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A RESOLVENT APPROACH TO STABILITY OF OPERATOR SEMIGROUPS

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ABSTRACT. It is proposed to characterize stability of strongly continuous and discrete operator semigroups in Banach and Hilbert spaces in terms of local resolvents of their "generators". Several such criterias are given and their relations to the previous results are discussed.

KEYWORDS: *Operator semigroups, stability, resolvent.*

MSC (2000): Primary 47D06; Secondary 34D05.

1. INTRODUCTION

Recently, the theory of asymptotic behaviour of operator semigroups, especially the study of the relations between their asymptotic and spectral properties, has received considerable attention. The revival started from the following Katznelson-Tzafriri result which we state together with its C_0-semigroup analogue obtained somewhat later.

THEOREM 1.1. (i) ([25]) *Let T be a linear bounded operator in a Banach space X such that $\sup\limits_{n\geqslant 0}\|T^n\|<\infty$. If the function*

$$f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n,\quad \lambda\in\mathbb{C},\,|\lambda|=1,\,\sum_{n=0}^{\infty}|a_n|<\infty,$$

is of spectral synthesis with respect to $\sigma(T)\cap\{\lambda\in\mathbb{C}:|\lambda|=1\}$, then

$$\lim_{n\to\infty}\|T^n\widehat{f}(T)\|=0,$$

where $\widehat{f}(T)x:=\sum\limits_{n=0}^{\infty}a_nT^nx$, $x\in X$.

(ii) ([17], [45]) *Let* $(T(t))_{t\geqslant 0}$ *be a bounded* C_0*-semigroup in a Banach space* X *with the generator* A*. If the function* $f \in L_1(\mathbb{R}^+)$ *is of spectral synthesis with respect to* $\mathrm{i}\sigma(A)\cap\mathbb{R}$*, then*

$$\lim_{t\to\infty} \|T(t)\,\widehat{f}(T)\| = 0$$

where $\widehat{f}(T)x := \int_0^\infty T(t)xf(t)\,\mathrm{d}t,\ x \in X$.

Note that, by definition, $f \in L_1(\mathbb{R}^+)$ is of spectral synthesis with respect to $\mathrm{i}\sigma(A)\cap\mathbb{R}$, if there exists $\{f_n : n \geqslant 1\} \subset L_1(\mathbb{R})$, $\lim_{n\to\infty} f_n = f$ in $L_1(\mathbb{R})$, such that the Fourier transform $\widehat{f_n} = 0$ in a neighbourhood of $\mathrm{i}\sigma(A)\cap\mathbb{R}$, $n \geqslant 1$. Thus, the set $\mathrm{i}\sigma(A)\cap\mathbb{R}$ is necessarily of Lebesgue measure zero. Similarly, under the conditions of Theorem 1.1 (i), the Lebesgue measure of $\sigma(T)\cap\{\lambda\in\mathbb{C} : |\lambda| = 1\}$ is zero.

After several papers devoted to various extensions of this result, the following famous Arendt-Batty-Lyubich-Vu ("ABLV-") theorem emerged in [2], [30] (see also [16], [17]).

THEOREM 1.2. ([2], [30]) *Let* $(T(t))_{t\geqslant 0}$ *be a bounded* C_0*-semigroup in a Banach space* X *with generator* A*. If*

(i) $\sigma(A)\cap\mathrm{i}\mathbb{R}$ *is countable;*

(ii) $\sigma_{\mathrm{p}}(A^*)\cap\mathrm{i}\mathbb{R} = \emptyset$*;*

then for every $x \in X$*:*

$$\lim_{t\to\infty}\|T(t)x\| = 0,$$

or in other words, $(T(t))_{t\geqslant 0}$ *is stable.*

Observe that Theorem 1.2 can be obtained from Theorem 1.1 (ii). If the conditions of Theorem 1.2 hold, then the set

$$M := \{\widehat{f}(T)x : x \in X,\ f \in L_1(\mathbb{R}^+),\ \operatorname{supp}\widehat{f}\cap\big(\mathrm{i}\sigma(A)\cap\mathbb{R}\big) = \emptyset\}$$

