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ADJOINTS, ABSOLUTE VALUES AND POLAR DECOMPOSITIONS

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ABSTRACT. Various questions about adjoints, absolute values and polar decompositions of operators are addressed from a constructive point of view. The focus is on bilinear forms. Conditions are given for the existence of an adjoint, and a general notion of a polar decomposition is developed. The Riesz representation theorem is proved without countable choice.

KEYWORDS: *Bounded operators, adjoints, absolute value, polar decomposition, Riesz representation theorem, constructive mathematics.*

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1. INTRODUCTION

Let H be an inner product space over the real or complex numbers. An operator is an everywhere defined linear transformation from H to H. If H is finite-dimensional, then every bounded operator has an adjoint, a theorem that can be proved in general using the Law of Excluded Middle. In the constructive framework of this paper, however, it cannot be shown that every bounded operator on an infinite-dimensional Hilbert space has an adjoint. In order to explain this by means of a Brouwerian example, we need a lemma whose proof is a straightforward application of the Cauchy-Schwarz inequality.

LEMMA 1.1. *Let $(\alpha_n), (\beta_n)$ be sequences of complex numbers such that $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges and $\sum_{n=1}^{\infty} |\beta_n|^2$ is bounded. Then $\sum_{n=1}^{\infty} |\alpha_n \beta_n|$ converges.*

A BROUWERIAN EXAMPLE Q. Let (a_n) be a binary sequence with $a_1 = 0$ and at most one term equal to 1, and let (e_n) be an orthonormal basis of an

infinite-dimensional Hilbert space. Lemma 1.1 enables us to define a bounded operator Q such that $Qe_n = a_n e_1$ for each n. That is

$$Qx = \left(\sum_{n=1}^{\infty} a_n x_n \right) e_1.$$

If Q has an adjoint, then either $\|Q^* e_1\| > 0$ or else $\|Q^* e_1\| < 1$. Since

$$\langle Q^* e_1, e_n \rangle = \langle e_1, Q e_n \rangle = a_n,$$

we see that $a_n = 1$ for some n in the first case, and $a_n = 0$ for all n in the second. Thus, the proposition

Every bounded operator on a separable Hilbert space has an adjoint,

entails the nonconstructive *limited principle of omniscience* (LPO), a countable form of the law of excluded middle:

> *For each binary sequence* (a_n), *there exists* n *such that* $a_n = 1$ *or else* $a_n = 0$ *for all* n.

In this paper, beginning with a choice-free proof of the Riesz Representation Theorem, we study conditions that are equivalent to the existence of an adjoint, beginning with a general result about when a bilinear form can be written as $\langle x, Ty \rangle$, a result that leads to a simple proof that a compact operator has an adjoint.

The usual definition of the absolute value of an operator, and the proof of the existence of the polar decomposition, both depend on the existence of an adjoint. As there is no guarantee that an adjoint exists, it is desirable to define the absolute value, and to construct a polar decomposition, without reference to an adjoint. In Section 4 we give adjoint-free definitions of the absolute value and of polar decompositions, and show that if the absolute value of an operator T has approximate polar decompositions, then T has an adjoint.

Background in constructive mathematics is available in [1], [2], [4], or [7]. In particular, the basics of constructive Hilbert space theory may be found in Chapter 7 of [2]. We do not restrict ourselves to *separable* Hilbert spaces, as is traditional in constructive mathematics; nor do we assume the countable axiom of choice. In order to deal with arbitrary Hilbert spaces, we need a couple of definitions.

An *orthonormal basis* for an inner product space is a set of pairwise orthogonal unit vectors that generate a dense subspace. Examples of this are provided by taking an arbitrary discrete set S, and considering the space of complex valued functions on S with finite support. This space has a natural inner product structure, and a basis consisting of those functions that are 1 on one element of S and 0 on the others. The completion is a Hilbert space, $L^2(S)$, with the same basis, which need not be separable. This definition is more traditional, even in the separable case, than the sequential one in [1] and [2] where basis elements must be allowed to be zero to achieve sufficient generality.

The second definition concerns sums over arbitrary index sets. If $(r_i)_{i \in I}$ is a family of nonnegative real numbers, then we define $\sum_{i \in I} r_i$ to be $\sup_F \sum_{i \in F} r_i$, where F

ranges over the finite subsets of I. This agrees with the standard definition when I is the set of positive integers.

