J. OPERATOR THEORY
45(2001), 209–229

C^*-EQUIVALENCES OF GRAPHS

D. DRINEN and N. SIEBEN

Communicated by Norberto Salinas

Abstract. Several relations on graphs, including primitive equivalence, explosion equivalence and strong shift equivalence, are examined and shown to preserve either the graph groupoid, a construction of Kumjian, Pask, Raeburn, and Renault, or the groupoid of a pointed version of the graph. Thus these relations preserve either the isomorphism class or the Morita equivalence class of the graph C^*-algebra, as defined by Kumjian, Pask, and Raeburn.

Keywords: *Graph C^*-algebra, graph groupoid, primitive equivalence, explosion, strong shift equivalence.*

MSC (2000): 46L05.

1. INTRODUCTION

Given any finite square matrix B with nonnegative integer entries and no zero rows or columns, Cuntz and Krieger defined a C^*-algebra $\mathcal{O}_B$, which is generated by partial isometries satisfying relations associated to B ([2]). Also, given any square matrix B with nonnegative integer entries, we can build a directed graph by putting B_{ij} edges from vertex i to vertex j. In [5], Kumjian, Pask, and Raeburn defined the graph C^*-algebra $C^*(E)$ of any countable row-finite directed graph E as the universal C^*-algebra generated by a family of projections and partial isometries which satisfy relations coming from E. Given a graph E with finitely many vertices and no sources or sinks, the vertex matrix B_E associated to the graph is finite and has no zero rows or columns. In this case it turns out that $C^*(E)$ coincides with $\mathcal{O}_{B_E}$. The graph thus becomes a useful tool for visualizing and generalizing the Cuntz-Krieger algebras.

In [6], Kumjian, Pask, Raeburn, and Renault defined the graph groupoid $\mathcal{G}_E$ of any countable row-finite directed graph E with no sinks. In this case, the C^*-algebra of the groupoid coincides with the C^*-algebra of the graph, so the graph groupoid is another tool for understanding the Cuntz-Krieger algebras and a large class of the graph algebras. In this paper, we examine several equivalence relations on graphs which preserve the graph groupoid or, in some cases, the groupoid of a pointed version of the graph.

In [3], Enomoto, Fujii, and Watatani defined primitive equivalence of finite directed graphs with no sources, no sinks, and no multiple edges, and showed that it is a sufficient condition for isomorphism of the resulting graph algebras. In Section 3 we generalize their result to countable row-finite graphs. Further, Enomoto, Fujii, and Watatani showed that primitive equivalence is also a necessary condition for isomorphism of the graph algebras of strongly connected graphs with three vertices. In Section 4, we show by counterexample that this does not hold in the four-vertex case.

Primitive equivalence involves changing the rows of a matrix. In graph-theoretical terms, this corresponds to changing the outgoing edges at a vertex. In Section 5, we define an equivalence relation which involves changing the columns of the matrix (alternatively, the incoming edges at a vertex). We call this reverse primitive equivalence, and show that it preserves the Morita equivalence class, though not the isomorphism class, of the graph algebras.

Primitive equivalence and reverse primitive equivalence only make sense for graphs with the same number of vertices. In Section 6, we define explosion and reverse explosion, operations which can change the size of the graph. The graph operation we call reverse explosion was defined and called explosion in [3]. We show that exploding a graph does not change the graph groupoid, hence does not change the isomorphism class of its graph C^*-algebra. Reverse exploding a graph does not preserve the graph groupoid, but the resulting graphs can be pointed in such a way that their groupoids are isomorphic. Thus, reverse exploding a graph preserves the Morita equivalence class of its C^*-algebra.

In Section 7, we recall from [1] the notion of elementary strong shift equivalence of graphs, and show that elementary strong shift equivalent graphs can be pointed in such a way that their groupoids are isomorphic. This is an alternate proof of a result of Cuntz and Krieger in [2], which states that elementary strong shift equivalent matrices correspond to Morita equivalent Cuntz-Krieger algebras. We then examine the relationship between elementary strong shift equivalence and explosion equivalence.

2. PRELIMINARIES

In [3], a C^*-algebra was associated to every connected finite directed graph with no multiple edges, no sources, and no sinks. We review this construction as we set up the notation. A *directed graph* E consists of a set E^0 of vertices, a set E^1 of edges and maps $s, r : E^1 \to E^0$ describing the source and range of each edge. Denote by E^j the set of paths in E of length j. Here, zero-length paths (i.e., vertices) are allowed. Denote by E^* the set of all finite paths in E and by E^∞ the infinite one-sided path space of E. We extend s and r to E^* and s also to E^∞. Associated to every directed graph E is an $E^0 \times E^0$ *vertex matrix* B_E, defined by $B_E(v,w) = \#\{e \in E^1 : s(e) = v,\, r(e) = w\}$. That is, the (v,w) entry of B_E is the number of edges in E from v to w. E has no multiple edges if and only if B_E is a 0-1 matrix. E has no sources (respectively sinks) if and only if B_E has no zero columns (respectively rows). A directed graph is said to be *strongly connected* if for every pair of vertices v and w, there is a path from v to w and a path from w to v. A directed graph is said to be *connected* if between each pair v, w there is an undirected path from v to w.

In this paper, all graphs are assumed to be countable, directed, connected, and to have no multiple edges.

Note that in [3], Enomoto, Fujii and Watatani worked with the adjacency matrix instead of the vertex matrix, whose transpose is the adjacency matrix. We choose to work with the vertex matrix in order to be consistent with the more recent graph algebra literature ([6]).

For any row-finite graph E, a Cuntz-Krieger E-family is a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections together with a set $\{S_e : e \in E^1\}$ of partial isometries satisfying:

(a) $S_e^* S_e = P_{r(e)}$;
(b) $P_v = \sum\limits_{s(e)=v} S_e S_e^*$, for $v \in s(E^1)$.

Kumjian, Pask, and Raeburn ([5]) defined the C^*-algebra of the graph, denoted by $C^*(E)$, to be the C^*-algebra generated by a universal Cuntz-Krieger E-family.

We now recall the construction of a groupoid from a row-finite graph ([6]). For $x, y \in E^\infty$, $k \in \mathbb{Z}$, say $x \sim_k y$ if and only if $x_i = y_{i-k}$ for large i, where x_i denotes the i-th edge of x. We remark that this definition differs slightly from the one given in [6], but it coincides with the currently accepted convention. Define $\mathcal{G}_E$, the path groupoid of E, by $\mathcal{G}_E = \{(x,k,y) : x \sim_k y\}$. The groupoid operations in $\mathcal{G}_E$ are

$$(x,k,y)^{-1} = (y,-k,x) \quad \text{and} \quad (x,k,y)\cdot(y,l,z) = (x,k+l,z).$$

Alternatively, one can define

$$\mathcal{G}_E = \{(\alpha, x, \beta) \in E^* \times E^\infty \times E^* : r(\alpha) = r(\beta) = s(x)\}/\sim,$$

where $\sim$ denotes the equivalence relation $(\alpha, \gamma x, \beta) \sim (\alpha\gamma, x, \beta\gamma)$. To see that the two definitions coincide, the reader may check that the map $[\alpha, x, \beta] \mapsto (\alpha x, |\alpha| -$