is dense in X, and $\|T(t)f(T)x\| \to 0$, $t\to\infty$ for every $x \in X$. This approach was realized in [16], [17]. There is also a discrete analogue of Theorem 1.2, with the same connection to Theorem 1.1 (i), see [2] and [17]. Since then a significant theory has been developed. The above theorem was generalized to wider classes of semigroups (and also Volterra equations) and to more "qualitative" forms (individual stability, estimates etc.). For an account of these developments see the book [34] and the surveys [4], [46]. But most of extensions follow the principle "the smaller the set $\sigma(A)\cap\mathrm{i}\mathbb{R}$, the better the asymptotic properties of the orbits of $(T(t))_{t\geqslant 0}$". On the other hand, as was already shown in [2], the ABLV-theorem is best possible in the sense that for every closed uncountable set $E \subset \mathbb{R}$ there exists a Banach space Y and an isometric C_0-group $(S(t))_{t\in\mathbb{R}}$ in Y such that the generator B of $(S(t))_{t\in\mathbb{R}}$ has the properties $\sigma(B) \subseteq \mathrm{i}E$ and $\sigma_{\mathrm{p}}(B^*)\cap\mathrm{i}\mathbb{R} = \emptyset$.

Recently, it has been observed in [12] and [21], that rather weak requirements to the boundary behaviour of $R(\lambda, A)$ in $\mathbb{C}_+ = \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > 0\}$ ensure the stability of $(T(t))_{t\geqslant 0}$. For example, Theorem 2.3 and the reasoning on page 63 from [12] imply that a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ in a Banach space X is stable provided the set of $x \in X$ with $R(\lambda, A)x$ continuously extendable in $\overline{\mathbb{C}}_+$ is dense in X.

At the same time, the "local" variant of the ABLV-theorem was obtained by C.J.K. Batty, J.M.A.M. van Neerven and F. Räbiger in [4].

THEOREM 1.3. ([4]) *Suppose $(T(t))_{t\geqslant 0}$ is a bounded C_0-semigroup in a Banach space X with generator A, $x \in X$ is fixed. Denote by $\sigma_{\mathrm{u}}(A,x)$ the set of $\mathrm{i}\beta \in i\mathbb{R}$ such that the local resolvent, $R(\alpha+\mathrm{i}\beta, A)x$, $\alpha > 0$, does not extend analytically in some neighbourhood of $\mathrm{i}\beta$. If*

(i) *$\sigma_{\mathrm{u}}(A,x)$ is countable;*

(ii) *for all β with $\mathrm{i}\beta \in \sigma_{\mathrm{u}}(A,x)$:*

$$\lim_{\alpha\to 0+} \alpha R(\alpha+\mathrm{i}\beta, A)x = 0,$$

then

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

(The second condition was stated in a slightly different form.)

Theorem 1.3 leads to the following generalization of the ABLV-theorem.

COROLLARY 1.4. ([4]) *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X with generator A. Suppose there exists a dense set $M \subset X$ such that:*

(i) *for every $x \in M$: $\sigma_{\mathrm{u}}(A,x)$ is countable;*

(ii) *for every $\beta \in -\mathrm{i}\sigma(A) \cap \mathbb{R}$ and every $x \in X$:*

$$\lim_{\alpha\to 0+} \alpha R(\alpha+\mathrm{i}\beta, A)x = 0,$$

then $(T(t))_{t\geqslant 0}$ is stable.

The conditions of Corollary 1.4 hold for the C_0-semigroup of left shifts in $L_1(\mathbb{R}^+)$ with the spectrum of the generator equal to $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$. But as was indicated in [6], they are not satisfied for the stable C_0-semigroup $(T(t))_{t\geqslant 0}$ defined by

$$(T(t)f)(s) = \begin{cases} f(s-t) & s \geqslant t, \\ 0 & s < t, \end{cases}$$

in the Banach space $L_1(\mathbb{R}^+, w)$ with an appropriate weight w (and the same spectrum of the generator).

So, for all the above statements we see the borders of the "spectral language" in characterization of semigroup stability.

In the course of these heuristical arguments, the aim of the present paper is to obtain the stability conditions for operator semigroups without appealing to their spectral properties. It turned out that the behaviour of $R(\lambda, A)x$ *near* to its set of singular points on $\mathrm{i}\mathbb{R}$, rather than the *absence* of this set, is crucial for the characterization of asymptotic properties of the orbit $T(t)x$. It also appeared that via this approach it is possible to explain some previous spectral stability criterias, and this was its unexpected byproduct.

