J. OPERATOR THEORY
45(2001), 265–301

ANALYTIC LEFT-INVARIANT SUBSPACES OF WEIGHTED HILBERT SPACES OF SEQUENCES

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Communicated by Nikolai K. Nikolskii

Abstract. Let ω be a weight on $\mathbb{Z}$, and assume that the translation operator $S : (u_n)_{n\in\mathbb{Z}} \to (u_{n-1})_{n\in\mathbb{Z}}$ is bounded on $\ell^2_\omega(\mathbb{Z})$, and that the spectrum of S equals the unit circle. A closed subspace G of $\ell^2_\omega(\mathbb{Z})$ is said to be left-invariant (respectively translation invariant, respectively right-invariant) if $S^{-1}(G) \subset G$ (respectively $S(G) = G$, respectively $S(G) \subset G$) and G is said to be analytic if G contains a nonzero sequence $(u_n)_{n\in\mathbb{Z}}$ such that $u_n = 0$ for $n < 0$. We show that if the weight $\omega(n)$ grows sufficiently fast as $n \to -\infty$, then all analytic left-invariant subspaces of $\ell^2_\omega(\mathbb{Z})$ are generated by their intersection with $\ell^2_\omega(\mathbb{Z}^+) := \{(u_n)_{n\in\mathbb{Z}} \in \ell^2_\omega(\mathbb{Z})\} : u_n = 0$ for $n < 0)$. Various concrete examples of weights ω for which this situation occurs are obtained by using sharp estimates of Matsaev-Mogulskii about the rate of growth of quotients of analytic functions in the disc.

We also discuss the existence of right-invariant subspaces of $\ell^2_\omega(\mathbb{Z}^+)$ having a specific division property needed to obtain analytic translation invariant subspaces of $\ell^2_\omega(\mathbb{Z})$.

Keywords: *Invariant subspaces, weighted shifts, quotients of analytic functions.*

MSC (2000): Primary 47A15; Secondary 46E99, 47B37.

1. INTRODUCTION

Let ω be a weight on $\mathbb{Z}$, i.e. a map from $\mathbb{Z}$ into $(0,\infty)$. Assume that

$$0 < \inf_{n\in\mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} \leqslant \sup_{n\in\mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} < +\infty,$$

and that

$$\lim_{|n|\to\infty} \widetilde{\omega}(n)^{1/n} = 1 \quad \text{where} \quad \widetilde{\omega}(n) = \sup_{p\in\mathbb{Z}} \frac{\omega(n+p)}{\omega(p)}.$$

Let

$$\ell_\omega := \ell^2_\omega(\mathbb{Z}) := \Big\{ u = (u_n)_{n\in\mathbb{Z}} : \|u\|_\omega := \Big[\sum_{n\in\mathbb{Z}} |u|^2\omega^2(n)\Big]^{1/2} < +\infty \Big\}.$$

Then the shift operator $S : (u_n)_{n\in\mathbb{Z}} \mapsto (u_{n-1})_{n\in\mathbb{Z}}$ is bounded on ℓ_ω, and its spectrum $\mathrm{Spec}(S)$ equals the unit circle $\mathbb{T}$. We will say that a closed subspace G of ℓ_ω is *translation invariant* (respectively *right-invariant*, respectively *left-invariant*) if $S(G) = G$ (respectively $S(G) \subset G$, respectively $S^{-1}(G) \subset G$), and we will say that a left-invariant subspace G is *analytic* if $G \cap \ell^+_\omega \neq \{0\}$, where $\ell^+_\omega = \{u = (u_n)_{n\in\mathbb{Z}} \in \ell_\omega \ : u_n = 0 (n < 0)\}$. Let $\tau = \omega|\mathbb{Z}^+$, and denote by $H_\tau := H^2_\tau(\mathbb{D})$ the usual weighted Hardy space

$$\Big\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_\tau := \Big[\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \cdot \omega^2(n)\Big]^{1/2} < +\infty \Big\}.$$

Here we denote by $\mathcal{H}(\mathbb{D})$ the space of functions holomorphic on the open unit disc $\mathbb{D}$, and for $f \in \mathcal{H}(\mathbb{D})$ we denote by $\widehat{f}(n)$ the n^{th} Taylor coefficient of f at the origin. We can identify ℓ^+_ω to $\ell^2_\tau(\mathbb{Z}^+)$ in the obvious way, and the Fourier transform $f \to (\widehat{f}(n))_{n\geqslant 0}$ is an isometry from H_τ onto ℓ^+_ω. Denote by $\check{u}$ the "inverse Fourier transform", so that

$$\check{u}(z) = \sum_{n=0}^{\infty} u_n \cdot z^n \quad \text{for } z \in \mathbb{D},\ u \in \ell^+_\omega.$$

Let G be an analytic left-invariant subspace of ℓ_ω, and set $F = G \cap \ell^+_\omega$. Then $\check{F}$ enjoys the "division property": if $f \in \check{F}$, and if $f(\lambda) = 0$, with $\lambda \in \mathbb{D}$, then $f_\lambda : \xi \to \frac{f(\xi)-f(\lambda)}{\xi-\lambda}$ is also an element of $\check{F}$. Section 2 is devoted to an elementary discussion of subspaces of H_τ having the division property. If M has the division property, then $\overline{M}$ also has the division property. Nontrivial closed subspaces of H_τ having the division property are characterized by the fact that $Z(M) = \emptyset$ and $\dim(M \ominus (M \cap zM)) = 1$. (Here we denote by $Z(M) = \{\lambda \in \mathbb{D} : f(\lambda) = 0,\ f \in M\}$ the zero-set of M in $\mathbb{D}$, and by zf the function $\xi \to \xi \cdot f(\xi)$ for $f \in H_\tau$.) In this case there exists a bounded operator U_M on $M^\perp$ satisfying the condition $U_M \cdot P \cdot zf = P \cdot f$, $(f \in H_\tau)$, where we denote by P the orthogonal projection from H_τ onto $M^\perp$. When $z \cdot M \subset M$ these conditions are equivalent to the fact that $\mathrm{Spec}(T_M) \subset \mathbb{T}$, where we denote by T_M the "compression" to $M^\perp$ of the usual unilateral shift $T : f \to z \cdot f$ on H_τ.

A subspace F of ℓ^+_ω is said to have the *division property* iff $\check{F}$ has the division property. In Section 3 we show that, if $\omega(-n)$ grows sufficiently fast as $n \to \infty$, the map $G \to G \cap \ell^+_\omega$ provides a bijection between the set of analytic left-invariant subspaces of ℓ_ω and the set of closed subspaces of ℓ^+_ω which have the division property. Let F be a nontrivial closed subspace of ℓ^+_ω having the division property and set $\omega_F(n) = \|U^n_{\check{F}} \cdot P \cdot 1\|_\tau$ for $n \geqslant 1$, with the same notation as above. The results of Section 3 are based on Theorem 3.5: if

$$\sum_{n=1}^{\infty} \frac{\omega^2_F(n)}{\omega^2(-n)} < +\infty, \text{ then } F = \Big(\bigvee_{n\leqslant 0} S^n \cdot F\Big) \cap \ell^+_\omega \text{ and } \ell^+_\omega + \Big(\bigvee_{n\leqslant 0} S^n \cdot F\Big) = \ell_\omega.$$

In other terms, the natural map from ℓ^+_ω/F into $\ell_\omega/\bigvee_{n\leqslant 0} S^n\cdot F$ is then a bijection, and the compression of S^{-1} to $\Big(\bigvee_{n\leqslant 0} S^n\cdot F\Big)^\perp$ is similar to $U_{\check F}$ (see Remark 3.7). The proof of Theorem 3.5, which is elementary and constructive, is based on the use of two nonorthogonal projections associated to F. A weaker condition, necessary and sufficient to have $\bigvee_{n\leqslant 0} S^n\cdot F\subsetneq\ell_\omega$, is given in Proposition 3.8.

When $\omega(-n)$ grows sufficiently fast as $n\to\infty$, the hypothesis of Theorem 3.5 are satisfied for all nontrivial closed subspaces of ℓ^+_ω having the division property. In this case $L=\bigvee_{n\leqslant 0} S^n\cdot(L\cap\ell^+_\omega)$ for every analytic left-invariant subspace L of ℓ_ω.

In Section 4 we use estimates about the rate of growth of quotients of analytic functions due to Matsaev-Mogulskii ([32]) to give various concrete examples of weights ω to which the results of Section 3 can be applied, see Theorem 4.5. For example the results of Section 3 apply to ω if it is log-convex,

$$\sum_{n=1}\frac{\log\omega^{-1}(n)}{n^{3/2}}<+\infty,\quad \liminf_{n\to\infty}\frac{\log\omega(-n)}{\sqrt{n}}=+\infty,$$

or if ω is log-convex, $\Big(\frac{\log\omega^{-1}(n)}{n^\alpha}\Big)_{n\geqslant 1}$ is eventually increasing for every $\alpha\in(0,1)$, and $\liminf_{n\to\infty}\frac{\log\omega(-n)}{\log\omega^{-1}(n)}>1$.

In the last section of the paper we discuss the existence of z-invariant subspaces of a weighted Hardy space H_τ having the division property. In the case of the usual Hardy space H^2, these subspaces are the subspaces $U\cdot H^2$ where U is a singular inner function. (The Hitt-Sarason theory ([28], [36]) of subspaces of H^2 "weakly invariant for the backward shift" gives a complete description of the lattice of closed subspaces of H^2 which have the division property, see Section 2.) A similar description of z-invariant subspaces having the division property, involving Korenblum's "Bergman-inner" functions ([29]), holds for the usual Bergman space B^2. Also it follows from a recent result of Borichev ([11]) that, if $\liminf_{n\to\infty}\tau(n)=0$, then H_τ possesses a nontrivial z-invariant subspace M such that $Z(M)=\emptyset$, but all the subspaces considered by Borichev satisfy $\dim(M\ominus zM)\geqslant 2$ and so these subspaces do not have the division property, and the general case remains open. We describe some partial results. The "abstract Keldysh method" developed by Nikolski in [34] is applied to construct explicit examples of functions without zeroes in $\mathbb{D}$ which are not z-cyclic in H_{τ_α}, where $\tau_\alpha(n)=\mathrm{e}^{-n^\alpha}$ for $n\geqslant 0$, $1/2<\alpha<1$, and of course the subspaces $M_f:=\bigvee_{n\geqslant 0} z^n\cdot f$ have then the division property. It follows also from a recent work of Atzmon ([4], [5]), based on sharp results about the growth of entire functions of zero exponential type, that H_τ possesses nontrivial z-invariant subspaces M, having the division property, for which $\mathrm{Spec}(T_M)=\{1\}$, when τ is log-convex and when

$$\sup\Big[\frac{\tau(n+1)\cdot\tau(n-1)}{\tau(n)^2}\Big]^{1/n}<+\infty.$$

A new method to produce non z-cyclic functions in H_τ without zeroes in $\mathbb{D}$ was introduced by Hedenmalm and the second author in [26].

This paper completes the first part of a program concerning analytic translation invariant subspaces of ℓ_ω. In a forthcoming paper, the authors will show that if $\omega(-n)$ grows "sufficiently fast and regularly" as $n \to \infty$ then *all* nonzero translation invariant subspaces of ℓ_ω are analytic (and for all nonzero left-invariant subspaces F of ℓ_ω there exists an integer $k \geqslant 0$, which depends on F, such that $S^k \cdot F$ is analytic). For example if $\omega(n) = 1$ for $n \geqslant 0$, and if $\omega(n) = \mathrm{e}^{\frac{|n|}{1+\log|n|}}$ for $n < 0$, then all nontrivial translation invariant subspaces of ℓ_ω have the form $\bigvee_{n\in\mathbb{Z}} S^n \cdot \widehat{U}$ where U is a singular inner function. A summary of these results, which are based on the theory of asymptotically holomorphic functions ([10], [13] and [38]) appeared in [23].

Since these results do not involve any regularity conditions on $\tau = \omega|\mathbb{Z}^+$ other than those of Section 3, we thus see that an example of a weight τ on $\mathbb{Z}^+$ for which H_τ does not possess any nontrivial z-invariant subspace having the division property would give an example of a weight ω on $\mathbb{Z}$ for which ℓ_ω does not have any nontrivial invariant subspace. This fact was the motivation for the detailed description given in Section 5.

We refer to the works of Nikolski ([35]) and Shields ([37]) for general properties of weighted shifts. Apostol ([2]) constructed translation invariant subspaces of ℓ_ω for weights ω having "irregular" behaviour at infinity. In fact, he reduced the question of existence of nontrivial invariant subspaces of ℓ_ω to the case where $\lim_{|n|\to\infty} \omega(n)^{1/n} = 1$ and where the spectral radius of S equals 1. Domar ([21]) constructed recently nontrivial invariant subspaces of ℓ_ω for the weights ω such that $\omega(n) \cdot \omega(-n) = 1$, $n \geqslant 0$ and

$$\sum_{n=1}^{\infty} |\log\omega(n+1) + \log\omega(n-1) - 2\log\omega(n)| < +\infty.$$

His methods, based on results about entire functions related to the Beurling-Malliavin theorem ([9]), are very different from the methods discussed here, and the translation invariant subspaces constructed in [21] are not analytic. We refer to Atzmon's paper ([4]) for a description of the state of the art concerning existence of translation-invariant subspaces of ℓ_ω.

2. Weighted Hardy spaces and the division property

We denote by $\mathcal{S}^+$ the set of weights $\tau : \mathbb{Z}^+ \to (0,\infty)$ such that

$$0 < \inf_{n\geqslant 0} \frac{\tau(n+1)}{\tau(n)} \leqslant \sup_{n\geqslant 0} \frac{\tau(n+1)}{\tau(n)} < +\infty,$$

and such that if we set for $n \geqslant 0$

$$\overline{\tau}(n) = \sup_{p\geqslant 0} \frac{\tau(p)}{\tau(n+p)}, \quad \widetilde{\tau}(n) = \sup_{p\geqslant 0} \frac{\tau(n+p)}{\tau(p)} \tag{2.1}$$

we have

$$\lim_{n\to\infty} \overline{\tau}(n)^{1/n} = \lim_{n\to\infty} \widetilde{\tau}(n)^{1/n} = 1. \tag{2.2}$$

Throughout this section we will denote by τ an element of $\mathcal{S}^+$. Since $\frac{\tau(0)}{\overline{\tau}(n)} \leqslant \tau(n) \leqslant \tau(0)\widetilde{\tau}(n)$ we obtain

$$\lim_{n\to\infty} \tau(n)^{1/n} = 1. \tag{2.3}$$

For $f \in \mathcal{H}(\mathbb{D})$, denote by $\widehat{f}(n)$ the n^{th} Taylor coefficient of f at the origin. Set

$$H_\tau = H^2_\tau(\mathbb{D}) := \Big\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_\tau := \Big[\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \tau^2(n)\Big]^{1/2} < +\infty \Big\}. \tag{2.4}$$

Also for $f \in \mathcal{H}(\mathbb{D})$, $\lambda \in \mathbb{D}$, set

$$f_\lambda(\zeta) = \frac{f(\zeta) - f(\lambda)}{\zeta - \lambda}, \quad \zeta \in \mathbb{D} \setminus \{\lambda\} \text{ and } f_\lambda(\lambda) = f'(\lambda). \tag{2.5}$$

The usual shift T and the backward shift R are given by the formulae

$$T(f)(\lambda) = \lambda \cdot f(\lambda),\ (\lambda \in \mathbb{D}) \quad \text{and} \quad Rf = f_0,\ (f \in H_\tau). \tag{2.6}$$

Clearly, $\|T^n\| = \widetilde{\tau}(n)$ and $\|R^n\| = \overline{\tau}(n)$, $n \geqslant 0$. In particular $1 - \lambda R$ is invertible for $\lambda \in \mathbb{D}$. Denote by z the identity map on $\mathbb{D}$.

An immediate verification shows that

$$(1 - \lambda R)\Big(\sum_{i=0}^{n-1} \lambda^{n-1-i} z^i\Big) = z^{n-1} = R \cdot z^n \quad n \geqslant 1.$$

Hence $(T-\lambda)(1-\lambda R)^{-1} \cdot R z^n = z^n - \lambda^n$, $n \geqslant 0$. We obtain $(T-\lambda)(1-\lambda R)^{-1} \cdot R \cdot f = f - f(\lambda)$ and so $f_\lambda \in H_\tau$ for $f \in H_\tau$, $\lambda \in \mathbb{D}$ and we have

$$f_\lambda = R(1 - \lambda R)^{-1} \cdot f, \quad f \in H_\tau. \tag{2.7}$$

A closed subspace M of H_τ will be said to be *z-invariant* if $M \in \operatorname{Lat} T$, and we will write $z \cdot A$ instead of $T(A)$ for $A \subset H_\tau$. We will also often write $\frac{f - f(\lambda)}{z - \lambda}$ instead of f_λ. For $f \in \mathcal{H}(\mathbb{D})$ set $Z(f) = \{\lambda \in \mathbb{D} : f(\lambda) = 0\}$ and for $A \subset \mathcal{H}(\mathbb{D})$ set $Z(A) = \bigcap_{f \in A} Z(f)$. If M is a linear subspace of H_τ we will denote by $\pi_M : f \to f + M$ the canonical surjection from H_τ onto H_τ/M. If $zM \subset M$, the map $T_M : H_\tau/M \to H_\tau/M$ is defined by the formula

$$T_M \circ \pi_M = \pi_M \circ T. \tag{2.8}$$

Let M be a linear subspace of H_τ, and let $\lambda \in \mathbb{D}$. We have

$$\text{the map } \pi_M \circ (T - \lambda) : H_\tau \to H_\tau/M \text{ is onto iff } \lambda \notin Z(M). \tag{2.9}$$

To see this, consider $\lambda \in \mathbb{D} \setminus Z(M)$. There exists $\varphi \in M$ such that $\varphi(\lambda) = 1$. Let $f \in H_\tau$. Then $\lambda \in Z(f - f(\lambda)\varphi)$, and

$$\pi_M(f) = \pi_M(f - f(\lambda)\varphi) = [\pi_M \circ (T - \lambda)][(f - f(\lambda)\varphi)_\lambda]$$

so that $\pi_M \circ (T - \lambda)$ is onto. Conversely, if $\pi_M \circ (T - \lambda)$ is onto, there exists $g \in H_\tau$ such that $1 - (z - \lambda)g \in M$, and so $\lambda \notin Z(M)$.

