)$, i.e. $z \notin \mathcal{Q}_n(\Sigma)$. Thus, the LHS of (ii) is a subset of the RHS. ∎

3. ENCLOSURES FOR HIGHER ORDER RELATIVE SPECTRA OF NORMAL AND SELF-ADJOINT OPERATORS

Theorem 3.1. *Let $\Sigma \subset \mathbb{C}$ be an arbitrary compact set and $n \in \mathbb{N}$.*

(i) *If H is a normal operator and $\operatorname{Spec}(H) \subset \Sigma$, then*

$$\operatorname{Spec}_n(H, \mathcal{L}) \subset \mathcal{Q}_n(\Sigma); \tag{3.1}$$

(ii) *For an arbitrary $z \in \mathcal{Q}_n(\Sigma)$ there exists a normal operator H acting on $\mathcal{H} = \mathbb{C}^3$ and a one-dimensional subspace $\mathcal{L} \subset \mathbb{C}^3$ such that*

$$\operatorname{Spec}(H) \subset \Sigma \quad \textit{and} \quad z \in \operatorname{Spec}_n(H, \mathcal{L}). \tag{3.2}$$

Proof. (i) Let us take any $z \in \mathbb{C} \setminus \mathcal{Q}_n(\Sigma)$. According to (2.7), $0 \notin \operatorname{conv}(\Sigma_z^n)$. Then Σ_z^n lies in some open half-plane $\{w \in \mathbb{C} : \operatorname{Re}(e^{i\theta_0} w) > 0\}$, $\theta_0 \in]-\pi, \pi]$ (see, e.g., [7], Theorem V.4 (c)). Therefore

$$\operatorname{Re}(e^{i\theta_0}(\lambda - z)^n) > 0, \quad \forall \lambda \in \Sigma$$

(see (2.6)). The compactness of Σ implies the existence of $c > 0$ such that

$$\operatorname{Re}(e^{i\theta_0}(\lambda - z)^n) \geqslant c > 0, \quad \forall \lambda \in \operatorname{Spec}(H) \subset \Sigma.$$

Now applying the spectral theorem we obtain

$$\begin{aligned} \operatorname{Re}(e^{i\theta_0} M_n(z)u, u) &= \operatorname{Re}(e^{i\theta_0}(H - zI)^n Pu, Pu) = \operatorname{Re}(e^{i\theta_0}(H - zI)^n u, u) \\ &= \int_{\operatorname{Spec}(H)} \operatorname{Re}(e^{i\theta_0}(\lambda - z)^n)\, d(E(\lambda)u, u) \\ &\geqslant c \int_{\operatorname{Spec}(H)} d(E(\lambda)u, u) = c\|u\|^2, \quad \forall u \in \mathcal{L}. \end{aligned}$$

It is well known that the inequality

$$\operatorname{Re}(e^{i\theta_0} M_n(z)u, u) \geqslant c\|u\|^2, \quad \forall u \in \mathcal{L}, \tag{3.3}$$

implies invertibility of $M_n(z)$ within $\mathcal{L}$. Thus, we have proved that if $z \notin \mathcal{Q}_n(\Sigma)$, then $z \notin \operatorname{Spec}_n(H, \mathcal{L})$.

(ii) According to the definition (2.7), $0 \in \operatorname{conv}(\Sigma_z^n)$. Then 0 lies in the convex hull of some subset of Σ_z^n that contains at most 3 points (see, e.g., [8], 3.25, Lemma), i.e. there exist the points $\lambda_1, \lambda_2, \lambda_3 \in \Sigma$ and $t_1, t_2, t_3 \geqslant 0$ such that $t_1 + t_2 + t_3 = 1$ and

$$\sum_{j=1}^{3} t_j(\lambda_j - z)^n = 0. \tag{3.4}$$

Let

$$H = \operatorname{diag}(\lambda_j)_{j=1}^3, \tag{3.5}$$

$$\mathcal{L} = \Big\{u \in \mathbb{C}^3 : u_j = t_j^{1/2} w,\, w \in \mathbb{C},\, j = 1, 2, 3\Big\}. \tag{3.6}$$

Then, for an arbitrary $u \in \mathcal{L}$, we have

$$((H - zI)^n u, u) = |w|^2 \sum_{j=1}^{3} t_j(\lambda_j - z)^n = 0$$

(see (3.4)). Therefore, $P(H - zI)^n P = 0$, where P is the orthogonal projection onto $\mathcal{L}$. Thus, $z \in \operatorname{Spec}_n(H, \mathcal{L})$. It is clear that $\operatorname{Spec}(H) = \{\lambda_1, \lambda_2, \lambda_3\} \subset \Sigma$. ∎

REMARK 3.2. Let $\Sigma \subset \mathbb{C}$ be an arbitrary compact set and let $z \in \mathcal{Q}_n(\Sigma)$ be such that 0 belongs to a segment $[(\lambda_1 - z)^n, (\lambda_2 - z)^n]$ for some $\lambda_1, \lambda_2 \in \Sigma$ (see Remark 2.5). Then the proof of Theorem 3.1 (ii) shows that there exist a normal operator H acting on $\mathcal{H} = \mathbb{C}^2$ and a one-dimensional subspace $\mathcal{L} \subset \mathbb{C}^2$ satisfying (3.2).

REMARK 3.3. If $\lambda_1, \lambda_2, \lambda_3$ from the proof of Theorem 3.1 (ii) belong to $\mathbb{R}$, then the operator (3.5) is self-adjoint. The same is true for the operator from the previous remark.