$|\beta|, \beta x)$ is a groupoid isomorphism. With this definition, we find that $[\alpha, x, \beta]^{-1} = [\beta, x, \alpha]$, that $[\alpha, x, \beta]$ and $[\gamma, y, \delta]$ are composable if and only if $\beta x = \gamma y$, and that

$$[\alpha, x, \beta][\gamma, y, \delta] = \begin{cases} [\alpha, x, \delta\varepsilon] & \text{if } \gamma\varepsilon = \beta \text{ and } y = \varepsilon x; \\ [\alpha\varepsilon, y, \delta] & \text{if } \beta\varepsilon = \gamma \text{ and } x = \varepsilon y. \end{cases}$$

With the topology generated by the sets $Z(\alpha, \beta) := \{[\alpha, x, \beta] : s(x) = r(\alpha)\}$, $\mathcal{G}_E$ is a locally compact Hausdorff r-discrete groupoid with Haar system. If E has no sinks, then $C^*(E)$, the C^*-algebra of E constructed in [5], coincides with $C^*(\mathcal{G}_E)$.

We now seek to remove the restriction on sinks. First, recall from [6] that a pair (E, S), where E is a row-finite graph with no sinks and S is a set of vertices of E, is called a *pointed graph.* If (E, S) is a pointed graph, then S determines a clopen subset $\{x \in E^\infty : s(x) \in S\}$ of the unit space of $\mathcal{G}_E$, which we also denote by S. If S is *cofinal*, meaning that given any $x \in E^\infty$ there exists $v \in S$ and a finite path from v to $s(x_i)$ for some i, then $C^*(\mathcal{G}_{(E,S)})$ is Morita equivalent to $C^*(\mathcal{G}_E)$, where $\mathcal{G}_{(E,S)}$ denotes $\mathcal{G}_E$ restricted to S ([6]).

Given a row-finite graph E with sinks at $\{v_i\}_{i\in I}$, define a pointed graph $\widetilde{E}$ as follows: the vertices of $\widetilde{E}$ are the vertices of E, along with the additional vertices $\{w_i^j\}$ for $i \in I$, $j = 1, 2, \ldots$. The edges in $\widetilde{E}$ are the edges in E along with an edge from v_i to w_i^1 for every $i \in I$, and an edge from w_i^j to w_i^{j+1} for every $i \in I$, $j = 1, 2, \ldots$. We have simply added a distinct infinite tail to each sink in E. Define the pointing set of $\widetilde{E}$ to be the original set of vertices of E. The reader may verify that $C^*(\mathcal{G}_{\widetilde{E}}) \cong C^*(E)$ by checking that both are generated by the same family of projections and partial isometries.

3. PRIMITIVE EQUIVALENCE

The following definition is due to Enomoto, Fujii, and Watatani. Let B be an $n \times n$, 0-1 matrix. For $1 \leqslant p \leqslant n$, denote the p-th row of B by B_p and denote the row $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the i^{th} position, by E_i. We apologize for any confusion caused by this multiple use of the letter E, but we are using standard conventions of [3] for primitive transfer and standard conventions of [5] for graphs.

Now suppose that there is a p such that B_p is not a zero row and

$$B_p = E_{k_1} + \cdots + E_{k_r} + B_{m_1} + \cdots + B_{m_s}$$

for some distinct $k_1, \ldots, k_r$, $m_1, \ldots, m_s$ such that $p \notin \{m_1, \ldots, m_s\}$ and B_{m_i} is not a zero row for any i. Define a new matrix C by

$$C_{ij} = \begin{cases} B_{ij} & \text{if } i \neq p; \\ 1 & \text{if } i = p \text{ and } j \in \{k_1, \ldots, k_r, m_1, \ldots, m_s\}; \\ 0 & \text{otherwise.} \end{cases}$$

That is, start with B, zero out the p-th row, and then put back 1's in positions $k_1, \ldots, k_r$, $m_1, \ldots, m_s$. C is called a *primitive transfer* of B at p. Note that this notion does not depend on finiteness of the matrix B.

DEFINITION 3.1. If B and C are 0-1, square matrices of the same (possibly infinite) size, we say B is *primitively equivalent* to C if and only if there exist matrices $D_1, \ldots, D_q$ such that $D_1 = B$, $D_q = C$, and for every $1 \leqslant i \leqslant q-1$, one of the following holds:

(i) D_i is a primitive transfer of D_{i+1};
(ii) D_{i+1} is a primitive transfer of D_i;
(iii) $D_i = PD_{i+1}P^{-1}$ for some permutation matrix P.

We say that two matrices, which satisfy the third condition above, are *permutations* of each other. Matrices which are permutations of each other should be primitively equivalent because we would like the vertex matrices of isormorphic graphs to be primitively equivalent. This is implicit but never stated in [3]. There are matrices which cannot be primitively transferred in any number of steps into a permutation of the same matrix. For example,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

are permuations of each other, but the reader may check that they would not be primitively equivalent if condition (iii) were removed from the definition.

Franks ([4], Corollary 2.2) defined a similar matrix operation. His operation applies to graph with multiple edges and it involves only two rows or columns. He used this move in finding a canonical form for the flow equivalence class of a matrix.

The primitive transfer has the following graph-theoretical interpretation: suppose vertex p points to the same vertices which are pointed to by vertices $m_1, \ldots, m_s$ (and only one of the vertices $m_1, \ldots, m_s$ points to each of those vertices) and, in addition, vertex p points to vertices $k_1, \ldots, k_r$. Then a *primitively transferred graph* can be obtained by erasing all the edges emanating from vertex p, except those pointing to vertices $k_1, \ldots, k_r$, and adding an edge from vertex p to each of the vertices $m_1, \ldots, m_s$. Note that this procedure is only allowed if we do not create any multiple edges. The following example shows that we may first erase and then redraw the same edge.

EXAMPLE 3.2. Consider the following graph E and its primitive transfer F:

Vertex p points to vertices p, v, w, m_3 and k_1. Together, vertices m_1, m_2 and m_3 point to vertices p, v, w and m_3. This means that, if B is the vertex matrix of E, then $B_p = E_{k_1} + B_{m_1} + B_{m_2} + B_{m_3}$. So row p of the vertex matrix of F has

1's in positions k_1, m_1, m_2 and m_3. Thus F has edges from vertex p to vertices k_1, m_1, m_2 and m_3.

DEFINITION 3.3. Two graphs with no multiple edges are said to be *primitively equivalent* if and only if their vertex matrices are primitively equivalent.

We will show that if F is a primitive transfer of E, then $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic. But first we need the following definitions, notation and lemmas.

Let E and F be row-finite graphs with no sinks. To simplify notation, we denote their vertex matrices by B and C, respectively. Suppose that F is a primitive tranfer of E. Without loss of generality, assume that $1 \in E^0$ and $B_1 = E_{k_1} + \cdots + E_{k_r} + B_{m_1} + \cdots + B_{m_s}$. We identify E^0 and F^0. Define $K := \{k_1, \ldots, k_r\}$ and $M := \{m_1, \ldots, m_r\}$. Note that $K \cap M = \emptyset$. Since E and F have no multiple edges we can use the notation e^{ij} to denote the unique edge in E with source v_i and range v_j, if there is one; that is, if $B_{ij} = 1$. Likewise, we will denote edges in F by f^{ij}.