Lemma 2.1. *Let $M \neq \{0\}$ be a linear subspace of H_τ, and let $\lambda \in \mathbb{D}$. Then the following conditions are equivalent:*

(i) *$f_\lambda \in M$ for every $f \in M$ such that $f(\lambda) = 0$;*

(ii) *There exists a map $U_M(\lambda) : H_\tau/M \to H_\tau/M$ such that*

$$U_M(\lambda) \circ \pi_M \circ (T - \lambda) = \pi_M;$$

(iii) *$\lambda \notin Z(M)$, and $\dim[M/M \cap (z - \lambda)M] = 1$.*

If these conditions are satisfied, the map $U_M(\lambda)$ satisfying (ii) *is unique and linear. If, further, $zM \subset M$ then the above conditions are equivalent to*

(iv) *$\lambda \notin \sigma(T_M)$,*

and in this case $U_M(\lambda) = (T_M - \lambda)^{-1}$.

Proof. Set $\pi = \pi_M$. If (i) holds then $\lambda \notin Z(M)$. Let $\varphi \in M$ such that $\varphi(\lambda) = 1$, and let $f \in M$. Then $f - f(\lambda)\varphi \in (z - \lambda)M \cap M$, $\varphi \notin (z - \lambda)M$ and so λ satisfies (iii).

If (iii) holds, consider again $\varphi \in M$ such that $\varphi(\lambda) = 1$. If $f \in M$ there exists $\gamma \in \mathbb{C}$ such that $f - \gamma\varphi \in M \cap (z - \lambda)M$. Then $\gamma = f(\lambda)$.

If $f(\lambda) = 0$ then $f \in M \cap (z - \lambda)M$ and so $f_\lambda \in M$. Hence (i) and (iii) are equivalent.

Assume again that (i) holds. Set, for $f \in H_\tau$

$$U_M(\lambda)[\pi(f)] = \pi[(f - f(\lambda)\varphi)_\lambda] \quad \text{where } \varphi \in M \text{ satisfies } \varphi(\lambda) = 1. \tag{2.10}$$

Let $\psi \in M$ and let $g = f + \psi$. Then $\psi - \psi(\lambda)\varphi \in M$, $\lambda \in Z(\psi - \psi(\lambda)\varphi)$ and so $(\psi - \psi(\lambda)\varphi)_\lambda)_\lambda \in M$. Hence $[g - g(\lambda)\varphi]_\lambda \in (f - f(\lambda) \cdot \varphi)_\lambda + M$ and $U_M(\lambda)$ is well-defined. Clearly, $U_M(\lambda) \circ \pi \circ (T - \lambda) = \pi$ and (ii) holds.

Now assume that (ii) holds. Then $U_M(\lambda)(0) = [U_M(\lambda) \circ \pi \circ (T - \lambda)](0) = \pi(0) = 0$. Now if $f \in M$, and if $f(\lambda) = 0$, then $\pi(f_\lambda) = U_M(\lambda)[\pi(f)] = U_M(\lambda)(0) = 0$, and $f_\lambda \in M$ so that (i) is satisfied.

Now if $V \circ \pi \circ (T - \lambda) = \pi$ then λ satisfies (ii) and so $\lambda \notin Z(M)$. It follows from (2.9) that $\pi \circ (T - \lambda)$ is onto, and so $V = U_M(\lambda)$. It follows from (2.10) that $U_M(\lambda)$ is linear. Assume that $zM \subset M$. If $\lambda \notin \sigma(T_M)$ then $(T_M - \lambda)^{-1} \circ \pi \circ (T - \lambda) = (T_M - \lambda)^{-1} \circ (T_M - \lambda) \circ \pi = \pi$ and (ii) is satisfied, with $U_M(\lambda) = (T_M - \lambda)^{-1}$. Conversely if the equivalent conditions (i), (ii), (iii) are satisfied then $U_M(\lambda) \circ (T_M - \lambda) \circ \pi = \pi$ and so $T_M - \lambda$ is one-to-one. Also $\lambda \notin Z(M)$ and so, by (2.9), $(T_M - \lambda) \circ \pi = \pi \circ (T - \lambda)$ is onto. Hence $T_M - \lambda$ is onto, and $\lambda \notin \sigma(T_M)$. ∎

The following lemma is an immediate consequence of (iv) when $zM \subset M$.

Lemma 2.2. *Let $M \neq \{0\}$ be a linear subspace of H_τ, and denote by $\Omega(M)$ the set of elements of $\mathbb{D}$ satisfying the equivalent conditions* (i), (ii), (iii) *of Lemma* 2.1 *with respect to M. If $\lambda \in \Omega(M)$ then*

$$\Omega(M) \setminus \{\lambda\} = \Big\{\mu \in \mathbb{D} \setminus \{\lambda\} : \frac{1}{\mu - \lambda} \notin \sigma(U_M(\lambda))\Big\},$$

and we have

$$U_M(\mu) = U_M(\lambda) \circ [1 - (\mu - \lambda)U_M(\lambda)]^{-1}, \quad \lambda \in \Omega(M),\ \mu \in \Omega(M).$$

Proof. If $\mu \in \mathbb{D} \setminus \Omega(M)$ there exists $f \in M$ such that $f(\mu) = 0$ and $f_\mu \notin M$. Set again $\pi = \pi_M$. Then $([1 - (\mu - \lambda)U_M(\lambda)] \circ \pi)((z - \lambda) \cdot f_\mu)) = \pi[(z - \lambda)f_\mu -$

$(\mu-\lambda)f_\mu] = \pi(f) = 0$. Since $\lambda \in \Omega(M)$, $(z-\lambda)f_\mu \notin M$ and $1-(\mu-\lambda)U_M(\lambda)$ is not one-to-one. Hence $\frac{1}{\mu-\lambda} \in \sigma(U_M(\lambda))$.

Now assume that $\mu \in \Omega(M)$. Then $[1-(\mu-\lambda)U_M(\lambda)] \circ \pi \circ (T-\lambda) = \pi \circ [(T-\lambda)] - (\mu-\lambda)\pi = \pi \circ (T-\mu)$. Since $\mu \notin Z(M)$, it follows then from (2.9) that $1-(\mu-\lambda)U_M(\lambda)$ is onto. Also $U_M(\mu) \circ [1-(\mu-\lambda)U_M(\lambda)] \circ \pi \circ (T-\lambda) = U_M(\mu) \circ \pi \circ (T-\mu) = \pi$. Hence $U_M(\mu) \circ [1-(\mu-\lambda)U_M(\lambda)] = U_M(\lambda)$, and $\ker[1-(\mu-\lambda)U_M(\lambda)] \subset \ker U_M(\lambda)$. Hence $1-(\mu-\lambda)U_M(\lambda)$ is one-to-one, and $\frac{1}{\mu-\lambda} \notin \sigma(U_M(\lambda))$ if $\mu \neq \lambda$. Also, $U_M(\mu) = U_M(\lambda) \circ [1-(\mu-\lambda)U_M(\lambda]^{-1}$. ∎

The following corollary is a standard result when M is z-invariant.

Corollary 2.3. *Let $M \neq \{0\}$ be a closed linear subspace of H_τ. Then either $\Omega(M) = \emptyset$ or $\Omega(M) = \mathbb{D} \setminus Z(M)$.*

Proof. It follows from (2.10) that $U_M(\lambda)$ is bounded on H_τ/M if $\lambda \in \Omega(M)$, and it follows then from Lemma 2.2 that $\Omega(M)$ is an open subset of $\mathbb{D} \setminus Z(M)$. Now assume that $\Omega(M) \neq \emptyset$, and let $\lambda \in \overline{\Omega(M)} \cap (\mathbb{D} \setminus Z(M))$. There exists $\varphi \in M$ such that $\varphi(\lambda) = 1$, and there exists a sequence $(\lambda_n)_{n\geqslant 1}$ of elements of $\Omega(M)$ such that $|\lambda - \lambda_n| \underset{n\to\infty}{\longrightarrow} 0$. We can assume that $\varphi(\lambda_n) \neq 0$ for $n \geqslant 1$. Let $f \in M$ such that $f(\lambda) = 0$, and set $f_n = f - \varphi(\lambda_n)^{-1} \cdot f(\lambda_n) \cdot \varphi$. Then $(f_n)_{\lambda_n} \in M$. It follows from (2.7) that the map $(\xi, g) \to g_\xi$ is continuous from $\mathbb{D} \times H_\tau$ into H_τ. Hence $f_\lambda = \lim_{n\to\infty} (f_n)_{\lambda_n} \in M$, and $\lambda \in \Omega(M)$. Since $\mathbb{D} \setminus Z(M)$ is connected, $\Omega_M = \mathbb{D} \setminus Z(M)$. ∎

Definition 2.4. A linear subspace M of H_τ has the *division property* if $f_\lambda \in M$ for every $f \in M$ and every $\lambda \in Z(f)$.

We will denote by $\mathcal{D}_\tau$ the set of closed subspaces of H_τ having the division property.

Clearly, if $(M_i)_{i\in I}$ is a family of linear subspaces of H_τ having the division property, then $\bigcap_{i\in I} M_i$ has the division property. Also every $M \in \operatorname{Lat} R$ has the division property. Notice that if M has the division property then $f \cdot M \cap H_\tau$ has the division property for every $f \in \mathcal{H}(\mathbb{D})$ such that $Z(f) = \emptyset$.

We also have

(2.11) if M has the division property, then $\overline{M}$ has the division property.

To see this, consider $f \in \overline{M}$ and $\lambda \in Z(f)$. There exists $\varphi \in M$ such that $\varphi(\lambda) = 1$. Let $(f_n)_{n\geqslant 1}$ be a sequence of elements of M such that $\|f - f_n\|_\tau \underset{n\to\infty}{\longrightarrow} 0$, and set $g_n = f_n - f_n(\lambda) \cdot \varphi$, so that $\|f - g_n\|_\tau \underset{n\to\infty}{\longrightarrow} 0$. It follows then from (2.7) that $f_\lambda = \lim_{n\to\infty} (g_n)_\lambda \in \overline{M}$, which proves (2.11).

If M is a closed linear subspace of H_τ, the map $\pi(f) \to P_{M^\perp} \cdot f$, where $P_{M^\perp}$ denotes the orthogonal projection of H_τ onto $M^\perp$, defines an isometry from H_τ/M onto $M^\perp$. If $M \in \mathcal{D}_\tau$ we can thus consider $U_M(\lambda)$ as a linear operator acting on $M^\perp$, which is characterized by the formula

$$U_M(\lambda) \cdot P_{M^\perp} \cdot (T-\lambda) = P_{M^\perp}. \tag{2.12}$$

Also if M is z-invariant, identifying as above H_τ/M and $M^\perp$ we can consider T_M as a linear operator acting on $M^\perp$, and we obtain

$$T_M = (T^*|M^\perp)^*. \tag{2.13}$$

We have the following characterization of elements of $\mathcal{D}_\tau$.

Proposition 2.5. *Let $M \neq \{0\}$ be a closed linear subspace of H_τ. The following conditions are equivalent:*

(i) *M has the division property;*

(ii) *$Z(M) = \emptyset$, and $dim(M \ominus (M \cap zM)) = 1$;*

(iii) *For every $\lambda \in \mathbb{D}$, there exists a map*

$$U_M(\lambda) : M^\perp \to M^\perp$$

such that

$$U_M(\lambda) \cdot P_{M^\perp} \cdot (T - \lambda) = P_{M^\perp}.$$

If these conditions are satisfied, then the map $U_M(\lambda)$ defined by (iii) *is unique, $U_M(\lambda)$ is a bounded linear operator,*

$$\sigma(U_M(0)) \subset \overline{\mathbb{D}}$$

and

$$U_M(\lambda) = U_M(0) \cdot [1 - \lambda U_M(0)]^{-1}$$

for $\lambda \in \mathbb{D}$.

If M is z-invariant, conditions (i), (ii), (iii) *are equivalent to:*

(iv) *$\sigma(T^*_{|M^\perp}) \subset \mathbb{T}$,*

and $U_M(\lambda) = (T_M - \lambda)^{-1}(\lambda \in \mathbb{D})$.

Proof. The proposition follows immediately from Lemma 2.1, Lemma 2.2, Corollary 2.3 and formulae (2.12) and (2.13). ∎

The lattice $\mathcal{D}_\tau$ is always very rich. For example $\mathbb{C}f \in \mathcal{D}_\tau$ for every $f \in H_\tau$ such that $Z(f) = \emptyset$. Also if $M \in \operatorname{Lat} R$, and if the map $f \to gf$ is continuous from M into H_τ for some $g \in \mathcal{H}(\mathbb{D})$ such that $Z(g)$ is empty, then $[gM]^- \in \mathcal{D}_\tau$. This construction provides all closed subspaces of the Hardy space $H^2 = H^2(\mathbb{D})$ having the division property. In fact, it follows from the work of Hitt and Sarason ([28] and [36]; see also [41]) that these subspaces of H^2 have the form

$$M = U \cdot F \cdot N^\perp \tag{2.14}$$

where U is a singular inner function, N is z-invariant (so that $N = \{0\}$, $N = H^2$ or $N = VH^2$ where V is an inner function) and where F is an outer function satisfying

$$\|Ff\|_2 = \|f\|_2, \quad f \in N^\perp. \tag{2.15}$$

Clearly, the nontrivial z-invariant subspaces of H^2 having the division property are the subspaces of the form $U \cdot H^2$, where U is a singular inner function. Also if $\tau \in \mathcal{S}^+$, and if $f \in H_\tau$ is not z-cyclic and if $Z(f) = \emptyset$, then $\bigvee_{n \geqslant 0} z^n \cdot f$ has the division property since the space of polynomial functions has the division property. We do not know a precise description of the lattice $\mathcal{D}_\tau$ when τ is not the constant weight. We do not even know whether there always exist nontrivial z-invariant subspaces of H_τ having the division property. This question will be discussed at the end of the paper.

The following notions will play an important role in the next sections.

Definition 2.6. Let $M \neq \{0\}$ be a closed subspace of H_τ which has the division property. We set

$$\tau_M(n) = \|U_M^n(0) \cdot P_{M^\perp} \cdot 1\|_\tau, \quad n \geqslant 1,$$

and

$$\tau_{[M]}(n) = \|U_M^n(0)\|, \quad n \geqslant 1.$$

We now wish to obtain some information about the growth of the sequence $(\tau_M(n))_{n\geqslant 0}$ when $M \in \mathcal{D}_\tau \setminus \{0\}$. Set, for $0 \leqslant r < 1$

$$K_{1,\tau}(r) = \sum_{n=0}^{\infty} \overline{\tau}(n+1) \cdot r^n \tag{2.16}$$

$$K_{2,\tau}(r) = \Big(\sum_{n=0}^{\infty} \tau^{-2}(n) \cdot r^{2n}\Big)^{1/2}. \tag{2.17}$$

For $p \geqslant 1$, denote by $\mathfrak{A}_{p,\tau}$ the set of functions $\varphi \in \mathcal{H}(\mathbb{D})$ which can be written as a quotient $\varphi = f/g$, where $f, g \in \mathcal{H}(\mathbb{D})$ satisfy the following conditions

$$|g(0)| \geqslant \frac{1}{p}, \quad |f(z)| \leqslant K_{1,\tau}(|z|), \quad |g(z)| \leqslant K_{2,\tau}(|z|), \quad z \in \mathbb{D}. \tag{2.18}$$

Now set, for $p \geqslant 1$, $0 \leqslant r < 1$

$$L_\tau^{(p)}(r) = \sup\{|\varphi(z)| : \varphi \in \mathfrak{A}_{p,\tau}, |z| = r\} \tag{2.19}$$

and, for $n \geqslant 0$,

$$\tau^{(p)}(n) = \inf_{0<r<1} r^{-n} \cdot L_\tau^{(p)}(r). \tag{2.20}$$

Proposition 2.7. *Let $M \neq \{0\}$ be a closed linear subspace of H_τ. If M has the division property, then there exists $p \geqslant 1$ such that*

$$\tau_M(n+1) \leqslant \tau^{(p)}(n), \quad n \geqslant 0.$$

Proof. Let $\varphi \in M$ such that $\varphi(0) \neq 0, \|\varphi\|_\tau = 1$. There exists $p \geqslant 1$ such that $|\varphi(0)| \geqslant \frac{1}{p}$. Using (2.12), we obtain for $\lambda \in \mathbb{D}$ that $\varphi(\lambda) \cdot U_M(\lambda) \cdot P_{M^\perp} \cdot 1 = U_M(\lambda) \cdot P_{M^\perp}[\varphi(\lambda) \cdot 1 - \varphi] = -U_M(\lambda) \cdot P_{M^\perp} \cdot [(z-\lambda) \cdot \varphi_\lambda] = -P_{M^\perp} \cdot \varphi_\lambda$. It follows from (2.7) and (2.16) that $|P_{M^\perp} \cdot \varphi_\lambda\|_\tau \leqslant \|\varphi_\lambda\|_\tau \leqslant K_{1,\tau}(|\lambda|)$. Also

$$|\varphi(\lambda)| \leqslant \Big[\sum_{n=0}^{\infty} |\widehat{\varphi}(n)|^2 \tau^2(n)\Big]^{1/2} \cdot \Big[\sum_{n=0}^{\infty} |\lambda|^{2n} \tau^{-2}(n)\Big]^{1/2} \leqslant K_{2,\tau}(|\lambda|).$$