Using the results of Section 2 one can formulate obvious corollaries of Theorem 3.1, which describe enclosures for higher order relative spectra of normal and self-adjoint operators in more clear geometric terms (see also Remarks 3.2 and 3.3). We will not dwell on this.

COROLLARY 3.4. *Let H be a self-adjoint operator and $z \in \operatorname{Spec}_2(H, \mathcal{L})$. Then*

$$\operatorname{Spec}(H) \cap [\operatorname{Re} z - |\operatorname{Im} z|, \operatorname{Re} z + |\operatorname{Im} z|] \neq \emptyset. \tag{3.7}$$

Proof. Suppose the LHS of (3.7) is empty. Then there exists $\varepsilon > 0$ such that

$$\operatorname{Spec}(H) \cap [\operatorname{Re} z - |\operatorname{Im} z| - \varepsilon, \operatorname{Re} z + |\operatorname{Im} z| + \varepsilon] = \emptyset.$$

According to Theorems 2.6 and 3.1

$$\operatorname{Int} B(\operatorname{Re} z - |\operatorname{Im} z| - \varepsilon, \operatorname{Re} z + |\operatorname{Im} z| + \varepsilon)$$

does not intersect $\operatorname{Spec}_2(H, \mathcal{L})$. On the other hand, z belongs to this open disk and to $\operatorname{Spec}_2(H, \mathcal{L})$. The obtained contradiction proves that the LHS of (3.7) cannot be empty. ∎

4. A REMARK ON UNBOUNDED SELF-ADJOINT OPERATORS

Above we dealt with bounded operators only. Let now H be an arbitrary self-adjoint operator acting on $\mathcal{H}$.

Lemma 4.1. *If $a_0, b_0 \in \mathbb{R}$ and*

$$]a_0, b_0[\, \cap \operatorname{Spec}(H) = \emptyset,$$

then

$$\operatorname{Int} B(a_0, b_0) \cap \operatorname{Spec}_2(H, \mathcal{L}) = \emptyset.$$

Proof. Let us take an arbitrary $z \in \operatorname{Int} B(a_0, b_0)$. The interval $]a_0, b_0[$ is seen from z at an angle equal to $\pi/2 + \varepsilon$, where $\varepsilon \in]0, \pi/2]$. It is clear that the set

$$F = \{(\lambda - z)^2 : \lambda \in \operatorname{Spec}(H)\}$$

lies in an angle with vertex at 0 and size $\pi - 2\varepsilon$. The distance from 0 to F is obviously positive. Therefore there exist $\theta_0 \in] - \pi, \pi]$ and $c > 0$ such that

$$\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta_0}(\lambda - z)^2) \geqslant c, \quad \forall\, \lambda \in \operatorname{Spec}(H).$$

The remaining part of the proof coincides with that of Theorem 3.1 (i). ∎

Corollary 4.2. *If $z \in \operatorname{Spec}_2(H, \mathcal{L})$, then*

$$\operatorname{Spec}(H) \cap [\operatorname{Re} z - |\operatorname{Im} z|, \operatorname{Re} z + |\operatorname{Im} z|] \neq \emptyset.$$

Proof. Cf. Corollary 3.4. ∎

5. AN ESTIMATE FOR HIGHER ORDER RELATIVE SPECTRA OF BOUNDED OPERATORS

We start by an auxiliary result.

Lemma 5.1. *Let $x_0 \geqslant 0$, $y_0 > 0$, $x_0/y_0 = \cot\beta$, $0 < k \leqslant \sin\beta$. Then*

$$\sup\left\{\frac{x_0 + x}{y_0 - y} : x, y \geqslant 0,\ x^2 + y^2 \leqslant k^2(x_0^2 + y_0^2)\right\} = \cot(\beta - \arcsin k). \tag{5.1}$$

Proof. For the fixed (x_0, y_0), the set

$$\{(x_0 + x, y_0 - y) : x^2 + y^2 \leqslant k^2(x_0^2 + y_0^2)\}$$

is obviously a disk with the centre (x_0, y_0). The condition $k \leqslant \sin\beta$ implies that the interior of this disk does not intersect the abscissae axis.

Let us draw the tangent to the corresponding circle passing through the origin and lying below the disk. Let (x_1, y_1) be the point of contact. An elementary geometric argument shows that the LHS of (5.1) equals

$$\frac{x_1}{y_1} = \cot(\beta - \arcsin k). \quad ∎$$

Theorem 5.2. (i) *Let H be an arbitrary bounded operator. Then*

$$|z| \leqslant \left(\sin\frac{\pi}{2n}\right)^{-1} \|H\|, \quad \forall\, z \in \operatorname{Spec}_n(H, \mathcal{L}). \tag{5.2}$$

(ii) *There exist a normal operator H acting on $\mathcal{H} = \mathbb{C}^2$ and a one-dimensional subspace $\mathcal{L}$ such that $\|H\| = 1$ and*

$$\sin^{-1} \frac{\pi}{2n} \in \operatorname{Spec}_n(H, \mathcal{L}). \tag{5.3}$$

Proof. (i) Let us fix an arbitrary $z \in \mathbb{C}$ such that

$$|z| > \left(\sin \frac{\pi}{2n}\right)^{-1} \|H\|. \tag{5.4}$$

It is convenient to rewrite $M_n(z)$ in the following form

$$M_n(z) = (-z)^n P(I - R)^n P, \tag{5.5}$$

where $R = z^{-1}H$,

$$\|R\| \leqslant \sin \frac{\pi(1-\varepsilon)}{2n} \tag{5.6}$$

for sufficiently small $\varepsilon > 0$.