The following notation for inequalities involving suprema will be convenient (see also [6]). If X is a subset of $\mathbb{R}$, and $\alpha, \beta \in \mathbb{R}$, then

$$\alpha < \sup X \leqslant \beta$$

means that $\alpha < x$ for some $x \in X$, and that $x \leqslant \beta$ for all $x \in X$, even if the supremum of X is not known to exist. For example, if T is an operator on H, then $\|T\| \leqslant 1$ means that $\|Tx\| \leqslant 1$ for all unit vectors x, and $\|T\| \geqslant 2$ means that for each $\varepsilon > 0$ there is a unit vector x such that $\|Tx\| \geqslant 2 - \varepsilon$. Other such notations will be used in the obvious, analogous ways. If S and T are operators on H, and $a > 0$, then $\|S\| \leqslant a\|T\|$ means that if $\|T\| \leqslant c$, then $\|S\| \leqslant ac$, and if $c \leqslant \|S\|$, then $c/a \leqslant \|T\|$.

2. LINEAR FUNCTIONALS

The space H' of bounded linear functionals on H is not quite a normed space because a bounded linear functional f need not have a norm. However, the statement $\|f\| \leqslant r$ has a meaning — namely that $|f(x)| \leqslant r\|x\|$ for all x in H. Also, the convex sets (closed balls) $S_r = \{f : \|f\| \leqslant r\}$ satisfy:

(i) $S_0 = \{0\}$,
(ii) $cS_r \subset S_{|c|r}$,
(iii) $S_r + S_s \subset S_{r+s}$,

and define a uniform structure on H'.

There is a natural embedding $\nu : H \to H'$ taking y to the linear functional $\langle \cdot\, , y\rangle$ whose norm is equal to $\|y\|$. The *Riesz representation theorem* says that if $f \in H'$ has a norm, then $f \in \nu(H)$. A proof of this is given in Section 2.3 of [2]. We give a direct proof here that avoids the countable axiom of choice and the prior verification that $\ker f$ is located if f is nonzero. First we have a lemma which will also be used later.

LEMMA 2.1. *Let $H = H_1 \oplus H_2$ be a decomposition of an inner product space H into orthogonal subspaces. If f is a linear functional on H, then*

$$\|f\|^2 = \|f\|_1^2 + \|f\|_2^2$$

where $\|f\|_i$ is the norm of the restriction of f to H_i.

Proof.

$$\begin{aligned}\|f\|^2 &= \sup_{\|u\|=1} |f(u)|^2 = \sup_{\|u_1\|^2+\|u_2\|^2=1} |f(u_1) + f(u_2)|^2 \\ &= \sup_{\substack{\|u_1\|^2=\|u_2\|^2=1 \\ |\alpha_1|^2+|\alpha_2|^2=1}} |\alpha_1 f(u_1) + \alpha_2 f(u_2)|^2,\end{aligned}$$

where the last equality holds because we may restrict to nonzero u_1 and u_2. In the last supremum, we may further restrict to u_1 and u_2 such that $f(u_1)$ and $f(u_2)$ are nonzero, so we may assume that $\alpha_1 f(u_1)$ and $\alpha_2 f(u_2)$ are positive. Hence

$$\sup_{\substack{\|u_1\|^2=\|u_2\|^2=1\\ |\alpha_1|^2+|\alpha_2|^2=1}} |\alpha_1 f(u_1)+\alpha_2 f(u_2)|^2 = \sup_{\substack{\|u_i\|^2=1\\ a_1^2+a_2^2=1\\ f(u_i)>0}} (a_1 f(u_1)+a_2 f(u_2))^2.$$

The supremum on the right, for fixed u_i, is realized when

$$a_i = f(u_i)/\sqrt{f(u_1)^2+f(u_2)^2},$$

so

$$\|f\|^2 = \sup_{\substack{\|u_i\|^2=1\\ f(u_i)>0}} (f(u_1)^2+f(u_2)^2) = \|f\|_1^2+\|f\|_2^2. \quad \blacksquare$$

It is important to observe that the norms in Lemma 2.1 need not exist as real numbers; the equation is an equality of expressions involving suprema.