The following lemma is an immediate consequence of the definition of the primitive transfer.

LEMMA 3.4. *If f^{1j} exists, then $j \in K \cup M$.*

If $f \in F^1$ is of the form f^{1m} for some $m \in M$, then we will call f a *new* (or *newly introduced*) edge.

LEMMA 3.5. *If f^{ij} is not new, then e^{ij} exists.*

Proof. $C_{ij} = 1$ because f^{ij} exists. Now, if f^{ij} is not new, then either $i \neq 1$ or $j \notin M$. If $i \neq 1$, then the primitive transfer does not change row i, so $B_{ij} = C_{ij} = 1$. If, on the other hand, $i = 1$ and $j \notin M$, then $j \in K$ by Lemma 3.4. So $B_i = \cdots + E_j + \cdots$ and thus $B_{ij} = 1$. In either case, $B_{ij} = 1$, so e^{ij} exists. ∎

LEMMA 3.6. *If f and f' are consecutive edges in F (that is, $r(f) = s(f')$), then f and f' cannot both be new.*

Proof. This follows from the definition of new and the fact that $1 \notin M$. ∎

LEMMA 3.7. *If $f^{1m} f^{mj} \in F^2$ for some $m \in M$, then e^{1j} exists, and $j \notin K$.*

Proof. Since f^{mj} is not new, e^{mj} exists by Lemma 3.5, and so $B_{mj} = 1$. Hence we have $B_{1j} = 1$, since $B_1 = \cdots + B_m + \cdots$. Thus e^{1j} exists.

Now suppose $j \in K$. Then we have $B_1 = \cdots + E_j + \cdots + B_m + \cdots$. But we know that $B_{mj} = 1$, and thus $B_{1j} > 1$, a contradiction. ∎

PROPOSITION 3.8. *If E is a row-finite graph and F is a primitive transfer of E, then $\mathcal{G}_E \cong \mathcal{G}_F$.*

Proof. The strategy of the proof is as follows: We use the properties of the primitive transfer to construct an s, r-preserving injective map φ from the edges in E to paths of length one or two in F. This will induce injective maps from finite paths in E to finite paths in F, from infinite paths in E to infinite paths in F, and from $\mathcal{G}_E$ to $\mathcal{G}_F$. The injective map between the groupoids will turn out to be a surjective homomorphism.

We use the same notation as above and assume, without loss of generality, that $B_1 = E_{k_1} + \cdots + E_{k_r} + B_{m_1} + \cdots + B_{m_s}$.

Note that if E has an edge from vertex 1 to vertex j for some $j \notin K$, then there is a unique $m \in M$ such that $B_{mj} = 1$. Since $C_{1m} = 1$, we can define $\varphi : E^1 \to F^1 \cup F^2$ by

$$\varphi(e^{ij}) = \begin{cases} f^{1m} f^{mj} & \text{if } i = 1 \text{ and } j \notin K; \\ f^{ij} & \text{else.} \end{cases}$$

For instance, in Example 3.2, $\varphi(e^{pp}) = f^{pm_1} f^{m_1 p}$, $\varphi(e^{pv}) = f^{pm_1} f^{m_1 v}$, $\varphi(e^{pw}) = f^{pm_3} f^{m_3 w}$ and $\varphi(e^{pm_3}) = f^{pm_2} f^{m_2 m_3}$. All the other edges would be mapped to their corresponding edges.

The function φ induces a map, which we also denote by φ, from $E^* \to F^*$ by

$$\varphi(\alpha_1 \alpha_2 \cdots \alpha_{|\alpha|}) = \varphi(\alpha_1)\varphi(\alpha_2) \cdots \varphi(\alpha_{|\alpha|}).$$

The function $\varphi : E^\infty \to F^\infty$ is defined similarly, and $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ is defined by

$$\varphi[\alpha, x, \beta] = [\varphi(\alpha), \varphi(x), \varphi(\beta)].$$

It is easily seen that $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ is a well-defined homomorphism.

In order to show that $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ is injective, we need to know that $\varphi : E^* \to F^*$ and $\varphi : E^\infty \to F^\infty$ are injective.

First note that K and M are disjoint. Hence if $\varphi(e)_1 = \varphi(e')_1$ for some $e, e' \in E^1$ then $|\varphi(e)| = |\varphi(e')|$. Now suppose that $\varphi(\alpha) = \varphi(\beta)$ for some finite or infinite paths α and β. It follows by induction that $\varphi(\alpha_i) = \varphi(\beta_i)$ for all i. Since $\varphi : E^1 \to F^1 \cup F^2$ is clearly injective, we have that $\varphi : E^* \to F^*$ and $\varphi : E^\infty \to F^\infty$ are injective.

We are now in position to show that $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ is injective. Suppose $\varphi[\alpha, x, \beta] = \varphi[\gamma, y, \delta]$. Then we can assume, without loss of generality, that $\varphi(\alpha) = \varphi(\gamma)\eta$, $\varphi(\beta) = \varphi(\delta)\eta$, and $\varphi(y) = \eta\varphi(x)$. We claim that $\eta \in \varphi(E^*)$. If $|\varphi(\alpha)| \leqslant 1$ then either $\eta = \varphi(\alpha)$ or $\eta = r(\varphi(\alpha)) = \varphi(r(\alpha))$, so we can suppose that $|\varphi(\alpha)| > 1$. Since $\varphi(\gamma)\eta = \varphi(\alpha_1) \cdots \varphi(\alpha_{|\alpha|})$ and $\varphi(\alpha_i)$ has length one or two for every i, it follows that for some k, either $\varphi(\gamma) = \varphi(\alpha_1) \cdots \varphi(\alpha_k)$ or $\varphi(\gamma) = \varphi(\alpha_1) \cdots \varphi(\alpha_k) f$, where $\varphi(\alpha_{k+1}) = fg$ $(f, g \in F^1)$. However, the latter case is not possible since, by definition of φ, the last edge of $\varphi(\gamma)$ cannot be a new edge, but f must be a new edge. In the former case, $\eta = \varphi(\alpha_{k+1} \cdots \alpha_{|\alpha|})$. Thus there exists $\mu \in E^*$ such that $\eta = \varphi(\mu)$. So $\varphi(\alpha) = \varphi(\gamma)\varphi(\mu) = \varphi(\gamma\mu)$. Similarly, $\varphi(\beta) = \varphi(\delta\mu)$, and $\varphi(y) = \varphi(\mu x)$. By injectivity of $\varphi : E^* \to F^*$ and $\varphi : E^\infty \to F^\infty$, we have $\alpha = \gamma\mu$, $\beta = \delta\mu$, and $y = \mu x$, and hence $[\alpha, x, \beta] = [\gamma, y, \delta]$. Thus $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ is injective.

We now show that $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ is onto. Since elements of the form $[f, y, r(f)]$, where $f \in F^1$, and $y \in F^\infty$ with $r(f) = s(y)$, generate $\mathcal{G}_F$, it suffices to show that each of them is in the range of φ.