Let us take an arbitrary $u \in \mathcal{L} \setminus \{0\}$ and consider the vectors

$$u_k = (I - R)^k u, \quad k = 0, 1, \ldots, n. \tag{5.7}$$

We have

$$\frac{\|Qu_k\|}{\|Pu_k\|} = \frac{\|Q(I-R)u_{k-1}\|}{\|P(I-R)u_{k-1}\|} \leqslant \frac{\|Qu_{k-1}\| + \|QRu_{k-1}\|}{\|Pu_{k-1}\| - \|PRu_{k-1}\|}, \quad k = 1, \ldots, n,$$

where $Q := I - P$. Taking into account that

$$\begin{aligned} \|QRu_{k-1}\|^2 + \|PRu_{k-1}\|^2 = \|Ru_{k-1}\|^2 &\leqslant \left(\sin \frac{\pi(1-\varepsilon)}{2n}\right)^2 \|u_{k-1}\|^2 \\ &= \left(\sin \frac{\pi(1-\varepsilon)}{2n}\right)^2 (\|Qu_{k-1}\|^2 + \|Pu_{k-1}\|^2) \end{aligned} \tag{5.8}$$

(see (5.6)) and applying Lemma 5.1 with

$$\begin{aligned} &x_0 = \|Qu_{k-1}\|, \quad y_0 = \|Pu_{k-1}\|, \\ &x = \|QRu_{k-1}\|, \quad y = \|PRu_{k-1}\|, \quad k = \sin \tfrac{\pi(1-\varepsilon)}{2n}, \end{aligned}$$

we obtain successively

$$\begin{aligned} &\operatorname{arccot} \frac{\|Qu_0\|}{\|Pu_0\|} = \operatorname{arccot} 0 = \frac{\pi}{2}, \\ &\operatorname{arccot} \frac{\|Qu_k\|}{\|Pu_k\|} \geqslant \operatorname{arccot} \frac{\|Qu_{k-1}\|}{\|Pu_{k-1}\|} - \frac{\pi(1-\varepsilon)}{2n} \geqslant \cdots \geqslant \frac{\pi}{2}\left(1 - \frac{k}{n}(1-\varepsilon)\right), \end{aligned}$$

$k = 1, \ldots, n$. Thus

$$\frac{\|Q(I-R)^n u\|}{\|P(I-R)^n u\|} \leqslant \cot \frac{\pi\varepsilon}{2}. \tag{5.9}$$

Further,

$$\frac{\|Q(I-R)^n u\|}{\|P(I-R)^n u\|} \geqslant \frac{\|(I-R)^n u\|}{\|P(I-R)^n u\|} - 1, \tag{5.10}$$

$$\begin{aligned} \|(I-R)^n u\| &\geqslant \|(I-R)^{n-1}u\| - \|R\|\,\|(I-R)^{n-1}u\| \\ &= (1-\|R\|)\|(I-R)^{n-1}u\| \geqslant \cdots \geqslant (1-\|R\|)^n\|u\| \\ &\geqslant \left(1-\sin\frac{\pi(1-\varepsilon)}{2n}\right)^n \|u\| \end{aligned} \tag{5.11}$$

(see (5.6)). It follows from (5.9)–(5.11) that

$$\|P(I-R)^n u\| \geqslant \left(1+\cot\frac{\pi\varepsilon}{2}\right)^{-1}\left(1-\sin\frac{\pi(1-\varepsilon)}{2n}\right)^n \|u\|.$$

So, there exists $c>0$ such that

$$\|M_n(z)u\| \geqslant c\|u\|, \quad \forall\, u \in \mathcal{L}\setminus\{0\}, \tag{5.12}$$

(see (5.4), (5.5)). This inequality implies that

$$\operatorname{Ker} M_n(z) = \{0\} \tag{5.13}$$

and the image of $M_n(z)$ is closed (see, e.g., [9], Chapter III, Theorem 5.1).

The image of the operator $P(H-zI)^nP : \mathcal{H}\to\mathcal{H}$ coincides with $\operatorname{Im} M_n(z)$ and, therefore, is closed. Consequently,

$$v \in \operatorname{Im} P(H-zI)^nP \iff (v,g)=0, \quad \forall\, g\in \operatorname{Ker} P(H^*-\bar{z}I)^nP \tag{5.14}$$

(see [9], Chapter III, Theorem 3.4 and note that in our case P is orthogonal, i.e. $P=P^*$).

We can prove analogously to (5.13) that

$$\operatorname{Ker} P(H^*-\bar{z}I)^nP = \operatorname{Ker} P. \tag{5.15}$$

Comparing (5.14), (5.15) with the relation

$$v\in \operatorname{Im} P \iff (v,g)=0, \quad \forall\, g \in \operatorname{Ker} P,$$

we obtain

$$\operatorname{Im} M_n(z) = \operatorname{Im} P(H-zI)^nP = \operatorname{Im} P = \mathcal{L}.$$

Thus according to the Banach theorem, $M_n(z)$ is invertible within $\mathcal{L}$ (see (5.13)).