THEOREM 2.2. *Let f be a linear functional on a Hilbert space H. If f has a norm, then $f = \langle \cdot, y\rangle$ for some (necessarily unique) element y in H.*

Proof. As H is complete, it suffices to show that f is in the closure of $\nu(H)$. Given $\delta > 0$, we construct $y \in H$ such that $\|f-\nu(y)\|^2 \leqslant 2\delta$. Either $\|f\|^2 < 2\delta$ or $\|f\|^2 > 0$. In the former case take $y = 0$. In the latter we may assume that $\|f\| = 1$. Pick y so that $\|y\| = 1$ and $f(y) \geqslant 1-\delta$ (so $f(y) \leqslant 1$ is real). Let $H = H_1 \oplus H_2$, where H_1 is the span of y, and H_2 is the orthogonal complement of y. Then

$$|(f-\nu y)(cy)| = |c||f(y)-1| \leqslant |c|\delta,$$

so $\|f-\nu(y)\|_1 \leqslant \delta$, in the notation of Lemma 2.1. By Lemma 2.1,

$$1 = \|f\|^2 = \|f\|_1^2+\|f\|_2^2 \geqslant (1-\delta)^2+\|f\|_2^2$$

so

$$\|f-\nu(y)\|_2^2 = \|f\|_2^2 \leqslant 2\delta-\delta^2.$$

By Lemma 2.1 again,

$$\|f-\nu(y)\|^2 = \|f-\nu(y)\|_1^2+\|f-\nu(y)\|_2^2 \leqslant \delta^2+2\delta-\delta^2 = 2\delta. \quad \blacksquare$$

Let B be a bilinear form, and let

$$\|B\| = \sup_{\substack{\|x\|\leqslant 1\\ \|y\|\leqslant 1}} |B(x,y)|.$$

We say that B is *left* (respectively, *right*) *representable* if there exists an operator T, necessarily unique, such that $B(x,y) = \langle Tx, y\rangle$ (respectively, $B(x,y) = \langle x, Ty\rangle$) for all x and y. Note that, in either case, $\|B\| = \|T\|$; so, in particular, B is bounded if and only if T is bounded. Note also that an operator T has an adjoint if and only if the (left representable) bilinear form $\langle Tx, y\rangle$ is right representable.

THEOREM 2.3. *Let B be a bilinear form on a Hilbert space. Then B is right representable if and only if the linear functional $B(\,\cdot\,, y)$ has a norm for each y.*

Proof. Denote the unique z such that $B(\,\cdot\,, y) = \langle\,\cdot\,, z\rangle$ by Ty. It is readily checked that T is an operator. The converse is trivial. ∎

COROLLARY 2.4. *Any compact operator on a Hilbert space has an adjoint.*

Proof. Let T be a compact operator and consider the bilinear form $B(x, y) = \langle Tx, y\rangle$. Because T is compact, the linear functional $B(\,\cdot\,, y\,)$ has a norm for each y, hence $B(\,\cdot\,, y) = \langle\,\cdot\,, T^*y\rangle$ for some bounded linear operator T^*. ∎

We give a criterion, in terms of the convergence of a series, for when a linear functional on a Hilbert space with an orthonormal basis has a norm. First we require a lemma about an arbitrary bounded linear mapping between Hilbert spaces.

LEMMA 2.5. *Let E be an orthonormal basis of a Hilbert space H, and P_F the projection on the span of the finite subset $F \subset E$. Let $T : H \to K$ be a bounded linear mapping. Then $\|TP_F\| \leqslant \|TP_{F'}\|$ if $F \subset F'$, and*

$$\|T\| = \sup_F \|TP_F\|.$$

In particular, T has a norm if and only if $\sup_F \|TP_F\|$ exists.

Proof. Note that TP_F has a norm, since it is a compact operator. If $F \subset F'$, then $\|TP_F\| \leqslant \|TP_{F'}\|$ because the supremum is over a smaller set. Clearly

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\| \geqslant \sup_{\|x\|\leqslant 1} \|TP_Fx\| = \|TP_F\|.$$

As T is bounded, $TP_Fx \to Tx$ for each x in H, so

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\| = \sup_F \sup_{\|x\|\leqslant 1} \|TP_Fx\| = \sup_F \|TP_F\|. \quad ∎$$

THEOREM 2.6. *Let E be an orthonormal basis of a Hilbert space H, and f a bounded linear functional on H. Then*

$$\sum_{e\in E} |f(e)|^2 = \|f\|^2.$$

In particular, f is bounded by $c \geqslant 0$ if and only if the finite partial sums of $\sum_{e\in E} |f(e)|^2$ are bounded by c^2, and f has norm c if and only if

$$\sup_{\substack{F\subset E\\ F\text{ finite}}} \sum_{e\in F} |f(e)|^2$$

exists and equals c^2.

