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A LIFTING THEOREM GIVING AN ISOMORPHISM OF KK-PRODUCTS IN BOUNDED AND UNBOUNDED KK-THEORY

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Abstract. We prove a generalization of the Kasparov technical theorem.

It is known that KK-theory cycles can be defined using unbounded operators ([3]), even in the equivariant case ([15]). We apply this generalized Kasparov technical theorem to a problem involving the Kasparov product of cycles defined by unbounded operators. In some earlier work ([15] and [16]) we showed that the Kasparov product can be defined directly in terms of unbounded cycles, provided that the cycles satisfy certain conditions. In this paper we show that these conditions are necessary as well as sufficient, after equivalence.

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1. INTRODUCTION

The starting point for Kasparov KK-theory ([12]) is an abstraction and axiomatization of the main properties of zero$^{\text{th}}$ order elliptic operators, whereas in unbounded Kasparov theory, here denoted Ψ, one axiomatizes the properties of first order operators instead.

Baaj and Julg showed in [3] that in one simple but important special case, the Kasparov product is a generalization of the "sharp product" introduced by Atiyah and Singer in their proof of an index theorem ([2]), and that is easier to define this product for first order operators than for zero$^{\text{th}}$ order operators. Specifically, they proved that the so-called external case of the Kasparov product reduces to a sharp product (a graded tensor product) when written in terms of unbounded operators: their result is that

$$[D_1] \otimes [D_2] = [D_1 \otimes 1 + 1 \otimes D_2]$$

in this very special case.

In an earlier paper ([16]) we established conditions for an element of $z \in \Psi(A,C)$ to be a product of $x \in \Psi(A,B)$ and $y \in \Psi(B,C)$, and showed that if z is a product of x and y in our sense, then $b(z)$ is a Kasparov product of $b(x)$ and $b(y)$. The conditions are:

THEOREM 1.1. ([16]) *Suppose that we are given three unbounded cycles: $(E_1 \widehat{\otimes} E_2, \varphi_1 \widehat{\otimes} \mathrm{Id}, D) \in \Psi(A,C)$, $(E_1, \varphi_1, D_1) \in \Psi(A,B)$ and $(E_2, \varphi_2, D_2) \in \Psi(B,C)$. They are a Kasparov product if*

(i) *for all x in some dense subset of $\varphi_1(A)E_1$, the operator*

$$\left[\begin{pmatrix} 0 & T_x^* \\ T_x & 0 \end{pmatrix}, \begin{pmatrix} D_2 & 0 \\ 0 & D \end{pmatrix}\right] \quad \text{where } T_x(e) \mapsto x \otimes e,$$

is bounded on $\mathfrak{Dom}\,(D_2 \oplus D)$;

(ii) *for λ sufficiently large, $\langle (D_1 \otimes 1)x, Dx\rangle + \langle Dx, (D_1 \otimes 1)x\rangle \geqslant -\lambda\langle x, x\rangle$ for all x in some dense subset of $\mathfrak{Dom}\, D \cap \mathfrak{Dom}\,(D_1 \otimes 1)$;*

(iii) *and the resolvent of D is compatible with $D_1 \otimes 1$ or the reverse.*

The first condition is called the (unbounded) connection condition, and the second is called the (unbounded) positivity condition. It is technically convenient to give the positivity condition in terms of quadratic forms, though this is not the original form of the condition. Compatibility is a mild technical condition to make the intersection of the domain of D and of D_1 large enough. It is satisfied if, for example, the domain of D is contained in the domain of $D_1 \otimes 1$:

LEMMA 1.2. (Compatibility lemma, [16]) *We say that the resolvent of D is* compatible *with D_1 if there is a dense submodule $\mathcal{W}$ such that $D_1(\mathrm{i}\mu + D)^{-1}(\mathrm{i}\mu_1 + D_1)^{-1}$ is defined on $\mathcal{W}$, for all $\mu, \mu_1 \in \mathbb{R} \setminus \{0\}$. Any of the following conditions are sufficient to imply compatibility:*

(i) $\mathfrak{Dom}\, D \subseteq \mathfrak{Dom}\, D_1$*;*

(ii) $(\mathrm{i}\mu + D)^{-1}$ *maps the submodule $C_{\mathrm{c}}^\infty(D_1)E$ into $\mathfrak{Dom}\, D_1$.*

By definition, cycles $[D_1]$ and $[D_2]$ are *composable* if there exists a $[D]$ that satisfies the conditions of Theorem 1.1. In this paper we show that such a D always exists, up to equivalence, by lifting a general Kasparov product from the bounded to the unbounded picture. This procedure also gives some information about the form of D. The theorem is that:

THEOREM 1.3. *If the three cycles $(E_1, \varphi_1, F_{E_1}) \in \mathrm{KK}_G(A,B)$, $(E_2, \varphi_2, F_{E_2}) \in \mathrm{KK}_G(B,C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, F) \in \mathrm{KK}_G(A,C)$ form a Kasparov product, they lift to unbounded cycles $(E_1, \varphi_1, F_{E_1}h_1^{-1}) \in \Psi_G(A,B)$, $(E_2, \varphi_2, F_{E_2}h_2^{-1}) \in \Psi_G(B,C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, Fh_{12}^{-1}) \in \Psi_G(A,C)$ that satisfy the conditions for an unbounded Kasparov product.*

If we suppose that F_{E_1} and F_{E_2} come from unbounded cycles, then the above theorem shows that a suitable pair of unbounded cycles can always be perturbed to equivalent cycles which are composable. This does not give a general and completely explicit construction for the product cycle, but the problem of giving such a construction has guided our work.

This basic problem of how to find a product cycle given two unbounded cycles is clearly of great interest. This paper proves the existence of the product cycle, but unfortunately in a somewhat indirect way, via bounded KK-theory. Even though there are many different unbounded cycles that are equivalent to any given cycle, it is probably impossible to write down a general formula for the product of two unbounded cycles, and therefore it is not likely that a completely constructive proof of the existence of the product cycle can be given. Of course, it may be both preferable and possible to prove the results of this paper entirely within the unbounded picture, for example by giving a way to perturb two unbounded cycles so that the product cycle is given by the sum of the perturbed operators. Nevertheless, the result we obtain is enough to suggest that in any concrete situation, one can look for a formula for the unbounded product cycle, presumably simpler and more informative than the bounded one.

Before giving the proof of the main result we discuss an example that helps, in hindsight, to motivate the proof. First of all, Baaj and Julg ([3]) gave a "lifting theorem" that they used to show that the natural map $b : \Psi(A, B) \to \mathrm{KK}(A, B)$ is surjective. Proposition 2.9 suggests that one could use the construction of Baaj and Julg to 'lift' a pair of bounded cycles satisfying the ordinary connection condition to a pair of unbounded cycles satisfying the unbounded connection condition. The original theorem of Baaj and Julg ([3]) is not directly applicable, but their proof establishes the following lemma (for unital A):

Lemma 1.4. *Given a strictly positive $h \in \mathcal{K}(E)$, a cycle $(E, \varphi, F) \in \mathrm{KK}(A, B)$ and a countable dense subset $(a_i) \subset A$, there is an $\ell \in C^*(h)$ such that $[F, \varphi(a_i)]\ell^{-1}$, $[\ell^{-1}, \varphi(a_i)]$, $[\ell^{-1}, F]$ and $(1-F^*F)^{1/2}\ell^{-1}$ are defined on the range of ℓ and coincide with bounded operators.*

This lemma is enough to show that $(E \oplus E_2, \widetilde{\varphi}, F \oplus F_2) \in \mathrm{KK}(A + \mathcal{K}(E_1 \oplus B), C)$ can be lifted to an equivalent unbounded nonselfadjoint cycle $(E \oplus E_2, \widetilde{\varphi}, (F \oplus F_2)\ell^{-1})$. We can choose an h, therefore an ℓ, which is a direct sum $\ell_0 \oplus \ell_2$, so that $F\ell_0^{-1}$ is an unbounded connection for $F_2\ell_2^{-1}$. This shows that unbounded connections always exist, which is something that could also be proven using the stabilization theorem. The main reason for the approach we have used is the possibility that one could lift not just a pair of cycles, but a triple of cycles. If this could be done in such a way that both the connection condition and the positivity condition are preserved, then we can show that the conditions for the unbounded Kasparov product can always be satisfied, and that the unbounded theory is formally completely equivalent to the usual theory. However, one can expect that certain products can be computed more easily in the unbounded picture, which is the motivation for introducing the unbounded picture in the first place.

Kasparov's original approach ([12]) to the product was to show the existence of operators M and N such that

$$[F_1] \otimes [F_2] = \bigl[M^{1/2}(F_1 \otimes 1) + N^{1/2}(1 \otimes F_2)\bigr].$$

The operators N and M have very special properties and are not easy to construct explicitly; furthermore, there are some technical complications coming from the fact that the tensor product $1 \otimes F_2$ is not well-defined in most cases of interest. One therefore has to stabilize the Hilbert modules involved and this makes it even more difficult to explicitly determine the product cycle. We call Kasparov's approach the noncommutative partition of unity approach.

The next development was the discovery of the connection approach to the Kasparov product ([7]). Connes and Skandalis found the following criterion for $(E_1 \otimes E_2, \varphi_1 \otimes 1, F)$ to be the Kasparov product of (E_1, φ_1, F_1) and (E_2, φ_2, F_2) :

(i) $FT_x - (-1)^{\partial x} T_x F_2$ is compact (where T_x is the tensoring operator $T_x : y \mapsto x \otimes y$).

(ii) $\varphi_1(a)[F, F_1 \otimes 1]\varphi_1(a)^*$ is positive modulo compact operators for all $a \in A$.

The author has shown that analoguous conditions hold for unbounded cycles. These conditions are given in Theorem 1.1. Roughly speaking, the result of using unbounded cycles instead of bounded ones is that, quite generally, bounded operators are replaced by unbounded ones, and compactness conditions are replaced by boundedness conditions. This has the advantage that not only is boundedness almost always easier to prove than compactness, but the Kasparov product can be simpler to compute.