Use the following procedure to find an inverse image for y. Note that every new edge is followed by an edge which is not new (Lemma 3.6). Lemma 3.7 says that we can find an inverse image for these new-not new pairs. The remaining edges are not new, and by Lemma 3.5 these edges can be pulled back individually.

Now we fix an edge $f = f^{ij} \in F^1$ and a path $y \in F^\infty$ with $s(y) = r(f)$. If f is not new, then clearly $\varphi[e^{ij}, \varphi^{-1}(y), r(e^{ij})] = [f, y, r(f)]$. If, on the other hand, $f = f^{1m}$ is a new edge, then there must be an edge $e' \in E^1$ such that $\varphi(e') = fy_1$. Further, if f is a new edge, then y_1 cannot be new, so there must be an edge $e'' \in E^1$ with $\varphi(e'') = y_1$. In this case,

$$\varphi[e', \varphi^{-1}(y_2y_3\cdots), e''] = [fy_1, y_2y_3\cdots, y_1] = [f, y, r(f)].$$

Thus φ is onto.

All that remains to show is that φ is continuous and open. To see that φ is open, the reader may check that $\varphi(Z(\alpha,\beta)) = Z(\varphi(\alpha),\varphi(\beta))$ for any finite paths α and β. Likewise, $\varphi^{-1}(Z(\gamma,\delta)) = Z(\varphi^{-1}(\gamma),\varphi^{-1}(\delta))$ for any γ,δ, so φ is continuous. ∎

The proposition immediately yields the following corollary, which was proved in [3] for the case where E and F are finite graphs which satisfy (L) and have no sources or sinks:

COROLLARY 3.9. *If E is a row-finite graph with no sinks and F is primitively equivalent to E, then $\mathcal{G}_E \cong \mathcal{G}_F$ and hence $C^*(E) \cong C^*(F)$.*

Recall that if E has sinks, we can build a graph $\widetilde{E}$ with no sinks by affixing a distinct infinite tail to each sink. By pointing $\widetilde{E}$ at all the original vertices of E, we obtain a pointed graph whose groupoid C^*-algebra coincides with $C^*(E)$.

LEMMA 3.10. *Let E be a row-finite graph, possibly with sinks, and let F be a primitive transfer of E. Then $\widetilde{F}$ is a primitive transfer of $\widetilde{E}$.*

Proof. Let $B, C, \widetilde{B}$, and $\widetilde{C}$ denote the vertex matrices of $E, F, \widetilde{E}$, and $\widetilde{F}$, respectively. Assume that $B_p = E_{k_1} + \cdots + E_{k_r} + B_{m_1} + \cdots + B_{m_s}$, so that $C_p = E_{k_1} + \cdots + E_{k_r} + E_{m_1} + \cdots + E_{m_s}$. Since for each i, the vertex m_i is not a sink in E, $B_{m_ij} = 1$ if and only if $\widetilde{B}_{m_ij} = 1$. Similarly, since the vertex p is not a sink in E, $B_{pj} = 1$ if and only if $\widetilde{B}_{pj} = 1$, Thus $\widetilde{B}_p = E_{k_1} + \cdots + E_{k_r} + \widetilde{B}_{m_1} + \cdots + \widetilde{B}_{m_s}$, and $\widetilde{C}_p = E_{k_1} + \cdots + E_{k_r} + E_{m_1} + \cdots + E_{m_s}$. Thus $\widetilde{F}$ is a primitive transfer of $\widetilde{E}$. ∎

THEOREM 3.11. *If F is primitively equivalent to the row-finite graph E, then $C^*(E) \cong C^*(F)$.*

Proof. By Lemma 3.10 and Corollary 3.9 we have $C^*(\mathcal{G}_{\widetilde{E}}) \cong C^*(\mathcal{G}_{\widetilde{F}})$, and hence $C^*(E) \cong C^*(F)$. ∎

4. CLASSIFICATION

A matrix B is said to be *irreducible* if for every i, j, there exists an $N \in \mathbb{N}$ such that $B^N(i,j) > 0$. In [3] a computer, along with some K-theory, was used to show that for all 3×3 irreducible matrices which are not permutation matrices, primitive equivalence is a necessary as well as sufficient condition for isomorphism of the Cuntz-Krieger algebras.

We used a similar method to see whether this result is true for irreducible 4×4 matrices. It is not. The following counterexample is one of many:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

By [8], $\mathrm{K}_0(\mathcal{O}_A)$ is the abelian group generated by $\{[P_i^A] : i = 1, \ldots, 4\}$ subject to the relations $[P_i^A] = \sum_j A(i,j)[P_j^A]$, and similarly for $\mathcal{O}_B$. One may check that

$$[P_1^A] \mapsto (0,2), \quad [P_2^A] \mapsto (0,1), \quad [P_3^A] \mapsto (1,4), \quad [P_4^A] \mapsto (1,1)$$

is a faithful representation of $\mathrm{K}_0(\mathcal{O}_A)$ as $\mathbb{Z}_2 \oplus \mathbb{Z}_6$, and that

$$[P_1^B] \mapsto (0,1), \quad [P_2^B] \mapsto (1,1), \quad [P_3^B] \mapsto (0,4), \quad [P_4^B] \mapsto (1,2)$$

is a faithful representation of $\mathrm{K}_0(\mathcal{O}_B)$ as $\mathbb{Z}_2 \oplus \mathbb{Z}_6$. Further, notice that $[1_{\mathcal{O}_A}] = \sum_1^4 [P_i^A] \mapsto (0,2)$, and $[1_{\mathcal{O}_B}] \mapsto (0,2)$, as well. Thus, since A and B are irreducible and there is an isomorphism between the K_0 groups which preserves the class of the identity, $\mathcal{O}_A \cong \mathcal{O}_B$ by [9].

However, it is easy to check by hand using the definition of the primitive transfer that each of these matrices is primitively equivalent only to its permutations, and that A and B are not permutations of each other.

5. REVERSE PRIMITIVE EQUIVALENCE

In this section, we define a modified version of primitive equivalence using column operations instead of row operations. Recall that a cofinal vertex is one from which any infinite path can be intercepted.

DEFINITION 5.1. Suppose that B and C are 0-1 (possibly infinite) square matrices. We say C is a *reverse primitive transfer* of B if C^{T} is a primitive transfer of B^{T} at a cofinal vertex. We say that B and C are *reverse primitively equivalent* if there is a sequence $B = D_1, D_2, \ldots, D_q = C$ such that for each $i < q$, D_{i+1} is a permutation of D_i, D_{i+1} is a reverse primitive transfer of D_i, or D_i is a reverse primitive transfer of D_{i+1}.

Two graphs E and F are *reverse* graphs if the vertex matrices B_E and B_F are transposes of each other, that is, E and F have the same vertices and their edges have opposite directions. Disregarding cofinality, two graphs are reverse primitive equivalent if their reverse graphs are primitively equivalent. The following example shows that reverse primitive equivalence of graphs E and F does not imply that $C^*(E) \cong C^*(F)$.

Example 5.2. If

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then C^{T} is a primitive transfer of B^{T} (via $B_3^{\mathrm{T}} = B_2^{\mathrm{T}}$). However, $\mathcal{O}_C \cong \mathcal{O}_3$ and $\mathcal{O}_B \cong \mathcal{O}_3 \otimes M_2$, so they are not isomorphic ([7]).