The paper is organized as follows. In Section 2 we will give some basic constructions and auxiliary statements, which will be used in the further reasonings, and outline the key idea of proofs. Section 3 is devoted to the study of the resolvent criterias for stability of operator semigroups in Hilbert spaces. The same questions for Banach space semigroups are studied in Section 4. Finally, we consider some examples to illustrate the obtained statements and analyze their relations to previous results in Section 5.

2. PRELIMINARY NOTIONS AND CONSIDERATIONS

This section contains essentially known material but arranged in the way most appropriate for our further purposes.

We start with some notation. Let X be a complex Banach space and H be a complex Hilbert space. Suppose $(T(t))_{t\geqslant 0}$ is a C_0-semigroup in X or in H, with generator A, T is a linear bounded operator in X or in H, and $(T^n)_{n\geqslant 0}$ is the corresponding discrete operator semigroup. Denote by $R(\lambda, A)$ the resolvent of A, by $\rho(A)$ the resolvent set of A, by $\sigma(A)$ and $\sigma_{\mathrm{p}}(A)$ the spectrum and the point spectrum of A respectively, and by $\mathrm{Im}(A)$ the image of the operator A. Let $\mathcal{D}(\mathbb{R})$ stand for the space of $C^\infty(\mathbb{R})$-functions with compact support. The symbol "$*$" always means conjugation of the corresponding space or operator. We fix the following definition for the sequel.

DEFINITION 2.1. A C_0-semigroup $(T(t))_{t\geqslant 0}$ (discrete operator semigroup $(T^n)_{n\geqslant 0}$) in X is called stable, if for every $x \in X$:

$$\lim_{t\to\infty} \|T(t)x\| = 0,$$

(respectively $\lim\limits_{n\to\infty} \|T^n x\| = 0$).

By definition it follows that a stable operator semigroup is necessarily bounded:

$$\sup_{t\geqslant 0} \|T(t)\| < \infty \quad (\sup_{n\geqslant 0} \|T^n\| < \infty).$$

Thus, without loss of generality, we can consider only bounded operator semigroups.

It is a remarkable fact and one of the cornerstones of the theory of asymptotic behaviour of semigroups that the study of stable semigroups is reduced essentially to the study of isometric semigroups (groups).

This correspondence is described in the following statement.

THEOREM 2.2. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach (Hilbert) space X. Then there exist a Banach (Hilbert) space Y, a linear bounded operator $j : X \to Y$, and an isometric C_0-group $(S(t))_{t\in\mathbb{R}}$ in Y such that:*

(i) *$S(t)j(x) = j(T(t)x)$, $x \in X$, $t \geqslant 0$;*

(ii) *for every $x \in X : j(x) = 0$ if and only if $\lim\limits_{t\to\infty} \|T(t)x\| = 0$;*

(iii) *$\bigcup\{S(-t)j(X) : t \geqslant 0\}$ is dense in Y.*

Such a group $(S(t))_{t\in\mathbb{R}}$ is unique up to similarity: if $(S'(t))_{t\in\mathbb{R}}$ is another isometric C_0-group in a Banach (Hilbert) space Y' satisfying the above properties, then $S'(t) = KS(t)K^{-1}$ for some linear bounded invertible operator $K : Y \to Y'$. Moreover, if $(T(t))_{t\geqslant 0}$ is a semigroup of Hilbert space contractions, then $(S(t))_{t\in\mathbb{R}}$ is unique up to unitary equivalence.

We will call the triple $\mathcal{S} = \big((S(t))_{t\in\mathbb{R}}, j, Y\big)$ the limit isometric C_0-group in accordance with established terminology in semigroup theory and will refer to it simply as $(S(t))_{t\in\mathbb{R}}$. Only the properties (i) and (ii) of $(S(t))_{t\in\mathbb{R}}$ will be used in the proofs. The property (iii) implies the similarity equivalence of the limit isometric groups, and we mentioned it to underline the "universality" of $(S(t))_{t\in\mathbb{R}}$ ([7]).