The starting point for Kasparov's original construction of the Kasparov product is the construction of a strictly positive compact ℓ with $\pm\mathrm{i}[F, \ell] \leqslant \ell^2$ for some given self-adjoint operator F (see [12], Chapter 3, Theorem 1). Applying ℓ^{-1} from left and right formally gives $\pm\mathrm{i}[\ell^{-1}, F] \leqslant \gamma$. Kasparov used this commutator construction to prove what is now known as Kasparov's technical theorem. There is now a simple proof of this theorem ([11]), but Kasparov's original proof was motivated by the pseudodifferential calculus, and it is not difficult to rewrite some parts of it in terms of unbounded operators. However, in this paper we will not be able to follow the same steps as in Kasparov's proof, because we have to satisfy more conditions on the operators that we construct. Our replacement for Kasparov's technical theorem will be:

THEOREM 1.5. *Suppose B_1 and B_0 are σ-unital C^*-algebras with $B_0 = \overline{B_1 B_0}$. Suppose $\Delta_i \subset \mathrm{Der}\, B_i$, and $\Delta_{10} \subset \mathrm{Der}\,(B_1, B_0)$ are pointwise norm compact sets of derivation. Suppose K_i is any norm-compact subset of B_i^{sa}. Then for each integer m there are strictly positive elements $h_i \in B_i$ and a scalar γ with*

(i) $h_0^m \leqslant h_1^m \leqslant 1$;

(ii) $\delta(h_k)\delta(h_k)^* \leqslant \gamma h_k^m$ *for all* $\delta \in \Delta_k$;

(iii) $\delta(h_1)\delta(h_1)^* \leqslant \gamma h_0^m$ *for all* $\delta \in \Delta_{10}$;

(iv) $(\pm\mathrm{i}[h_1, h_0])^2 \leqslant \gamma h_0^m$;

(v) $\pm K_i \leqslant \gamma h_i^m$.

2. DEFINITIONS AND LEMMAS

The adjointable operators $\mathcal{L}$ and the compact operators on a Hilbert module E form C^*-algebras, with the corresponding norm topology. In general, when we speak of the range of an operator in $\mathcal{L}(E)$, we mean its Hilbert module range. Recall that given a representation $\varphi : A \to \mathcal{L}(E)$, the right ideal $J_\varphi^{\mathrm{R}} \subset \mathcal{L}(E)$ is $\{L \in \mathcal{L}(E) : L\varphi(A) \subseteq \mathcal{K}(E)\}$ and J_φ is the self-adjoint subalgebra $J_\varphi^{\mathrm{R}} \cap (J_\varphi^{\mathrm{R}})^*$, which is itself an ideal in $I_\varphi := \{L \in \mathcal{L}(E) : [\varphi(A), L] \subseteq \mathcal{K}(E)\}$. The compact operators $\mathcal{K}(E)$ are a special case of J_φ, obtained if $\varphi(A)$ is unital. Given a family of operators from one Banach space to another, the *pointwise norm topology* on the family is the weakest topology in which the maps defined by evaluation at a point are continuous with respect to the norm topology on the range space. By the Tychonoff theorem, a set of operators $\{\mathcal{F}_{\lambda\in\Lambda} : A \to B\}$ is *pointwise norm compact* if and only if the image under evaluation at a point, $\{\mathcal{F}_\lambda(a) : \lambda \in \Lambda\}$, is norm-compact in B for each a in A. Finally, an unbounded operator T is said to be *bounded on a domain* $\mathcal{D}$ if $\|Tx\| \leqslant M\|x\|$ for all x in $\mathcal{D}$.

In general, we will be looking at unbounded operators of the form $F\ell^{-1}$, where ℓ is a strictly positive compact operator and F is a bounded self-adjoint operator with $\pm\mathrm{i}[F, \ell] \leqslant \ell^2$. This implies that F and ℓ^{-1} commute up to a bounded operator, and that there is a formal adjoint, $(F\ell^{-1})^* := \ell^{-1}F$, with good properties. However, at the moment this is only a formal adjoint. One of the subtleties of working with unbounded operators on Hilbert modules is that to show that $F\ell^{-1}$ has an adjoint, which means that we have to verify that it is *regular*. Regularity means primarily that the graph is an orthogonally complemented submodule of $E \oplus E$, though some minor conditions on density of the domain are also assumed:

Definition 2.1. A closed unbounded operator $D : E \to E$ is *regular* with adjoint T if $\Gamma(D) \oplus M_2\Gamma(T) = E \oplus E$, in the sense of an orthogonal direct sum, where M_2 is the mirror image operator $M_2(x, y) := (-y, x)$. We assume that T and D are both closed and densely defined.

In the case of $F\ell^{-1}$, we can prove the following lemma:

Lemma 2.2. *Let E be a Hilbert module, and suppose $\ell, F \in \mathcal{L}(E)$. Suppose ℓ is injective and has dense range, $\ell^{*-1}F^*$ is densely defined, and $F^*F + \ell^*\ell$ is invertible. Then $F\ell^{-1}$ is regular with adjoint $\ell^{*-1}F^*$. If ℓ is in $J_\varphi(E)$ then $F\ell^{-1}$ has resolvent in J_φ.*

Proof. Since ℓ has dense range and is injective, $F\ell^{-1}$ is an unbounded operator and is densely defined. Consider the operator $T : x \mapsto (\ell x, Fx)$ whose range is the graph of $F\ell^{-1}$. There is a bounded adjointable right inverse given by

$$V : (p, q) \mapsto (F^*F + \ell^*\ell)^{-1}(\ell^*p + F^*q)$$

and therefore $\mathfrak{Im}\,T$ is closed, so that $E \oplus E = \mathfrak{Im}\,T \oplus \mathfrak{Ker}\,T^*$ (see [21], 15.3.8), and $F\ell^{-1}$ has orthocomplemented graph. The kernel of T^* is the set $\{(y, w) : \ell^*y + F^*w = 0\}$, which is the mirror image of the graph of $\ell^{*-1}F^*$, and the adjoint of $F\ell^{-1}$ is $\ell^{*-1}F^*$, as expected. The adjoint is densely defined, so we have regularity.

If ℓ is in J_φ, then

$$(1+(\ell^{-1})^*F^*F\ell^{-1})^{-1} = \ell(\ell^*\ell + F^*F)^{-1}\ell^*$$

is in J_φ. ∎

LEMMA 2.3. (i) *If T is regular and T, T^* have dense range, then T^{-1} is regular and $(T^{-1})^* = (T^*)^{-1}$.*

(ii) *Suppose that $S, T \in \mathcal{L}(E)$ have dense range and adjoints with dense range. Then $S^{-1}T^{-1}$ is regular with adjoint $(T^*)^{-1}(S^*)^{-1}$.*

Proof. T^{-1} and $(T^*)^{-1}$ are densely defined by hypothesis, and the mirror image map $M_2 : (x, y) \mapsto (-y, x)$ takes the graph of T to the graph of T^{-1}, so that T^{-1} is regular with adjoint $(T^*)^{-1}$. This proves part (i).

A continuity argument shows that ST and $(ST)^*$ have dense range, and certainly $ST \in \mathcal{L}(E)$ is regular, so the rest follows from the first part. ∎

We are going to use non-selfadjoint unbounded operators to define KK-cycles, under the hypothesis that the operators only differ from a self-adjoint operator by a bounded operator, so we extend the usual definition of unbounded KK-theory to allow this.

DEFINITION 2.4. The set of *unbounded Kasparov modules* $\Psi(A, B)$ is given by triples (E, φ, D) where E is a Hilbert B-module, $\varphi : A \to \mathcal{L}(E)$ is a $*$-homomorphism, and D is an unbounded regular densely defined degree one operator on E, such that:

(i) $\mathfrak{Dom}\, D = \mathfrak{Dom}\, D^*$ and $D - D^*$ is bounded;

(ii) the operator $(\lambda + D)^{-1}$ is in J_φ for some $\lambda \in \mathbb{C}$;

(iii) for all a in some dense subalgebra of A, the graded commutator $[D, \varphi(a)]$ is bounded on the domain of D.

We say that two cycles are equivalent if and only if the self-adjoint parts are equivalent. To justify this definition we need to check that the self-adjoint part of the operator is a cycle in the ordinary sense, so we show that the compactness of the resolvent is stable under perturbation by bounded operators:

LEMMA 2.5. *Let $T = D + B$ where D is self-adjoint and regular, and B is a bounded adjointable operator. Then the following are equivalent:*

(i) $(1 + T^*T)^{-1} \in J_\varphi$;

(ii) *for some* $\lambda \notin \operatorname{Sp} T$, $(\lambda - T)^{-1} \in J_\varphi^{\mathrm{R}}$;

(iii) *for all* $\lambda \notin \operatorname{Sp} T$, $(\lambda - T)^{-1} \in J_\varphi$;

(iv) *for all* $\lambda \notin \operatorname{Sp} D$, $(\lambda - D)^{-1} \in J_\varphi$.

Proof. First we prove the equivalence of (ii), (iii) and (iv). We have a resolvent equation:

$$\begin{aligned}(T-\lambda)^{-1} - (D-\mu)^{-1} &= -(D-\mu)^{-1}(B-\mu+\lambda)(T-\lambda)^{-1}\\ &= -(T-\lambda)^{-1}(B-\mu+\lambda)(D-\mu)^{-1},\end{aligned}$$

which immediately implies the equivalence of (iii) and (iv). Since D is self-adjoint, we can replace J_φ by the one-sided ideal J_φ^{R} when establishing condition (iv), and then it is clear that (ii) is equivalent to (iv). For the next part of the proof, it is convenient to suppose that $\|B\| < \frac{1}{2}$, so that

$$(1 + \mathrm{i}(B^* - B) + T^*T)^{-1} = (\mathrm{i} + T)^{-1}(-\mathrm{i} + T^*)^{-1}.$$

The equivalence (iv) $\Leftrightarrow$ (iii) still holds if we replace D by T^*T and T by $\mathrm{i}(B^* - B) + T^*T$ in the previous proof, so condition (i) is therefore equivalent to

(i′) $(\mathrm{i} + T)^{-1}(-\mathrm{i} + T^*)^{-1} \in J_\varphi$.