(ii) Let

$$\lambda_1 = \sin\frac{\pi}{2n} - \mathrm{i}\cos\frac{\pi}{2n}, \quad \lambda_2 = \sin\frac{\pi}{2n} + \mathrm{i}\cos\frac{\pi}{2n}.$$

It is clear that

$$|\lambda_1| = |\lambda_2| = 1. \tag{5.16}$$

Further,

$$\begin{aligned} \frac{\lambda_1 - \sin^{-1}\frac{\pi}{2n}}{\lambda_2 - \sin^{-1}\frac{\pi}{2n}} &= \frac{(\sin\frac{\pi}{2n} - \sin^{-1}\frac{\pi}{2n}) - \mathrm{i}\cos\frac{\pi}{2n}}{(\sin\frac{\pi}{2n} - \sin^{-1}\frac{\pi}{2n}) + \mathrm{i}\cos\frac{\pi}{2n}} \\ &= \frac{-\cos^2\frac{\pi}{2n} - \mathrm{i}\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}}{-\cos^2\frac{\pi}{2n} + \mathrm{i}\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}} = \frac{\cos\frac{\pi}{2n} + \mathrm{i}\sin\frac{\pi}{2n}}{\cos\frac{\pi}{2n} - \mathrm{i}\sin\frac{\pi}{2n}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{n}}. \end{aligned}$$

Consequently, for $z = \sin^{-1}\frac{\pi}{2n}$, we have $(\lambda_1 - z)^n = -(\lambda_2 - z)^n$. Let

$$H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

and

$$\mathcal{L} = \left\{ u \in \mathbb{C}^2 : u_1 = u_2 \right\}.$$

It is clear that $\|H\| = 1$ (see (5.16)) and

$$((H - zI)^n u, u) = |u_1|^2((\lambda_1 - z)^n + (\lambda_2 - z)^n) = 0, \quad \forall\, u \in \mathcal{L}.$$

Therefore, $P(H - zI)^n P = 0$ and (5.3) holds. ∎

6. LIMIT BEHAVIOUR OF HIGHER ORDER RELATIVE SPECTRA

Above we studied the "statics" of higher order relative spectra, i.e. the case when the subspace $\mathcal{L}$ was fixed. Now we are going to consider their "dynamics". It is interesting to know what may happen to $\mathrm{Spec}_n(H, \mathcal{L})$ when $\mathcal{L}$ "tends" to $\mathcal{H}$. This question is connected to the theory of projection methods.

Let Λ be the set of all sequences of closed linear subspaces $\mathcal{L}_k \subset \mathcal{H}$, $k \in \mathbb{N}$, such that the corresponding orthogonal projections $P_k : \mathcal{H} \to \mathcal{L}_k$ converge strongly to the identity operator I as $k \to +\infty$. Let Λ_0 be the subset of Λ consisting of increasing sequences: $\mathcal{L}_k \subset \mathcal{L}_{k+1}$, $\forall\, k \in \mathbb{N}$.

For a sequence of sets $M_k \subset \mathbb{C}$, $k \in \mathbb{N}$, we will use the following notation

$$\lim_{k\to+\infty}{}_{*} M_k := \Big\{ z \in \mathbb{C} \mid \exists\, z_k \in M_k : \lim_{k\to+\infty} z_k = z \Big\},$$
$$\lim_{k\to\infty}{}^{*} M_k := \{ z \in \mathbb{C} \mid \exists\, k_m \in \mathbb{N}, \exists\, z_{k_m} \in M_{k_m} : k_m \to +\infty \text{ and } z_{k_m} \to z \text{ as } m \to +\infty \}.$$

Simple examples show that the limit behaviour of $\mathrm{Spec}_n(H, \mathcal{L}_k)$ depends in general on the choice of the sequence $(\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda$. So, it is natural to unite limit sets corresponding to different sequences $(\mathcal{L}_k)_{k=1}^{\infty}$ in order to obtain an invariant set depending only on H. It turns out that this set depends on the spectrum and the essential spectrum of H.

In this paper we will adopt the following definition of the essential spectrum. Remove from $\mathrm{Spec}(H)$ all isolated points which are eigenvalues with finite multiplicities; the remaining set $\mathrm{Spec}_{\mathrm{e}}(H)$ is the essential spectrum of H.

THEOREM 6.1. *Let H be a bounded normal operator. Then*

$$\bigcup_{(\mathcal{L}_k)\in\Lambda} \lim_{k\to\infty}{}^{*}\, \mathrm{Spec}_n(H, \mathcal{L}_k) = \mathrm{Spec}(H) \cup \mathcal{Q}_n(\mathrm{Spec}_{\mathrm{e}}(H)). \tag{6.1}$$

This equality will remain valid if we replace "$\lim^*$" by "$\lim_*$". The same is true for Λ_0 instead of Λ.

Proof. Let us fix an arbitrary $z \notin \mathrm{Spec}(H) \cup \mathcal{Q}_n(\mathrm{Spec}_{\mathrm{e}}(H))$. There exists a closed neighbourhood F of $\mathrm{Spec}_{\mathrm{e}}(H)$ such that $z \notin \mathcal{Q}_n(F)$. The set $S = \mathrm{Spec}(H) \cap$

$(\mathbb{C}\setminus F)$ consists of an at most finite number of eigenvalues with finite multiplicities. The operator

$$P_S := E(S), \tag{6.2}$$

where $E(\cdot)$ is the spectral measure corresponding to the normal operator H, commutes with H and is the orthogonal projection on the finite-dimensional subspace spanned by eigenvectors corresponding to the eigenvalues belonging to S (see, e.g., [6], Section 4.5).