Let $g \in M^\perp$ such that $\|g\|_\tau = 1$. The analytic function $\lambda \to \langle U_M(\lambda) \cdot P_{M^\perp} \cdot 1, g\rangle$ belongs to $\mathfrak{A}_{p,\tau}$, and we obtain

$$\|U_M(\lambda) \cdot P_{M^\perp} \cdot 1\|_\tau \leqslant L_\tau^{(p)}(|\lambda|), \quad \lambda \in \mathbb{D}. \tag{2.21}$$

Since $U_M(\lambda) = U_M(0)[1 - \lambda U_M(0)]^{-1} = \sum\limits_{n=0}^{\infty} \lambda^n \cdot U_M^{n+1}(0)$, the result follows from the vector-valued version of Cauchy's inequalities. ∎

Remark 2.8. (i) Let $M \in \mathcal{D}_\tau \setminus \{0\}$. We have, for $n \geqslant 0$, $f \in H_\tau$,

$$\begin{aligned} U_M^n(0) \cdot P_{M^\perp} \cdot f &= \sum_{p=0}^{\infty} \widehat{f}(p) \cdot U_M^n(0) \cdot P_{M^\perp} \cdot z^p \\ &= \sum_{p=0}^{n-1} \widehat{f}(p) \cdot U_M^{n-p}(0) \cdot P_{M^\perp} \cdot 1 + \sum_{p=n}^{\infty} \widehat{f}(p) \cdot P_{M^\perp} \cdot z^{p-n} \\ &= \sum_{p=0}^{n-1} \widehat{f}(p) \cdot U_M^{n-p}(0) \cdot P_{M^\perp} \cdot 1 + P_{M^\perp} \cdot R^n \cdot f, \end{aligned}$$

where R is the backward shift introduced in 2.6. Since $\|R^n\| = \overline{\tau}(n+1)$, we obtain

$$\|U_M^n(0)\| \leqslant \overline{\tau}(n+1) + \Big(\sum_{p=0}^{n-1} \frac{\tau_M^2(n-p)}{\tau^2(p)}\Big)^{1/2} \tag{2.22}$$

and so Proposition 2.7 can be applied to obtain estimates on the growth of the sequence

$$(\tau_{[M]}(n))_{n\geqslant 0} = (\|U_M^n(0)\|_{n\geqslant 0}).$$

(ii) Let $M \in \mathcal{D}_\tau \setminus \{0\}$, $\varphi \in M \setminus \{0\}$, $\lambda \in \mathbb{D}$. As in the proof of Proposition 2.7, we see that we have

$$\varphi(\lambda) \cdot U_M(0) \cdot [1 - \lambda U_M(0)]^{-1} \cdot P_{M^\perp} \cdot 1 = -P_{M^\perp} \cdot \varphi_\lambda. \tag{2.23}$$

Hence the vector-valued analytic function $\theta_M : \lambda \to -\frac{P_{M^\perp} \cdot \varphi_\lambda}{\varphi(\lambda)}$ does not depend on the choice of $\varphi \in M \setminus \{0\}$ and we have

$$U_M^{n+1}(0) \cdot P_{M^\perp} \cdot 1 = \frac{1}{n!} \theta_M^{(n)}(0), \quad n \geqslant 0. \tag{2.24}$$

(iii) We have $\tau_M(n) = 0$ for every $n \geqslant 1$, or, equivalently, $\tau_M(1) = 0$, if and only if $M \in \operatorname{Lat} R$. To see this, assume that M is R-invariant, and let $\varphi \in M$ such that $\varphi(0) = 1$. Then

$$\begin{aligned} \tau_M(n) &= \|U_M^n(0) \cdot P_{M^\perp} \cdot (1-\varphi)\|_\tau = \|U_M^n(0) \cdot P_{M^\perp} \cdot zR\varphi\|_\tau \\ &= \|U_M^{n-1}(0) \cdot P_{M^\perp} \cdot R\varphi\|_\tau = 0 \qquad \text{for } n \geqslant 1. \end{aligned}$$

Conversely, if $\tau_M(1) = 0$, let $f \in H_\tau$. Then

$$U_M(0) \cdot P_{M^\perp} \cdot f = \widehat{f}(0) \cdot U_M(0) \cdot P_{M^\perp} \cdot 1 + P_{M^\perp} \cdot Rf$$

and we have

$$U_M(0) \cdot P_{M^\perp} = P_{M^\perp} \cdot R. \tag{2.25}$$

This shows that M is R-invariant. Also $U_M(0) = (R^*|M^\perp)^*$ is the compression of R to $M^\perp$.

(iv) For $p \geqslant 1$ denote by $\mathfrak{P}_{p,\tau}$ the set of functions $\psi \in \mathcal{H}(\mathbb{D})$ which can be written as a quotient $\psi = g/h$ where $g, h \in \mathcal{H}(\mathbb{D})$ satisfy the following properties

$$|h(0)| \geqslant \frac{1}{p}, \quad |h(\lambda)| \leqslant K_{2,\tau}(|\lambda|), \quad |g(\lambda)| \leqslant 2K_{1,\tau} \cdot K_{2,\tau}(|\lambda|), \quad \lambda \in \mathbb{D}. \tag{2.26}$$

Let

$$L_\tau^{[p]}(r) = \sup\{|\psi(z)| : \psi \in \mathfrak{P}_{p,\tau}, |z| = r\}, \quad 0 \leqslant r < 1 \tag{2.27}$$

$$\tau^{[p]}(n) = \inf_{0<r<1} r^{-n} \cdot L_\tau^{[p]}(r), \quad n \geqslant 0. \tag{2.28}$$

We have, for some $p \geqslant 1$

$$\tau_{[M]}(n+1) \leqslant \tau^{[p]}(n), \quad n \geqslant 0. \tag{2.29}$$

To see this, consider $\varphi \in M$ such that $|\varphi(0)| \neq 0$, $\|\varphi\|_\tau = 1$ and let $f \in M^\perp$ such that $\|f\|_\tau = 1$.

We have for $\lambda \in \mathbb{D}$

$$\begin{aligned}|\varphi(\lambda)| \cdot \|U_M(\lambda) \cdot f\|_\tau &= \|U_M(\lambda) \cdot P_{M^\perp} \cdot [\varphi(\lambda) \cdot f - f(\lambda) \cdot \varphi]\|_\tau \\ &= \|P_{M^\perp} \cdot [\varphi(\lambda) \cdot f - f(\lambda) \cdot \varphi]_\lambda\|_\tau \\ &\leqslant K_{1,\tau}(|\lambda|)[|f(\lambda)| + |\varphi(\lambda)|] \leqslant 2K_{1,\tau}(|\lambda|) \cdot K_{2,\tau}(|\lambda|).\end{aligned}$$

Hence

$$|\varphi(\lambda)| \cdot \|U_M(\lambda)\|_\tau \leqslant 2K_{1,\tau}(|\lambda|) \cdot K_{2,\tau}(|\lambda|).$$

If $p \cdot |\varphi(0)| \leqslant 1$, $|U_M(\lambda)\| \leqslant L_\tau^{[p]}(|\lambda|)$, $\lambda \in \mathbb{D}$.

Inequality (2.29) follows then from Cauchy's inequalities.

We will give in Section 4 some estimates on the weights $(\tau^{(p)}(n))_{n\geqslant 1}$ and $(\tau^{[p]}(n))_{n\geqslant 1}$ (defined in (2.20), respectively (2.28)) in concrete cases, using known results concerning the growth of quotients of analytic functions.

3. ANALYTIC LEFT-INVARIANT SUBSPACES OF WEIGHTED HILBERT SPACES SEQUENCES

We will denote by $\mathcal{S}$ the class of weights $\omega : \mathbb{Z} \to (0, \infty)$ satisfying the two following conditions

$$0 < \inf_{p\in\mathbb{Z}} \frac{\omega(n+p)}{\omega(p)} \leqslant \sup_{p\in\mathbb{Z}} \frac{\omega(n+p)}{\omega(p)} < +\infty \tag{3.1}$$

$$\omega_+ \in \mathcal{S}^+, \quad \text{where } \omega_+ = \omega_{|\mathbb{Z}^+}. \tag{3.2}$$

Throughout this section we will denote by ω an element of $\mathcal{S}$. Let

$$\ell_\omega = \ell^2_\omega(\mathbb{Z}) := \left\{u = (u_n)_{n\in\mathbb{Z}} : \|u\|_\omega := \Big[\sum_{n\in\mathbb{Z}} |u_n|^2 \cdot \omega^2(n)\Big]^{1/2} < +\infty\right\}. \tag{3.3}$$

The usual bilateral shift on ℓ_ω is the bounded invertible operator defined by the formula

$$S \cdot u = (u_{n-1})_{n\in\mathbb{Z}}, \quad u = (u_n)_{n\in\mathbb{Z}} \in \ell_\omega. \tag{3.4}$$

Let

$$\ell^+_\omega = \{(u_n)_{n\in\mathbb{Z}} \in \ell_\omega : u_n = 0, n < 0\}, \tag{3.5}$$

$$\ell^-_\omega = \{(u_n)_{n\in\mathbb{Z}} \in \ell_\omega : u_n = 0, n \geqslant 0\}, \tag{3.6}$$

$$S^+ = S|\ell^+_\omega, \tag{3.7}$$

$$e_p = (\delta_{p,n})_{n\in\mathbb{Z}} \tag{3.8}$$

where we denote by $\delta_{p,n}$ the Kronecker symbol.

Identifying ℓ_ω^+ to

$$\ell^2_{\omega_+}(\mathbb{Z}) := \Big\{u = (u_n)_{n\geqslant 0} : \|u\|_{\omega_+} := \Big[\sum_{n=0}^{\infty} |u_n|^2\omega^2(n)\Big]^{1/2} < +\infty\Big\}$$

in the obvious way, we see that the Fourier transform $\mathcal{F} : f \to (\widehat{f}(n))_{n\geqslant 0}$ is an isometry from the weighted Hardy space H_{ω_+} onto ℓ_ω^+, which defines a unitary equivalence between the shift T given by (2.6) and S^+.

For $u \in \ell_\omega^+$ set

$$\overset{\vee}{u} = \mathcal{F}^{-1}(u) \tag{3.9}$$

and for $\lambda \in \mathbb{D}$, define $u_\lambda \in \ell_\omega^+$ by the formula

$$u_\lambda = \mathcal{F}[\,[\mathcal{F}^{-1}(u)]_\lambda] \tag{3.10}$$

so that

$$(S-\lambda)\cdot u_\lambda = u - \overset{\vee}{u}(\lambda)e_0. \tag{3.11}$$

Definition 3.1. Let F be a linear subspace of ℓ_ω^+. We will say that F has the *division property* if $u_\lambda \in F$ for every $u \in F$ and every $\lambda \in \mathbb{D}$ such that $\overset{\vee}{u}(\lambda) = 0$.

Clearly, F has the division property iff $\overset{\vee}{F} := \mathcal{F}^{-1}(F)$ has the division property in H_{ω_+} in the sense of Definition 2.4.

If F is a closed subspace of ℓ_ω^+, denote by $P^+_{F^\perp}$ the orthogonal projection of ℓ_ω^+ onto $\ell_\omega^+ \cap F^\perp$. It follows immediately from Proposition 2.5 that nonzero closed subspaces of ℓ_ω^+ having the division property are characterized by the following equivalent conditions

$$\dim[F \ominus (F \cap S\cdot F)] = 1, \quad \text{and} \quad Z(\overset{\vee}{F}) = \emptyset. \tag{3.12}$$

For every $\lambda \in \mathbb{D}$, there exists then a map $V_F(\lambda) : F^\perp \cap \ell_\omega^+ \to F^\perp \cap \ell_\omega^+$ satisfying

$$V_F(\lambda)\cdot P^+_{F^\perp}\cdot(S^+ - \lambda) = P^+_{F^\perp}. \tag{3.13}$$

The map $V_F(\lambda)$ defined by (3.13) is then unique, linear and bounded, and we have

$$V_F(\lambda) = V_F(0)\cdot[1-\lambda V_F(0)]^{-1}, \quad \lambda \in \mathbb{D}. \tag{3.14}$$

If F has the division property, set $\omega_F(n) = (\omega_+)_{\overset{\vee}{F}}(n)$ for $n \geqslant 1$ (see Definition 2.6).

We obtain

$$\omega_F(n) = \|V_F^n(0)\cdot P^+_{F^\perp}\cdot e_0\|_\omega, \quad n \geqslant 1. \tag{3.15}$$

Let L be a closed subspace of ℓ_ω. We will say that L is *right-invariant* (respectively *left-invariant*, respectively *translation invariant*) if $S\cdot L \subset L$ (respectively $S^{-1}\cdot L \subset L$, respectively $S\cdot L = L$).

If L is left-invariant, we define the *compression* S_L^{-1} of S^{-1} to $L^\perp$ by the formula

$$S_L^{-1}\cdot P_{L^\perp} = P_{L^\perp}\cdot S^{-1}. \tag{3.16}$$

Notice that in this situation we have also

$$S_L^{-1} = ((S^{-1})^*|L^\perp)^*. \tag{3.17}$$

The following easy result was the motivation to introduce the division property.

PROPOSITION 3.2. *Let $\omega \in \mathcal{S}$, and assume that*

$$\lim_{n\to-\infty}\left[\sup_{p\geqslant 0}\frac{\omega(n+p)}{\omega(p)}\right]^{1/n} = 1.$$

Then $L^+ := L\cap\ell_\omega^+$ has the division property for every left-invariant subspace L of ℓ_ω. Also if $L^+ \neq \{0\}$ then

$$|v_{-n}|\cdot\omega^2(-n) \leqslant \|v\|_\omega\cdot\omega_{L^+}(n), \quad n\geqslant 1$$

for every $v\in L^\perp$.

Proof. Let $v\in L^+$ and $\lambda\in\mathbb{D}$ such that $\check{v}(\lambda)=0$. Since

$$\|S^n|\ell_\omega^+\| = \sup_{p\geqslant 0}\frac{\omega(n+p)}{\omega(p)} \quad \text{for } n\in\mathbb{Z},$$

the series $\sum_{n=0}^{\infty}\lambda^n\cdot S^{-n-1}\cdot v$ is convergent, and $w=\sum_{n=0}^{\infty}\lambda^n\cdot S^{-n-1}\cdot v\in L$. Hence $(S-\lambda)\cdot w = v = v-\check{v}(\lambda)\cdot e_0 = (S-\lambda)\cdot v_\lambda$, and so $v_\lambda = w\in L\cap\ell_\omega^+ = L^+$.

Now set $P=P_{L^\perp}$ and denote by P^+ the orthogonal projection of ℓ_ω^+ onto $\ell_\omega^+\ominus L^+$, so that $P\cdot P^+\cdot v = P\cdot v$ for $v\in\ell_\omega^+$. Denote by $\widetilde{P}$ the restriction of P to $\ell_\omega^+\ominus L^+$, set $V=V_{L^+}(0)$ and let $u\in L^+$ such that $\check{u}(0)=1$. Let $v\in\ell_\omega^+$. We have

$$\begin{aligned}P\cdot V\cdot P^+\cdot v &= P\cdot V\cdot P^+\cdot S\cdot S^{-1}\cdot[v-\check{v}(0)\cdot u] = P\cdot P^+\cdot S^{-1}\cdot[v-\check{v}(0)\cdot u]\\ &= P\cdot S^{-1}\cdot[v-\check{v}(0)\cdot u] = P\cdot S^{-1}\cdot v\\ &= S_L^{-1}\cdot P\cdot v = S_L^{-1}\cdot P\cdot P^+\cdot v.\end{aligned}$$

Hence $\widetilde{P}\cdot V = S_L^{-1}\cdot\widetilde{P}$ and we obtain

$$S_L^{-n}\cdot\widetilde{P} = \widetilde{P}\cdot V_{L^+}^n(0), \quad n\geqslant 0. \tag{3.18}$$

Now let $v\in L^\perp$. We have, for $n\geqslant 1$

$$\begin{aligned}|v_{-n}|\cdot\omega^2(-n) &= |\langle e_{-n}, v\rangle| = |\langle e_{-n}, P\cdot v\rangle| = |\langle P\cdot e_{-n}, v\rangle|\\ &= |\langle S_L^{-n}\cdot P\cdot P^+\cdot e_0, v\rangle| = |\langle P\cdot V^n\cdot P^+\cdot e_0, v\rangle|\\ &\leqslant \|P\|\cdot\|v\|_\omega\cdot\|V^n(0)\cdot P^+\cdot e_0\|_\omega = \|v\|_\omega\cdot\omega_{L^+}(n). \quad ∎\end{aligned}$$

Notice that since $L\cap\ell_\omega^+ = L^+$, $\widetilde{P}$ is one-to-one. Also for $n\geqslant 1$ we have, with the notation above, $P\cdot e_{-n} = S_L^{-n}\cdot P\cdot e_0 = P\cdot V_{L^+}^n(0)\cdot P^+\cdot e_0\in P\cdot\ell_\omega^+$ and so $\widetilde{P}\cdot[\ell_\omega^+\ominus L^+] = P\cdot\ell_\omega^+$ is dense in $L^\perp$, so that $\ell_\omega^+ + L$ is dense in ℓ_ω. If, further, $\ell_\omega^+ + L = \ell_\omega$, then $\widetilde{P}$ is also onto and we obtain

$$\|S_L^{-n}\| \leqslant \|\widetilde{P}\|\cdot\|\widetilde{P}^{-1}\|\cdot\|V_{L^+}^n(0)\|, \quad n\geqslant 0. \tag{3.19}$$

Notice also that if we only assume that

$$0 < \inf_{p\in\mathbb{Z}}\frac{\omega(p+1)}{\omega(p)} \leqslant \sup_{p\in\mathbb{Z}}\frac{\omega(p+1)}{\omega(p)} < +\infty,$$

then Proposition 3.2 remains true in a weaker sense: in this general situation L^+ has the property that $S^{-1} \cdot u \in L^+$ for every $u \in L^+$ such that $u_0 = \check{u}(0) = 0$ (such subspaces are called by Sarason-Hitt "weakly invariant for the backward shift"). The map $V_{L^+}(0)$ can be defined in the same way and the inequality of Proposition 3.2 holds in this general situation if $L^+ \neq \{0\}$.