Proof. For F a finite subset of E, let P_F be the projection on the span of F. It follows from Lemma 2.1 that

$$\|fP_F\|^2 = \sum_{e\in F} |f(e)|^2.$$

The result now follows from Lemma 2.5. ∎

3. THE EXISTENCE OF ADJOINTS

We are interested in conditions equivalent to the existence of an adjoint. In this connection, we refer the reader to the work of Ishihara ([5]).

Note that if T^* is an adjoint of T, then

$$\|T\| = \sup_{\substack{\|x\|=1\\ \|y\|=1}} |\langle Tx, y\rangle| = \sup_{\substack{\|x\|=1\\ \|y\|=1}} |\langle x, T^*y\rangle| = \|T^*\|.$$

THEOREM 3.1. *Let E be an orthonormal basis of a Hilbert space H, and let B be a bounded bilinear form on H. Then the following are equivalent:*

(i) *The linear functional $B(\,\cdot\,, y)$ has a norm for each y in H (in other words, B is right representable).*

(ii) $\sum_{e\in E} |B(e, e')|^2$ *converges for each* $e' \in E$.

(iii) $\sum_{e\in E} |B(e, y)|^2$ *converges for each* $y \in H$.

Proof. Theorem 2.6 shows that (i) is equivalent to (iii). Clearly (iii) implies (ii).

Now assume (ii), and let $c > 0$ be a bound for B. The linear functional $B(\,\cdot\,, e')$ is bounded by c, so $\sum_{e\in E} |B(e, e')|^2 \leqslant c^2$, by Theorem 2.6.

Given $x \in H$ and $\varepsilon > 0$, choose a finite subset F' of E such that $\|x - \sum_{e'\in F'} x_{e'}e'\| < \varepsilon$, where $x_{e'} = \langle x, e'\rangle$. Then, using (ii), choose a finite subset F of E such that

$$\sum_{e'\in F'} \sum_{e\in E\setminus F} |B(e, e')|^2 \leqslant \varepsilon.$$

Write $x = y + z$ where $y = \sum_{e'\in F'} x_{e'}e'$. So

$$|B(e, y)|^2 = \left|B\Big(e, \sum_{e'\in F'} x_{e'}e'\Big)\right|^2 = \left|\sum_{e'\in F'} \overline{x}_{e'} B(e, e')\right|^2 \leqslant \|x\|^2 \sum_{e'\in F'} |B(e, e')|^2$$

for $e \in E \setminus F$ (by the Cauchy-Schwarz inequality) so

$$\sum_{e\in E\setminus F} |B(e, y)|^2 \leqslant \|x\|^2\varepsilon \quad \text{and} \quad \sum_{e\in E\setminus F} |B(e, z)|^2 \leqslant c^2\|z\|^2 \leqslant c^2\varepsilon.$$

Then

$$\begin{aligned}\sum_{e\in E\setminus F} |B(e, x)|^2 &= \sum_{e\in E\setminus F} |B(e, y + z)|^2 = \sum_{e\in E\setminus F} |B(e, y) + B(e, z)|^2\\ &\leqslant (\sqrt{\|x\|^2\varepsilon} + \sqrt{c^2\varepsilon})^2 = \|x\|^2\varepsilon + \varepsilon c^2 + 2\|x\|\varepsilon c = (\|x\| + c)^2\varepsilon\end{aligned}$$

(the triangle inequality in $L^2(E)$). Hence $\sum_{e\in E} |B(e, x)|^2$ converges, so (ii) implies (iii). ∎

Note that in the last theorem the boundedness of B is only required to prove that (ii) $\Rightarrow$ (iii).

It is tempting to look for weaker conditions that entail the existence of an adjoint. One natural candidate is

$$\inf_F \sup_{e \in E \setminus F} |\langle Te, x\rangle| = 0, \tag{3.1}$$

which suffices to construct the adjoint of Q, the Brouwerian example in Section 1. This condition is not generally sufficient, even when T has a norm, as the following Brouwerian example shows.