It is clear that (iii) implies (i′). Factoring $(\mathrm{i} + T)^{-1}$ in $\mathcal{L}(E)$ as $u|(\mathrm{i} + T)^{-1}|^{1/2} = u((\mathrm{i} + T)^{-1}(-\mathrm{i} + T^*)^{-1})^{1/4}$ shows that (i′) implies (ii). ∎

COROLLARY 2.6. *A* KK*-cycle* (E, φ, D) *with* D *self-adjoint is equivalent to* $(E, \varphi, D + B)$ *for all* $B \in \mathcal{L}(E)$.

We collect some known results about connections in KK-theory:

LEMMA 2.7. *Suppose that* $F \in \mathcal{L}(E_1 \otimes E_2)$ *is an* F_2*-connection.*

(i) *If* $L \in \mathcal{L}(E_1)$*, the commutator* $[L \otimes 1, F]$ *is a* 0*-connection.*

(ii) *If* $K \in \mathcal{K}(E_1)$*, the commutator* $[K \otimes 1, F]$ *is compact.*

(iii) *If* f *is a continuous function and* F, F_2 *are normal, then* $f(F)$ *is an* $f(F_2)$*-connection.*

(iv) *If* $K \in \mathcal{K}(E_1 \otimes E_2)$ *then* $F + K$ *is an* F_2*-connection.*

The following factorization result will be used.

THEOREM 2.8. (Cohen, [5]) *Let* A *be a Banach algebra with left approximate unit bounded by* d*. Let* E *be a left Banach* A*-module. Then for all* $z \in \overline{AE}$ *and* $\varepsilon > 0$*, there is an* $a \in A$ *and* $y \in E$ *with:*

(i) $z = ay$;

(ii) $\|a\| \leqslant d$; *and*

(iii) $\|y - z\| < \varepsilon$.

PROPOSITION 2.9. *Let* $E := E_1 \widehat{\otimes}_{\varphi_2} E_2$*. Then* $(E \oplus E_2, \widetilde{\varphi}, F \oplus F_2)$ *is a* $\mathrm{KK}(A + \mathcal{K}(E_1 \oplus B), C)$ *cycle if and only if:*

(i) F *is an* F_2*-connection;*

(ii) $(E, \varphi_1 \otimes 1, F)$ *is a* $\mathrm{KK}(A, C)$ *cycle; and*

(iii) (E_2, φ_2, F_2) *is a* $\mathrm{KK}(B, C)$ *cycle.*

If conditions (i) *and* (iii) *hold then* $1 - F^2$ *is a* 0*-connection.*

REMARK 2.10. We define the representation $\widetilde{\varphi}$ of $\mathcal{K}(E_1 \oplus B)$ on $E \oplus E_2$ as follows:

$$\widetilde{\varphi}\begin{pmatrix} k & e_4 \\ \langle e_3, \cdot\rangle & \beta \end{pmatrix} := \begin{pmatrix} k \otimes 1 & T_{e_4} \\ T^*_{e_3} & \varphi_2(\beta) \end{pmatrix} \in \mathcal{L}(E \oplus E_2)$$

where the e_i are in E_1, k is in $\mathcal{K}(E_1)$ and β is in B. That this action defines a homomorphism follows from $T^*_\alpha T_\beta = \varphi_2(\langle \alpha, \beta\rangle)$ and $T_\alpha T^*_\beta = \alpha\langle \beta, \rangle \otimes 1$. On A, $\widetilde{\varphi}$ is $\varphi_1 \otimes 1 \oplus 0$, and we can extend $\widetilde{\varphi}$ to $\widetilde{\varphi} : A + \mathcal{K}(E_1 \oplus B) \to \mathcal{L}(E \oplus E_2)$.

Proof. First we show that if (i) and (iii) hold then $1 - F^2$ is an 0-connection. Consider

$$G := \begin{pmatrix} 0 & T_x(1 - F_2^2) \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(E \oplus E_2).$$

This operator has a factorization in $\mathcal{L}(E\oplus E_2)$ of the form $G=u|G|^{1/2}$, where

$$u=\begin{pmatrix} u_{11} & u_{12}\\ u_{21} & u_{22}\end{pmatrix} \qquad \text{and} \qquad |G|^2=\begin{pmatrix} 0 & 0\\ 0 & (1-F_2^2)T_x^*T_x(1-F_2^2)\end{pmatrix}.$$

But $|G|^{1/2}$ is compact since $T_x^*T_x(1-F_2^2)=\varphi_2(\langle x,x\rangle)(1-F_2^2)$ is, and therefore $T_x(1-F_2^2)=u_{12}|G|^{1/2}$ is in $\mathcal{K}(E_2,E)$. Now it follows from the connection property that $(1-F^2)T_x$ is compact, as claimed.

Next we show that (i), (ii) and (iii) imply that $\widetilde{\varphi}(a)\big(1-(F\oplus F_2)^2\big)$ is compact. Since F and F_2 come from KK-cycles, $(\varphi_1(a)\otimes 1)(1-F^2)$ and $\varphi_2(b)(1-F_2^2)$ are compact. Because $1-F^2$ is a 0-connection, $(\mathcal{K}(E_1)\otimes 1)(1-F^2)$ and $\begin{pmatrix} 0 & T_x\\ T_x^* & 0\end{pmatrix}\begin{pmatrix} 1-F^2 & 0\\ 0 & 1-F_2^2\end{pmatrix}$ are also compact, so $\widetilde{\varphi}(a)\begin{pmatrix} 1-F^2 & 0\\ 0 & 1-F_2^2\end{pmatrix}$ is compact for all self-adjoint a in $A+\mathcal{K}(E_1\oplus B)$.

Next we show that if (i), (ii), and (iii) hold, then $[\widetilde{\varphi}(a),F\oplus F_2]$ is compact. It is again enough to consider a self-adjoint a, in which case the commutator

$$\left[\begin{pmatrix} (\varphi_1(a)\otimes 1)+k & T_x\\ T_x^* & \varphi_2(b)\end{pmatrix},\begin{pmatrix} F & 0\\ 0 & F_2\end{pmatrix}\right]$$

(where $a\in A$, $k\in\mathcal{K}(E_1)$, $b\in B$) is clearly compact.

Therefore we have shown that $(E\oplus E_2,\widetilde{\varphi},F\oplus F_2)$ is a $\mathrm{KK}(A+\mathcal{K}(E_1\oplus B),C)$ cycle if the conditions (i), (ii), and (iii) hold. The converse is easily shown. ∎

We can prove an interesting lemma for commutator estimates:

PROPOSITION 2.11. *Suppose that a and c are commuting elements of a ($\mathbb{Z}_2$-graded) C^*-algebra, D is an unbounded operator, and a is a self-adjoint operator that preserves the domain of D. Let f be operator monotone on an interval containing the spectrum of a. If $\pm[a,D]\leqslant c$, then $f(a)$ preserves the domain of D and $\pm[f(a),D]\leqslant f'(a)c$.*

Proof. Replacing $f(x)$ by $f(x-\lambda)$ and a by $a+\lambda$ for a suitable λ, we can assume that a is positive. If f is operator monotone on $[0,\|a\|]$, then f can be represented there by

$$f(x)=\int_{\mathbb{R}^+} x(1+\alpha x)^{-1}\,\mathrm{d}\mu(\alpha),$$

where μ is a positive Riemann-Stieltjes measure ([10]). The remark about preservation of the domain by $f(a)$ follows from the next lemma, applied to $1+\alpha a$. Also,

$$\begin{aligned}[a(1+\alpha a)^{-1},D]&=-\frac{[(1+\alpha a)^{-1},D]}{\alpha}=-(1+\alpha a)^{-1}[D,1+\alpha a]\frac{(1+\alpha a)^{-1}}{\alpha}\\ &=-(1+\alpha a)^{-1}[D,a](1+\alpha a)^{-1}.\end{aligned}$$

Hence for x in the domain of D,

$$\begin{aligned}\big\langle x,[f(a),D]x\big\rangle&=\int_{\mathbb{R}^+}\big\langle x,(1+\alpha a)^{-1}\big((-1)^{\partial D\partial a}[a,D]\big)(1+\alpha a)^{-1}x\big\rangle\,\mathrm{d}\mu(\alpha)\\ &\leqslant\int_{\mathbb{R}^+}\big\langle x,c(1+\alpha a)^{-2}x\big\rangle\,\mathrm{d}\mu(\alpha)\end{aligned}$$

and the last expression is equal to $\big\langle x,cf'(a)x\big\rangle$. ∎

LEMMA 2.12. *If a is a positive and invertible bounded operator that preserves the domain of a closed operator D, and if $[D,a]$ is bounded on that domain, then a is a bijection of the domain.*

Proof. Let $A := \|a\|$. The spectrum of a is contained in some disk about A of radius $A-\varepsilon$. Let

$$\sum_0^\infty c_n(x-A)^n$$

be the Taylor series expansion about A of $\frac{1}{x}$. The partial sums of this series, $f_n(a)$, clearly preserve $\mathfrak{Dom}\, D$, so we only need to show that $[D, a^{-1} - f_n(a)]x$ converges to zero in norm for every $x \in \mathfrak{Dom}\, D$. Note that

$$\big\|[(a-A)^n, D]\big\| \leqslant n\|a-A\|^{n-1}\big\|[a-A,D]\big\| \leqslant n(A-\varepsilon)^{n-1}\big\|[a,D]\big\|,$$

so

$$\big\|c_n[(a-A)^n, D]\big\| \leqslant \frac{n}{A^2}\Big(\frac{A-\varepsilon}{A}\Big)^{n-1}\big\|[a,D]\big\|$$

for $n \geqslant 1$, and the sequence $N \mapsto \Big[D, \sum\limits_{N+1}^\infty c_n(a-A)^n\Big]x$ converges to zero in norm. We see that the sequence $Df_n(a)x$ converges in norm to $Da^{-1}x$ for each x in the domain of D. D being closed, it follows that $a^{-1}x$ is in the domain of D. Therefore a maps the domain of D onto itself. ∎