Of course, we may have $L \cap \ell_\omega^+ = \{0\}$ if L is a left-invariant subspace of ℓ_ω (consider the case where $L = \ell_\omega^-$), and even $S^k \cdot L \cap \ell_\omega^+ = \{0\}$ for every $k \geqslant 0$. Also we may have $\bigvee_{n \leqslant 0} S^n \cdot F = \ell_\omega$ if F is a closed subspace of ℓ_ω^+ which has the division property. For example if $\omega(n) = 1$ for $n \in \mathbb{Z}$ this is the case for all nontrivial translation invariant subspaces L of $\ell_\omega = \ell^2$ and for all subspaces F of $(\ell^2)^+ \simeq H^2(\mathbb{D})$ of the form $F = \bigvee_{n \geqslant 0} S^n \cdot \widehat{U}$ where U is a non constant singular inner function, by Wiener's classical characterization of translation invariant subspaces of ℓ^2 ([40] or [27], Chapter 1).

The following result, which is the main result of the paper, gives a sufficient condition on F and ω which guarantees that $\bigvee_{n \leqslant 0} S^n \cdot F$ is a proper subspace of ℓ_ω if F is a nontrivial closed subspace of ℓ_ω^+ having the division property.

Theorem 3.3. *Let $\omega \in \mathcal{S}$, and let $F \neq \{0\}$ be a closed subspace of ℓ_ω^+ which has the division property. If $\sum_{n=1}^{\infty} \frac{\omega_F^2(n)}{\omega^2(-n)} < +\infty$, then $\Big(\bigvee_{n \leqslant 0} S^n \cdot F\Big) \cap \ell_\omega^+ = F$, and $\ell_\omega = \ell_\omega^+ + \Big(\bigvee_{n \leqslant 0} S^n \cdot F\Big)$.*

Proof. Set $G = \bigvee_{n \leqslant 0} S^n \cdot F$, $V = V_F(0)$, $P^+ = P^+_{F^\perp}$ where we denote as above by $P^+_{F^\perp}$ the orthogonal projection of ℓ_ω^+ onto $F^\perp \cap \ell_\omega^+$. Also denote by Q^+ (respectively Q^-) the orthogonal projection of ℓ_ω onto ℓ_ω^+ (respectively ℓ_ω^-) and set

$$v^+ = Q^+ \cdot v, \quad v^- = Q^- \cdot v, \quad v \in \ell_\omega. \tag{3.20}$$

Set

$$K = \Big[\sum_{n=1}^{\infty} \frac{\omega_F^2(n)}{\omega^2(-n)}\Big]^{1/2}$$

and let $v \in \ell_\omega^-$.

We have

$$\sum_{n<0} \|v_n \cdot V^{-n} \cdot P^+ \cdot e_0\|_\omega \leqslant \Big[\sum_{n<0} |v_n|^2 \omega^2(n)\Big]^{1/2} \Big[\sum_{n=1}^{\infty} \frac{\omega_F^2(n)}{\omega^2(-n)}\Big]^{1/2} = K \cdot \|v\|_\omega.$$

Now set

$$\Delta \cdot v = \sum_{n<0} v_n \cdot V^{-n} \cdot P^+ \cdot e_0, \quad v \in \ell_\omega^-. \tag{3.21}$$

We first consider the (not necessarily orthogonal) projection $Q^- - \Delta \cdot Q^-$ and we want to show that its range is contained in G.

Let $u \in F$ be such that $u_0 = \overset{\vee}{u}(0) = 1$, and set $\varphi = \overset{\vee}{u}$, $M = \overset{\vee}{L}$. Since $H_{\omega^+} = \mathbb{C} \cdot \varphi + z \cdot H_{\omega^+}$, we see by an immediate induction that there exists a sequence $(a_p)_{p \geqslant 0}$ of complex numbers, with $a_0 = 1$, and a sequence $(\psi_p)_{p \geqslant 1}$ of elements of H_{ω^+} satisfying

$$1 = \Big(\sum_{k=0}^{p-1} a_k \cdot z^k\Big) \cdot \varphi + z^p \cdot \psi_p, \quad p \geqslant 1. \tag{3.22}$$

Set $w_p = \widehat{\psi}_p$, so that $w_p \in \ell^+_\omega$. We obtain

$$e_0 = \Big(\sum_{k=0}^{p-1} a_k S^k\Big) \cdot u + S^p \cdot w_p, \quad p \geqslant 1. \tag{3.23}$$

It follows from (3.13) that $V^p \cdot P^+ \cdot S^p \cdot w_p = P^+ \cdot w_p$ and that $V^p \cdot P^+ \cdot S^k \cdot u = V^{p-k} \cdot P^+ \cdot u = 0$, for $0 \leqslant k \leqslant p-1$. We obtain

$$\Delta \cdot e_{-p} = V^p \cdot P^+ \cdot e_0 = P^+ \cdot w_p, \quad p \geqslant 1. \tag{3.24}$$

Also

$$\begin{aligned} e_{-p} = S^{-p} \cdot e_0 &= w_p + \Big(\sum_{k=0}^{p-1} a_k \cdot S^{k-p}\Big) \cdot u \\ &= P^+ w_p + (w_p - P^+ \cdot w_p) + \Big(\sum_{k=0}^{p-1} a_k \cdot S^{k-p}\Big) \cdot u. \end{aligned}$$

Since $w_p - P^+ w_p \in F \subset G$, we see that $e_{-p} - \Delta e_{-p} \in G$ for $p \geqslant 1$. By continuity we obtain

$$(Q^- - \Delta \cdot Q^-) \cdot \ell_\omega \subset G. \tag{3.25}$$

Hence $v = (v^- - \Delta \cdot v^-) + (v^+ + \Delta \cdot v^-) \in G + \ell^+_\omega$ for every $v \in \ell_\omega$, which proves the second assertion of the theorem.

To prove the first assertion we consider the projection $1 - Q^- + \Delta \cdot Q^- = Q^+ + \Delta \cdot Q^-$ and we will show that it maps G into F.

Consider u as above, and for $p \geqslant 1$ suppose that $(Q^+ + \Delta \cdot Q^-) \cdot S^{-k} \cdot u \in F$, $1 \leqslant k \leqslant p-1$. Then, by (3.23) and (3.24),

$$\begin{aligned} (Q^+ + \Delta Q^-) S^{-p} u &= (Q^+ + \Delta Q^-)\Big(S^{-p} e_0 - w_p - \sum_{1 \leqslant k \leqslant p-1} a_k S^{k-p} u\Big) \\ &\in \Delta e_{-p} - w_p + F = P^+ w_p - w_p + F = F. \end{aligned}$$

It follows by induction that for every $u \in F$ with $u_0 = 1$ we have

$$(Q^+ + \Delta \cdot Q^-) \cdot S^{-k} \cdot u \in F, \quad k \geqslant 1. \tag{3.26}$$

Let $k > 0$. Since every element of $S^{-k} \cdot F \setminus F$ is equal to $S^{-p} \cdot u$ for some $p > 0$, $u \in F$ with $u_0 \neq 0$, relation (3.26) gives us $(Q^+ + \Delta Q^-)(S^{-k} \cdot F \setminus F) \subset F$. Finally, $(Q^+ + \Delta Q^-)F = Q^+ \cdot F = F$ and we obtain

$$(\Delta \cdot Q^- + Q^+) \cdot G \subset F. \tag{3.27}$$

Now let $v \in \Big(\bigvee_{n \leqslant 0} S^n \cdot F\Big) \cap \ell^+_\omega = G \cap \ell^+_\omega$. Then $Q^- \cdot v = 0$, $(\Delta \cdot Q^- + Q^+) \cdot v = v$ and so $v \in F$. Hence $\Big(\bigvee_{n \geqslant 0} S^n \cdot F\Big) \cap \ell^+_\omega = F$. ∎

Definition 3.4. Let $\omega \in \mathcal{S}$. A left-invariant subspace L of ℓ_ω is said to be *analytic* if $L^+ = L \cap \ell_\omega^+ \neq \{0\}$.

We now deduce from Proposition 2.7, Proposition 3.2 and Theorem 3.3 the following result:

Corollary 3.5. *Let $\omega \in \mathcal{S}$, and assume that*

$$\sum_{n=1}^{\infty} \left[\frac{\omega_+^{(p)}(n)}{\omega(-n)}\right]^2 < +\infty \qquad \text{for every } p \geqslant 1.$$

Then $F = \Big(\bigvee_{n\leqslant 0} S^n \cdot F\Big) \cap \ell_\omega^+$ and $\ell_\omega = \ell_\omega^+ + \bigvee_{n\leqslant 0} S^n \cdot F$ for every nonzero closed subspace F of ℓ_ω^+ having the division property. If, further

$$\lim_{n\to-\infty} \left[\sup_{p\geqslant 0} \frac{\omega(n+p)}{\omega(p)}\right]^{1/n} = 1,$$

then $L = \bigvee_{n\leqslant 0} S^n \cdot (L \cap \ell_\omega^+)$ for every analytic left-invariant subspace L of ℓ_ω.

Proof. The first assertion is immediate. Now assume that

$$\lim_{n\to-\infty} \left[\sup_{p\geqslant 0} \frac{\omega(n+p)}{\omega(p)}\right]^{1/n} = 1,$$

and let L be an analytic left-invariant subspace of ℓ_ω. Then $L^+ = L \cap \ell_\omega^+$ has the division property. Let $G = \bigvee_{n\leqslant 0} S^n \cdot L^+$. Then $G \subset L$, and $\ell_\omega = G + \ell_\omega^+$. Let $u \in L$. Then $u = v + w$, where $v \in G$, $w \in \ell_\omega^+$. Hence $w \in L \cap \ell_\omega^+ = L^+ \subset G$, and so $L = G$. ∎

Notice that, of course, $\bigvee_{n\leqslant 0} S^n \cdot F_1 = \bigvee_{n\leqslant -k} S^n \cdot F_2$ if $F_2 = S^k \cdot F_1$. Conversely, we have the following result:

Corollary 3.6. *Let $\omega \in \mathcal{S}$, and assume that*

$$\sum_{n=1}^{\infty} \left[\frac{\omega_+^{(p)}(n)}{\omega(-n)}\right]^2 < +\infty \qquad \text{for every } p \geqslant 1.$$

Let F_1 and F_2 be two closed subspaces of ℓ_ω^+ having the division property and let $k \geqslant 0$. Then $\bigvee_{n\leqslant 0} S^n \cdot F_1 = \bigvee_{n\leqslant -k} S^n \cdot F_2$ if and only if F_1 and F_2 satisfy one of the two following equivalent conditions:

(i) $F_2 = \operatorname{Span}\{S^p \cdot F_1\}_{0\leqslant p\leqslant k}$;

(ii) $F_1 = \{u \in \ell_\omega^+ : S^k \cdot u \in F_2\}$.

Proof. If (ii) holds, let $u \in S^p \cdot F_1$ with $0 \leqslant p \leqslant k$, and let $v = S^{-p} \cdot u$. Then $S^k \cdot v \in F_2$, and so $u = S^{p-k} \cdot S^k \cdot v \in F_2$, since F_2 has the division property. Hence $\operatorname{Span}\{S^p \cdot F_1\}_{0\leqslant p\leqslant k} \subset F_2$, and, in particular, $F_1 \subset F_2$. Conversely, let $v \in F_2$, and let $u \in F_1$ such that $u_0 = 1$. We see by induction that there exist $b_0, \dots, b_{k-1} \in \mathbb{C}$ and $w \in F_2$ satisfying

$$v = \sum_{p=0}^{k-1} b_p \cdot S^p \cdot u + S^k \cdot w. \tag{3.28}$$

Then $S^k \cdot w \in F_2$, so that $w \in F_1$, and $F_2 \subset \mathrm{Span}\{S^p \cdot F_1\}_{0 \leqslant p \leqslant k}$.

Now assume that (i) holds. Then $S^k \cdot u \in F_2$ for every $u \in F_1$. Conversely, if $u \in \ell_\omega^+$, and if $S^k \cdot u \in F_2$, then there exist $u_0, \dots, u_k \in F_1$ such that $S^k \cdot u = u_0 + Su_1 + \cdots + S^k \cdot u_k$. Hence $u_0 \in (S \cdot \ell_\omega^+) \cap F_1 = S \cdot F_1$, and $S^{k-1} \cdot u \in \mathrm{Span}\{S^p \cdot F_1\}_{0 \leqslant p \leqslant k-1}$.

By an immediate induction we see that $u \in F_1$, and so (i) and (ii) are equivalent.

Now assume that

$$\sum_{n=1}^{\infty} \left[\frac{\omega_+^{(p)}(n)}{\omega(-n)}\right]^2 < +\infty, \quad p \geqslant 1.$$

If $\bigvee_{n \leqslant 0} S^n \cdot F_1 = \bigvee_{n \leqslant -k} S^n \cdot F_2$, let $u \in F_1$. Then $S^k \cdot u \in \Big(\bigvee_{n \leqslant 0} S^n \cdot F_2\Big) \cap \ell_\omega^+ = F_2$. Conversely, if $u \in \ell_\omega^+$, and if $S^k \cdot u \in F_2$, then $u \in \Big(\bigvee_{n \leqslant 0} S^n \cdot F_1\Big) \cap \ell_\omega^+ = F_1$. Hence (ii) holds.

Conversely, (i) and (ii) imply that $\bigvee_{n \leqslant -k} S^n \cdot F_2 = \bigvee_{n \leqslant 0} S^n \cdot F_1$. ∎

We shall gives examples of weights satisfying the conditions of Corollary 3.5 in the next section. We now give a few comments concerning Theorem 3.3.

Remark 3.7. (i) The conclusion of Theorem 3.3 holds, without any restriction on the sequence $(\omega(-n))_{n \geqslant 1}$, when F is invariant for the backward shift $\widehat{R} : u \to S^{-1} \cdot (u - \check{u}(0) \cdot e_0)$ (in this situation we have $\omega_F(n) = 0$ for $n \geqslant 1$; see Remark 2.8). But in fact the result is trivial in this case. An immediate induction shows that $e_{-n} \in \bigvee_{p \leqslant 0} S^p \cdot F$ for $n \geqslant 1$, and so $\bigvee_{p \leqslant 0} S^n \cdot F = F \oplus \ell_\omega^-$. Hence $\Big(\bigvee_{n \leqslant 0} S^n \cdot F\Big) \cap \ell_\omega^+ = F$ and $\Big(\bigvee_{n \leqslant 0} S^n \cdot F\Big) + \ell_\omega^+ = \ell_\omega$. Notice also that in this situation we have $\Big(\bigvee_{n \leqslant 0} S^n \cdot F\Big)^\perp = F^\perp \cap (\ell_\omega^-)^\perp = F^\perp \cap \ell_\omega^+$.

(ii) Now consider a closed subspace $F \neq \{0\}$ of ℓ_ω^+ having the division property, and set again $G = \bigvee_{n \leqslant 0} S^n \cdot F$. Denote by P the orthogonal projection of ℓ_ω onto $G^\perp$ and denote by P^+ the orthogonal projection of ℓ_ω^+ onto $F^\perp \cap \ell_\omega^+$. One way to interpret the conclusion of Theorem 3.3 consists in saying that the natural map from ℓ_ω^+/F into ℓ_ω/G is a bijection or, equivalently, that $\widetilde{P} = P|\ell_\omega^+ \cap F^\perp$ is a bijection from $\ell_\omega^+ \cap F^\perp$ onto $G^\perp$. Now define the compression S_G^{-1} of S^{-1} to $G^\perp$ by (3.16). Using (3.18), we obtain

$$S_G^{-1} = \widetilde{P} \cdot V_F(0) \cdot \widetilde{P}^{-1}. \tag{3.29}$$

In other words, S_G^{-1}, which acts on $G^\perp$, is similar to $V_F(0)$, which acts on $F^\perp \cap \ell_\omega^+$, and, heuristically, all the information concerning S_G^{-1} is already given by $V_F(0)$.

(iii) The notation being as above, assume that the conditions of Theorem 3.3 are satisfied, and set $\rho = Q^+|G^\perp$ so that $\rho(G^\perp) \subset F^\perp \cap \ell_\omega^+$. Since $\ell_\omega = \ell_\omega^+ + G$, ρ is one-to-one. Now let $\Delta : \ell_\omega^- \to \ell_\omega^+ \cap F^\perp$ be the map defined by (3.21). By (3.27) we have$(1 + \Delta^*) \cdot (\ell_\omega^+ \cap F^\perp) = (Q^+ + \Delta \cdot Q^-)^*(\ell_\omega^+ \cap F^\perp) \subset G^\perp$ and, obviously, $\rho \cdot (1+\Delta^*)$ is the identity map on $\ell_\omega^+ \cap F$. Hence ρ is a bijection, and $\rho^{-1} = 1+\Delta^*$.