Suppose that E and F are row-finite graphs and that F is a reverse primitive transfer of E at vertex 1. That is, $B_1^{\mathrm{T}} = E_{k_1} + \cdots + E_{k_r} + B_{m_1}^{\mathrm{T}} + \cdots + B_{m_s}^{\mathrm{T}}$. Where appropriate, we use the notation established prior to Proposition 3.4.

We have analogues of some of the preliminary lemmas for Proposition 3.8, and their proofs are similar:

Lemma 5.3. *If f^{i1} exists, then $i \in K \cup M$.*

The definition of *new* must be altered slightly. The edge $f \in F^1$ is said to be *new* if it is of the form f^{m1} for some $m \in M$.

Lemma 5.4. *If $f^{im} f^{m1} \in F^2$ for some $m \in M$, then e^{i1} exists.*

Note that, with the modified definition of new, Lemma 3.5 and Lemma 3.6 are true exactly as stated.

Proposition 5.5. *If the row finite graph F is a reverse primitive transfer of E at v, then $\mathcal{G}_{(E,\{v\})} \cong \mathcal{G}_{(F,\{v\})}$.*

Proof. Let B be the vertex matrix of E and C the vertex matrix of F. Without loss of generality, assume $B_1^{\mathrm{T}} = E_{k_1} + \cdots + E_{k_r} + B_{m_1}^{\mathrm{T}} + \cdots + B_{m_s}^{\mathrm{T}}$.

We again define a map $\varphi : E_1 \to F^1 \cup F^2$ by

$$\varphi(e^{ij}) = \begin{cases} f^{im} f^{m1} & \text{if } j = 1 \text{ and } i \notin K; \\ f^{ij} & \text{else;} \end{cases}$$

and extend it to a map $\varphi : \mathcal{G}_E \to \mathcal{G}_F$. By arguments similar to those in the proof of Proposition 3.8, φ is an injective groupoid homomorphism. In this case, however, it fails to be onto, and this is because of the difference between Lemma 3.7 and Lemma 5.4. Thus it is necessary to point.

It is not hard to see that φ restricts to an injective groupoid homomorphism from $\mathcal{G}_{(E,\{v\})}$ to $\mathcal{G}_{(F,\{v\})}$ which, of course, we also denote by φ.

We claim that any finite or infinite path whose first edge is not new can be pulled back through φ. First, find all the new edges in the path. These are preceded by edges which are not new, and these (not new)-new pairs can be pulled back. The remaining edges are all not new and can be pulled back individually.

Now, fix any $[\alpha, x, \beta]$ with $s(\alpha) = s(\beta) = 1$. Since α and β start at 1, their first edge cannot be new (because $1 \notin M$). Hence they can be pulled back through φ. Now, if x_1 is not new, x can be pulled back as well, so $[\alpha, x, \beta]$ has an inverse image. Since φ preserves source and range, this inverse image will be in $\mathcal{G}_{(E,\{v\})}$. If, on the other hand, x_1 is new, then we know x_2 is not new, so we pull back the triple $[\alpha x_1, x_2 x_3 \cdots, \beta x_1]$. This shows that $\varphi : \mathcal{G}_{(E,\{v\})} \to \mathcal{G}_{(F,\{v\})}$ is onto.

It is not hard to check that φ is continuous. We show that it is open. Clearly, $\varphi(Z(\alpha, \beta)) \subseteq Z(\varphi(\alpha), \varphi(\beta))$. We show the reverse inclusion: let $[\varphi(\alpha), y, \varphi(\beta)] \in$

$Z(\varphi(\alpha), \varphi(\beta))$. If y_1 is not new, then y has an inverse image and hence $[\varphi(\alpha), y, \varphi(\beta)] \in \varphi(Z(\alpha, \beta))$. If, on the other hand, y_1 is a new edge, then note that y_2 is not new, so the path $y_2 y_3 \cdots$ has an inverse image. Since

$$Z(\varphi(\alpha), \varphi(\beta)) = \bigcup_{s(f)=r(\alpha)} Z(\varphi(\alpha)f, \varphi(\beta)f),$$

$[\varphi(\alpha), y, \varphi(\beta)] \in \varphi(Z(\alpha, \beta))$. Thus $\varphi(Z(\alpha, \beta)) = Z(\varphi(\alpha), \varphi(\beta))$, which shows that φ is open. ∎

COROLLARY 5.6. *If E is a graph with no sinks and F is reverse primitively equivalent to E, then $C^*(E)$ is Morita equivalent to $C^*(F)$.*

Proof. If F is a reverse primitive transfer of E, then the result follows easily from the previous proposition and the fact that pointing a graph at a cofinal vertex does not change the Morita equivalence class of its C^*-algebra ([6]). Since reverse primitive equivalence is generated by the reverse primitive transfer and permutation, the result follows. ∎

6. EXPLOSIONS

Given a graph E, its *edge matrix* A_E is an $E^1 \times E^1$ matrix defined by

$$A_E(e, f) = \begin{cases} 1 & \text{if } r(e) = s(f); \\ 0 & \text{otherwise.} \end{cases}$$

The *adjoint graph* of E is the graph whose vertex matrix is the edge matrix of E. In [3] the explosion of a graph was defined as a generalization of the adjoint graph, and it was shown that exploding a graph does not change its C^*-algebra. Since we work with the vertex matrix instead of the adjacency matrix, our edges are backwards. To be consistent with our earlier terminology, we shall call *reverse explosion* what Enomoto, Fujii, and Watatani called explosion, and we develop a very similar notion, which we shall call *explosion.*

Let E be a graph. Let $v \in E^0$ satisfy $|s^{-1}(v)| > 1$, and fix an edge e whose source is v. First assume that e is not a loop, that is, $v \neq r(e)$. Denote the set of non-loop edges pointing to v by $K = \{k_1, k_2, \ldots\}$ and the set of non-loop edges different from e starting at v by $M = \{m_1, m_2, \ldots\}$. The *edge explosion* F of E at the edge e is defined as follows. Split the vertex v into two vertices v' and v''. The source of e is replaced by v'. The source of each edge in M is replaced by v''. Every edge in K is replaced by a pair of edges k', k'' having the same source as k and pointing to v' and v'' respectively. If there is a loop edge f at v, then it is replaced by a loop f'' at v'' and an edge f' pointing from v'' to v'. The following picture shows an example of E and its explosion at e.

Next assume that e is a loop. To get the explosion at e, split the vertex v into v' and v'' and change the edges in M and K as before. Also, replace e by a loop e' at v' and an edge e'' pointing from v' to v''. The following picture shows an example of E and its explosion at a loop e.

DEFINITION 6.1. Two graphs G and E are said to be *explosion equivalent* if there is a finite sequence $E = F_0, F_1, \ldots, F_n = G$ of graphs such that for every i, either F_i is an edge explosion of F_{i+1} or F_{i+1} is an edge explosion of F_i.