Because our reasonings will depend essentially on the existence of $(S(t))_{t\in\mathbb{R}}$, we will outline its construction following mainly [7], but with some modifications taken from [4], [5]. It will be important for us, that a C_0-semigroup $(T(t))_{t\geqslant 0}$ in an initial Hilbert space X transforms to a (part of) isometric C_0-group $(S(t))_{t\in\mathbb{R}}$ in a space Y, which is also *Hilbert*.

Let L be an arbitrary Banach limit over $L_\infty(\mathbb{R}^+)$, that is a linear functional $L : L_\infty(\mathbb{R}^+) \to \mathbb{C}$ such that $L(1) = 1$, $\|L\| = 1$, and $L(f(t+s)) = L(f(t))$, $\{t, s\} \subset \mathbb{R}^+$, for every $f \in L_\infty(\mathbb{R}^+)$. Then, define a new seminorm $p_1(x) := L(\|T(\cdot)x\|)$ on X, if X is a Banach space, and a new semiinner product $p_2(x, y) := L(T(\cdot)x, T(\cdot)y)$ on X, if X is a Hilbert space. The rest of the reasonings is the same for both cases and the symbol $p = p(x)$ will refer to $p_1(x)$ or to $p_2(x) := \sqrt{p_2(x, x)}$ depending on the original space X.

Consider the algebraic quotient space $X/\operatorname{Ker} p$, and the corresponding quotient operator $j : X \to X/\operatorname{Ker} p$. Observe that in both Banach space and Hilbert space cases,

$$j(x) = 0 \quad \text{if and only if } \lim_{t\to\infty} \|T(t)x\| = 0, \quad x \in X.$$

This follows directly from the next inequality for Banach limits

$$\liminf_{t\to\infty} f(t) \leqslant L(f) \leqslant \limsup_{t\to\infty} f(t), \quad f \in L_\infty(R^+),\ f \geqslant 0,$$

and the fact that for fixed $x \in X$ the properties

(1) $\lim_{t\to\infty} \|T(t)x\| = 0$;

(2) $\lim_{t\to\infty} \|T(t)x\|^2 = 0$;

(3) $\liminf_{t\to\infty} \|T(t)x\| = 0$,

are equivalent (see also [7]).

Let Y_0 be the completion of $X/\operatorname{Ker} p$ under the norm p . Then it is easy to show that the C_0-semigroup $(S_0(t))_{t\geqslant 0}$, defined by

$$S_0(t)j(x) = j(T(t)x), \quad x \in X,\ t \geqslant 0,$$

is isometric and extends by continuity to the whole Banach (Hilbert) space Y_0.

Having defined the isometric C_0-semigroup $(S_0(t))_{t\geqslant 0}$ on Y_0, we can extend it to an isometric (unitary) C_0-group in the original Banach (Hilbert) space Y_0, if $\sigma(A) \not\supseteq \mathrm{i}\mathbb{R}$ ([5]), and in a larger Banach (Hilbert) space otherwise ([4], [5], [7], [15]). Namely, in the last case, if $(S_0(t))_{t\geqslant 0}$ is an isometric C_0-semigroup in a Banach (Hilbert) space Y_0, then there is a Banach (Hilbert) space Y and an isometric (unitary) C_0-group $(S(t))_{t\in\mathbb{R}}$ in it such that

$$S_0(t)j(x) = S(t)j(x) \quad \text{for every } x \in X \text{ and every } t \in \mathbb{R}^+,$$

where we identified the C_0-semigroup $(S_0(t))_{t\geqslant 0}$ with its isometric image. This statement was proved first in [15].

For the formal definition of a C_0-group $(S(t))_{t\in\mathbb{R}}$ as well as for the details we refer the reader to [15], [7], [5]. The proof of uniqueness (up to similarity) of $(S(t))_{t\in\mathbb{R}}$, based on the property (iii), Theorem 2.2, can be found in [7] and for a less general situation in [5], [45].

Word by word, repeating the above construction for the discrete operator semigroup $(T^n)_{n\geqslant 0}$ up to replacement of $\mathbb{R}^+$ by $\mathbb{N} \cup \{0\}$, we obtain an analogue of Theorem 2.2 (see [7], and also [26], [5]).