Let us take an arbitrary $z_0 \in F$ and consider the operator

$$H_0 := (I - P_S)H + z_0 P_S = (I - P_S)H(I - P_S) + z_0 P_S. \tag{6.3}$$

It is easy to prove that $\operatorname{Spec}(H_0) \subset F$ (see [6], Section 4.5). It follows from the normality of H that the operators H, H^* and $P_S = P_S^*$ commute with each other. Thus, H_0 is a normal operator. Then the proof of Theorem 3.1 (i) (see (3.3)) implies that for an arbitrary orthogonal projection P

$$\|P(H_0 - zI)^n P u\| \geqslant c\|Pu\|, \quad \forall\, u \in \mathcal{H}, \tag{6.4}$$

where c does not depend on P.

For the operator H we have the representation $H = H_0 + T$, where $T = P_S H - z_0 P_S$ is a finite-dimensional and, consequently, a compact operator. So, the operator

$$(H - zI)^n - (H_0 - zI)^n \tag{6.5}$$

is compact. Taking into account that the operator $(H - zI)^n$ is invertible and applying (6.4) we obtain from [3], Chapter II, Theorems 2.1 and 3.1, that for an arbitrary sequence $(\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda$ the corresponding operator $P_k(H - zI)^n P_k$ is invertible within $\mathcal{L}_k$ if k is sufficiently large. Thus, z does not belong to the LHS of (6.1), i.e. the LHS is a subset of the RHS.

Now we have to prove the opposite inclusion. Let us fix an arbitrary $w \in \operatorname{Spec}(H) \cup \mathcal{Q}_n(\operatorname{Spec_e}(H))$. If $w \in \operatorname{Spec}(H) \setminus \operatorname{Spec_e}(H)$, then w is an eigenvalue of H. Taking an arbitrary sequence $(\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda_0$ such that all of the subspaces $\mathcal{L}_k$ contain an eigenvector of H corresponding to w, we obtain

$$w \in \bigcup_{(\mathcal{L}_k) \in \Lambda_0} \lim_{k\to+\infty}{}^{*} \operatorname{Spec}_n(H, \mathcal{L}_k). \tag{6.6}$$

Since

$$\operatorname{Spec}(H) \cup \mathcal{Q}_n(\operatorname{Spec_e}(H)) = (\operatorname{Spec}(H) \setminus \operatorname{Spec_e}(H)) \cup \mathcal{Q}_n(\operatorname{Spec_e}(H)),$$

it is left to consider the case $w \in \mathcal{Q}_n(\operatorname{Spec_e}(H))$. We can establish as in the proof of Theorem 2.2 that there exist (not necessarily different) points $\lambda_1, \lambda_2, \lambda_3 \in \operatorname{Spec_e}(H)$ such that $0 \in \operatorname{conv}(\{(\lambda_j - w)^n\}_{j=1}^{3})$, i.e.

$$\exists\, t_1, t_2, t_3 \in [0,1] : t_1 + t_2 + t_3 = 1, \quad \sum_{j=1}^{3} t_j(\lambda_j - w)^n = 0. \tag{6.7}$$

Let $E(\cdot)$ be the spectral measure corresponding to the normal operator H. Using the fact that $\lambda_j \in \mathrm{Spec_e}(H)$, we can construct closed linear subspaces $\mathcal{H}_l^j \subset \mathcal{H}$, $l \in \mathbb{N}$, $j = 1, 2, 3$, having the following properties:

$$\mathcal{H}_l^j \neq \{0\}, \quad H\mathcal{H}_l^j \subset \mathcal{H}_l^j, \quad l \in \mathbb{N},\ j = 1, 2, 3, \tag{6.8}$$

$$\mathcal{H}_l^j \perp \mathcal{H}_{l'}^{j'} \quad \text{if} \quad l \neq l' \text{ or } j \neq j', \tag{6.9}$$

and for an arbitrary neighbourhood W^j of λ_j the inclusions

$$\mathcal{H}_l^j \subset \operatorname{Im} E(W^j) = E(W^j)\mathcal{H} \tag{6.10}$$

hold for sufficiently large l. Indeed, if λ_j is an isolated point of $\mathrm{Spec}(H)$, then it is an eigenvalue of infinite multiplicity and we may take $\mathcal{H}_l^j := \mathrm{span}\{e_l^j\}$, where e_l^j are mutually orthogonal eigenvectors of H corresponding to λ_j. If λ_j is not an isolated point, then we may take $\mathcal{H}_l^j := \operatorname{Im} E(W_l^j) = E(W_l^j)\mathcal{H}$, where $(W_l^j)_{l=1}^{\infty}$ is a suitable sequence of pairwise disjoint closed sets "tending" to λ_j.