Consider the following more general notion of explosion, which we call *vertex explosion*. Fix $v \in E^0$ with $|s^{-1}(v)| > 1$. Instead of exploding at an edge whose source is v, we explode at a subset of edges whose source is v. Write $s^{-1}(v) = M_1 \cup M_2$, where M_1 and M_2 are disjoint and nonempty. Again we split the vertex v into two vertices v' and v'' and put an edge from every vertex in $s(r^{-1}(v))$ to both v' and v''. Also, put an edge from v' to every vertex in $r(M_1)$ and from v'' to every vertex in $r(M_2)$. If there is a loop edge in M_1, we add an edge from v' to v''. If there is a loop edge in M_2, we add an edge from v'' to v'. If F is an explosion of E at v, we always identify $E^0 \setminus \{v\}$ with $F^0 \setminus \{v', v''\}$.

The following lemma gives a characterization of vertex explosion in terms of the vertex matrix. The proof follows immediately from the definition.

LEMMA 6.2. *Let F be an explosion of E at vertex v with respect to the decomposition $s^{-1}(v) = M_1 \cup M_2$. Let B and C respectively denote the vertex matrices of E and F. Then we have the following:*

(i) $B_{uw} = C_{uw}$ *for every* $u \in E^0 \setminus \{v\}$ *and every* $w \in F^0 \setminus \{v', v''\}$;

(ii) $C_{wv'} = C_{wv''}$ *for every* $w \in F^0$;

(iii) $C_{v''w} = 1 \Leftrightarrow e^{vw} \in M_2$ *and* $C_{v'w} = 1 \Leftrightarrow e^{vw} \in M_1$ *for every* $w \in F^0 \setminus \{v', v''\}$.

Further, if B *and* C *are* 0-1 *matrices satisfying* (i)–(iii)*, then the graph of* C *(i.e. the graph whose vertex matrix is* C*) is an explosion at vertex* v *of the graph of* B.

Definition 6.3. Let E be a graph and suppose that $v \in E^0$ satisfies $|s^{-1}(v)| = k > 1$. Order the vertices of E so that v is vertex 1, and let $B^{(1)}$ denote the vertex matrix of E. Now, for $m = 1, 2, \dots, k-1$, perform the following procedure. First find the largest j such that $B^{(m)}_{1j} = 1$. Next, insert the row E_j between rows 1 and 2 of $B^{(m)}$. Then duplicate the first column of the resulting matrix. Finally, change the 1 in the $(1, j)$ position to a 0. Name this new matrix $B^{(m+1)}$. The *complete explosion of* E *at* v is defined to be the graph of the matrix $B^{(k)}$.

Note that, by the previous lemma, each of the $k-1$ steps in the above procedure corresponds to an explosion at an edge. We now offer the following example to guide the reader through the definition of complete explosion.

Example 6.4. The graph on the left can be completely exploded in two steps by exploding at the dotted edge each time. The dotted edge becomes the dashed edge in the exploded graph at each stage. In the matrices, $*$ denotes the unaffected parts of the matrix.

$$B^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & * & * \\ 1 & * & * \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & * & * \\ 1 & 1 & * & * \end{pmatrix}, \quad B^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & * & * \\ 1 & 1 & 1 & * & * \end{pmatrix}.$$

The reader may check that if E is a graph with no sinks, the complete explosion of E at every vertex with more than one edge emanating from it yields the adjoint graph of E.

Lemma 6.5. *Edge explosion and vertex explosion generate the same equivalence relation.*

Proof. Since edge explosion is a special case of vertex explosion, it suffices to show that an arbitrary graph E and any vertex explosion F of E can be edge exploded into a common graph.

Now if E is any graph and F is a vertex explosion of E at v, the reader may verify by examining the vertex matrices that the complete explosion of E at vertex v coincides with the graph obtained by performing a complete explosion of F at v' and v''. ∎

The following is closely related to the notion of explosion defined in [3].

Definition 6.6. A graph F is a *reverse explosion* of the graph E at a vertex v if the reverse graph of F is the explosion of the reverse graph of E at v and v is cofinal. Two graphs are said to be *reverse explosion equivalent* if there is a finite sequence of reverse explosions connecting them.

Proposition 6.7. *If E is a row-finite graph and F is an explosion of E, then the groupoids of E and F are isomorphic.*

Proof. By Lemma 6.5, it suffices to prove the case when F is the explosion of E at an edge e. First we assume that e is not a loop edge and we use the notation f, M and K as in the definition of explosion. We define a map $\varphi : E^\infty \to F^\infty$ as follows. If $x \in E^\infty$ then φ makes the following replacements on path segments of x:

$$\begin{aligned}
\varphi(\cdots ke \cdots) &= \cdots k'e \cdots \\
\varphi(\cdots k\overbrace{f\cdots ff}^{n+1}e\cdots) &= \cdots k''\overbrace{f''\cdots f''}^{n}f'e\cdots \\
\varphi(\cdots k\overbrace{f\cdots f}^{n}m\cdots) &= \cdots k''\overbrace{f''\cdots f''}^{n}m\cdots \\
\varphi(\overbrace{f\cdots ff}^{n+1}e\cdots) &= \overbrace{f''\cdots f''}^{n}f'e\cdots \\
\varphi(\overbrace{f\cdots f}^{n}m\cdots) &= \overbrace{f''\cdots f''}^{n}m\cdots \\
\varphi(\cdots k\overbrace{fff\cdots}^{\text{all } f}) &= \cdots k''\overbrace{f''f''f''\cdots}^{\text{all } f''} \\
\varphi(\overbrace{fff\cdots}^{\text{all } f}) &= \overbrace{f''f''f''\cdots}^{\text{all } f''}
\end{aligned}$$

where $k \in K$, $m \in M$ and n is a non-negative integer. It is easy but tedious to check that $\varphi : \mathcal{G}_E \to \mathcal{G}_F$ defined by $\varphi(x, k, y) = (\varphi(x), k, \varphi(y))$ is a bijective homomorphism.

It remains to check that it is open and continuous. First we extend φ to a subset of E^*. If $\alpha \in E^*$ and $r(\alpha) \neq v$, then we define $\varphi(\alpha)$ similarly to the above.

To show that φ is open it suffices to check that $\varphi(Z(\alpha,\beta))$ is open. First suppose that $r(\alpha) = r(\beta) \neq v$. In this case,

$$\begin{aligned}\varphi(Z(\alpha,\beta)) &= \{(\varphi(\alpha x, |\alpha| - |\beta|, \beta x) : x \in E^\infty,\ s(x) = r(\alpha)\} \\ &= \{(\varphi(\alpha)\varphi(x), |\varphi(\alpha)| - |\varphi(\beta)|, \varphi(\beta)\varphi(x)) : x \in E^\infty,\ s(x) = r(\alpha)\} \\ &= \{(\varphi(\alpha)y, |\varphi(\alpha)| - |\varphi(\beta)|, \varphi(\beta)y) : y \in F^\infty,\ s(y) = r(\varphi(\alpha))\} \\ &= Z(\varphi(\alpha), \varphi(\beta)).\end{aligned}$$

Now, if $r(\alpha) = r(\beta) = v$, we have three cases. If $\alpha = \alpha' k$ and $\beta = \beta' l$ for some $k, l \in K$, then $\varphi(Z(\alpha,\beta)) = Z(\varphi(\alpha')k', \varphi(\beta')l') \cup Z(\varphi(\alpha')k'', \varphi(\beta')l'')$. If $\alpha = \alpha' k$ and $\beta = \beta' lff \cdots f$, then $\varphi(Z(\alpha,\beta)) = Z(\varphi(\alpha')k'', \varphi(\beta')l''f''f'' \cdots f'')$. Finally, in the case where $\alpha = \alpha' kff \cdots f$ and $\beta = \beta' lff \cdots f$, we have $\varphi(Z(\alpha,\beta)) = Z(\varphi(\alpha')k''f''f'' \cdots f'', \varphi(\beta')l''f''f'' \cdots f'')$. Continuity of φ follows from a similar argument.