LEMMA 2.13. ([1], p. 332) *Given a continuous function $f : [-1,1] \to \mathbb{R}$, there is a continuous function δ with $\delta(0) = 0$, such that $\big\|[S, f(T)]\big\|$ is bounded by $\delta\big(\|\,[S,T]\,\|\big)$ for all S and all self-adjoint T in the unit ball.*

LEMMA 2.14. *If K is a norm-compact subset of a C^*-algebra B, there is $b \in B$ and a norm-compact subset C such that $K = Cb$. Given $\varepsilon > 0$, we can choose to have $\|C\| \leqslant (1+\varepsilon)\|K\|$ and $\|b\| \leqslant 1$.*

Proof. The algebra B acts on $C(K,B)$ from the right. Since K is compact and B has a canonical approximate unit, there is an approximate unit for $C(K,B)$ in B. By Cohen's theorem, the identity function $\iota : K \to K \subset B$ can be factored as $\iota(k) = f(k)b$ with $\|b\| \leqslant 1$ and

$$\sup_{x\in K}\|f(x)\| \leqslant (1+\varepsilon)\sup_{x\in K}\|x\|.$$

Then, define C to be the image of f. The set C is norm-compact because f is continuous, and clearly is bounded by $(1+\varepsilon)\|K\|$. ∎

LEMMA 2.15. *Given $\varepsilon > 0$, a strictly positive element p of a C^*-algebra B, and a compact subset of self-adjoint operators $K \subset B^{\mathrm{sa}}$, there is a function $f \in C_0(\operatorname{Sp} p)$ with $K \leqslant f(p)$ and $0 \leqslant f \leqslant (1+\varepsilon)\|K\|$.*

Proof. We can as well consider only the positive part of the operators in K. By Lemma 2.14 applied to $K^{1/2}$, there is $b \in B$ with $\|b\| \leqslant 1$ such that $K = b^*Cb$ and $\|C\| \leqslant (1+\varepsilon)^{1/2}\|K\|$. Since $C^*(p)$ contains an approximate unit for B, Cohen's theorem gives g with $b = b'g(p)$, where $\|b'\| \leqslant 1$ and $|g| \leqslant (1+\varepsilon)^{1/4}$. Therefore, $K = g(p)b'^*Cb'g(p)$ and $b'^*Cb' \leqslant (1+\varepsilon)^{1/2}\|K\|$, implying that

$$K \leqslant g(p)^2(1+\varepsilon)^{1/2}\|K\| =: f(p). \quad ∎$$

The next lemma is a special case of the one given in [4], Remark 4.2.2, and is based on Arveson's construction of quasi-central approximate units. We will see later that it extends to the case of a group action, in which case we obtain approximate equivariance under the action of a suitable set of group elements.

LEMMA 2.16. *Suppose that $p \leqslant 1$ in a C^*-algebra A and that $f_n(p) \subset C^*(p)$ is an increasing approximate unit. Given $\varepsilon > 0$, and a pointwise norm compact subset $\Delta \subset \operatorname{Der} A$, a norm-compact subset $U \subset A$ and $g \in C_0(0,1]$ with $g < 1$, there is $h \in \overline{\operatorname{Conv}(f_n)}$ such that $\big\|\Delta(h(p))\big\| < \varepsilon$, $\big\|(1-h(p))U\big\| < \varepsilon$ and $h \geqslant g$.*

Proof. The convex set

$$T := \overline{\{z \in \operatorname{Conv}\{f_n(p)\} : z \geqslant g(p)\}}$$

contains an approximate unit for A. Hence 1 is in the strict closure of T. Define the affine map $\mathcal{F} : A \to C(\Delta, A) \oplus C(U, A)$ by $x \mapsto \ell(x) \oplus (1-x)u$, $u \in U$, $\ell \in \Delta$. This map has a strictly continuous extension to the multiplier algebras, so the strict closure of $\mathcal{F}(T)$ contains $\mathcal{F}(1) = 0$. In particular, the C^*-weak closure of $\mathcal{F}(T)$ contains 0. But $\mathcal{F}(T)$ is convex, so the norm closure coincides with the weak closure ([8]), and there is a sequence in $\mathcal{F}(T)$ that converges to zero in norm. We choose an $h(p) \in T$ with $\|\mathcal{F}(h(p))\| < \varepsilon$. ∎

3. A LIFTING THEOREM THAT PRESERVES POSITIVITY

From the proof of Theorem 1 in [12], Section 3, we extract the following lemma:

LEMMA 3.1. *Suppose A is a unital C^*-algebra and B is a closed subalgebra. Let (u_r) be an increasing commutative approximate unit in B, and let $C \subset A$ be a self-adjoint subset of operators, satisfying the norm inequalities:*

(i) $\big\|(1-u_{r+1})^{1/4}u_r^{1/4}\big\| \leqslant 2^{-r}$;

(ii) $\big\|(1-u_{r+1})^{1/4}cu_r^{1/4}\big\| \leqslant 2^{-r}$ *for all* $c \in C$.

Then, for any positive integer n, the operator $h := \sum(r^{-1/n} - (r+1)^{-1/n})u_r$ satisfies the operator inequality

$$[h,c][h,c]^* \leqslant \gamma(n)h^{2n+2} \quad \textit{for all } c \in C,$$

where $\gamma(n)$ is a scalar depending only on n.

REMARK 3.2. Given two C^*-algebras B_0 and B_1 with $B_0 = \overline{B_1B_0}$ inside some larger C^*-algebra A, Cohen's theorem shows that B_1 is in the multiplier algebra of B_0, so we can take $A = \mathcal{M}(B_0)$. If B_0 is actually contained in B_1, then the condition $B_0 = \overline{B_1B_0}$ holds if and only if B_0 is an ideal in B_1.

Also recall that compactness in the pointwise norm topology simply means that the image of a given set of operators, after evalution at a point, is norm compact.

PROPOSITION 3.3. *Suppose B_1 and B_0 are C^*-algebras with strictly positive elements ℓ_i, and suppose $B_0 = \overline{B_1 B_0}$. Let $\Delta_i \subseteq \operatorname{Der} B_i$ and $\Delta_{10} \subseteq \operatorname{Der}(B_1, B_0)$ be pointwise norm compact sets of derivations. Let $K_i \subset B_i^{\mathrm{sa}}$ be some given norm-compact subset of selfadjoint operators. Then for every integer m there is a scalar γ and a function $f \in C_0(\mathbb{R}^+)$, with $h_i := f(\ell_i)$ satisfying:*

(i) $\delta(h_i)\delta(h_i)^* \leqslant \gamma h_i^m$ *for all* $\delta \in \Delta_i$;
(ii) $\delta(h_1)\delta(h_1)^* \leqslant \gamma h_0^m$ *for all* $\delta \in \Delta_{10}$;
(iii) $[h_1, h_0][h_1, h_0]^* \leqslant \gamma h_i^m$;
(iv) $K_i \leqslant \gamma h_i^m$; *and*
(v) f^m *is operator monotone.*

Proof. As pointed out in the remark, Cohen's theorem (Theorem 2.8) implies that $B_0 = \overline{B_1 B_0} = B_1 B_0$. The positive elements ℓ_i can be used to produce abelian countable approximate units, $\big(\ell_i^{1/jm}\big)_{j=1}^{\infty} \subset B_i$, and because B_0 factors through B_1, $\ell_1^{1/jm}$ is an approximate unit for $B_0 + B_1$.

Let us suppose that K_i and Δ_i are bounded by 1. We use Lemma 2.16 with $A := B_1 + B_0$ to find a sequence of functions g_i in the convex hull of $\big\{t^{1/im} : i \in \mathbb{N}\big\}$, such that $g_i(t) \geqslant t^{1/im}$, $\big\|\Delta_0(g_i(\ell_0))\big\| \leqslant 2^{-i}$, $\big\|\Delta_1(g_i(\ell_1))\big\| \leqslant 2^{-i}$, and $\big\|\Delta_{10}(g_i(\ell_1))\big\| \leqslant 2^{-i}$. The condition $g_i(t) \geqslant t^{1/im}$ implies that $(g_i(\ell_1))$ is still an approximate unit for $B_0 + B_1$. Define $D_k := \{0\} \cup \{\delta(g_i(\ell_k)) : i \in \mathbb{N},\, \delta \in \Delta_k\}$. The function g_i were chosen so that the sequence $i \mapsto \Delta_k(g_i(\ell_k))$ would converge uniformly to zero, implying that D_k is compact. Since $D_k \times D_k$ is compact, the set

$$\{0\} \cup \{\delta(g_i(\ell_k))\delta(g_j(\ell_k))^* + \delta(g_j(\ell_k))\delta(g_i(\ell_k))^* : i, j \in \mathbb{N},\, \delta \in \Delta_k\}$$

is also compact, is bounded by 1, and is in B_k by hypothesis. Applying the same argument to Δ_{10}, we see that the sets:

$$\begin{aligned} K_0' &:= K_0 \cup \{0\} \cup \{\delta(g_i(\ell_1))\delta(g_j(\ell_1))^* + \delta(g_j(\ell_1))\delta(g_i(\ell_1))^* : i, j \in \mathbb{N},\, \delta \in \Delta_{10}\} \\ &\qquad\qquad \cup \{\delta(g_i(\ell_0))\delta(g_j(\ell_0))^* + \delta(g_j(\ell_0))\delta(g_i(\ell_0))^* : i, j \in \mathbb{N},\, \delta \in \Delta_0\} \\ K_1' &:= K_1 \cup \{0\} \cup \{\delta(g_i(\ell_1))\delta(g_j(\ell_1))^* + \delta(g_j(\ell_1))\delta(g_i(\ell_1))^* : i, j \in \mathbb{N},\, \delta \in \Delta_1\} \end{aligned}$$

are norm-compact and bounded by 1.