The map $\rho^{-1} : \ell_\omega^+ \cap F^\perp \to G^\perp$ can be described in a concrete way. Let $v \in \ell_\omega^+ \cap F^\perp$, and set $w = \rho^{-1} \cdot v$. Then $w_n = v_n$ for $n \geqslant 0$. Also, for $n < 0$, we have

$$\overline{w}_n \cdot \omega^2(n) = \langle e_n, w\rangle = \langle e_n, v + \Delta^* \cdot v\rangle = \langle \Delta \cdot e_n, v\rangle = \langle V_F^{-n}(0) \cdot P_{F^\perp}^+ \cdot e_0, v\rangle.$$

Using (2.24), we obtain for every $u \in F \setminus \{0\}$

$$\sum_{n=0}^{\infty} \lambda^n \overline{w_{-n-1}} \omega^2(-n-1) = -\frac{1}{\check{u}(\lambda)} \langle P^+ \cdot u_\lambda, v\rangle, \quad \lambda \in \mathbb{D}. \tag{3.30}$$

Hence $\overline{w_{-n-1}} \cdot \omega^2(-n-1)$ is for $n \geqslant 0$ the n^{th} Taylor coefficient at the origin of the analytic function $\lambda \to -\frac{1}{\check{u}(\lambda)} \langle P^+ \cdot u_\lambda, v\rangle$.

Notice that the map ρ^{-1} defined above and the map $\widetilde{P} = P|F^\perp \cap \ell_\omega^+$ coincide if and only if F is $\widehat{R}$-invariant. To see this assume that $\rho^{-1} = \widetilde{P}$.

Then for every $v \in \ell_\omega^+ \cap F^\perp$ we have $P \cdot v = v + \Delta^* \cdot v$, and so $P \cdot \Delta^* \cdot v = 0$ and $Q^- \cdot P \cdot v = \Delta^* \cdot v \in G$. Since $\widetilde{P} : \ell_\omega^+ \cap F^\perp \to G^\perp$ is onto, this means that $Q^- \cdot G^\perp \subset G$. Hence $\|Q^- \cdot w\|_\omega^2 = \langle Q^- \cdot w, w\rangle = 0$ for every $w \in G^\perp$, and so $G^\perp \subset \ell_\omega^+$ and $\ell_\omega^- \subset G$. In this case

$$\widehat{R} \cdot u = S^{-1} \cdot (u - \check{u}(0) \cdot e_0) = S^{-1} \cdot u - \check{u}(0) e_{-1} \in G \cap \ell_\omega^+ = F$$

for every $u \in F$, and so F is $\widehat{R}$-invariant.

Conversely, if F is $\widehat{R}$-invariant, then $\Delta = 0$ (see Remark 2.8). Hence $\Delta^* = 0$ and $\rho^{-1} = \widetilde{P}$ is the identity map on $\ell_\omega^+ \cap F^\perp = G^\perp$.

The results of this section concern a “bijective” correspondence between F and $\bigvee_{n \leqslant 0} S^n \cdot F$, where F is a closed subspace of ℓ_ω^+ having the division property. In order to obtain positive results concerning existence of translation invariant subspaces, we just need to know when $\bigvee_{n \leqslant 0} S^n \cdot F$ is a proper subspace of ℓ_ω. In order to state such a condition, it is natural to introduce the “dual weight” of $\omega \in \mathcal{S}$, defined by

$$\omega^*(n) = \omega(-n-1)^{-1}, \quad n \in \mathbb{Z}. \tag{3.31}$$

We can identify ℓ_{ω^*} to the dual of ℓ_ω by using the formula

$$(u, v) = \sum_{n \in \mathbb{Z}} u_n \cdot v_{-n-1}, \quad u = (u_n)_{n \in \mathbb{Z}} \in \ell_\omega, \quad v = (v_n)_{n \in \mathbb{Z}} \in \ell_{\omega^*}. \tag{3.32}$$

For $u \in \ell_\omega, v \in \ell_{\omega^*}$, define the sequence $u * v$ by the usual convolution formulae on $\mathbb{Z}$. We obtain

$$(u * v)_n = (S^{-n-1}u, v), \quad n \in \mathbb{Z},\, u \in \ell_\omega,\, v \in \ell_{\omega^*}. \tag{3.33}$$

We thus see that $(S^n \cdot u, v) = 0$ for $n < 0$ iff $(u * v)_n = 0$ for $n \geqslant 0$, and that $(S^n \cdot u, v) = 0$ for $n \in \mathbb{Z}$ iff $u * v = 0$. Now let $\omega \in \mathcal{S}$. Set, for $w \in \ell_\omega$

$$w^* = (\langle e_{-n-1}, w\rangle)_{n\in\mathbb{Z}} = (\overline{w}_{-n-1} \cdot \omega^2(-n-1))_{n\in\mathbb{Z}}. \tag{3.34}$$

The map $w \to w^*$ is clearly an isometry from ℓ_ω onto ℓ_{ω^*}, and $\langle v, w\rangle = (v, w^*)$ for $v \in \ell_\omega$, $w \in \ell_\omega$. Now let $w \in \ell^+_\omega$. Then $w^* \in \ell^-_{\omega^*}$. For $u \in \ell^+_\omega$, $\lambda \in \mathbb{D}$ we have

$$\begin{aligned}\langle u_\lambda, w\rangle &= \sum_{n=1}^{\infty} u_n \Big(\sum_{k=0}^{n-1} \lambda^k \langle e_{n-1-k}, w\rangle\Big) = \sum_{n=1}^{\infty} u_n \Big(\sum_{k=0}^{n-1} \lambda^k w^*_{k-n}\Big)\\ &= \sum_{k=0}^{\infty} \lambda^k \Big(\sum_{n=k+1}^{\infty} u_n \cdot w^*_{k-n}\Big).\end{aligned}$$

We obtain

$$\langle u_\lambda, w\rangle = \sum_{k=0}^{\infty} \lambda^k (u * w^*)_k, \quad \lambda \in \mathbb{D},\, u \in \ell^+_\omega,\, w \in \ell^+_\omega. \tag{3.35}$$

Recall that if $F \neq \{0\}$ is a closed subspace of ℓ^+_ω having the division property, and if $w \in \ell^+_\omega \ominus F$, then the function $\lambda \to -\frac{\langle u_\lambda, w\rangle}{\check{u}(\lambda)}$ extends analytically to $\mathbb{D}$ for $u \in F \setminus \{0\}$ and does not depend on the choice of u.

PROPOSITION 3.8. *Let $\omega \in \mathcal{S}$, and let $F \neq \{0\}$ be a closed subspace of ℓ^+_ω having the division property. Then the following conditions are equivalent:*

(i) $\bigvee_{n\leqslant 0} S^n \cdot F$ *is a proper subspace of* ℓ_ω*;*

(ii) *there exists $w \in \ell^+_\omega \ominus F$ such that the function $\lambda \to \frac{\langle u_\lambda, w\rangle}{\check{u}(\lambda)}$, $\lambda \in \mathbb{D}$ belongs to $H_{\omega^*_+}$ for $u \in F \setminus \{0\}$.*

Proof. Denote again by Q^+ (respectively Q^-) the orthogonal projection of ℓ_ω onto ℓ^+_ω (respectively ℓ^-_ω). Assume that $G = \bigvee_{n\leqslant 0} S^n \cdot F$ is a proper subspace of ℓ_ω. Then G does not contain ℓ^+_ω, and so there exists $v \in G^\perp$ such that $Q^+(v) \neq 0$. Let $w = Q^+(v)$, so that $w \in \ell^+_\omega \ominus F$, and let $u \in F \setminus \{0\}$. Let $s = v^* - w^*$, so that $s \in \ell^+_{\omega^*}$. Since $(u * v^*)_n = 0$ for $n \geqslant 0$, we have for $\lambda \in \mathbb{D}$, by (3.35)

$$\langle u_\lambda, w\rangle = -\sum_{n=0}^{\infty} \lambda^n (u * s)_n = -\check{u}(\lambda) \cdot \Big(\sum_{n=0}^{\infty} \lambda^n \cdot s_n\Big).$$

Since $w \neq 0$, (ii) is satisfied.

Now assume that (ii) is satisfied for some nonzero $w \in \ell^+_\omega \ominus F$. Let $u \in F \setminus \{0\}$ and denote by φ the function $\lambda \to -\frac{\langle u_\lambda, w\rangle}{\check{u}(\lambda)}$. Let $s = -\widehat{\varphi}$. Then $s \in \ell^+_{\omega^*}$, and so $w^* - s \in \ell_{\omega^*}$. By (3.35), we have, for $\lambda \in \mathbb{D}$

$$\sum_{n=0}^{\infty} \lambda^n (u * w^*)_n = \langle u_\lambda, w\rangle = -\check{u}(\lambda) \cdot \varphi(\lambda) = \sum_{n=0}^{\infty} \lambda^n (u * s)_n.$$

Hence $s - w^* \perp S^n \cdot u$ for $n < 0$. Since $s \in \ell^+_{\omega^*}$, $(u, s) = 0$.

Also $(u, w^*) = \langle u, w \rangle = 0$, and so $(v, s - w^*) = 0$ for every $v \in \bigvee_{n \leqslant 0} S^n \cdot F$. ∎

We conclude this section with an example. Assume that $\omega(n) = 1$ for $n \geqslant 0$, let U be a singular inner function and set $F = \widehat{U \cdot H^2}$, where $H^2 = H^2(\mathbb{D})$ is the usual Hardy space. Let $u = \widehat{U}$, $w = \widehat{R \cdot U}$ where $R = S^*$ is the backward shift on H^2.

For $\lambda \in \mathbb{D}$ we have

$$\langle u_\lambda, w \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}t} \cdot \overline{U}(\mathrm{e}^{\mathrm{i}t}) \cdot \frac{U(\mathrm{e}^{\mathrm{i}t}) - U(\lambda)}{\mathrm{e}^{\mathrm{i}t} - \lambda} \, \mathrm{d}t = 1 - \overline{U}(0) \cdot U(\lambda).$$

Hence $\frac{\langle u_\lambda, w \rangle}{\check{u}(\lambda)} = U^{-1}(\lambda) - \overline{U}(0)$ and we see that $\bigvee_{n \in \mathbb{Z}} S^n \cdot \widehat{U}$ is a proper subspace of ℓ_ω if $U^{-1} \in H_{\omega^+_*}$. This observation was used by the first author in [22] to construct nontrivial translation invariant subspaces of ℓ_ω when ω is nonincreasing and satisfies $\omega(n) = 1$, for $n \geqslant 0$, $\omega(n) \underset{n \to -\infty}{\longrightarrow} \infty$.

4. EXAMPLES

In this section we shall give in concrete cases some upper bounds for the weights $(\tau^{[p]}(n))_{n \geqslant 1}$ and $(\tau^{(p)}(n))_{n \geqslant 1}$ introduced in Section 2 for $\tau \in \mathcal{S}^+$. This will provide various examples of weights $\omega \in \mathcal{S}$ satisfying the hypothesis of Corollary 3.5.

A standard way, which goes back to the last century, to obtain information about the growth of the quotient of two functions analytic in the disc, when this quotient is also analytic in the disc, consists in applying Jensen's formula (see for example [3], Lemma 5). This method gives, for $\tau \in \mathcal{S}^+$, $p \geqslant 1$

$$\log L_\tau^{(p)}(r) \leqslant \log K_{1,\tau}(\rho) + 2\,\frac{\rho + r}{\rho - r} \log K_{2,\tau}(\rho) + \frac{\rho + r}{\rho - r} \log p, \quad 0 \leqslant r < \rho < 1, \tag{4.1}$$

$$\log L_\tau^{[p]}(r) \leqslant \log K_{1,\tau}(\rho) + \frac{3\rho + r}{\rho - r} \log K_{2,\tau}(\rho) + \frac{\rho + r}{\rho - r} \log p, \quad 0 \leqslant r < \rho < 1. \tag{4.2}$$

Following the works of Cartwright ([16]) and Linden ([30] and [31]) concerning the growth of inverses or quotients of functions satisfying estimates of the type $\log |f(z)| = \mathrm{O}\left(\frac{1}{(1-|z|}\right)^\alpha\right)$, more sophisticated methods were developed in the seventies. We refer to Nikolskii ([34]) and Hayman-Korenblum ([24]) for estimates of inverses of analytic functions in the disc. Concerning analytic functions of the form $\varphi = f/g$, where g is allowed to have zeroes, the best result known to the authors is due to Matsaev-Mogulskii ([32]) (see also [33]).

Theorem 1 in [32] concerns functions ψ analytic on the half-plane.

For $\varepsilon > 0$ set

$$C(\varepsilon) = \frac{54}{\pi} \varepsilon^{-3} (1 + \varepsilon) \Big(1 + \frac{2\varepsilon}{3}\Big)^2 \Big(1 + \frac{44}{5} \mathrm{e}^{(26\pi + 3/2)(2 + \varepsilon^{-1})}\Big). \tag{4.3}$$

By considering the function $\psi(z) = \frac{f_1(\mathrm{e}^{-z})}{f_2(\mathrm{e}^{-z})}$, $\mathrm{Re}\, z > 0$ we obtain, after a change of variables in the integral, the following version of Theorem 1 of [32] for quotients of functions analytic in the open unit disc.

Theorem 4.1. (Matsaev-Mogulskii) *Let M be a positive, continuous, increasing function on $[0,1)$ and let $f_1, f_2 \in \mathcal{H}(\mathbb{D})$ satisfying the following conditions:*

(i) $f_2(0) = 1$;

(ii) $\log|f_i(\lambda)| \leqslant M(|\lambda|)$, *with* $\lambda \in \mathbb{D}$, $i = 1, 2$.

If f_1/f_2 is analytic on $\mathbb{D}$, then, for every $\varepsilon > 0$, we have

$$\text{(iii)}\ \log|f_1(\lambda)/f_2(\lambda)| \leqslant \frac{C(\varepsilon)}{\log 1/|\lambda|}\left[\int_0^{|\lambda|^{\frac{1}{1+\varepsilon}}} \sqrt{\frac{M(t)}{\log 1/t}}\ \frac{\mathrm{d}t}{t}\right]^2,$$

where $C(\varepsilon)$ is given by (4.3).

In concrete situations, the integral in the right hand side of (iii) may be the divergent at 0, so it is more convenient for applications to use the following asymptotic estimate:

Corollary 4.2. *Let M be a positive, continuous, increasing function on $[0,1)$ and let $\varphi \in \mathcal{H}(\mathbb{D})$. Assume that there exist $f_1, f_2 \in \mathcal{H}(\mathbb{D})$, with $f_2(0) \neq 0, f_2 \cdot \varphi = f_1$, satisfying the following condition:*

(i) $\overline{\lim}_{|\lambda| \to 1^-} \log|f_i(\lambda)| - M(|\lambda|) < +\infty$.

If $\int_0^1 \sqrt{\frac{M(t)}{1-t}}\,\mathrm{d}t < +\infty$, *then we have*

(ii) $\log|\varphi(\lambda)| = \mathrm{O}\Big(\frac{1}{1-|\lambda|}\Big)$, $|\lambda| \to 1^-$.

If $\int_0^1 \sqrt{\frac{M(t)}{1-t}}\,\mathrm{d}t = +\infty$, *then we have for every* $\varepsilon > 0$

$$\text{(iii)}\ \overline{\lim_{|\lambda|\to 1^-}} (1-|\lambda|)\log|\varphi(\lambda)| \cdot \left[\int_0^{|\lambda|^{1/(1+\varepsilon)}} \sqrt{\frac{M(t)}{1-t}}\,\mathrm{d}t\right]^{-2} \leqslant C(\varepsilon),$$

where $C(\varepsilon)$ is given by (4.3).

Proof. This variant of Theorem 4.1 is certainly well-known, but we give the details for the sake of completeness. Without loss of generality, assume that M is unbounded, that $|f_1(0)| \leqslant 1$, $|f_2(0)| = 1$ and that for some $b > 0$, $\log|f_i(\lambda)| \leqslant M(|\lambda|) + b$. Then, for some positive a, $|f_i(\lambda)| \leqslant 1 + a|\lambda|$, $\log|f_i(\lambda)| \leqslant a|\lambda|$, $|\lambda| \leqslant 1/2$, $i = 1, 2$. We can then construct a positive continuous function N increasing on $[0,1)$ such that $N(r) = ar$ on a neighborhood of 0, $N(r) = M(r) + b$ for $r_0 \leqslant r < 1$ and such that $\log|f_i(\lambda)| \leqslant N(|\lambda|)$, $\lambda \in \mathbb{D}$, $i = 1, 2$.

Let $\varepsilon > 0$. Using Theorem 4.1, we obtain, for $r_1 \in [r_0, 1)$ and $|\lambda| \in [r_1, 1)$

$$\log|f_1(\lambda)/f_2(\lambda)| \leqslant \alpha(r_1)\frac{C(\varepsilon)}{1-|\lambda|} \cdot \left[\int_0^{r_1} \sqrt{\frac{N(t)}{\log 1/t}}\,\frac{\mathrm{d}t}{t} + \beta(r_1)\int_0^{|\lambda|^{1/(1+\varepsilon)}} \sqrt{\frac{M(t)}{1-t}}\,\mathrm{d}t\right]^2 \tag{4.4}$$

where

$$\alpha(r_1) = \sup_{r_1 \leqslant r < 1} \frac{1-r}{\log 1/r}, \quad \beta(r_1) = \sup_{r_1 \leqslant r < 1}\left[\frac{1}{r}\sqrt{\frac{1-r}{\log 1/r}} \cdot \sqrt{\frac{M(r)+b}{M(r)}}\right].$$

Since

$$\lim_{r_1\to 1^-}\alpha(r_1)=\lim_{r_1\to 1^-}\beta(r_1)=1,$$

the corollary follows immediately from (4.4). ∎

Let $\tau\in\mathcal{S}^+$. We have

$$K_{2,\tau}(r)\leqslant\tau(0)^{-1}\cdot\widetilde{\tau}(1)\cdot K_{1,\tau}(r),\quad 0<r<1. \tag{4.5}$$

The following result follows easily from Corollary 4.2.