The case when e is a loop edge is handled similarly, using a slightly different definition for φ. It now makes the following replacements on $x \in E^\infty$:

$$\begin{aligned}\varphi(\cdots km \cdots) &= \cdots k''m \cdots \\ \varphi(\cdots k\overbrace{e \cdots ee}^{n+1} m \cdots) &= \cdots k'\overbrace{e' \cdots e'}^{n} e''m \cdots \\ \varphi(\overbrace{e \cdots ee}^{n+1} m \cdots) &= \overbrace{e' \cdots e'}^{n} e''m \cdots \\ \varphi(\cdots k\overbrace{ee \cdots}^{\text{all } e}) &= \cdots k\overbrace{e'e' \cdots}^{\text{all } e'} \\ \varphi(\overbrace{ee \cdots}^{\text{all } e}) &= \overbrace{e'e' \cdots}^{\text{all } e'}\end{aligned}$$

where $k \in K$, $m \in M$ and n is a non-negative integer. ∎

Corollary 6.8. *If E is a graph and F is explosion equivalent to E, then $C^*(E) \cong C^*(F)$.*

Proof. If E has no sinks, this follows immediately from the proposition. If E has sinks, then one can readily verify that $\widetilde{F}$ is an explosion of $\widetilde{E}$ and so $C^*(E) \cong C^*(\mathcal{G}_{\widetilde{E}}) \cong C^*(\mathcal{G}_{\widetilde{F}}) \cong C^*(F)$. ∎

The following two results are proven similarly to Proposition 6.7 and Corollary 6.8.

Proposition 6.9. *If E is a row-finite graph and F is the reverse explosion of E at an edge whose source is v, then the groupoid of E pointed at v and the groupoid of F pointed at v'' are isomorphic.*

Corollary 6.10. *If E is a row-finite graph with no sinks and F is a reverse explosion of E then $C^*(E)$ and $C^*(F)$ are Morita equivalent.*

7. ELEMENTARY STRONG SHIFT EQUIVALENCE

A matrix A is *elementary strong shift equivalent* to a matrix B if there are matrices R and S such that $A = RS$ and $B = SR$ ([10]). Note that, for any permutation matrix P, PBP^{-1} and B are elementary strong shift equivalent via $R = PB$, $S = P^{-1}$. Thus, any two vertex matrices of the same graph are elementary strong shift equivalent. Two graphs E and F are said to be elementary strong shift equivalent if their vertex matrices B_E and B_F are elementary strong shift equivalent.

Note that elementary strong shift equivalence is not an equivalence relation. The equivalence relation generated by elementary strong shift equivalence is called *strong shift equivalence*.

Note that if $B_E = RS$ and $B_F = SR$ then the rows and columns of R can be indexed by E^0 and F^0 respectively. Also the rows and columns of S can be indexed by F^0 and E^0 respectively. Using this property we define a bipartite *imprimitivity graph* X as follows: the set of vertices of X is the disjoint union of E^0 and F^0 and the vertex matrix of X is

$$B_X = \begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix}.$$

The construction of X is due to Ashton ([1]).

EXAMPLE 7.1. Using $R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$ we have E, X and F

with vertex matrices

$$B_E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B_X = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad B_F = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

PROPOSITION 7.2. *If E and F are elementary strong shift equivalent, row-finite graphs and X is the imprimitivity graph, then the groupoid of X pointed at E^0 is isomorphic to $\mathcal{G}_E$ and the groupoid of X pointed at F^0 is isomorphic to $\mathcal{G}_F$.*

Proof. First note that E^0 and F^0 are automatically cofinal pointing sets. By symmetry it suffices to show that $\mathcal{G}_{(X,E^0)} \cong \mathcal{G}_E$. By the construction of X we have a unique bijection $\varphi : E^1 \to X^2$ such that $s = s \circ \varphi$ and $r = r \circ \varphi$. We extend φ to $E^* \cup E^\infty$. It is easy to check that $\varphi : \mathcal{G}_E \to \mathcal{G}_{(X,E^0)}$ defined by

$$\varphi[\alpha, x, \beta] = [\varphi(\alpha), \varphi(x), \varphi(\beta)]$$

is an isomorphism. ∎

The C^*-algebras of strong shift equivalent graphs are not necessarily isomorphic (see Example 8.1), but we have:

Corollary 7.3. *If E and F are strong shift equivalent, row-finite graphs with no sinks then $C^*(E)$ and $C^*(F)$ are Morita equivalent.*

Following Ashton ([1]), we say a 0-1 matrix is column subdivision if each of its columns contains at most one 1. Two matrices A and B are said to be elementary strong shift equivalent with column subdivision if $A = RS$, $B = SR$, and either R or S is column subdivision. Likewise, two graphs are said to be elementary strong shift equivalent with *column subdivision* if their vertex matrices are. It was shown in both [1] and [3] that if two finite 0-1 matrices A and B are elementary strong shift equivalent with column subdivision, then $\mathcal{O}_A \cong \mathcal{O}_B$. The following result, combined with Corollary 6.8, provides an alternate proof of a special case of this fact:

Proposition 7.4. *Let E and F be graphs with no sinks. Suppose that E and F have n and $n+1$ vertices respectively. Then E and F are elementary strong shift equivalent with column subdivision if and only if F is an explosion of E.*

Proof. Denote the vertex matrix of E by B and the vertex matrix of F by C. Now suppose that F is an explosion of E, with vertex v splitting into v' and v''. Without loss of generality, we may assume that v is vertex 1 in E and v' and v'' are the first two vertices of F. That is, the first row and column of B corresponds to v and the first two rows and columns of C correspond to v' and v''. Define S to be C with the first column deleted. That is, $S_{ij} = C_{ij}$ for $i = 1, \ldots, n+1$, $j = 1, \ldots, n$. If R is the following $n \times (n+1)$ column subdivision matrix

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then $B = RS$ and $C = SR$.

Now suppose that E and F are elementary strong shift equivalent with column subdivision. That is, for any choices B and C of vertex matrices of E and F, there exist R, S such that $B = RS$, $C = SR$, and either S or R is column subdivision. Now, S is an $(n+1) \times n$ matrix, so in order for it to be column subdivision, it must have a zero row. But a zero row in S would yield a zero row in C, and hence a sink in F. Thus it must be R which is column subdivision.

Since R is an $n \times (n+1)$ matrix which is column subdivision and has no zero rows, there exist an $n \times n$ permutation matrix P and an $(n+1) \times (n+1)$ permutation matrix Q such that

$$PRQ = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So by replacing B with PBP^{-1} and C with QCQ^{-1}, we may assume that R has this form.