By Lemma 2.15, there is a function $\kappa \in C_0(\mathbb{R}^+)$ such that $1 > \kappa$ and $2\kappa(\ell_0 \oplus \ell_1) \geqslant K_0' \oplus K_1'$.

Clearly the $(g_n(\ell_i))_{n=1}^{\infty}$ are approximate units. We inductively choose a sequence of functions f_r from the convex hull of g_n such that:

(i) $f_{r+1} \geqslant \kappa^{1/m}$;
(ii) $f_{r+1} \geqslant f_r$ and $f_{r+1} \geqslant g_{r+1}$;
(iii) $\big\|(1 - f_{r+1})^{1/4} f_r^{1/4}\big\| \leqslant 2^{-(r+1)}$;
(iv) $\big\|[f_{r+1}(\ell_i), f_n^{1/4}(\ell_j)]\big\| \leqslant 2^{-(r+1)}$ for all $n \leqslant r$;
(v) $\big\|[(1 - f_{r+1}(\ell_i))^{1/4}, f_n(\ell_j)]\big\| \leqslant 2^{-(r+1)}$ for all $n \leqslant r$,

where $j = 1 - i$ and $i \in \{0, 1\}$. Suppose that f_r is known and we choose f_{r+1} as follows. Let $f \in C_0(0, 1]$ be a function which is less than 1, but larger than $\kappa^{1/m}$, larger than f_r and larger than g_{r+1}, and is such that $(1 - f)^{1/4} f_r^{1/4} \leqslant 2^{-(r+1)}$.

Apply Lemma 2.16 with $\ell_0 \oplus \ell_1$ as the strictly positive element, f as the function which is there referred to as g, and Δ given by the commutators

$$x \mapsto [x, f_n(\ell_0 \oplus \ell_1)], \quad x \mapsto \big[x, f_n(\ell_0 \oplus \ell_1)^{1/4}\big].$$

Finally, ε in Lemma 2.16 is chosen small enough so that all the inequalities are satisfied, using the fact that $t \mapsto t^{1/4}$ acting on $\mathcal{M}(B_i)$ is strictly continuous, and preserves quasicentrality by Lemma 2.13.

Looking at the last three of the above five conditions, we see that

(vi) $\big\|(1 - f_{r+1}(\ell_i))^{1/4} f_n(\ell_j) f_r^{1/4}(\ell_i)\big\| \leqslant 2^{-r}$ for all n.

If $n \leqslant r$, this follows from (iii) and (v) of the above list; if $n - 1 \geqslant r$, it follows from (iii) and (iv). If we then define

$$f := \sum \mu_r f_r \quad \text{ and } \quad h_i := f(\ell_i)$$

where μ_r is defined to be $r^{-1/s} - (r+1)^{-1/s}$ for some integer s larger than $\frac{m-2}{2}$, then

(vi′) $\big\|(1 - f_{r+1}(\ell_i))^{1/4} h_j f_r^{1/4}(\ell_i)\big\| \leqslant 2^{-r}$ for all n.

This and conditions (ii) and (iii) allow us to apply Lemma 3.1 with $f_k(\ell_i)$ as the approximate unit, B_i as B, and $\{h_j\}$ as the subset of operators. We obtain a scalar γ such that:

(i) $[h_j, h_i][h_j, h_i]^* \leqslant \gamma h_i^m$.

Since $f_r \geqslant \kappa^{1/m}$ for all r, we have $h_i \geqslant \kappa(\ell_i)^{1/m}$ in $C^*(\ell_i)$, implying that $h_0^m \oplus h_1^m \geqslant \kappa(\ell_0 \oplus \ell_1) \geqslant \frac{1}{2}(K_0' \oplus K_1')$; therefore:

(ii) $2h_i^m \geqslant K_i$;

(iii) $2h_k^m \geqslant \delta(g_i(\ell_k))\delta(g_j(\ell_k))^* + \delta(g_j(\ell_k))\delta(g_i(\ell_k))^*$ for all $\delta \in \Delta_k$ and $i, j \in \mathbb{N}$, and

(iv) $2h_0^m \geqslant \delta(g_i(\ell_1))\delta(g_j(\ell_1))^* + \delta(g_j(\ell_1))\delta(g_i(\ell_1))^*$ for all $\delta \in \Delta_{10}$ and $i, j \in \mathbb{N}$.

Because h_1 is in the convex hull of $\{g_i(\ell_1)\}$, we have $h_1 = \sum \omega_i g_i(\ell_1)$. Summing $\delta(g_i(\ell_1))^* \delta(g_j(\ell_1))$ with coefficients $\omega_i \omega_j$, we find that $2h_0^m \geqslant \delta(h_1)^* \delta(h_1)$ for all $\delta \in \Delta_{10}$. Similarly, $2h_k^m \geqslant \delta(h_k)^* \delta(h_k)$ for all $\delta \in \Delta_k$.

Finally, recalling that the f_i are in the convex hull of $(t^{1/im})_{i=1}^{\infty}$, we see that the m-fold products $f_{r_1} f_{r_2} \cdots f_{r_m}$ are convex combinations of operator monotone functions, so that $f^m = \big(\sum \mu_r f_r\big)^m$ is also operator monotone. ∎

Corollary 3.4. *Suppose B_1 and B_0 are σ-unital C^*-algebras with $B_0 = \overline{B_1 B_0}$. Suppose $\Delta_i \subset \operatorname{Der} B_i$, and $\Delta_{10} \subset \operatorname{Der}(B_1, B_0)$ are pointwise norm compact sets of derivations. Suppose K_i is any norm-compact subset of B_i^{sa}. Then for each positive integer m there are strictly positive elements $h_i \in B_i$ and a scalar γ such that:*

(i) $h_0^m \leqslant h_1^m \leqslant 1$;

(ii) $\delta(h_k)\delta(h_k)^* \leqslant \gamma h_k^m$ *for all* $\delta \in \Delta_k$;

(iii) $\delta(h_1)\delta(h_1)^* \leqslant \gamma h_0^m$ *for all* $\delta \in \Delta_{10}$;

(iv) $[h_1, h_0][h_1, h_0]^* \leqslant \gamma h_0^m$;

(v) $\pm K_i \leqslant \gamma h_i^m$.

REMARK 3.5. If $m > 4$, then we can multiply h_i by a suitable scalar in order to replace γ by 1, but then we can no longer expect that $h_1^m \leqslant 1$.

Proof. By Cohen's theorem, the hypothesis $B_0 = \overline{B_1 B_0}$ implies $B_0 = B_1 B_0 = B_0 B_1$. The existence of countable approximate units implies that there are strictly positive elements $p_i \in B_i$. Suppose that $\|p_i\| \leqslant 1$, and define $\ell_1^2 := p_1$, $\ell_0^2 = \ell_1 p_0 \ell_1$, so that $\ell_0^2 \leqslant \ell_1^2$ in $\mathcal{M}(B_0)$, and ℓ_0 is in B_0. Because $\ell_1 B_0 = (\ell_1 B_1) B_0$ and $\ell_1 B_1$ is dense in B_1, we have $\ell_1 B_0$ dense in B_0. A continuity argument shows that the composition of two operators with dense range has dense range, so $\ell_1 p_0 \ell_1 B_0$ is dense in B_0; and ℓ_0^2, hence ℓ_0, is strictly positive in B_0. Replacing K_i by $(-K_i) \cup K_i$, we apply Proposition 3.2, and multiply the h_i by a suitable positive scalar to make $h_1 \leqslant 1$. The operator monotonicity of f^m implies that $h_0^m \leqslant h_1^m \leqslant 1$ in $\mathcal{M}(B_0)$. Finally, note that the operators h_i are in fact strictly positive, because it can be easily verified that $h_i \geqslant \alpha \ell_i > 0$ for some scalar α. ∎

4. UNBOUNDED KASPAROV PRODUCTS

We now apply the above results to prove a theorem concerning unbounded Kasparov products.

We begin with some lemmas about Hilbert module ranges that fit the situation of Proposition 3.2. In Hilbert space case, the range of an operator $A \in \mathcal{L}_{\mathbb{C}}$ contains the range of another operator $B \in \mathcal{L}_{\mathbb{C}}$ if and only if there exists a factorization $B = AD$. The difficulty with the corresponding result for a Hilbert module is that the bounded operator D does not necessarily have to be adjointable, and questions about the extension of partially defined bounded operators to adjointable operators arise. Hence range inclusion results, which are really statements about the existence of certain, possibly nonadjointable, bounded operators, are easier to obtain than the corresponding factorization results, and are quite sufficient for our application. In the next lemma, a different proof allows replacing the exponent 4 by $2 + \varepsilon$, but the exact value of the exponent does not matter for our application.