Let $v_l^j \in \mathcal{H}_l^j \setminus \{0\}$, $l \in \mathbb{N}$, $j = 1, 2, 3$, be arbitrary vectors and $\mathcal{H}_0$ be the closed linear subspace spanned by $\mathcal{H}_l^j$, $l \in \mathbb{N}$, $j = 1, 2, 3$. Let $\mathcal{L}_k$ be the subspaces spanned by $\mathcal{H}_l^j$, $l < k$, $j = 1, 2, 3$, the orthogonal complement $\mathcal{H}_0^{\perp}$ of $\mathcal{H}_0$ and the vector

$$u_k := \sum_{j=1}^{3} \sqrt{t_j}\ \|v_k^j\|^{-1} v_k^j. \tag{6.11}$$

Note that $\|u_k\| = 1$ (see (6.7), (6.9)). It is easily seen that $(\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda_0$. The conditions (6.8) and (6.9) imply the equality

$$((H - \zeta I)^n u_k, u) = 0, \quad \forall\, u \in \mathcal{H}_0^{\perp} \cup \bigcup_{\substack{j=1,2,3\\ l<k}} \mathcal{H}_l^j, \quad \forall\, \zeta \in \mathbb{C}.$$

Therefore

$$P_k(H - \zeta I)^n P_k u_k = P_k(H - \zeta I)^n u_k = ((H - \zeta I)^n u_k, u_k) u_k. \tag{6.12}$$

It is clear that

$$p_k(\zeta) := ((H - \zeta I)^n u_k, u_k) \tag{6.13}$$

is a polynomial of degree n with respect to ζ. The spectral theorem and (6.8)–(6.11) imply that its coefficients tend to the corresponding coefficients of the polynomial

$$p(\zeta) := \sum_{j=1}^{3} t_j(\lambda_j - \zeta)^n$$

as $k \to +\infty$. According to (6.7), $p(w) = 0$. Consequently, $p_k(\zeta)$ has a zero w_k arbitrarily close to w if k is sufficiently large (see, e.g., [4], Theorem 4.10c). Taking into account that $P_k(H - w_k I)^n P_k u_k = 0$ (see (6.12) and (6.13)), we obtain (6.6). Thus

$$\mathrm{Spec}(H) \cup \mathcal{Q}_n(\mathrm{Spec_e}(H)) \subset \bigcup_{(\mathcal{L}_k) \in \Lambda_0} \lim_{k \to +\infty}{}_{*}\, \mathrm{Spec}_n(H, \mathcal{L}_k).$$

Applying the following obvious relations

$$\bigcup_{(\mathcal{L}_k) \in \Lambda_0} \lim_{k \to +\infty}{}_{*} \cdots = \bigcup_{(\mathcal{L}_k) \in \Lambda_0} \lim_{k \to \infty}{}^{*} \cdots \subset \bigcup_{(\mathcal{L}_k) \in \Lambda} \lim_{k \to \infty}{}^{*} \cdots = \bigcup_{(\mathcal{L}_k) \in \Lambda} \lim_{k \to +\infty}{}_{*} \cdots \tag{6.14}$$

we conclude the proof. ∎

Let now $\mathcal{H}$ be a separable Hilbert space. Then it is natural to consider subsets Λ^f and Λ_0^f of Λ and Λ_0 correspondingly, consisting of sequences of finite-dimensional subspaces.

THEOREM 6.2. *If H is a bounded normal operator acting on a separable Hilbert space $\mathcal{H}$, then* (6.1) *will remain valid if we replace Λ by Λ^f or Λ_0^f. The same is true for* "$\lim_*$" *instead of* "$\lim^*$".

Proof. Theorem 6.1 implies:

$$\bigcup_{(\mathcal{L}_k)\in\Lambda^f} \lim_{k\to\infty}{}^* \operatorname{Spec}_n(H,\mathcal{L}_k) \subset \operatorname{Spec}(H)\cup\mathcal{Q}_n(\operatorname{Spec}_{\mathrm{e}}(H)).$$

So, it is sufficient to prove that

$$\bigcup_{(\mathcal{L}_k)\in\Lambda_0^f} \lim_{k\to+\infty}{}_* \operatorname{Spec}_n(H,\mathcal{L}_k) \supset \operatorname{Spec}(H)\cup\mathcal{Q}_n(\operatorname{Spec}_{\mathrm{e}}(H))$$

(cf. (6.14)). The only thing we have to change in the proof of Theorem 6.1 is the definition of $\mathcal{L}_k$. Let $e_{l,m}^j$, $m=1,\dots,$ be an orthonormal basis of $\mathcal{H}_l^j$ and e_m, $m=1,\dots,$ be an orthonormal basis of $\mathcal{H}_0^\perp$ (see the proof of Theorem 6.1). These basses exist because all subspaces of $\mathcal{H}$ are separable. Let $\mathcal{L}_k$ be the subspace spanned by the vectors e_m, $e_{l,m}^j$, $l,m<k$, $j=1,2,3$ and

$$u_k := \sum_{j=1}^{3} \sqrt{t_j}\, e_{k,1}^j .$$

The remaining part of the proof repeats that of Theorem 6.1. ∎

REMARK 6.3. Let Λ_1^f be the subset of Λ_0^f consisting of sequences such that $\dim \mathcal{L}_{k+1}/\mathcal{L}_k = 1$. This class of sequences is often used in the theory of projection methods (see, e.g., [3], Chapter II, Sections 4–6). It is clear that

$$\bigcup_{(\mathcal{L}_k)\in\Lambda_1^f} \lim_{k\to\infty}{}^* \cdots = \bigcup_{(\mathcal{L}_k)\in\Lambda_0^f} \lim_{k\to\infty}{}^* \cdots .$$

Thus, Theorem 6.2 implies the equality

$$\bigcup_{(\mathcal{L}_k)\in\Lambda_1^f} \lim_{k\to\infty}{}^* \operatorname{Spec}_n(H,\mathcal{L}_k) = \operatorname{Spec}(H)\cup\mathcal{Q}_n(\operatorname{Spec}_{\mathrm{e}}(H)). \tag{6.15}$$

It would be interesting to investigate the set

$$\bigcup_{(\mathcal{L}_k)\in\Lambda_1^f} \lim_{k\to+\infty}{}_* \operatorname{Spec}_n(H,\mathcal{L}_k)$$

and, in particular, to know whether (6.15) is true for "$\lim_*$" instead of "$\lim^*$".