Let (a_n) be a binary sequence with $a_1 = a_2 = 0$ and at most one term equal to 1. Define a linear mapping $T : H \to H$ as follows:

(i) $Te_1 = 0$, $Te_2 = e_2$;

(ii) if $a_k = 0$ for $3 \leqslant k \leqslant n$, then $Te_n = 0$;

(iii) if $a_n = 1$, then $Te_n = \cdots = Te_{2n-1} = \frac{1}{\sqrt{n}} e_1$ and $Te_k = 0$ for all $k \geqslant 2n$.

It is straightforward to show that $\|T\| = 1$. Clearly, T satisfies (3.1). But, although the partial sums $\sum_{n=1}^{m} |\langle Te_n, e_1\rangle|^2$ are bounded by 1, the series $\sum_{n=1}^{\infty} |\langle Te_n, e_1\rangle|^2$ does not converge. If it converges to s, then either $s > 0$, in which case $\langle Te_N, e_1\rangle \neq 0$ for some N, and therefore $a_n = 1$ for some $n \leqslant N$; or else $s < 1$ and therefore $a_n = 0$ for all n.

Next we show that if a bilinear form is approximately right representable, then it is right representable.

PROPOSITION 3.2. *Let B be a bilinear form on a Hilbert space H, and suppose that for each $\varepsilon > 0$ there exists an operator T_ε such that*

$$|B(x, y) - \langle x, T_\varepsilon y\rangle| < \varepsilon$$

for all x, y in the unit ball of H. Then B is right representable.

Proof. For all $\varepsilon, \varepsilon' > 0$ and all x, y in the unit ball of H we have

$$|\langle x, (T_\varepsilon - T_{\varepsilon'})y\rangle| \leqslant \varepsilon + \varepsilon',$$

so $\|T_\varepsilon - T_{\varepsilon'}\| \leqslant \varepsilon + \varepsilon'$. By the completeness of H, there exists an operator T, which T_ε approximates within ε, such that $B(x, y) = \langle x, Ty\rangle$. ∎

4. ABSOLUTE VALUES AND POLAR DECOMPOSITIONS

An *absolute value* of an operator T is a (necessarily unique) positive selfadjoint operator $|T| : H \to H$ such that $\langle |T|x, |T|y\rangle = \langle Tx, Ty\rangle$ for all $x, y \in H$. If T has an adjoint, then $|T|^2 = T^*T$, and this equation may be used to define $|T|$.

An operator need not have an absolute value. Let Q be the Brouwerian example of the introduction, and P_1 the projection on the span of e_1. The operator $T = P_1 + Q$ does not have an absolute value, for if there were a linear mapping $S : H \to H$ such that

$$\langle Se_1, e_n\rangle = \langle Te_1, Te_n\rangle = \langle e_1, Qe_n\rangle = a_n$$

for each $n > 1$, then the sequence (a_n) would be square-summable, so we would be able to decide whether there exists n such that $a_n = 1$. Note that $\operatorname{ran} T$ is one-dimensional.

On the other hand, although we cannot construct the adjoint of Q, we can construct its absolute value: $|Q|x = \sum_{n=1}^{\infty} a_n x_n e_n$. Note that $|Q|$ is a projection.

To clarify the difference between having an adjoint and having an absolute value, consider any bounded operator T that maps H into the one-dimensional subspace spanned by e_1. If $\lambda_n = \langle Te_n, e_1\rangle$, then $\sum_{n=1}^{\infty} |\lambda_n|^2$ has bounded partial sums, but converges if and only if T has a norm. Suppose T has an absolute value. If $S = |T|^2$, then

$$\langle Sx, e_n\rangle = \langle Tx, Te_n\rangle = \Bigl(\sum_{k=1}^{\infty} \lambda_k x_k\Bigr)\lambda_n^*,$$

so the series

$$\sum_{n=1}^{\infty} \Bigl|\lambda_n^* \sum_{k=1}^{\infty} \lambda_k x_k\Bigr|^2 \tag{4.1}$$

converges to $\|Sx\|^2$. Conversely, if (4.1) converges, then T has an absolute value.