LEMMA 4.1. *Let $S, T \in \mathcal{L}(E)$ be self-adjoint, with S having dense range. If $0 \leqslant S^4 \leqslant T^4$ then $\mathfrak{Ran}\, S \subseteq \mathfrak{Ran}\, T$.*

Proof. Let $f_\alpha(\lambda) := (\lambda^{1/4} + \alpha)^{-1}$. Let $x \in \mathfrak{Ran}\, S$. Observing that $\frac{(\beta-\alpha)^2}{\alpha^2\beta^2+g(\lambda)}$ is operator monotone decreasing if g is operator monotone increasing, we see that

$$(f_\alpha(\lambda) - f_\beta(\lambda))^2 = \frac{(\beta - \alpha)^2}{(\alpha + \lambda^{1/4})^2(\beta + \lambda^{1/4})^2}$$

is operator monotone decreasing for all $\alpha, \beta > 0$. Hence

$$(f_\alpha(T^4) - f_\beta(T^4))^2 \leqslant (f_\alpha(S^4) - f_\beta(S^4))^2.$$

Apply $L \mapsto \big\|\langle Lx, x\rangle\big\|^{1/2}$ to both sides, obtaining

$$\big\|(\alpha + T)^{-1}x - (\beta + T)^{-1}x\big\| \leqslant \big\|(\alpha + S)^{-1}x - (\beta + S)^{-1}x\big\|.$$

We claim that $\alpha \mapsto (\alpha+S)^{-1}x$ converges as $\alpha \mapsto 0+$, in which case $\alpha \mapsto (\alpha+T)^{-1}x$ also converges, and hence

$$\mathfrak{Ran}\,T \ni T\lim(\alpha+T)^{-1}x = x - \lim\alpha(\alpha+T)^{-1}x = x.$$

To verify the claim, we use Cohen's theorem to factor $x \in \mathfrak{Ran}\,S$ into $Sf(S)y$ with $f \in C_0(\operatorname{Sp} S)$, and then it follows that

$$\left\|S^{-1}x - (\alpha+S)^{-1}x\right\| \leqslant \left\|y\right\| \sup_{\lambda\in\operatorname{Sp}S}\left|\frac{\alpha}{\alpha+\lambda}f(\lambda)\right|$$

goes to zero as α does. ∎

Corollary 4.2. *If $0 \leqslant S^4 \leqslant T^4$ and S has dense range, then $T^{-1}S$ is bounded and ST^{-1} is bounded on $\mathfrak{Dom}\,T^{-1}$.*

Proof. The closed graph theorem implies that $T^{-1}S$ is bounded. Since $\langle S^2y,y\rangle \leqslant \langle T^2y,y\rangle$, for all x in the domain of T^{-1} we have $\langle ST^{-1}x, ST^{-1}x\rangle \leqslant \langle x,x\rangle$ and $\|ST^{-1}x\|^2 \leqslant \|x\|$, hence ST^{-1} is bounded on its domain. ∎

Again, in the next lemma, the exponents can be improved; for example, 3 can be replaced by $2+\varepsilon$.

Lemma 4.3. *In a C^*-algebra A,*

(i) *if $CC^* \leqslant S^3$ then $C = Sa$ for some $a \in A$;*

(ii) *if C is self-adjoint and $\pm C \leqslant S^3$ then $C = SbS$ for some selfadjoint $b \in A$;*

(iii) *if C is self-adjoint and $\pm C \leqslant S^8$ and $S^3 \leqslant T^3$, then $C = STcTS$ and $S^2 = Td$ for some selfadjoint $c \in A$.*

Proof. The first statement is standard ([17], 1.4.5). The second statement follows if we note that

$$\left(\sqrt{\frac{S^3 \pm C}{2}}\right)^2 \leqslant S^3,$$

so $\sqrt{\frac{S^3\pm C}{2}} = Sa_\pm$ by the first part, and $C = S(a_+a_+^* - a_-a_-^*)S$. Now we prove the third part. By part (i), $S^{3/2} = Ta$. By part (ii), the operator inequality $\pm C \leqslant \|S^{1/2}\|S^{15/2}$ implies that $C = S^{5/2}bS^{5/2} = STaba^*TS$. ∎

Lemma 4.4. *Suppose that S and T are strictly positive, and d is selfadjoint in $\mathcal{L}(E)$. If*

(i) $\pm\mathrm{i}[S,T] \leqslant S^8$;

(ii) $S^3 \leqslant T^3$;

(iii) $\pm\mathrm{i}[d,T] \leqslant S^8$,

then $[T^{-1},d]S^{-1}$ is bounded on $\mathfrak{Ran}\,TS$.

Proof. By Lemma 4.3 there are bounded adjointable operators $p,u,r \in \mathcal{L}(E)$ such that

$$[d,T] = STuTS = TrT \quad \text{and} \quad [T,S] = STpTS.$$

By Corollary 4.2 there is a possibly nonadjointable bounded operator $L^\sharp := T^{-1}S$, and

$$\begin{aligned}(L^\sharp Tu + rTp)x &= (T^{-1}[d,T]S^{-1}T^{-1} + T^{-1}[d,T](T^{-1}S^{-1} - S^{-1}T^{-1}))x\\ &= ([T^{-1},d]S^{-1})x\end{aligned}$$

for all x in $\mathfrak{Ran}\,TS$. ∎

LEMMA 4.5. *Suppose that S and T are strictly positive operators on a Hilbert B-module, and M is a bounded self-adjoint operator. If*
(i) $\pm\mathrm{i}[S,T] \leqslant S^8$;
(ii) $S^4 \leqslant T^4$;
(iii) $\pm\mathrm{i}[M,T] \leqslant S^4$ *and* $\pm\mathrm{i}[M,S] \leqslant S^8$,
then
(a) $S^{-1}M^2T^{-1} - S^{-1}T^{-1}M^2$ *is bounded on* $\mathfrak{Ran}\, TS$; *and*
(b) $S^{-1}T^{-1}M^2 - MT^{-1/2}S^{-1}T^{-1/2}M$ *is bounded on* $\mathfrak{Ran}\, TS$.

Proof. STEP 1. First we show that M preserves $\mathcal{D} := \mathfrak{Ran}\, TS$.

Since $[M,S] = STuTS$ for some $u \in \mathcal{L}(E)$, the unbounded operator $T^{-1}[M, S^{-1}]$ is in fact bounded on $\mathfrak{Ran}\, S \supseteq \mathcal{D}$. (This range inclusion comes from the factorization $TS - ST = STbTS$.) By Lemma 4.4, $[M,T^{-1}]S^{-1}$ is bounded on $\mathfrak{D}$, so by the derivation property of commutators, $[M,T^{-1}S^{-1}]$ is bounded on $\mathfrak{D}$. Since on this domain $S^{-1}T^{-1} - T^{-1}S^{-1}$ is bounded, by the previous factorization, $[M,S^{-1}T^{-1}]$ is also bounded. In fact, we have the consequence that

$$M^2S^{-1}T^{-1} \sim MS^{-1}T^{-1}M \sim S^{-1}T^{-1}M^2 \sim M^2T^{-1}S^{-1}$$

where the equivalence signs denote equivalence modulo bounded operators on the domain $\mathcal{D}$.

STEP 2. Now we show that $S^{-1}M^2T^{-1} \sim S^{-1}T^{-1}M^2$.

By Lemma 4.4, the first and the last terms on the right hand side of

$$[T^{-1},M^2]S^{-1} = M[T^{-1},M]S^{-1} + [T^{-1},M][M,S^{-1}] + [T^{-1},M]S^{-1}M$$

are bounded on $\mathfrak{D}$. By Lemma 4.3, $[M,S^{-1}] = TuT$ and $[M,T] = TrT$, so the middle term is equal to the bounded operator $rTuT$ on $\mathcal{D}$.

STEP 3. The proof of the last part of the lemma is based on the proof of Proposition 2.11. Recall that for all x in $\mathfrak{Ran}\, S$,

$$[T^{1/2},S^{-1}]x = \int_{\mathbb{R}^+} (1+\alpha T)^{-1}[T,S^{-1}](1+\alpha T)^{-1}x\, \mathrm{d}\mu(\alpha),$$

for some positive Riemann-Stiltjes measure μ. Because of the factorization $[T,S] = STuTS$, we see that

$$[T^{1/2},S] = STvTS \quad \text{where } v := \int_{\mathbb{R}^+} (1+\alpha T)^{-1}u(1+\alpha T)^{-1}\, \mathrm{d}\mu(\alpha).$$

To verify the convergence of this integral, we note that $\pm\mathrm{i}u \leqslant \lambda T^2$, implying

$$\pm\mathrm{i}\int_{\mathbb{R}^+} (1+\alpha T)^{-1}u(1+\alpha T)^{-1}\, \mathrm{d}\mu(\alpha) \leqslant \lambda T^{3/2}.$$

Hence $T^{-1/2}[T^{-1/2},S^{-1}]T^{-1/2}$ is bounded on $\mathcal{D}$, so that in particular, $[T^{-1/2},S^{-1}] T^{-1/2}$ is bounded on $\mathcal{D}$.

STEP 4. Combining Steps 1 and 3 of the above proof, we see that the operators $MS^{-1}T^{-1}M - S^{-1}T^{-1}M^2$ and $MT^{-1/2}S^{-1}T^{-1/2}M - MS^{-1}T^{-1}M$ are bounded on $\mathfrak{D}$, as desired. ∎

Suppose that we are given three cycles satisfying the usual conditions for a Kasparov product: $(E_1, \varphi_1, F_{E_1}) \in \mathrm{KK}(A, B)$, $(E_2, \varphi_2, F_{E_2}) \in \mathrm{KK}(B, C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, F) \in \mathrm{KK}(A, C)$ where F is an F_2-connection and $[F, F_{E_1}]$ is positive in $I_{\varphi_1 \otimes 1}$ modulo $J_{\varphi_1 \otimes 1}$. We shall find three unbounded operators and lift each of these cycles, in such a way that the unbounded connection conditions are satisfied. We assume that the algebras A, B and C are σ-unital.

In the following theorem, the conditions for an unbounded Kasparov product are those given in Introduction, modified in a minor way to allow for nonselfadjoint unbounded cycles, as in Definition 2.4.

THEOREM 4.6. *If the three cycles $(E_1, \varphi_1, F_{E_1}) \in \mathrm{KK}(A, B)$, $(E_2, \varphi_2, F_{E_2}) \in \mathrm{KK}(B, C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, F) \in \mathrm{KK}(A, C)$ form a Kasparov product, they lift to unbounded cycles $(E_1, \varphi_1, F_{E_1} h_1^{-1}) \in \Psi(A, B)$, $(E_2, \varphi_2, F_{E_2} h_2^{-1}) \in \Psi(B, C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, F h_{12}^{-1}) \in \Psi(A, C)$ that satisfy the conditions for an unbounded Kasparov product.*

Proof. We state the proof for unital algebras A, B and C, which simplifies the exposition by replacing I_{φ_1} and J_{φ_1} with $\mathcal{L}$ and $\mathcal{K}$, respectively. However, it is the same proof in either case, since it is easily verified that all operators that appear are in I_{φ_1}.