For arbitrary $r\geqslant 0$ and a bounded linear operator H we will use the following notation

$$B(r) := \{\zeta\in\mathbb{C} : |\zeta|\leqslant r\}\,,$$
$$|||H||| := \inf\{\|H+T\| : T \text{ is compact on } \mathcal{H}\}\,.$$

The last quantity is called *the essential norm* of the operator H.

THEOREM 6.4. *Let H be an arbitrary bounded operator. Then*

$$\bigcup_{(\mathcal{L}_k)\in\Lambda} \limsup_{k\to\infty}{}^{*} \operatorname{Spec}_n(H,\mathcal{L}_k) \subset \operatorname{Spec}(H) \cup B\left(\left(\sin\frac{\pi}{2n}\right)^{-1} |||H|||\right). \tag{6.16}$$

Proof. According to the definition of the essential norm, for an arbitrary $\varepsilon > 0$, there exists a compact operator T such that $\|H + T\| < |||H||| + \varepsilon$. Let us take an arbitrary

$$z \notin \operatorname{Spec}(H) \cup B\left(\left(\sin\frac{\pi}{2n}\right)^{-1} (|||H||| + \varepsilon)\right).$$

It follows from the proof of Theorem 5.2 (see (5.12)) that for an arbitrary orthogonal projection P, (6.4) holds with $H_0 := H + T$. Taking into account that the operator (6.5) is compact and $(H - zI)^n$ is invertible, we conclude from [3], Chapter II, Theorems 2.1 and 3.1, that for an arbitrary sequence $(\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda$ the corresponding operator $P_k(H - zI)^n P_k$ is invertible within $\mathcal{L}_k$ if k is sufficiently large. Consequently z does not belong to the LHS of (6.16). Thus, the LHS of (6.16) is a subset of

$$\operatorname{Spec}(H) \cup B\left(\left(\sin\frac{\pi}{2n}\right)^{-1} (|||H||| + \varepsilon)\right), \quad \forall\,\varepsilon > 0. \quad \blacksquare$$

7. FINAL REMARKS

We conclude this paper by some remarks concerning projection methods of finding of spectra of bounded operators. We will not discuss the case when the operator under consideration is compact, because it is well understood (see, e.g. [5], Section 18).

Theorems 6.1, 6.2 and 2.8 (i) imply the following result.

THEOREM 7.1. *Let H be a bounded self-adjoint operator and n be an even number. Then*

$$\begin{aligned}\bigcup_{(\mathcal{L}_k)\in\Lambda} \limsup_{k\to\infty}{}^{*} \operatorname{Spec}_n(H,\mathcal{L}_k) \cap \mathbb{R} &= \bigcup_{(\mathcal{L}_k)\in\Lambda} \lim_{k\to+\infty}{}_{*} \operatorname{Spec}_n(H,\mathcal{L}_k) \cap \mathbb{R} \\ &= \operatorname{Spec}(H).\end{aligned} \tag{7.1}$$

The same is true for Λ_0 instead of Λ (and for Λ^f, Λ_0^f if $\mathcal{H}$ is separable).

Theorems 6.1 and 7.1 tell us what may happen for all sequences from Λ, but they do not provide sufficient information about

$$\lim_{k\to\infty}{}^{*} \operatorname{Spec}_n(H,\mathcal{L}_k) \quad \text{and} \quad \lim_{k\to+\infty}{}_{*} \operatorname{Spec}_n(H,\mathcal{L}_k) \tag{7.2}$$

for a given $(\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda$. The most interesting question is whether these sets cover $\operatorname{Spec}(H)$. For normal operators the answer is in general negative. For example, if H is the simple bilateral shift operator acting on $\mathcal{H} = l^2(\mathbb{Z})$:

$$(Hx)_m = x_{m+1}, \quad \forall\, m \in \mathbb{Z}, \forall\, x = (x_m) \in l^2(\mathbb{Z}),$$

then H is a unitary and, therefore, a normal operator and its spectrum coincides with the unit circle. A simple proof of this well known fact follows from the representation of H as the operator of multiplication by the function $\exp(\mathrm{i}\theta)$ acting on $L_2([0,2\pi])$ with the orthonormal basis $\left((2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}m\theta}\right)_{m\in\mathbb{Z}}$. On the other hand, let us consider the following sequence $(\mathcal{L}_k)_{k=1}^{\infty}\in\Lambda_1^f$ (see Remark 6.3):

$$\mathcal{L}_k=\left\{x\in l^2(\mathbb{Z}) : x_m=0 \text{ if } m<-[k/2] \text{ or } m>k-[k/2]\right\}, \quad k\in\mathbb{N}.$$

It is obvious that if $P_k : l^2(\mathbb{Z})\to\mathcal{L}_k$ is the orthogonal projection, then the matrix corresponding to the operator $P_kH^lP_k$, $l\in\mathbb{N}$ in the standard basis is nilpotent, i.e. is an upper triangular matrix with 0 on the diagonal. Thus

$$\mathrm{Spec}_n(H,\mathcal{L}_k)=\{0\}, \quad \forall\, k\in\mathbb{N}, \forall\, n\in\mathbb{N}.$$

So, $\mathrm{Spec}(H)$ and $\mathrm{Spec}_n(H,\mathcal{L}_k)$ are totally unrelated.