Proposition 4.3. *Let $\tau\in\mathcal{S}^+$, and assume that $\log\overline{\tau}(n)=\mathrm{O}(n^\alpha)$ as $n\to\infty$.*

(i) *If $\alpha<1/2$, then $\log\tau^{[p]}(n)=\mathrm{O}(\sqrt{n})$, $p\geqslant 1$.*

(ii) *If $\alpha=1/2$, then $\log\tau^{[p]}(n)=\mathrm{O}(\sqrt{n}\cdot\log(n+1))$, $p\geqslant 1$.*

(iii) *If $1/2<\alpha<1$, then $\log\tau^{[p]}(n)=\mathrm{O}(n^\alpha)$, $p\geqslant 1$.*

Proof. The notation being as in Remark 2.8, let $p\geqslant 1$, let $\varphi\in\mathfrak{P}_{p,\tau}$ and let $f_1,f_2\in\mathcal{H}(\mathbb{D})$ satisfying (2.26), with $\varphi=f_1/f_2$. We have

$$\varlimsup_{|\lambda|\to 1^-}\log|f_i(\lambda)|-2\log K_{1,\tau}(|\lambda|)<+\infty\quad i=1,2. \tag{4.6}$$

Now assume that $\log\overline{\tau}(n)=\mathrm{O}(n^\alpha)$ with $0<\alpha<1$. There exists $a>0$ such that $(n+1)^2\cdot\overline{\tau}(n+1)\leqslant \mathrm{e}^{a(n+1)^\alpha}$ for $n\geqslant 0$.

Let

$$M(r)=\sup_{n\geqslant 0}\mathrm{e}^{a(n+1)^\alpha}\cdot r^n.$$

It is a well-known fact that

$$\log M(r)=\mathrm{O}\Big(\frac{1}{(1-r)^{\frac{\alpha}{1-\alpha}}}\Big)$$

(see for example [39]).

Since $K_{1,\tau}(r)\leqslant\frac{\pi^2}{6}\cdot M(r)$, it follows from (4.6) and Corollary 4.2 that for every $\varepsilon>0$ we have

$$\varlimsup_{r\to 1^-}(1-r)\cdot\log L_\tau^{[p]}(r)\cdot\Bigg[\int_0^{r^{1/(1+\varepsilon)}}\frac{\mathrm{d}t}{(1-t)^{1/(2(1-\alpha))}}\Bigg]^{-2}<+\infty,\quad p\geqslant 1. \tag{4.7}$$

We obtain

$$\log L_\tau^{[p]}(r)=\mathrm{O}\Big(\frac{1}{1-r}\Big)\quad\text{for }\alpha<\frac{1}{2},$$

$$\log L_\tau^{[p]}(r)=\mathrm{O}\Big(\frac{\log^2(1-r)}{1-r}\Big)\quad\text{for }\alpha=\frac{1}{2},$$

and

$$\log L_\tau^{[p]}(r)=\mathrm{O}\Big(\frac{1}{(1-r)^{\frac{\alpha}{1-\alpha}}}\Big)\quad\text{for }\frac{1}{2}<\alpha<1.$$

Now for, $n\geqslant 1$, set $r_n=1-\frac{1}{\sqrt{n}}$ for $\alpha<1/2$, $r_n=1-\frac{\log(n+1)}{\sqrt{n}}$ for $\alpha=1/2$, $r_n=1-\frac{1}{n^{1-\alpha}}$ for $1/2<\alpha<1$. It follows from (2.28) that $\tau^{[p]}(n)\leqslant r_n^{-n}\cdot L_\tau^{[p]}(r_n)$ for $n\geqslant 1$, $p\geqslant 1$, and the result follows. ∎

Recall that a sequence $(u_n)_{n\geqslant p}$ of positive real numbers is said to be log-convex if the sequence $(u_{n+1}/u_n)_{n\geqslant p}$ is nondecreasing. We will say that $\tau \in \mathcal{S}^+$ is *eventually log-convex* if the sequence $(\tau(n))_{n\geqslant p}$ is log-convex for some $p \geqslant 0$. In this situation, by modifying if necessary the set $\{\tau(0), \dots, \tau(p-1)\}$, we can always assume that τ is log-convex.

Assume that $\tau \in \mathcal{S}^+$ is log-convex. Clearly, $\tau(n+1) \leqslant \tau(n)$ for $n \geqslant 0$, and we have

$$\overline{\tau}(n) = \tau(0)\cdot\tau^{-1}(n), \quad n \geqslant 0. \tag{4.8}$$

Let

$$\Delta_\tau(r) = \sup_{n\geqslant 0}(n+1)^2\cdot\tau^{-1}(n)\cdot r^n, \quad 0 \leqslant r < 1. \tag{4.9}$$

Since $\overline{\tau}(n+1) \leqslant \overline{\tau}(1)\cdot\overline{\tau}(n)$ for $n \geqslant 0$, we have

$$K_{1,\tau}(r) \leqslant \frac{\pi^2}{6}\cdot\tau(0)\cdot\overline{\tau}(1)\cdot\Delta_\tau(r), \quad 0 \leqslant r < 1. \tag{4.10}$$

Also, since the weight $n \mapsto \frac{\tau(n)}{(n+1)^2}$ is log-convex, we have, by standard properties of the Legendre transform

$$(n+1)^2\cdot\tau^{-1}(n) = \inf_{0<r<1}\Delta_\tau(r)\cdot r^{-n}, \quad n \geqslant 0. \tag{4.11}$$

We will say that a sequence $(u_n)_{n\geqslant 1}$ is *eventually increasing* if there exists $p \geqslant 1$ such that $u_{n+1} \geqslant u_n$ for $n \geqslant p$. We now deduce from Corollary 4.2 the following result:

Proposition 4.4. *Let $\tau \in \mathcal{S}^+$, and assume that τ is eventually log-convex.*

(i) *If $\sum\limits_{n=1}^{\infty}\frac{\log\tau^{-1}(n)}{n^{3/2}} < +\infty$, then $\log\tau^{[p]}(n) = \mathrm{O}(\sqrt{n})$, $p \geqslant 1$.*

(ii) *If $s > 1$, and if the sequence $\left(\frac{\log\tau^{-1}(n)}{n^\alpha}\right)_{n\geqslant 1}$ is eventually increasing for some $\alpha > \frac{1+2s^{-1}\sqrt{2C(s-1)}}{2+2s^{-1}\sqrt{2C(s-1)}}$, where $C(s-1)$ is given by (4.3), then $\overline{\lim\limits_{n\to\infty}}\frac{\log\tau^{[p]}(n)}{\log\tau^{-1}(n)} \leqslant s$, $p \geqslant 1$.*

(iii) *If the sequence $\left(\frac{\log\tau^{-1}(n)}{n^\alpha}\right)_{n\geqslant 1}$ is eventually increasing for every $\alpha < 1$, then $\frac{\log\tau^{[p]}(n)}{\log\tau^{-1}(n)} \underset{n\to\infty}{\longrightarrow} 1$, $p \geqslant 1$.*

Proof. We can assume without loss of generality that τ is log-convex. We have

$$K_{1,\tau}(r)\cdot K_{2,\tau}(r) \leqslant \frac{\pi^4}{36}\cdot\tau(0)\cdot\overline{\tau}(1)^2\cdot\widetilde{\tau}(1)\cdot\Delta_\tau^2(r), \quad 0 \leqslant r < 1. \tag{4.12}$$

Assume that $\sum\limits_{n=1}^{\infty}\frac{\log\tau^{-1}(n)}{n^{3/2}} < +\infty$, so that $\sum\limits_{n=1}^{\infty}\frac{\log[(n+1)^2\cdot\tau^{-1}(n)]}{n^{3/2}} < +\infty$. Since $\log\Delta_\tau$ is convex on $[0,1)$, it follows from a classical result ([34], Section 2.6, Lemma 2) that

$$\int_0^1\sqrt{\frac{\log\Delta_\tau(t)}{1-t}}\,\mathrm{d}t < +\infty.$$

Using (4.12), and applying Corollary 4.2 as in the proof of Proposition 4.3, we obtain (i).

Assume that

$$\int_0^1 \sqrt{\frac{\log \Delta_\tau(t)}{1-t}}\,\mathrm{d}t = +\infty.$$

Applying again Corollary 4.2 as in the proof of Proposition 4.3, we obtain, for every $\varepsilon > 0$

$$\overline{\lim_{r\to 1^-}}\,(1-r)\cdot \log L_\tau^{[p]}(r)\cdot \left[\int_0^{r^{\frac{1}{1+\varepsilon}}} \sqrt{\frac{\log \Delta_\tau(t)}{1-t}}\,\mathrm{d}t\right]^{-2} \leqslant 2C(\varepsilon). \tag{4.13}$$

Now assume that the sequence $\left(\frac{\log \tau^{-1}(n)}{n^\alpha}\right)_{n\geqslant 1}$ is eventually increasing, with $\alpha > 1/2$. A routine verification shows hat the sequence $\left(\frac{\log[(n+1)^2\tau^{-1}(n)]}{n^\beta}\right)_{n\geqslant 1}$ is eventually increasing for $0 < \beta < \alpha$. Since the sequence $\left(\frac{\tau(n)}{(n+1)^2}\right)_{n\geqslant 0}$ is log-convex it is a standard fact (see for example [39]) that there exists then $r_0 \in [0,1)$ such that the function $r \to (1-r)^{\frac{\beta}{1-\beta}}\cdot \log \Delta_\tau(r)$ is increasing on $[r_0, 1)$. If $\beta \in (1/2, \alpha)$, we obtain, for $r \in [r_0, 1)$

$$\left[\int_{r_0}^{r} \sqrt{\frac{\log \Delta_\tau(t)}{1-t}}\,\mathrm{d}t\right]^2 \leqslant (1-r)^{\frac{\beta}{1-\beta}}\cdot \log \Delta_r(r)\cdot \left[\int_{r_0}^{r} \sqrt{\frac{1}{(1-t)^{\frac{1}{1-\beta}}}}\,\mathrm{d}t\right]^2$$
$$\leqslant \frac{4(1-\beta)^2}{(2\beta-1)^2}\cdot (1-r)^{\frac{\beta}{1-\beta}}\cdot \log \Delta_r(r)\cdot (1-r)^{\frac{1-2\beta}{1-\beta}}.$$

We obtain

$$\frac{1}{1-r}\left[\int_{r_0}^{r} \sqrt{\frac{\log \Delta_\tau(t)}{1-t}}\,\mathrm{d}t\right]^2 \leqslant \frac{4(1-\beta)^2}{(2\beta-1)^2}\cdot \log \Delta_\tau(r), \quad r_0 \leqslant r < 1. \tag{4.14}$$

Fix $s > 1$. Applying (4.13) with $\varepsilon = s-1$, we obtain

$$\overline{\lim_{r\to 1^-}}\,(1-r^s)\cdot \log L_\tau^{[p]}(r^s)\cdot \left[\int_{r_0}^{r} \sqrt{\frac{\log \Delta_\tau(t)}{1-t}}\,\mathrm{d}t\right]^{-2} \leqslant 2\cdot C(s-1). \tag{4.15}$$

Since $\frac{1-r^s}{1-r} \underset{r\to 1^-}{\longrightarrow} s$, we obtain, using (4.14)

$$\overline{\lim_{r\to 1^-}}\,\frac{s^{-1}\cdot \log L_\tau^{[p]}(r^s)}{\log \Delta_\tau(r)} \leqslant \frac{8(1-\beta)^2}{(2\beta-1)^2} s^{-2}\cdot C(s-1). \tag{4.16}$$

Now assume that $\alpha > \frac{1+2s^{-1}\sqrt{2C(s-1)}}{2+2s^{-1}\sqrt{2C(s-1)}}$. There exists $\beta \in (1/2, \alpha)$ such that $\beta > \frac{1+2s^{-1}\sqrt{2C(s-1)}}{2+2s^{-1}\sqrt{2C(s-1)}}$. Using the fact that the function $t \to \frac{1-t}{2t-1}$ is decreasing on

$(1/2, 1)$, we obtain

$$\overline{\lim_{r\to 1^-}} \frac{s^{-1}\cdot \log L_\tau^{[p]}(r^s)}{\log \Delta_\tau(r)} < 1. \tag{4.17}$$

Fix $p \geqslant 1$. We have, for $n \geqslant 0$, by (2.28)

$$\begin{aligned} s^{-1}\cdot \log \tau^{[p]}(n) &= \inf_{0<r<1}[s^{-1}\cdot \log L_\tau^{[p]}(r) - n\log r^{1/s}] \\ &= \inf_{0<r<1}[s^{-1}\cdot \log L_\tau^{[p]}(r^s) - n\log r]. \end{aligned}$$

Hence there exists $n_p \geqslant 1$ such that for $n \geqslant n_p$ we have

$$s^{-1}\cdot \log \tau^{[p]}(n) \leqslant \inf_{0<r<1}[\log \Delta_\tau(r) - n\log r].$$

Using (4.11), this gives

$$s^{-1}\cdot \log \tau^{[p]}(n) \leqslant 2\log(n+1) + \log \tau^{-1}(n), \quad n \geqslant n_p. \tag{4.18}$$

Since $\frac{\log(n+1)}{\log \tau^{-1}(n)} \underset{n\to\infty}{\longrightarrow} 0$, this gives (ii), and (iii) follows immediately from (ii). ∎

Notice that the estimates of (ii) hold for the weights $\tau^{(p)}$ under the slightly weaker condition

$$\alpha > \frac{1 + 2s^{-1}\cdot \sqrt{C(s-1)}}{2 + 2s^{-1}\cdot \sqrt{C(s-1)}}.$$

We conclude this section by a few examples.

Theorem 4.5. *Let $\omega \in \mathcal{S}$. Assume that ω satifies one of the following conditions:*

(i) $\log \overline{\omega}_+(n) = \mathrm{O}(n^\alpha)$, *with $\alpha < 1/2$, and* $\lim_{n\to\infty} \frac{\log \omega(-n)}{\sqrt{n}} = +\infty$*;*

(ii) $\log \overline{\omega}_+(n) = \mathrm{O}(\sqrt{n})$, *and* $\lim_{n\to\infty} \frac{\log \omega(-n)}{\sqrt{n}\cdot \log(n+1)} = +\infty$*;*

(iii) $\log \overline{\omega}_+(n) = \mathrm{O}(n^\alpha)$, *with $1/2 < \alpha < 1$, and* $\lim \frac{\log \omega(-n)}{n^\alpha} = +\infty$*;*

(iv) ω_+ *is eventually log-convex,* $\sum_{n=1}^{\infty} \frac{\log \omega^{-1}(n)}{n^{3/2}} < +\infty$, *and* $\lim_{n\to\infty} \frac{\log \omega(-n)}{\sqrt{n}} = +\infty$*;*

(v) ω_+ *is eventually log-convex,* $\lim_{n\to\infty} \frac{\log \omega(-n)}{\log \omega^{-1}(n)} > s$, *with $s > 1$ and* $\left(\frac{\log \omega^{-1}(n)}{n^\alpha}\right)_{n\geqslant 1}$ *is eventually increasing for some*

$$\alpha > \frac{1 + 2s^{-1}\sqrt{2C(s-1)}}{2 + 2s^{-1}\sqrt{2C(s-1)}};$$

(vi) ω_+ *is eventually log-convex,*

$$\liminf_{n\to\infty} \frac{\log \omega(-n)}{\log \omega^{-1}(n)} > 1, \quad \textit{and} \quad \left(\frac{\log \omega^{-1}(n)}{n^\alpha}\right)_{n\geqslant 1}$$

is eventually increasing for every $\alpha < 1$.

Then

$$F = \Big(\bigvee_{n\leqslant 0} S^n \cdot F\Big) \cap \ell_\omega^+, \qquad \textit{and} \qquad \ell_\omega = \ell_\omega^+ + \bigvee_{n\leqslant 0} S^n \cdot F$$

for every closed subspace $F \neq \{0\}$ *of* ℓ_ω^+ *having the division property.*

If, further,

$$\lim_{n\to-\infty}\bigg[\sup_{p\geqslant 0}\frac{\omega(n+p)}{\omega(p)}\bigg]^{1/n} = 1,$$

then every nonzero analytic left invariant susbpace L *of* ℓ_ω *has the form* $L = \bigvee_{n\leqslant 0} S^n \cdot L^+$*, where* $L^+ = L \cap \ell_\omega^+$ *has the division property. Also,* $\log\|S_L^{-n}\| = \mathrm{O}(\sqrt{n})$ *if* ω *satifies* (i) *or* (iv)*;* $\log\|S_L^{-n}\| = \mathrm{O}(\sqrt{n}\cdot\log(n+1))$ *if* ω *satisfies* (ii)*;* $\log\|S_L^{-n}\| = \mathrm{O}(n^\alpha)$ *if* ω *satisfies* (iii)*;* $\overline{\lim\limits_{n\to\infty}}\,\frac{\log\|S_L^{-n}\|}{\log\omega^{-1}(n)} \leqslant s$ *if* ω *satisfies* (v)*; and* $\overline{\lim\limits_{n\to\infty}}\,\frac{\log\|S_L^{-n}\|}{\log\omega^{-1}(n)} \leqslant 1$ *if* ω *satisfies* (vi).