We define $E = E_1 \otimes_{\varphi_2} E_2$, $\widetilde{E} = E \oplus E_2$, $F_1 := F_{E_1} \otimes 1 \oplus 0$, $F_0 := F \oplus F_{E_2}$, and $[F_0, F_1] = [F, F_{E_1} \otimes 1] \oplus 0 = M^2 + Y$, where F is the given F_2-connection, M is positive and Y is compact. Define two subalgebras of $\mathcal{L}(\widetilde{E})$ to be $B_1 := \mathcal{K}(E_1) \otimes 1 \oplus \mathcal{K}(E_2)$ and $B_0 := \mathcal{K}(E_1 \otimes E_2) \oplus \mathcal{K}(E_2)$. By Lemma 2.7, the square root of the positive part of $[F, F_{E_1} \otimes 1]$ is a 0-connection, so M derives B_1 into B_0. Since F is a connection, F_0 also derives B_1 into B_0.

Recall that the connection part of the conditions for a Kasparov product implies, by Proposition 2.9, that there exists a cycle $(E \oplus E_2, \widetilde{\varphi}, F_0) \in \mathrm{KK}(A + \mathcal{K}(E \oplus B), C)$ with $F_0 = F \oplus F_{E_2}$, and the unbounded connection condition is then taken care of simply by lifting this cycle to any unbounded cycle $(E \oplus E_2, \widetilde{\varphi}, F_0 \ell_0^{-1})$ with $\ell_0 = h_{12} \oplus h_2 \in B_0$. To see this, note that if $(E \oplus E_2, \widetilde{\varphi}, F_0 \ell_0^{-1})$ is in $\Psi(A + \mathcal{K}(E \oplus E), C)$, then from the definition of $\widetilde{\varphi}$ in Remark 2.10 we have that:

(i) $(E, \varphi_1 \otimes 1, F h_{12}^{-1})$ is in $\Psi(A, C)$;

(ii) $(E_2, \varphi_2, F_{E_2} h_2^{-1})$ is in $\Psi(B, C)$; and,

(iii) the two cycles satisfy the first unbounded connection condition since $\widetilde{\varphi}\begin{pmatrix} 0 & e \\ \langle e, \cdot \rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix}$.

This shows that the first unbounded connection condition of Theorem 1.1 (i) can be obtained quite easily, only by using the fact that the operator ℓ_0^{-1} is a direct sum. At the same time, we want to satisfy the unbounded positivity condition by lifting $(E_1, \varphi_1, F_{E_1})$ to an unbounded cycle $(E_1, \varphi_1, F_{E_1} h_1^{-1}) \in \Psi(A, B)$ with the sesquilinear form given by commutator of the self-adjoint parts of $F_0 \ell_0^{-1}$ and of $F_{E_1} h_1^{-1} \otimes 1 \oplus 1$ semibounded below. In order to do this, we will have to use our generalization of the Kasparov technical theorem, as given in Corollary 3.4.

Let $\widetilde{\mathcal{A}}$ be a norm-compact set of self-adjoint operators with dense linear span in $A+\mathcal{K}(E\oplus B)$, and let $\mathcal{A}$ be a norm-compact set of self-adjoint operators with dense linear span in A. Consider

$$K_0 := \{Y, (1-F_0^2), \mathrm{i}[\widetilde{\varphi}(\widetilde{\mathcal{A}}), F_0]\}.$$

Recall that $F_0 = F \oplus F_{E_2}$, where F defines a $\mathrm{KK}(A,C)$ cycle and F_{E_2} defines a $\mathrm{KK}(B,C)$ cycle. Since A and B are unital, $1-F_0^2$ is in $\mathcal{K}(E\oplus E_2)$. Hence K_0 is in B_0 and is norm-compact. Let

$$\begin{aligned}
\Delta_{10} &:= \{x\mapsto [F_0,x],\ x\mapsto [M,x]\},\\
K_1 &:= \{(1-F_1^2),\ \mathrm{i}[\varphi_1(\mathcal{A})\otimes 1, F_1]\},\\
\Delta_0 &:= \{x\mapsto [F_1,x],\ x\mapsto [M,x],\ x\mapsto [F_0,x],\ x\mapsto [\widetilde{\varphi}(\widetilde{\mathcal{A}}),x]\},
\end{aligned}$$

and

$$\Delta_1 := \{x\mapsto [F_1,x],\ x\mapsto [\varphi(\mathcal{A})\otimes 1,x]\}.$$

Now apply Corollary 3.4 with $m=16$ to obtain $\ell_i \in B_i$ with:

(i) $\pm\mathrm{i}[M,\ell_0] \leqslant \ell_0^8$, $\pm\mathrm{i}[F_0,\ell_0]\leqslant \ell_0^8$;

(ii) $[\widetilde{\varphi}(\widetilde{a}), F_0\ell_0^{-1}]$ and $[\varphi_1(a)\otimes 1, F_1\ell_1^{-1}]$ compact for all a in $\mathcal{A}\subset A$ and $\widetilde{a}$ in $\widetilde{\mathcal{A}}$;

(iii) $\ell_0^{-2}Y\ell_0^{-2}$, $\ell_0^{-1}(1-F_0^2)\ell_0^{-1}$ and $\ell_1^{-1}(1-F_1^2)\ell_1^{-1}$ bounded, by Lemma 4.3 combined with part (v) of our version of Kasparov's technical theorem;

(iv) $\ell_0^8 \leqslant \ell_1^8$;

(v) $\pm\mathrm{i}[\ell_1,M]\leqslant \ell_0^8$, $\pm\mathrm{i}[\ell_1,F_0]\leqslant\ell_0^8$;

(vi) $\pm\mathrm{i}[\ell_1,F_1]\leqslant \ell_1^8$, $\pm\mathrm{i}[\ell_0,F_1]\leqslant\ell_0^8$; and

(vii) $\pm\mathrm{i}[\ell_1,\ell_0]\leqslant \ell_0^8$.

The special operators $\ell_i\in\mathcal{L}(E\oplus E_2)$ that we have obtained from the theorem are in $B_1 = (\mathcal{K}(E_1)\otimes 1)\oplus\mathcal{K}(E_2)$ and $B_0=\mathcal{K}(E_1\otimes E_2)\oplus\mathcal{K}(E_2)$, so as before, we can write $\ell_0 =: h_{12}\oplus h_2$, and $\ell_1 =: h_1\otimes 1\oplus h_2$. Properties (i) and (vi) of the above list show that the unbounded operators $F_i\ell_i^{-1}$ are almost self-adjoint.

Since we have $\lambda\ell_i^2 \geqslant 1-F_i^2$ for some λ, the operators $\lambda\ell_i^2+F_i^2$ are invertible, and so Lemma 2.2 implies that the operators $F_i\ell_i^{-1}$ are regular and that $F_0\ell_0^{-1}$ and $F_{E_1}h_1^{-1}$ have compact resolvents. Since $\ell_0^4\leqslant\ell_1^4$, we have $h_{12}^4\leqslant h_1^4\otimes 1$ and the domain of Fh_{12}^{-1} is contained in the domain of $(F_{E_1}h_1^{-1})\otimes 1$ which implies compatibility.

Finally, we take care of the positivity condition by showing that the sesquilinear form associated with the graded commutator of the self-adjoint parts of $F_1\ell_1^{-1}$ and $F_0\ell_0^{-1}$ is the sum of a positive unbounded form and a bounded form, so the commutator is semibounded below on its domain as a form. Recall that $[F_1,F_0]=M^2+Y$ with M positive. The commutator that we are interested in can be expanded using the derivation property:

$$\begin{aligned}
[F_1\ell_1^{-1}+\ell_1^{-1}F_1, F_0\ell_0^{-1}+\ell_0^{-1}F_0] &= F_1[\ell_1^{-1},F_0]\ell_0^{-1}+F_1F_0[\ell_1^{-1},\ell_0^{-1}]-F_0[F_1,\ell_0^{-1}]\ell_1^{-1}\\
&\quad + [F_1,F_0]\ell_0^{-1}\ell_1^{-1} - F_0\ell_1^{-1}[F_1,\ell_0^{-1}] + F_0[\ell_1^{-1},\ell_0^{-1}]F_1 + [\ell_1^{-1},F_0]F_1\ell_0^{-1}\\
&\quad + \ell_1^{-1}[F_1,F_0]\ell_0^{-1} + \text{eight more terms, the adjoints of the above.}
\end{aligned}$$

We consider the above expression, term by term.

(1) The form given by $[F_1, F_0]\ell_0^{-1}\ell_1^{-1}$ plus $\ell_1^{-1}[F_1, F_0]\ell_0^{-1}$ is equivalent to a positive (unbounded) form plus a form that is bounded on its domain.

To see this, we recall that $[F_1, F_0] = M^2 + Y$ and consider the case of M and the case of Y separately. Lemma 4.5 shows that the terms involving M^2 are equivalent to a positive unbounded form:

$$2\big\langle \ell_0^{-1/2}\ell_1^{-1/2}Mx, \ell_0^{-1/2}\ell_1^{-1/2}My\big\rangle.$$

The terms involving Y are bounded, because it is easy to deduce from the factorization Lemma 4.3 and Corollary 4.2 that $Y\ell_0^{-1}\ell_1^{-1}$ and $\ell_1^{-1}Y\ell_0^{-1}$ are bounded.

(2) $F_1F_0[\ell_1^{-1}, \ell_0^{-1}]$ and $F_0[\ell_1^{-1}, \ell_0^{-1}]F_1$ are bounded because $[\ell_1^{-1}, \ell_0^{-1}]$ is.

(3) $-F_0[F_1, \ell_0^{-1}]\ell_1^{-1} - F_0\ell_1^{-1}[F_1, \ell_0^{-1}]$ is bounded because $\pm\mathrm{i}[F_1, \ell_0] \leqslant \ell_0^8$ implies (by Lemma 4.3 (iii)) that $\ell_1^{-1}[F_1, \ell_0^{-1}]$ is bounded.