Certainly (4.1) converges if $\sum_{n=1}^{\infty} |\lambda_n|^2$ converges — that is, if T has a norm. But the operator Q (with $\lambda_n = a_n$) shows that the series can converge even if $\sum_{n=1}^{\infty} |\lambda_n|^2$ only has bounded partial sums. Explicitly, for any $x \in H$, and any n, we clearly have

$$\Bigl|a_n \sum_{k=1}^{\infty} a_k x_k\Bigr|^2 \leqslant |x_n|^2$$

so

$$\sum_{n=N+1}^{\infty} \Bigl|a_n \sum_{k=1}^{\infty} a_k x_k\Bigr|^2 \leqslant \sum_{n=N+1}^{\infty} |x_n|^2.$$

The right hand side goes to zero as N goes to ∞, so $\sum_{n=1}^{\infty} \Bigl|a_n \sum_{k=1}^{\infty} a_k x_k\Bigr|^2$ converges.

Rather than simply study the operators T and $|T|$, it is convenient to study operators T and R such that $\langle Tx, Ty\rangle = \langle Rx, Ry\rangle$ for all x, y. Call two such operators *isometric*. Note that in that case:

(i) if either R or T has an absolute value, then so does the other and the absolute values are equal;
(ii) if either R or T is bounded, then so is the other; and
(iii) if either R or T has a norm, then so does the other.

The first of these observations has a converse: two operators with absolute values are isometric if their absolute values are equal.

Theorem 4.1. *If the operators T and R are isometric, then there is an isometry U from* $\operatorname{ran} R$ *to* $\operatorname{ran} T$ *such that $T = UR$. Hence T is compact if and only if R is compact.*

Proof. The equation $\langle Rx, Ry\rangle = \langle Tx, Ty\rangle$ allows us to define U by $URx = Tx$, and shows that U is an isometry. If X is the image of the unit ball under R, then UX is the image of the unit ball under T. As U is an isometry, X is totally bounded if and only if UX is. ∎

Theorem 4.1 is a version of polar decomposition. Normally, one wants to extend U in a canonical way to all of H. Classically, this is done by extending U (uniquely) to the closure of $\operatorname{ran} R$ and defining U to be zero on the orthogonal complement of $\operatorname{ran} R$. However, this does not define U on all of H unless $\operatorname{ran} R$ is located. (A subset K of H is *located* if

$$\rho(x, K) = \inf\{\|x - y\| : y \in K\}$$

exists for each $x \in H$.) A subspace of H is located if and only if its closure is the image of a projection.

Let P be a projection and U an operator. Then the following conditions are equivalent:

(i) $|U|$ exists and equals P;
(ii) U is an isometry on the range of P and is 0 on the kernel of P.

If these hold for some projection P, we say that U is a *partial isometry*. Note that if U is a partial isometry, then $U|U| = U$. The Brouwerian example Q is a partial isometry.

Proposition 4.2. *An adjoint of a partial isometry is a partial isometry.*

Proof. Let U be a partial isometry with adjoint U^*. Then

$$(UU^*)^2 = U|U|^2U^* = U|U|U^* = UU^*.$$

Taking square roots of both sides, we have $|U^*|^2 = |U^*|$, so $|U^*|$ is a projection. ∎

LEMMA 4.3. *If $T = UR$, where U is a partial isometry, then the following are equivalent:*

(i) *T and R are isometric operators;*
(ii) *$|U|$ is an isometry on* $\operatorname{ran} R$*;*
(iii) $|U|R = R$*;*
(iv) $\ker U \subset (\operatorname{ran} R)^{\perp}$.

Proof. As

$$\langle |U|Rx, |U|Ry\rangle = \langle URx, URy\rangle = \langle Tx, Ty\rangle,$$

(i) and (ii) are equivalent. As $|U|$ is a projection, (ii) and (iii) are equivalent.

Note that $\ker U = \ker |U|$. Suppose that (iii) holds. If $x \in \ker U$, then

$$\langle x, Ry\rangle = \langle x, |U|Ry\rangle = \langle |U|x, Ry\rangle = 0.$$

So (iv) holds. Conversely, suppose that (iv) holds. Then $\ker |U| \subset (\operatorname{ran} R)^{\perp}$, so

$$\operatorname{ran} R \subset (\operatorname{ran} R)^{\perp\perp} \subset (\ker |U|)^{\perp} = \operatorname{ran} |U|$$

and (iii) holds. ∎

If T, R and U are as in the above lemma, then we say that $T = UR$ is a *polar decomposition*. If U has an adjoint, then for all $x, y \in H$,

$$\langle U^*Tx, y\rangle = \langle Tx, Uy\rangle = \langle URx, Uy\rangle = \langle |U|Rx, y\rangle = \langle Rx, y\rangle,$$

so $R = U^*T$ is a polar decomposition.