It is not difficult to prove that the sets (7.2) cover $\mathrm{Spec}(H)$ if H is self-adjoint and $n=1$ (see, e.g., [1], Theorem 2.3).

Lemma 7.2. *Let H be a bounded self-adjoint operator. Then*

$$\mathrm{Spec}(H)\subset \lim_{k\to+\infty}{}_{*}\, \mathrm{Spec}_1(H,\mathcal{L}_k)\subset \lim_{k\to\infty}{}^{*}\, \mathrm{Spec}_1(H,\mathcal{L}_k), \quad \forall\, (\mathcal{L}_k)_{k=1}^{\infty}\in\Lambda.$$

Proof. Let $(\mathcal{L}_k)_{k=1}^{\infty}$ be an arbitrary sequence from Λ and $E(\cdot)$ be the spectral measure corresponding to H. Let us fix an arbitrary $\lambda\in\mathrm{Spec}(H)$ and $\varepsilon>0$. We will use the following notation

$$d(\lambda,\varepsilon,k)=\inf\{\|u-v\| : u\in\mathrm{Im}\,E([\lambda-\varepsilon,\lambda+\varepsilon]),\ v\in\mathcal{L}_k,\ \|u\|=\|v\|=1\}. \tag{7.3}$$

For an arbitrary $\delta>0$, there exist $u_k\in\mathrm{Im}\,E([\lambda-\varepsilon,\lambda+\varepsilon])$ and $v_k\in\mathcal{L}_k$ such that

$$\|u_k-v_k\|\leqslant d(\lambda,\varepsilon,k)+\delta, \quad \|u_k\|=\|v_k\|=1.$$

It follows from the spectral theorem that $\|(H-\lambda I)u_k\|\leqslant\varepsilon$. Further,

$$\begin{aligned}\|P_k(H-\lambda I)P_kv_k\| &= \|P_k(H-\lambda I)v_k\|\leqslant\|(H-\lambda I)v_k\|\\ &\leqslant \|(H-\lambda I)u_k\|+\|(H-\lambda I)(v_k-u_k)\|\\ &\leqslant \varepsilon+2\|H\|\,\|u_k-v_k\|\leqslant\varepsilon+2\|H\|(d(\lambda,\varepsilon,k)+\delta).\end{aligned} \tag{7.4}$$

Taking into account that the restriction of $P_k(H-\lambda I)P_k$ to $\mathcal{L}_k$ is a self-adjoint operator, we conclude from (7.4) that its spectrum intersects the segment

$$[-\varepsilon-2\|H\|(d(\lambda,\varepsilon,k)+\delta),\ \varepsilon+2\|H\|(d(\lambda,\varepsilon,k)+\delta)]$$

(see, e.g., [8], 12.24). Consequently,

$$\mathrm{dist}(\lambda,\mathrm{Spec}_1(H,\mathcal{L}_k))\leqslant\varepsilon+2\|H\|(d(\lambda,\varepsilon,k)+\delta), \quad \forall\,\delta>0,$$

i.e.

$$\mathrm{dist}(\lambda,\mathrm{Spec}_1(H,\mathcal{L}_k))\leqslant\varepsilon+2\|H\|d(\lambda,\varepsilon,k). \tag{7.5}$$

The strong convergence of the orthogonal projections $P_k:\mathcal{H}\to\mathcal{L}_k$ to the identity operator implies

$$\begin{gathered}0\leqslant d(\lambda,\varepsilon,k)\leqslant\inf\{\|u-v\| : v\in\mathcal{L}_k,\ \|v\|=1\}\to 0\\ \text{as } k\to+\infty,\ \forall\, u\in\mathrm{Im}\,E([\lambda-\varepsilon,\lambda+\varepsilon]) : \|u\|=1,\ \forall\,\varepsilon>0.\end{gathered}$$

Thus

$$\lambda\in\lim_{k\to+\infty}{}_{*}\,\mathrm{Spec}_1(H,\mathcal{L}_k), \quad \forall\,\lambda\in\mathrm{Spec}(H). \quad ∎$$

Now from Theorem 6.1 and (2.10) we obtain the following result.

COROLLARY 7.3. *Let H be a bounded self-adjoint operator such that* $\mathrm{Spec}_{\mathrm{e}}(H)$ *is a segment (or a point). Then*

$$\lim_{k\to\infty}{}^{*} \mathrm{Spec}_1(H, \mathcal{L}_k) = \lim_{k\to+\infty}{}_{*} \mathrm{Spec}_1(H, \mathcal{L}_k) = \mathrm{Spec}(H), \quad \forall\ (\mathcal{L}_k)_{k=1}^{\infty} \in \Lambda,$$

and (7.5) *holds for any* $\lambda \in \mathrm{Spec}(H)$ *and* $\varepsilon > 0$ *(see* (7.3)*).*

Theorem 7.1 shows that projection methods of approximate computation of spectra of self-adjoint operators which use even order relative spectra do not lead to spurious points. The method involving $\mathrm{Spec}_2(H, \mathcal{L}_k)$ is probably the simplest one among these methods and thus seems to be the most attractive one (see also Corollaries 3.4 and 4.2). It would be interesting to know whether this method allows to find the whole spectrum, i.e. whether the sets (7.2) cover $\mathrm{Spec}(H)$ in the case when H is self-adjoint and $n = 2$ (cf. (7.1)).

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