THEOREM 2.3. *Let $(T^n)_{n\geqslant 0}$ be a bounded discrete operator semigroup in a Banach (Hilbert) space X (or, in another terminology, T is a power-bounded operator in X). Then there exists a Banach (Hilbert) space Y containing X, a linear bounded operator $j : X \to Y$, and an isometric group $(S^n)_{n\in\mathbb{Z}}$ in Y such that*

(i) *$S^n j(x) = j(T^n x)$, $x \in X$;*

(ii) *for every $x \in X : j(x) = 0$ if and only if $\lim_{n\to\infty} \|T^n x\| = 0$;*

(iii) *$\bigcup\{S^{-n} j(X) : n \geqslant 0\}$ is dense in Y.*

Such a group $(S^n)_{n\geqslant 0}$ is unique up to similarity in the same sense as in Theorem 2.2.

As above, $\mathcal{S} = \big((S^n)_{n\in\mathbb{Z}}, j, Y\big)$ will be called the limit isometric group.

Remark that, it is possible to combine both "continuous" and "discrete" cases here via the language of representations of locally compact groups, as it was done in several works. But since our reasonings will depend essentially on the theory of analytic functions, we treat these cases separately with emphasis on the continuous one.

Thus, in view of Theorem 2.2, to obtain the stability of $(T(t))_{t\geqslant 0}$, it is sufficient to prove that the corresponding limit isometric C_0-group $(S(t))_{t\in\mathbb{R}}$ is zero on the image $j(X)$ of the Banach (Hilbert) space X. Let here and in the sequel B denote the generator of the limit isometric C_0-group $(S(t))_{t\in\mathbb{R}}$. If X is a Hilbert space, then by Stone's theorem

$$B = \mathrm{i}\mathcal{C},$$

for some selfadjoint operator $\mathcal{C}$ in Y. So, the stability of $(T(t))_{t\geqslant 0}$ follows from the formal equality

$$j(X) = E(\mathbb{R})j(X) = \{0\},$$

where $E(\cdot)$ is the spectral measure corresponding to $\mathcal{C}$, reflecting the way of the further reasonings.

Similarly, the equality $E^{\mathrm{d}}(\mathbb{T})j(X) = 0$, where $E^{\mathrm{d}}(\cdot)$ is the spectral measure corresponding to the unitary operator S, implies the stability of the discrete operator semigroup $(T^n)_{n\geqslant 0}$.

If X is a Banach space, then we need to invoke the substitutes of Stone's theorem in this more general setting. One of them is its distributional counterpart, provided by the next theorem and used in the Banach space setting in [18], [22].

THEOREM 2.4. ("The edge of the wedge theorem", [39]) *Let $P := \{\alpha + \mathrm{i}\beta : |\alpha| < a,\ |\beta| < b\}$, $P^+ := \{\alpha + \mathrm{i}\beta : 0 < \alpha < a,\ |\beta| < b\}$, $P^- := \{\alpha + \mathrm{i}\beta : -a < \alpha < 0,\ |\beta| < b\}$. Suppose that the functions $f^+, f^- : P^\pm \to X$ are analytic in P^+, P^- respectively, and satisfy the condition*

$$\sup_{\beta\in(-b,b)} \|f^\pm(\pm\alpha + \mathrm{i}\beta)\| = \mathrm{O}(\alpha^{-n}), \quad \alpha \to 0+,\ n \in \mathbb{N}. \tag{2.1}$$

Then $f^\pm$ have boundary distributional values

$$Q^\pm(g) = \lim_{\alpha\to 0+} \int_{-b}^{b} f^\pm(\pm\,\alpha + \mathrm{i}\beta) g(\beta)\,\mathrm{d}\beta, \quad g \in \mathcal{D}((-b,b)), \tag{2.2}$$

in $P^{\pm}$ respectively. If the distributions $Q^{\pm}$ coincide on $(-b, b)$, then there exists $f : P \to X$, analytic in P, such that $f|P^+ = f^+$, $f|P^- = f^-$.

This Banach space version can be obtained from its scalar counterpart by a standard application of the uniform boundedness and the Hahn-Banach principles.