Proof. The theorem follows immediately from Proposition 4.3, Proposition 4.4, Corollary 3.5, Remark 2.8 (iv) and Remark 3.7 (ii).

Notice that condition (vi) of Theorem 4.5 is satisfied by the weights ω of the form

$$\omega(n) = \mathrm{e}^{\frac{-an}{(\log n+1)^c}}, \quad \omega(-n) = \mathrm{e}^{\frac{b|n|}{\log(|n|+1)^c}}, \quad n \geqslant 0$$

where $b > a > 0$, $c > 0$.

5. ON THE EXISTENCE OF z-INVARIANT SUBSPACES HAVING THE DIVISION PROPERTY

In order to construct analytic nontrivial translation invariant susbpaces of ℓ_ω, we need to be able to find nontrivial z-invariant subspaces of the weighted Hardy space H_{ω_+} which have the division property. The existence of such subspaces of H_τ is not known for arbitrary $\tau \in \mathcal{S}^+$. In the case of the Hardy space $H^2 = H^2(\mathbb{D})$, these subspaces are the subspaces $U \cdot H^2$ where U is a singular inner function. When $\tau(n) = (n+1)^{-1/2}$, for $n \geqslant 0$, the space H_τ is the usual Bergman space

$$B^2 = B^2(\mathbb{D}) = \Big\{f \in \mathcal{H}(\mathbb{D}) : \iint\limits_{\mathbb{D}} |f(x+\mathrm{i}y)|^2\,\mathrm{d}x\,\mathrm{d}y < +\infty\Big\}.$$

Korenblum ([29]) proposed a notion of outer and inner functions suitable for B^2. It turns out that Korenblum's "Bergman-outer" functions are exactly the z-cyclic elements of B^2 ([29] and [1]). Korenblum's "Bergman inner" functions are characterized by the conditions $\|U\|_{B^2} = 1$, $\langle U, z^n \cdot U\rangle = 0$ for $n \geqslant 1$. We will say that a Bergman-inner function U is singular if U has no zeroes in $\mathbb{D}$.

Using the Aleman-Richter-Sundberg theorem ([1]) we can characterize nontrivial z-invariant susbpaces of B^2 having the division property.

Proposition 5.1. *The nontrivial z-invariant susbpaces of B^2 having the division property are the subspaces of the form*

$$M = \bigvee_{n\geqslant 0} z^n \cdot U$$

where U is a nonconstant singular Bergman-inner function.

Proof. It is easy to check, and well-known, that a nonconstant Bergman-inner function U is not z-cyclic. If U is singular, $Z(U) = \emptyset$ and it follows from Proposition 2.5 that $\bigvee_{n\geqslant 0} z^n \cdot U$ is a nontrivial z-invariant susbpace of B^2 which has the division property.

Now assume that M is a nontrivial z-invariant subspace of B^2 having the division property, and let $U \in M$ be the Hedenmalm extremal function for M ([25]). This means that $\|U\|_{B^2} = 1$ and that $\operatorname{Re} f(0) \leqslant \operatorname{Re} U(0)$ for every $f \in M$ such that $\|f\|_{B^2} = 1$. Then $U \perp z \cdot M$ (see [25]), and so $\langle U, z^n \cdot U\rangle = 0$, $n \geqslant 1$, so that U is Bergman-inner.

It follows from the Aleman-Richter-Sundberg theorem ([1]) that $M = \bigvee_{n\geqslant 0} z^n(M \ominus zM)$. Since $\dim(M\ominus zM) = 1$, $M \ominus zM = \mathbb{C}\,U$, and so $M = \bigvee_{n\geqslant 0} z^n \cdot U$. Since $Z(M) = \emptyset$, $Z(U) = \emptyset$, and U is a Bergman-inner singular function having the division property. ∎

There is another situation where the lattice of z-invariant subspaces of $H_\tau(\mathbb{D})$ can be described. Assume that $\tau \in \mathcal{S}^+$ is increasing, and that $(n^\alpha \cdot \tau^{-1}(n))_{n\geqslant 1}$ is eventually log-convex for every $\alpha > 0$.

In this situation $H_\tau(\mathbb{D})$ is a Banach algebra of functions which are smooth on the closed disc. Carleson ([16]) showed that if, further,

$$\sum_{n=1}^{\infty} \frac{\log \tau(n)}{n^{3/2}} = +\infty,$$

then the zero set $\{z \in \overline{\mathbb{D}} : f(z) = 0\}$ is finite for every nonzero $f \in H_\tau(\mathbb{D})$, and Domar ([20]) showed that all z-invariant subspaces (here all ideals) of $H_\tau(\mathbb{D})$ are determined by their zero set, taking (finite) multiplicities into account. If $\omega \in \mathcal{S}$, and if ω_+ satisfies the above conditions, we obtain analytic translation invariant subspaces of ℓ_ω if

$$\sum_{n=1}^{\infty} \omega^{-2}(-n) < +\infty,$$

and every z-invariant subspace having the division property generates a nontrivial translation invariant subspace of ℓ_ω if $n^p \cdot \omega(-n) \underset{n\to\infty}{\longrightarrow} 0$ for every $p \geqslant 1$. But there is nothing surprising here. In the first situation ℓ_ω is continuously contained in $\mathcal{C}(\mathbb{T})$ and these analytic translation invariant subspaces have the form $\{f \in \ell_\omega : f|S = 0\}$ where S is a finite subset of $\mathbb{T}$. In the second case ℓ_ω is continuously contained in $\mathcal{C}^\infty(\mathbb{T})$ and these analytic translation invariant subspaces have the form

$$\bigcap_{1\leqslant i\leqslant k} \{f \in \ell_\omega : f^{(n)}|S_i = 0, n \leqslant d_i\}$$

where $S_1, \ldots, S_k$ are finite subsets of π and where $0 \leqslant d_1 < d_2 \cdots < d_k < +\infty$.

We now discuss the case where $\tau \in \mathcal{S}^+$ is log-convex. The simplest way to construct z-invariant subspaces of H_τ having the division property consists in considering spaces of the form $\bigvee_{n\geqslant 0} z^n \cdot f$ where $f \in H_\tau$ is not z-cyclic and has no zeroes in $\mathbb{D}$.

The question of z-cyclicity (which he called weak invertibility) has been extensively studied by Nikolskii in [34], Chapter 2, for various Banach spaces of analytic functions in the disc. It follows from a theorem of Beurling ([8]) that if $\tau(n) = \mathrm{O}(1/n^\alpha)$ for some $\alpha > 0$, $\tau^2(k) \leqslant \tau(n)$ for $n \leqslant 2k$ and if

$$\sum_{n=1}^{\infty} \frac{\log \tau^{-1}(n)}{n^{3/2}} < +\infty, \tag{5.1}$$

then the inner function $\exp\left(\frac{z+1}{z-1}\right)$ is not z-cyclic in $H_\tau(\mathbb{D})$.

Conversely, it follows from [34], Chapter 2, Section 2.6, Theorem 2, that all singular inner functions are z-cyclic in $H_\tau(\mathbb{D})$ if $\log \tau(n)$ is convex with respect to $\log n$ and if τ does not satisfy the "one-sided quasianalytic condition" (5.1).

We now describe the three methods known to the authors to produce z-invariant subspaces having the division property when τ is a log-convex weight satisfying (5.1). The first one, developed by Nikolskii in [34], Chapter 2, Section 2.8 is called the "abstract Keldysh method". It is based on the following lemma, which we formulate in the context of weighted Hardy spaces.

Lemma 5.2. ([34], Chapter 2, Section 2.8, Lemma 2) *Let $\tau \in \mathcal{S}^+$, and let γ be a closed Jordan curve which is symmetric with respect to the real axis, such that $\gamma \subset \overline{\mathbb{D}}$, $\gamma \cap \mathbb{T} = \{1\}$. Let $f \in H_\tau(\mathbb{D})$ and assume that there exists an outer function G on $\operatorname{Int}\gamma$ such that:*

(i) $K_{2,\tau}(|\xi|) \cdot |f^{-1}(\xi)| \leqslant |G(\xi)|$, $\xi \in \gamma$;

(ii) $|f(x)| = \mathrm{o}(|G^{-1}(x)|)$, $(x \to 1^-)$.

Then f is not z-cyclic in $H_\tau(\mathbb{D})$.

We refer to [34] for a proof. Here by an outer function on $\operatorname{Int}\gamma$ we mean a function G such that $G \circ \theta$ is outer in $\mathbb{D}$, where $\theta : \mathbb{D} \to \operatorname{Int}\gamma$ is a conformal mapping. Also

$$K_{2,\tau}(r) = \Big[\sum_{n=0}^{\infty} \tau^{-2}(n) r^{2n}\Big]^{1/2} = \|g_\xi\|_\tau,$$

where

$$g_\xi(\lambda) = \sum_{n=0}^{\infty} \tau^{-2}(n)\xi^n\lambda^n \quad \text{for } \lambda \in \mathbb{D},\ |\xi| = r$$

is the function defined in (2.17). Hence $K_{2,\tau}(|\xi|)$ is the norm of the functional $f \to f(\xi) = \langle f, g_\xi\rangle$ on $H_\tau(\mathbb{D})$.

It is possible to deduce from [34], Chapter 2, Section 2.8, Theorem 3, that there exists log-convex weights $\tau \in \mathcal{S}^+$ decreasing arbitrarily fast at infinity for which $H_\tau(\mathbb{D})$ contains non z-cyclic functions without zeroes in $\mathbb{D}$, but the weights for which this construction works are not explicit.

For $x \geqslant 0$, denote as usual by $[x]$ the nonnegative integer such that $[x] \leqslant x < [x] + 1$. Using the same tools as in the proof of [34], Section 2.8, Theorem 2, we obtain the following concrete result:

Theorem 5.3. *Let $\alpha \in (1/2, 1)$ and set*

$$\tau_\alpha(n) = \mathrm{e}^{-n^\alpha} \text{ for } n \geqslant 0, \quad \text{and} \quad p_\alpha = \Big[\frac{\alpha}{1-\alpha}\Big].$$

There exist $a_0, \dots, a_{p_\alpha} \in \mathbb{R}$, with $a_0 < 0$, such that the functions

$$\varphi_c : \lambda \to \exp\Big[\sum_{k=0}^{p_\alpha} \frac{a_k}{(1-\lambda)^{\frac{\alpha}{1-\alpha}-k}} - \frac{1}{(1-\lambda)^c}\Big]$$

are not z-cyclic elements of $H_{\tau_\alpha}(\mathbb{D})$ for $0 < c < 1$.

Proof. Let $\tau_\alpha(x) = \mathrm{e}^{-x^\alpha}$ for $x > 0$ and set

$$M_\alpha(r) = \sup_{n \geqslant 0} \tau_\alpha^{-1}(n) \cdot r^n, \quad N_\alpha(r) = \sup_{x \geqslant 0} \tau_\alpha^{-1}(x) \cdot r^x. \tag{5.2}$$

A routine verification shows that if $|\theta(x)| \leqslant 1$ for $x \geqslant 0$, then

$$\lim_{x\to\infty} \frac{\tau_\alpha^{-1}(x+\theta(x))}{\tau_\alpha^{-1}(x)} = 1.$$

We obtain

$$\log M_\alpha(r) = \log N_\alpha(r) + \mathrm{o}(1), \quad r \to 1^-. \tag{5.3}$$

The function $x \to x\log r + x^\alpha$ attains its maximum value on $[0,\infty)$ for $x = \big(\frac{1}{\alpha}\log\frac{1}{r}\big)^{\frac{1}{\alpha-1}}$. We obtain

$$\log N_\alpha(r) = \big(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\big)\left(\log\frac{1}{r}\right)^{\frac{\alpha}{\alpha-1}}. \tag{5.4}$$

Using an asymptotic expansion of $(\log\frac{1}{r})^{\frac{\alpha}{1-\alpha}}$ as $r \to 1^-$, we see that there exist $b_0, \dots b_{p_\alpha}$ with $b_0 = \alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} > 0$ such that we have

$$\log M_\alpha(r) = \sum_{k=0}^{p_\alpha} \frac{b_k}{(1-r)^{\frac{\alpha}{1-\alpha}-k}} + \mathrm{o}\Big(\frac{1}{(1-r)^{\frac{\alpha}{1-\alpha}-p_\alpha}}\Big), \quad r \to 1^-. \tag{5.5}$$

We now apply [34], Chapter 2, Section 2.8, Lemma 2.

Since $b_0 > 0$, there exists $a_0, \dots, a_{p_\alpha}$, $a_0 < 0$ such that if we set

$$p(\lambda) = \sum_{k=0}^{p_\alpha} \frac{a_k}{(1-\lambda)^{\frac{\alpha}{1-\alpha}-k}} \tag{5.6}$$

$$L(r) = \max_{|\lambda|=r} \operatorname{Re} p(\lambda), \quad 0 \leqslant r < 1, \tag{5.7}$$

then the maximum in (5.7) is attained on a curve γ satisfying the conditions of Lemma 5.2, and we have

$$\lim_{\substack{|\lambda|\to 1^- \\ \lambda\in\gamma}} |\arg(1-\lambda)| = \pi(1-\alpha) \tag{5.8}$$

$$L(r) = \log M_\alpha(r) + \mathrm{O}(1), \quad r \to 1^-. \tag{5.9}$$

Now set $\Delta(r) = \sup_{x \geqslant 0}(x+1)\cdot\tau_\alpha^{-1}(x)\cdot r^x$. The function $x \to x\log r + x^\alpha + \log(x+1)$ attains its maximum at x_r, where x_r satisfies the equation $\alpha x_r^{\alpha-1} + \frac{1}{1+x_r} = -\log r$. Hence $\lim_{r\to 1^-} x_r \cdot \left(\frac{\alpha}{1-r}\right)^{\frac{1}{\alpha-1}} = 1$.

We obtain

$$\Delta(r)\cdot M_\alpha^{-1}(r) = \mathrm{O}\Big(\frac{1}{(1-r)^{\frac{1}{1-\alpha}}}\Big) \quad \text{as } r \to 1^-.$$

Since $K_{2,\tau_\alpha}(r) \leqslant \frac{\pi}{\sqrt{6}}\cdot\Delta(r)$ for $0 \leqslant r < 1$, this gives

$$K_{2,\tau_\alpha}(r)\cdot M_\alpha^{-1}(r) = \mathrm{O}\Big(\frac{1}{(1-r)^{\frac{1}{1-\alpha}}}\Big), \quad r \to 1^-. \tag{5.10}$$

Fix $c \in (0,1)$, and set

$$f(\lambda) = \mathrm{e}^{p(\lambda) - \frac{1}{(1-\lambda)^c}}, \quad \lambda \in \mathbb{D}. \tag{5.11}$$

We have $|f'(\lambda)| = \mathrm{o}(\mathrm{e}^{L(|\lambda|)})$, $|\lambda| \to 1^-$. Hence, by (5.9), $|f'(\lambda)| = \mathrm{o}(M_\alpha(|\lambda|)$. By standard properties of the Legendre transform, $\tau_\alpha^{-1}(n) = \inf_{0<r<1} M_\alpha(r)\cdot r^{-n}$.

It follows then from Cauchy's inequalities that $\overline{\lim_{n\to\infty}}\, n|\widehat{f}(n+1)|\tau_\alpha(n) < +\infty$, and so $f \in H_{\tau_\alpha}(\mathbb{D})$.

Let $F(\lambda) = \mathrm{e}^{\frac{2}{(1-\lambda)^c}}$ for $\lambda \in \mathbb{D}$. The function F is outer on $\mathbb{D}$, since $0 < c < 1$. This implies as well-known that $F|\operatorname{Int}\gamma$ is outer on $\operatorname{Int}\gamma$.

It follows from (5.10) that we have

$$\log|K_{2,\tau_\alpha}(|\lambda|)| - \log|f(\lambda)| = \log M_\alpha(|\lambda|) - \mathrm{Re}\ p(\lambda) + \mathrm{Re}\,\frac{1}{(1-\lambda)^c} + \mathrm{o}\Big(\log\frac{1}{1-|\lambda|}\Big)$$

as $|\lambda| \to 1^-$. Using (5.9), we obtain

$$\overline{\lim_{\substack{|\lambda|\to 1\\ \lambda\in\gamma}}}\, K_{2,\tau_\alpha}(|\lambda|)|f^{-1}(\lambda)|\cdot|F^{-1}(\lambda)| = 0. \tag{5.12}$$

Clearly, $|f(x)| = \mathrm{o}(F^{-1}(x))$ as $x \to 1^-$, and the theorem follows then from Lemma 5.2. ∎

Notice that since the function $\lambda \to \mathrm{e}^{\frac{\lambda+1}{\lambda-1}}$ is outer on the angle $\{\xi \in \mathbb{C} : |\arg(1-\xi)| < \beta\pi\}$ for every $\beta < 1/2$, we can show using similar arguments that the function

$$f : \lambda \to (1-\lambda)^{\frac{1}{1-\alpha}}\cdot\exp\Big(\sum_{k=0}^{p_\alpha}\frac{a_k}{(1-\lambda)^{\frac{\alpha}{1-\alpha}-k}}\Big)\cdot\exp\frac{\lambda+1}{\lambda-1}$$

is an element of $H_{\tau_\alpha}(\mathbb{D})$ which is not z-cyclic. We leave the details to the reader.