PROPOSITION 4.4. *If R is an operator that has an adjoint and a polar decomposition $R = UT$, then T^* exists and equals R^*U.*

Proof. Using part (iii) of Lemma 4.3, we have

$$\langle Tx, y\rangle = \langle |U|Tx, y\rangle = \langle UTx, Uy\rangle = \langle x, R^*Uy\rangle. \quad ∎$$

If T and R are isometric operators such that $\operatorname{ran} T$ and $\operatorname{ran} R$ are located — a classically trivial condition — then there exists a partial isometry U with an adjoint such that

$$T = UR \quad \text{and} \quad R = U^*T.$$

To construct U, extend the U of Theorem 4.1 to H by setting it equal to zero on the complement of $\operatorname{ran} R$. Then U^* is the inverse of U on $\operatorname{ran} T$ and zero on the complement of $\operatorname{ran} T$.

If Q is the Brouwerian example of the introduction, then $Q = Q|Q|$ is a polar decomposition. So a polar decomposition $T = U|T|$ does not guarantee that T^* exists. However, a polar decomposition $|T| = UT$ does entail that T^* exists (Proposition 4.4); in fact, approximate polar decompositions suffice (Theorem 4.6).

Let R and T be isometric operators. We say that UR is an *ε-approximate polar decomposition* of T if U is a partial isometry and $\|T - UR\| < \varepsilon$. It is shown in [3] (Theorem 1.1) that if T is a bounded operator with an adjoint, then for each $\varepsilon > 0$ there exists a partial isometry U, with an adjoint, such that $\||T| - UT\| < \varepsilon$ and $\|T - U^*|T|\| < \varepsilon$. It follows that if R and T are isometric operators with adjoints, then $|R| = |T|$ and for each $\varepsilon > 0$ there is a partial isometry U, with an adjoint, such that $\|R - UT\| < \varepsilon$ and $\|T - U^*R\| < \varepsilon$. The following lemma will be used to prove a converse of that result.

LEMMA 4.5. *Let R and T be isometric bounded operators. Suppose further that R has an adjoint. For each $\varepsilon > 0$, there exists $\delta > 0$ such that if UT is a δ-approximate polar decomposition of R, then*

$$|\langle Tx, y\rangle - \langle x, R^*Uy\rangle| \leqslant \varepsilon$$

for all x, y in the unit ball.

Proof. Choose $\delta > 0$ so that $\sqrt{2(\|T\| + \delta)\delta} + \delta < \varepsilon$. Let $P = |U|$ and $S = UT - R$, so $\|R\| \leqslant \delta$. For all x, y in the unit ball

$$\begin{aligned}\langle PTx, PTy\rangle &= \langle UTx, UTy\rangle = \langle (R+S)x, (R+S)y\rangle \\ &= \langle Tx, Ty\rangle + \langle Sx, Ry\rangle + \langle Rx, Sy\rangle + \langle Sx, Sy\rangle\end{aligned}$$

so

$$|\langle PTx, PTy\rangle - \langle Tx, Ty\rangle| \leqslant (2\|T\| + \delta)\delta\|x\|\,\|y\|.$$

Since

$$\|(PT - T)x\|^2 = \langle PTx - Tx, PTx - Tx\rangle = -\langle PTx, PTx\rangle + \langle Tx, Tx\rangle,$$

it follows that $\|PT - T\|^2 \leqslant (2\|T\| + \delta)\delta$. Then

$$\begin{aligned}|\langle Tx, y\rangle - \langle x, R^*Uy\rangle| &\leqslant |\langle Tx, y\rangle - \langle PTx, y\rangle| + |\langle UTx, Uy\rangle - \langle Rx, Uy\rangle| \\ &\leqslant \|T - PT\|\,\|x\|\,\|y\| + \|UT - R\|\,\|x\|\,\|y\| \\ &\leqslant \sqrt{(2\|T\| + \delta)\delta} + \delta < \varepsilon\end{aligned}$$

completing the proof. ∎

THEOREM 4.6. *Let R and T be bounded isometric operators such that R has an adjoint. If for each $\varepsilon > 0$ there is an ε-approximate polar decomposition UT of R, then T has an adjoint.*

Proof. The result follows immediately from Lemma 4.5 and Proposition 3.2. ∎

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