If $(S(t))_{t\in\mathbb{R}}$ is an isometric C_0-group, then the resolvent of its generator, $R(\lambda, B)$, satisfies the inequality

$$\|R(\lambda, B)\| \leqslant \frac{1}{|\operatorname{Re}\lambda|}, \quad \operatorname{Re}\lambda \neq 0. \tag{2.3}$$

Define the distribution $Q : \mathcal{D}(\mathbb{R}) \to L(Y)$ by the formula

$$Q(g) := \lim_{\alpha\to 0+} \frac{1}{2\pi} \int\limits_{\mathbb{R}} \big(R(\alpha + \mathrm{i}\beta, B) - R(-\alpha + \mathrm{i}\beta, B)\big) g(\beta)\, \mathrm{d}\beta, \quad g \in \mathcal{D}(\mathbb{R}). \tag{2.4}$$

(For use of $Q(\cdot)$ in the spectral theory of bounded operator semigroups see, for example, [18], [22], [33].) In view of the inequality (2.3) and Theorem 2.4, the distribution $Q(\cdot)$ is well defined, and so are the distributions $Q(\cdot)y : \mathcal{D}(\mathbb{R}) \to Y$, $y \in Y$. Note that $R(\lambda, B)y$ extends to an entire function if and only if $y = 0$ ([37] p. 23). So, by Theorem 2.4, the distributions $Q(\cdot)y$, $y \neq 0$ are different from zero. It is a classical fact that a different from zero distribution has nonempty support. Since this property of $Q(\cdot)y$ will be used frequently, we will state it separately.

LEMMA 2.5. *For every $y \in Y$, $y \neq 0$, we have*

$$\operatorname{supp} Q(\cdot)y \neq \emptyset. \tag{2.5}$$

Discrete operator semigroups $(T^n)_{n\geqslant 0}$ in Banach spaces can be considered following the same scheme as the C_0-semigroups. Let T be a bounded linear operator in X with $\sup\limits_{n\geqslant 0} \|T^n\| < \infty$. The resolvent $R(\lambda, T) := (\lambda I - T)^{-1}$ of T is defined (a priori) in $U := \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ by the expansion

$$R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n, \quad \lambda \in U. \tag{2.6}$$

If $(S^n)_{n\in\mathbb{Z}}$ is the limit unitary group corresponding to the bounded semigroup $(T^n)_{n\geqslant 0}$, then the distribution $Q_d : \mathcal{D}(\mathbb{T}) \to L(Y)$ defined by the formula

$$Q^{\mathrm{d}}(g) := \lim_{r\to 1-} \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathbb{T}} \left(\frac{1}{r} R\left(\frac{\xi}{r}, S\right) - rR(r\xi, S)\right) g(\xi)\, \mathrm{d}\xi, \quad g \in \mathcal{D}(\mathbb{T}), \tag{2.7}$$

has a property similar to the case of C_0-semigroups.

LEMMA 2.6. *For every $y \in Y$, $y \neq 0$,*

$$\operatorname{supp} Q^d(\cdot)y \neq \emptyset. \tag{2.8}$$

This is the simple consequence of Liouville's theorem.
The support of $Q^{\mathrm{d}}(\cdot)$ can be studied either by the "reduction" to the halfplane case (as, for instance, in [44], p. 220–225) or directly. The latter approach for the distribution $Q^{\mathrm{d}}(\cdot)$ was employed, in particular, in [31], [35]. In the sequel we reduce the proofs of the statements on stability of Banach space discrete operator

semigroups to their "continuous" analogues, and deal with the distribution $Q(\cdot)$ instead of $Q^{\mathrm{d}}(\cdot)$. Nevertheless it is instructive to have in mind both Lemma 2.5 and Lemma 2.6.

Summarizing the above reasonings, if for every $x \in X$:

$$Q(\cdot)j(x) = 0 \quad (Q^{\mathrm{d}}(\cdot)j(x) = 0),$$

then $j(X) = \{0\}$, and the C_0-semigroup $(T(t))_{t\geqslant 0}$ (or, respectively, discrete operator semigroup $(T^n)_{n\geqslant 0}$) is stable.

Finally, we note the following simple statement which is the key for the rest of the paper.