Very recently, Atzmon ([3], [4]) obtained important new results concerning existence of translation invariant subspaces of ℓ_ω. In order to describe his results we need to introduce some notation. Let φ be a nonnegative, piecewise smooth, concave function on $[0,\infty)$ such that $\varphi(t) = \mathrm{O}(t)$ as $t \to 0^+$ and $\varphi(t) = \mathrm{o}(t)$ as $t \to \infty$. Set

$$J(\varphi) = \int_1^\infty \frac{\varphi(t)}{t^{3/2}}\,\mathrm{d}t. \tag{5.13}$$

Now define $\varphi_c^*(x)$ for $x \geqslant 0$ by the formulae

$$\varphi_c^*(x) = c\sqrt{x} \quad \text{if } J(\varphi) < +\infty,\ c > 0; \tag{5.14}$$

$$\varphi_c^*(x) = \frac{x^{3/2}}{\pi}\int_0^\infty \frac{\varphi(t)}{t^{3/2}(x+t)}\,\mathrm{d}t \quad \text{if } J(\varphi) = +\infty,\ c \in \mathbb{R}. \tag{5.15}$$

Denote by $\mathcal{H}_0(\mathbb{C})$ the space of entire functions of exponential type and — φ_c^* being defined by (5.14) or (5.15) — set

$$\begin{aligned} B_\varphi(c) = \Big\{ f \in \mathcal{H}_0(\mathbb{C}) : \|f\|_\varphi := \Big[\int_0^\infty |f(t)|^2 \cdot \mathrm{e}^{2\varphi(t)} \cdot \mathrm{d}t\Big]^{1/2} < +\infty \\ \text{and } |f(-t)| = \mathrm{O}(\mathrm{e}^{\varphi_c^*(t)}) \text{ as } t \to \infty \Big\}. \end{aligned} \tag{5.16}$$

It follows from [3], [4] that the spaces $B_\varphi(c)$ are infinite dimensional Hilbert spaces with respect to the norm $\|\cdot\|_\varphi$, and that convergence in these spaces implies uniform convergence on compact subsets of $\mathbb{C}$.

Also, if $f \in B_\varphi(c)$ and $s \in \mathbb{R}$ then $f_s : z \to f(z+s)$ belongs to $B_\varphi(c)$. Now assume that φ satisfies the following condition

$$|\varphi(x+2) + \varphi(x) - 2\varphi(x+1)| = \mathrm{O}(x^{-1})(x \to \infty). \tag{5.17}$$

Then there exists $a > 0, b > 0$ such that we have for $f \in B_\varphi(c)$

$$a\Big[\sum_{n=0}^\infty |\widehat{f}(n)|^2 \mathrm{e}^{2\varphi(n)}\Big]^{1/2} \leqslant \|f\|_\varphi \leqslant b\Big[\sum_{n=0}^\infty |f(n)|^2 \cdot \mathrm{e}^{2\varphi(n)}\Big]^{1/2}. \tag{5.18}$$

Also, if (5.17) is satisfied then the spaces $B_\varphi(c)$ are stable under differentiation, and the differentiation operator $D : f \to f'$ is bounded and quasinilpotent on $B_\varphi(c)$.

Let $\omega \in \mathcal{S}^+$ such that $\omega(0) = 1$. Assume that $\omega^{-1}|\mathbb{Z}^+$ is log-convex. Let φ be the function continuous on $[0,\infty)$ and affine on each interval $[n, n+1]$ which satisfies $\varphi(n) = \log\omega(n)$ for $n \geqslant 0$. The notation being as above, set $\omega_c^*(n) = \mathrm{e}^{\varphi_c^*(n)}$ for $n \geqslant 1$. Atzmon ([3], Theorem 4.2) showed that ℓ_ω possesses nontrivial translation invariant subspaces if ω satisfies the two following conditions

$$\sup_{n\geqslant 1}\Big[\frac{\omega(n-1)\cdot\omega(n+1)}{\omega^2(n)}\Big]^{-n} < +\infty, \tag{5.19}$$

$$\liminf_{n\to\infty} \frac{\log\omega^{-1}(-n)}{\log w_c^*(n)} > 1 \quad \text{for some } c. \tag{5.20}$$

Here $c > 0$ if $J(\varphi) < +\infty$, and $c \in \mathbb{R}$ if $J(\varphi) = +\infty$. The translation invariant subspaces constructed by Atzmon have the form $\{(g(n))_{n\in\mathbb{Z}} : g \in B_\varphi(c)\}$. We refer to [3] and [4] for various examples of weights satisfying these conditions.

Using this construction, Atzmon showed that if $\tau \in \mathcal{S}^+$ is log-convex and satisfies:

$$\sup_{n\geqslant 0}\Big[\frac{\tau(n+1)\cdot\tau(n-1)}{\tau(n)^2}\Big]^n < +\infty \tag{5.21}$$

then H_τ contains a proper z-invariant subspace M such that $Z(M) = \emptyset$.

Let M be a z-invariant subspace of H_τ. Define $T_M : M^\perp \to M^\perp$ by the following formula analogous to (2.8), where we denote by P the orthogonal projection of H_τ onto $M^\perp$

$$T_M \cdot P = P \cdot T. \tag{5.22}$$

Then $T_M = (T^*|M^\perp)^*$, according to (2.13), and M has the division property iff $\sigma(T_M) \subset \mathbb{T}$. In fact, Atzmon's construction gives a result much stronger that the existence of zero-free z-nvariant subspaces.

Theorem 5.4. *Let $\tau \in \mathcal{S}^+$ be log-convex. If τ satifies* (5.21)*, then H_τ contains proper z-invariant subspaces M having the division property such that $\sigma(T_M) = \{1\}$.*

Proof. More general results will be proved in [5], but we give the details for the sake of completeness. We can assume that $\tau(0) = 1$. Let φ be a function continuous on $[0,\infty)$ and affine on each interval $[n, n+1]$ such that $\varphi(n) = \log\tau^{-1}(n)$ for $n \geqslant 0$. Then φ is log-concave on $[0,\infty)$, and we can apply to φ Atzmon's construction. Since τ satisfies (5.19), φ satisfies (5.17).

We can identify H_τ^* to $\ell_{\tau^{-1}} := \ell^2_{\tau^{-1}}(\mathbb{Z}^+)$, the duality being implemented by the formula

$$(f, v) = \sum_{n=0}^{\infty}\widehat{f}(n)\cdot\overline{v_n}, \quad f \in H_\tau,\ v = (v_n)_{n\geqslant 0} \in \ell_{\tau^{-1}}. \tag{5.23}$$

Now set $N_c = \{(g(n))_{n\geqslant 0} : g \in B_\varphi(c)\}$, and set $M_c = \{f \in H_\tau : (f, v) = 0, v \in N_c\}$. It follows from (5.18) that N_c is a closed subspace of $\ell_{\tau^{-1}}$.

Now define the backward shift R on $\ell_{\tau^{-1}}$ by the formula

$$R\cdot(v_n)_{n\geqslant 0} = (v_{n+1})_{n\geqslant 0}, \quad (v_n)_{n\geqslant 0} \in \ell_{\tau^{-1}}. \tag{5.24}$$

We have

$$(zf, v) = (f, Rv), \quad f \in H_\tau,\ v \in \ell_{\tau^{-1}}. \tag{5.25}$$

For $g \in B_\varphi(c)$, $s \in \mathbb{R}$, set $V_s \cdot g = g_s$, where $g_s(z) = g(z+s)$, $z \in \mathbb{C}$. Then $V_s \cdot g \in B_\varphi(c)$ for $g \in B_\varphi(c)$, $s \in \mathbb{R}$. In particular $R(N_c) \subset N_c$, and so, by (5.25), M_c is z-invariant.

The differentiation operator $D : g \to g'$ is quasinilpotent on $B_\varphi(c)$. We have, for $g \in B_\varphi(c), z \in \mathbb{C}$

$$(\mathrm{e}^D\cdot g)(z) = \sum_{n=0}^{\infty}\frac{g^{(n)}(z)}{n!} = g(z+1).$$

Hence $e^D = V_1$, and $\sigma(V_1) = \{1\}$. Let $\theta : g \to g|\mathbb{Z}^+$ be the restriction map from $B_\varphi(c)$ onto N_c. Then θ is bijective and continuous, and $R_c := R|N_c$ satisfies $R_c = \theta \circ V_1 \circ \theta^{-1}$. Hence $\sigma(R_c) = \sigma(V_1) = \{1\}$.

Now denote by $M_c^\perp$ the orthogonal of M_c in H_τ, taken in the usual Hilbert space sense.

For $h \in H_\tau$ set $\rho(h) = (\tau^2(n) \cdot \widehat{h}(n))_{n \geqslant 0}$. Then $\rho : H_\tau \to \ell_{\tau^{-1}}$ is unitary, and we have, for $f \in H_\tau$ and $h \in H_\tau$.

$$\langle f, h\rangle = (f, \rho(h)). \tag{5.26}$$

Denote again by $T : f \mapsto z \cdot f$ the usual shift on H_τ. Let $f, h \in H_\tau$. Using (5.24), we obtain $(f, (\rho \circ T^*)(h)) = \langle f, T^* \cdot h\rangle = \langle z \cdot f, h\rangle = (z \cdot f, \rho(h)) = (f, (R \circ \rho)(h))$. Hence $\rho \circ T^* = R \circ \rho$. Clearly, $\rho(M_c^\perp) = N_c$. Let $\rho_c = \rho|M_c^\perp$. Then $\rho_c : M_c^\perp \to N_c$ is unitary. Since $\rho_c \circ (T^*|M_c^\perp) = R_c \circ \rho_c$, $T^*|M_c^\perp$ is unitarily equivalent to R_c. Hence $\sigma(T^*|M_c^\perp) = \sigma(R_c) = \{1\}$. Since $T_{M_c} = (T^*|M_c^\perp)^*$, $\sigma(T_{M_c}) = \{1\}$, which concludes the proof of the theorem. ∎

Notice that, by slightly modifying the notation of the proof of Theorem 5.4, we can identify $(H_\tau)^*$ to $H_{\tau^{-1}}$ by using the formula

$$(f, g) = \sum_{n=0}^{\infty} \widehat{f}(n) \cdot \overline{\widehat{g}(n)}, \quad f \in H_\tau,\ g \in H_{\tau^{-1}}. \tag{5.27}$$

Assume that M is a z-invariant subspace of H_τ which has the division property, and let $M_\perp$ be the orthogonal of M in $H_{\tau^{-1}}$ with respect to formula (5.27). Denote by R the backward shift on $H_{\tau^{-1}}$, and set $R_M = R|M_\perp$. We see as above that R_M is unitarily equivalent to T_M^*.

For $g \in H_{\tau^{-1}}$ we have

$$(1, (1 - \overline{\lambda}R)^{-1} \cdot g) = ((1 - \lambda T)^{-1} \cdot 1, g) = \sum_{n=0}^{\infty} \lambda^n \cdot \overline{\widehat{g}(n)} = \overline{g(\overline{\lambda})}, \quad \lambda \in \mathbb{D}.$$

For $g \in M_\perp$ we obtain

$$g(\lambda) = \overline{(1, (1 - \lambda R_M)^{-1} \cdot g)}, \quad \lambda \in \mathbb{D}. \tag{5.28}$$

Formula (5.28) defines an analytic extension of every $g \in M_\perp$ to $\mathbb{C}_\infty \setminus \sigma(T_M)$, where we denote by $\mathbb{C}_\infty$ the Riemann sphere. The link between g and its extension to $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ given by (5.28) is unclear when $\sigma(T_M) = \mathbb{T}$. When $\sigma(T_M) = \{1\}$, of course, g extends analytically to $\mathbb{C}_\infty \setminus \{1\}$.

Assume again that $\sigma(T_M) = \{1\}$. We can then define $\log R_M = \log(1 + (R_M - 1))$ by the usual series, and $\log R_M$ is a quasinilpotent operator. Set for $z \in \mathbb{C}$

$$R_M^z = e^{z \log R_M}, \tag{5.29}$$

$$G(z) = \overline{(1, R_M^z \cdot g)}. \tag{5.30}$$

Clearly, G is an entire function of zero exponential type, and $G(n) = \overline{(T^n \cdot 1, g)} = \widehat{g}(n)$ for $n \in \mathbb{Z}^+$. This well-known argument gives a way to associate to each z-invariant subspace M of H_τ such that $\sigma(T_M) = \{1\}$ a Hilbert space $\widetilde{M}$ of entire functions of zero exponential type, and it is easy to see that the differentiation operator is a bounded, quasinilpotent operator on $\widetilde{M}$. This theory, due to Atzmon, will be developed in [5].

We did not investigate the spectrum of T_M where $M = \bigvee_{n\geqslant 0} z^n \cdot \varphi_c$ is the singly generated z-invariant subspace having the division property constructed in Theorem 5.3. In the "abstract Keldysh method" the behavior of some functions $f \in H_\tau$ near 1 is the only ingredient of the construction, and there are good reasons to think that Nikolskii's method and Atzmon's method play a dual role: Atzmon constructs closed subspaces of $H_{\tau^{-1}}$ invariant for the backward shift, and their orthogonal complements give z-invariant subspaces of H_τ having the divison property, while Nikolskii constructs directly functions in H_τ without zeroes in $\mathbb{D}$ which are not z-cyclic in H_τ.

The comparison between these two methods clearly deserves more investigations.

Another method was introduced recently by Borichev and Hedenmalm ([12]) to construct functions without zeroes in the Bergman space B^2 which are not z-cyclic. Answering negatively a question of Korenblum, they constructed a function $f \in B^2$ which is not z-cyclic and has no zeroes in $\mathbb{D}$, such that $1/f \in B^2$. This method was developed by Hedenmalm and the second author ([26]) to produce functions without zeroes in $\mathbb{D}$ which are not z-cyclic in the Banach space

$$E_1 = \Big\{ f \in \mathcal{H}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| \mathrm{e}^{-\frac{1}{1-|z|}} < +\infty \Big\}.$$

This construction, which can certainly be adapted to weighted Hardy spaces, is rather different from the Keldysh method: the points where $|f|$ attains "extremal rates of increase" and "extremal rates of decrease" accumulate on the whole circle. This direction seems promising to construct z-invariant subspaces M of H_τ having the division property for which the sequence $(\tau_M(n))_{n\geqslant 1}$ introduced in Definition 2.7 grows as slowly as possible (notice that in the case of the Hardy space H^2 the fastest rate of growth for $\tau_M(n)$ is given by the spaces $M_c = \mathrm{e}^{c\frac{z+1}{z-1}} \cdot H^2$, $c > 0$).

If $(\tau(n))_{n\geqslant 0}$ is decreasing, the shift T on $H_\tau(\mathbb{D})$ belongs to the class $\mathbb{A}$ of Brown-Chevreau-Pearcy (and $T \in \mathbb{A}_{\aleph_0}$ if $\tau(n) \underset{n\to\infty}{\longrightarrow} 0$, see [6], [7], [15], [18] and [19]). In the second case the lattice of z-invariant subspaces of H_τ is very rich: given any bounded operator V on the separable Hilbert space such that $\|V\| < 1$, there exists two z-invariant subspaces M and N of H_τ with $N \subset M$, such that V is unitarily equivalent to the compression to $M \ominus N$ of the shift T on H_τ. This method also shows that given any inner function U, there exists $f, g \in H_\tau$ such that $\langle f, g\rangle = 1$ and $\langle z^n \cdot U \cdot f, g\rangle = 0$ for $n \geqslant 0$, so that the spaces $\bigvee_{n\geqslant 0} z^n \cdot f$ and $\bigvee_{n\geqslant 0} z^n \cdot U \cdot f$ are distinct, but have the same zero set if U is singular. It was not possible so far to perform this construction in order to obtain a function f without zeroes in $\mathbb{D}$.

We conclude the paper by mentioning an interesting result of Borichev ([11]).

Applied to the weighted Hardy spaces $H_\tau, \tau \in \mathcal{S}^+$, his construction, based on lacunary series, shows that if $\liminf_{n\to\infty} \tau(n) = 0$ then for every p, $2 \leqslant p \leqslant \infty$ there exists a z-invariant subspace M_p of H_τ such that $Z(M_p) = \emptyset$ and such that $\dim(M_p \ominus zM_p) = p$. Unfortunately, we need $\dim(M \ominus zM) = 1$ to have the division property, and it follows from the discussion in Section 2 that for the spaces M_p constructed by Borichev we have $\sigma(T_{M_p}) = \overline{\mathbb{D}}$, where T_{M_p} is defined by (2.13) (while $\sigma(T_M) \subset \mathbb{T}$ if M is a z-invariant subspace of H_τ having the division property).

Note added in proof. After this paper was submitted Borichev, Hedenmalm and the second author developped in "Large Bergman spaces: invertibility, cyclicity, and subspaces of arbitrary index" a new method to produce noncyclic elements without zeroes in the open unit disc for a very large class of weighted Bergman spaces. They obtain in particular in this preprint nontrivial analytic translation invariant subspaces for some quasianalytic weights with odd logarithm, which are completely different from the translation invariant subspaces constructed by Domar in [21]. The question of existence of nontrivial translation invariant subspaces having the division property in arbitrary weighted Bergman spaces (or equivalently, in weighted Hardy spaces associated to arbitrary log-convex weights) remains open. In a very different direction A. Atzmon constructed non trivial translation invariant subspaces of l_ω for all even weights on $\mathbb{Z}$ (see "On the existence of translation invariant subspaces of symmetric self-adjoint sequence paces on $\mathbb{Z}$", to appear in J. Funct. Analysis). His short and elegant proof is indirectly related to some recent developments of the method of Lomonossov (see "An extension of Lomonosov's techniques to non-compact operators", Trans. Amer. Math. Soc. 348(1996), 975–995, by A. Simonic).

Acknowledgements. The authors wish to thank A. Atzmon and N.K. Nikolskii for fruitful discussions during the preparation of this paper, and A. Borichev and the referee for their pertinent comments, which helped to improve the final version and shorten several proofs.

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Received September 21, 1998; revised April 20, 1999.