(4) $F_1[\ell_1^{-1}, F_0]\ell_0^{-1} + [\ell_1^{-1}, F_0]F_1\ell_0^{-1}$ is bounded because, first of all, by Lemma 4.4, $[\ell_1^{-1}, F_0]\ell_0^{-1}$ is bounded, and if we write

$$[\ell_1^{-1}, F_0]F_1\ell_0^{-1} = [\ell_1^{-1}, F_0][F_1, \ell_0^{-1}] + [\ell_1^{-1}, F_0]\ell_0^{-1}F_1,$$

then Lemma 4.3 shows the middle term is bounded. ∎

This completes the proof of the main result, Theorem 1.3, in the case of no group action. For simplicity of exposition, we have focused on the equivariant case. However, the equivariant case presents no new problems. Recall that a locally compact group G is said to have an action on a graded C^*-algebra B if there is a homomorphism α from G into the degree zero $*$-automorphisms of B. The group acts on a Hilbert B-module E by a homomorphism, also denoted α, into the invertible bounded linear transformations on E as a Banach space, with $\alpha_g(eb) = \alpha_g(e)\alpha_g(b)$, $\alpha_g\langle x, y\rangle = \langle \alpha_g x, \alpha_g y\rangle$. It will be convenient to consider $\widetilde{E} := C(K, E)$, where $K \subset G$ is compact, and then there are two natural ways to transfer operators from E to $\widetilde{E}$, defined by $(\widetilde{L}f)(g) := Lf(g)$ and $(\alpha(\widetilde{L})f)(g) := \alpha_g(L\alpha_g^{-1}(f(g))$. For the sake of completeness we give the definition of Ψ_G, as first given by the author in [15]:

DEFINITION 4.7. The set of unbounded equivariant Kasparov modules $\Psi_G(A, B)$ is given by triples (E, φ, D) where E is a Hilbert B-module with G-action; $\varphi : A \to \mathcal{L}(E)$ is a $*$-homomorphism satisfying $\alpha_g(\varphi(a)e) = \varphi(\alpha_g(a))e$; and D is an unbounded regular degree one self-adjoint operator on E, such that:

(i) $g \mapsto \big\{D - \alpha_g(D)\big\}$ is continous as a map from G into the bounded operators on E with the pointwise norm topology;

(ii) the operator $(\mathrm{i} + D)^{-1}$ is in J_φ;

(iii) for all a in some dense subalgebra of A, the commutator $[D, \varphi(a)]$ is bounded on the domain of D.

It is perhaps surprising that pointwise norm continuity (strong continuity in the Hilbert module sense) is all that is needed in part (i) of the above definition. It has been shown by the author that the unbounded connection conditions for a Kasparov product still hold and have the same form in the equivariant case.

THEOREM 4.8. *Suppose that G is a locally compact second countable topological group. If the three cycles $(E_1, \varphi_1, F_{E_1}) \in \mathrm{KK}_G(A, B)$, $(E_2, \varphi_2, F_{E_2}) \in \mathrm{KK}_G(B, C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, F) \in \mathrm{KK}_G(A, C)$ form a Kasparov product, they lift to unbounded cycles $(E_1, \varphi_1, F_{E_1} h_1^{-1}) \in \Psi_G(A, B)$, $(E_2, \varphi_2, F_{E_2} h_2^{-1}) \in \Psi_G(B, C)$, and $(E_1 \otimes_{\varphi_2} E_2, \varphi_1 \otimes 1, F h_{12}^{-1}) \in \Psi_G(A, C)$ that satisfy the conditions for an unbounded equivariant Kasparov product.*

Proof. We indicate the changes necessary for adapting the proof of Theorem 4.6. Let K be a precompact neighbourhood of $e \in G$ with $K^{-1} = K$. Replace E_1 and E_2 by $C(K, E_1)$ and $C(K, E_2)$ so that, for example, B_1 and B_0 become $B_1 := C(K, (\mathcal{K}(E_1) \otimes 1) \oplus \mathcal{K}(E_2))$ and $B_0 := C(K, \mathcal{K}(E_1 \otimes E_2) \oplus \mathcal{K}(E_2))$. Let $\alpha \in \mathcal{M}(B_i)$ be given by the action of K on E_i. Now note that the only place in the proof of Proposition 3.3 where we use the fact that the Δ_i are sets of derivations is in Lemma 2.16. Therefore the only change needed to add the "quasiderivation" $L \mapsto \alpha(L) - L$ to the sets Δ_i in Proposition 3.3 is to do so in the lemma. Deferring the proof that this can be done to Lemma 4.9, we next define

$$
\begin{aligned}
K_0 &:= \{\alpha(F_0) - F_0, Y, (1 - F_0^2), \mathrm{i}[\widetilde{\varphi}(\widetilde{\mathcal{A}}), F_0]\}, \\
K_1 &:= \{\alpha(F_1) - F_1, (1 - F_1^2), \mathrm{i}[\varphi_1(\mathcal{A}) \otimes 1, F_1]\}, \\
\Delta_0 &:= \{x \mapsto \alpha(x) - x,\, x \mapsto \alpha^{-1}(x) - x,\, x \mapsto [F_1, x], \\
&\qquad\qquad x \mapsto [M, x], x \mapsto [F_0, x],\, x \mapsto [\widetilde{\varphi}(\widetilde{\mathcal{A}}), x]\}, \\
\Delta_1 &:= \{x \mapsto \alpha(x) - x,\, x \mapsto \alpha^{-1}(x) - x,\, x \mapsto [F_1, x],\, x \mapsto [\varphi(\mathcal{A}) \otimes 1, x]\},
\end{aligned}
$$

and Δ_{10} is chosen in the same way as before.

Now we choose strictly positive constant sections of B_i and proceed as before to obtain operators ℓ_i^{-1}, having the additional properties that $\alpha(\ell^{-1}) - \ell^{-1}$ and $(\alpha(F_i) - F_i)\ell_i^{-1}$ are uniformly bounded and strongly continuous as functions of $g \in K$, using Lemma 4.11. Therefore,

$$g \mapsto F_i \ell_i^{-1} - \alpha_g(F_i \ell_i^{-1})$$

is pointwise continuous and uniformly bounded over $K \subset G$. Since the group G is σ-compact, any $g \in G$ can be written as a finite product of elements in K, and it follows that $F_i \ell_i^{-1} - \alpha_g(F_i \ell_i^{-1})$ is locally bounded and strongly continuous everywhere in G. We then proceed as in the proof of Theorem 4.6 and obtain elements of unbounded equivariant KK-theory. ∎

We now prove the required equivariant generalization of Lemma 2.16.

LEMMA 4.9. *Suppose that $p \leqslant 1$ in a C^*-algebra A and that $f_n(p) \subset C^*(p)$ is an increasing approximate unit. Given $\varepsilon > 0$, and a pointwise norm compact subset $\Delta \subset \operatorname{Der} A$, a norm-compact subset $U \subset A$ and $g \in C_0(0, 1]$ with $g < 1$, there is an $h \in \overline{\operatorname{Conv}(f_n)}$ such that $\|\Delta(h(p))\| < \varepsilon$, $\|(1 - h(p))U\| < \varepsilon$ and $h \geqslant g$. If there is a group action on A, and if we are given a pointwise norm compact set of group elements K, then $\|h(p) - \alpha_k(h(p))\| < \varepsilon$.*

REMARK 4.10. Pointwise norm compactness of the action of a subset K of the group means that the sets $\{\alpha_k a : k \in K\}$ are norm-compact in A for each $a \in A$. The corresponding topology on the group is the unique weakest topology in which all the maps $\pi_a : G \to A$ given by $\pi_a : g \mapsto \alpha_g a$ are continuous. In general this topology is not even T_0.

Proof. The convex set

$$T := \overline{\{z \in \mathrm{Conv}\,\{f_n(p)\} : z \geqslant g(p)\}}$$

contains an approximate unit for A. Hence 1 is in the strict closure of T. Define the affine map $\mathcal{F} : A \to C(\Delta, A) \oplus C(U, A) \oplus C(K, A)$ by $x \mapsto \ell(x) \oplus (1-x)u \oplus (x - \alpha_k x)$, $u \in U$, $\ell \in \Delta, k \in K$. This map has a strictly continuous extension to the multiplier algebras, so the strict closure of $\mathcal{F}(T)$ contains $\mathcal{F}(1) = 0$, and the C^*-weak closure of $\mathcal{F}(T)$ contains 0. But $\mathcal{F}(T)$ is convex, so the norm closure coincides with the weak closure ([8]), and there is a sequence in $\mathcal{F}(T)$ that converges to zero in norm. We choose an $h(p) \in T$ with $\|\mathcal{F}(h(p))\| < \varepsilon$. ∎

LEMMA 4.11. *If*

$$\pm(\alpha(\ell) - \ell) \leqslant \gamma\ell^3 \quad \text{and} \quad \pm(\alpha^{-1}(\ell) - \ell) \leqslant \gamma\ell^3$$

on $C(K, E)$ then $\alpha(\ell^{-1}) - \ell^{-1}$ is bounded on $\mathfrak{Ran}\,\ell$.

Proof. By Lemma 4.3 there are bounded operators u and v such that $\alpha(\ell) - \ell = \ell u \ell$ and $\alpha^{-1}(\ell) - \ell = -\ell v \ell$. Hence

$$\alpha(\ell)(\ell^{-1} - \alpha(\ell^{-1}))\ell = \alpha(\ell) - \ell = \ell u \ell = \alpha(\ell)\alpha(v)\alpha(\ell) = \alpha(\ell)\alpha(v)(\ell + \ell u \ell),$$

which implies that $\ell^{-1} - \alpha(\ell^{-1}) = \alpha(v)(1 + \ell u)$ on $\mathfrak{Ran}\,\ell$.

This completes the proof of Theorem 1.3. ∎

5. FINAL REMARKS

There is the natural question of whether our arguments generalize to ideals other than $J_\varphi \lhd I_\varphi$. Most of what we have done is quite general, but we do need the existence of approximate units having various special properties, in particular, quasicentrality in I_φ. If we specialize to the case of K-homology, then Voiculescu ([20]) shows that the existence of quasicentral approximate units is a necessary condition for the existence of an unbounded K-homology cycle with respect to a given ideal. The most important ideals to consider are the Schatten-von Neumann p-classes, since K-homology with Schatten classes instead of compact operators is the starting point for noncommutative geometry ([6]).

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