Now, if we index the rows of R and the columns of S by $\{1, 2, \ldots, n\}$ and the columns of R and the rows of S by $\{0, 1, \ldots, n\}$, then we have the following:

(i) $B_{ij} = C_{ij}$ for $i = 2, \ldots, n$, $j = 1, \ldots, n$;

(ii) $C_{i0} = C_{i1}$ for $i = 0, \ldots, n$;

(iii) $C_{0j} + C_{1j} \leqslant 1$ for $j = 0, \ldots, n$.

(i) and (ii) are easy to check. To see (iii), suppose that for some j, C_{0j} and C_{1j} are both 1. If $j > 1$, then

$$2 = C_{0j} + C_{1j} = \sum_k S_{0k}R_{kj} + \sum_l S_{1l}R_{lj} = S_{0j} + S_{1j} = \sum_m R_{1m}S_{mj} = B_{1j},$$

which is a contradiction. If, on the other hand, $j \leqslant 1$, then similar calculations show that $B_{11} = 2$.

These three facts imply that B and C satisfy the three conditions of Lemma 6.2 for some suitable choice of v, M_1 and M_2. Thus E is an explosion of F. ∎

REMARK 7.5. Note that the restriction on sinks is necessary in the preceding proposition, since if

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then B and C are elementary strong shift equivalent with column subdivision via

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

but the graph of C is not an explosion of the graph of B.

8. COUNTEREXAMPLES

In this section we collect several examples which show that neither primitive equivalence nor reverse primitive equivalence is implied by any of the other equivalence relations discussed in this paper.

EXAMPLE 8.1. *Elementary strong shift equivalence does not imply primitive equivalence.* This is trivially true because two matrices which are primitively equivalent must be the same size, while elementary strong shift equivalence may change the size. But this example, taken from [3], shows that elementary strong shift equivalent matrices need not be primitively equivalent, even if they have the same size: if

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

then $\mathcal{O}_{RS} \cong \mathcal{O}_3$ and $\mathcal{O}_{SR} \cong \mathcal{O}_3 \otimes M_2$ are not isomorphic ([7]). Hence the graph corresponding to RS cannot be primitively equivalent to the graph corresponding to SR. This example also answers negatively a question posed in [1], namely, do elementary strong shift equivalent graphs always yield isomorphic algebras?

EXAMPLE 8.2. *Elementary strong shift equivalence does not imply reverse primitive equivalence.* Using R and S from the previous example, $S^{\mathrm{T}}R^{\mathrm{T}} = (RS)^{\mathrm{T}}$ and $R^{\mathrm{T}}S^{\mathrm{T}} = (SR)^{\mathrm{T}}$ are not primitively equivalent. This can be verified from the table on page 450 of [3]. We remark in passing that the second graph in the final row of that table is misprinted. There should not be a loop on the top vertex, and there should be a loop added to the lower left vertex.

EXAMPLE 8.3. *Reverse primitive equivalence does not imply primitive equivalence.* Example 5.2 shows two matrices which are reverse primitively equivalent, but whose Cuntz-Krieger algebras are not isomorphic. Hence they are not primitively equivalent.

EXAMPLE 8.4. *Explosion equivalence does not imply primitive equivalence and reverse explosion equivalence does not imply reverse primitive equivalence.* Consider the matrices

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Both are explosions of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ so they are explosion equivalent. But they are not primitively equivalent. The reader with a spare afternoon may verify this by checking that there are 60 elements in the primitive equivalence class of C, and B is not one of them. Also note that, since A is irreducible, every vertex is cofinal. Thus B^{T} and C^{T} are reverse explosion equivalent, but not reverse primitively equivalent.

EXAMPLE 8.5. *Reverse explosion equivalence does not imply primitive equivalence and explosion equivalence does not imply reverse primitive equivalence.* If

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

B and C are reverse explosion equivalent because their transposes are explosions of $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ but they are not primitively equivalent. Unfortunately, it requires a computer program to verify this (the primitive equivalence class of C has 183,204 elements), and we could not find a more manageable example.

EXAMPLE 8.6. *Primitive equivalence does not imply reverse primitive equivalence.* It can be verified using Section 4 of [3] that

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

are primitively equivalent, but not reverse primitively equivalent.

It is an open question whether or not primitive equivalence and explosion equivalence together are enough to characterize graph groupoid isomorphism. That is, given two graphs with isomorphic groupoids, are they explosion-primitive equivalent? There are difficulties on both ends. In particular, given two graphs, determining whether or not their groupoids are isomorphic is highly non-trivial. Also, we currently do not have an efficient algorithm for determining whether or not two graphs are explosion-primitive equivalent. There are two difficulties here. First, even in the 5×5 case, some matrices have primitive equivalence classes with an unmanageable number of elements, so the computer time required to check whether two matrices are primitively equivalent becomes an issue. Second, we cannot say with certainty that two matrices are not explosion equivalent. For example, consider the matrices A and B from Section 4. We have checked that no explosion of A to a 5×5 matrix is primitively equivalent to any explosion of B to a 5×5 matrix. However, it may be possible, for example, that A and B can be exploded into 6×6 (or larger) matrices which are primitively equivalent.

It would be desirable to have more graph transformations that preserve the isomorphism class of the groupoid or the C^*-algebra, or the Morita equivalence class of the C^*-algebra, of the graph. Having more operations would increase our chances of finding a canonical form.

Acknowledgements. The authors would like to thank Alex Kumjian, David Pask, John Quigg, and Jack Spielberg for many helpful discussions.

REFERENCES

1. B. Ashton, Morita equivalence of graph C^*-algebras, Honors Thesis, University of Newcastle, 1996.
2. J. Cuntz, W. Krieger, *A class of C^*-algebras and topological Markov chains*, *Invent. Math.* **56**(1980), 251–268.
3. M. Enomoto, M. Fujii, Y. Watatani, K_0-groups and classifications of Cuntz-Krieger algebras, *Math. Japon.* **26**(1981), 443–460.
4. J. Franks, Flow equivalence of subshifts of finite type, *Ergodic Theory Dynamical Systems* **4**(1984), 53–66.
5. A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, *Pacific J. Math* **184**(1998), 161–174.
6. A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids, and Cuntz-Krieger algebras, *J. Funct. Anal.* **144**(1997), 505–541.
7. W.L. Paschke, N. Salinas, Matrix algebras over $\mathcal{O}_n$, *Michigan Math. J.* **26**(1979), 3–12.
8. D. Pask, I. Raeburn, On the K-theory of Cuntz-Krieger algebras, *Publ. Res. Inst. Math. Sci.* **32**(1996), 415–443.
9. M. Rørdam, Classification of Cuntz-Krieger algebras, *K-theory* **9**(1995), 31–58.

10. R. WILLIAMS, Classification of subshifts of finite type, *Ann. of Math.* **98**(1973), 120–153; Errata **99**(1974), 380–381.

D. DRINEN
Department of Mathematics
Dartmouth College
6188 Bradley Hall
Hanover, NH 03755-3551
USA

E-mail: drinen@dartmouth.edu

N. SIEBEN
Department of Mathematics
Arizona State University
Tempe, AZ 85287
USA

E-mail: nandor.sieben@asu.edu

Received October 20, 1998; revised January 14, 1999.