LEMMA 2.7. ("Majorization lemma") *Let B be the generator of the limit isometric C_0-group $(S(t))_{t\in\mathbb{R}}$. Then:*

(i) *the halfplane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ is contained in $\rho(A) \cap \rho(B)$, and for every $x \in X$ and $\lambda \in \mathbb{C}_+$:*

$$\|R(\lambda, B)j(x)\| \leqslant \|j\|\,\|R(\lambda, A)x\|,$$
$$\|R^2(\lambda, B)j(x)\| \leqslant \|j\|\,\|R^2(\lambda, A)x\|;$$

(ii) *for every $x \in X$ and $\alpha > 0$:*

$$\|(R(\alpha+\mathrm{i}\beta, B) - R(-\alpha+\mathrm{i}\beta, B))j(x)\| \leqslant 6\alpha\|j\|\,\|R^2(\alpha+\mathrm{i}\beta, A)x\|;$$

(iii) *for every $x \in X$ and $\alpha > 0$:*

$$\frac{1}{2}\|\alpha R^2(\alpha+\mathrm{i}\beta, A)x\| \leqslant \|(R(\alpha+\mathrm{i}\beta, A) - R(2\alpha+\mathrm{i}\beta, A))x\| \leqslant 2\|\alpha R^2(\alpha+\mathrm{i}\beta, A)x\|.$$

Proof. (i) By Theorem 2.2, $(S(t))_{t\geqslant 0}$ and $(T(t))_{t\geqslant 0}$ are both bounded semigroups. Moreover, if $\operatorname{Re}\lambda > 0$, then

$$\|R^2(\lambda, B)x\| = \Big\|\int_0^\infty \mathrm{e}^{-\lambda t} t\, S(t)j(x)\,\mathrm{d}t\Big\| = \Big\|j\Big(\int_0^\infty \mathrm{e}^{-\lambda t} t\, T(t)x\,\mathrm{d}t\Big)\Big\| \leqslant \|j\|\,\|R^2(\lambda, A)x\|.$$

Similarly,

$$\|R(\lambda, B)j(x)\| \leqslant \|j\|\,\|R(\lambda, A)x\|, \quad \operatorname{Re}\lambda > 0.$$

(ii) By the first resolvent identity,

$$R(-\alpha+\mathrm{i}\beta, B) - R(\alpha+\mathrm{i}\beta, B) = 2\alpha R(\alpha+\mathrm{i}\beta, B)R(-\alpha+\mathrm{i}\beta, B),$$

so

$$\begin{aligned} &R(\alpha+\mathrm{i}\beta, B) - R(-\alpha+\mathrm{i}\beta, B)\\ &\quad= 2\alpha R(\alpha+\mathrm{i}\beta, B)(2\alpha R(\alpha+\mathrm{i}\beta, B)R(-\alpha+\mathrm{i}\beta, B) + R(\alpha+\mathrm{i}\beta, B))\\ &\quad= 2\alpha R^2(\alpha+\mathrm{i}\beta, B)(2\alpha R(-\alpha+\mathrm{i}\beta, B) + I). \end{aligned}$$

In view of the resolvent estimate (2.3) and (i),

$$\begin{aligned} \|(R(\alpha+\mathrm{i}\beta, B) - R(-\alpha+\mathrm{i}\beta, B))j(x)\| &\leqslant 6\alpha\|R^2(\alpha+\mathrm{i}\beta, B)j(x)\|\\ &\leqslant 6\alpha\|j\|\,\|R^2(\alpha+\mathrm{i}\beta, A)x\|. \end{aligned}$$

(iii) We will prove only the left inequality. The reasoning for the right inequality is similar. Using the first resolvent identity we obtain

$$\alpha R^2(\alpha+\mathrm{i}\beta,A)-\alpha R(2\alpha+\mathrm{i}\beta,A)=\alpha^2R^2(\alpha+\mathrm{i}\beta,A)R(2\alpha+\mathrm{i}\beta,A),$$

and then for $x\in X$

$$\begin{aligned}\|\alpha R^2(\alpha+\mathrm{i}\beta,A)x\|&\leqslant\|(\alpha R(\alpha+\mathrm{i}\beta,A)+I)(R(\alpha+\mathrm{i}\beta,A)-R(2\alpha+\mathrm{i}\beta,A))x\|\\&\leqslant 2\|(R(\alpha+\mathrm{i}\beta,A)-R(2\alpha+\mathrm{i}\beta,A))x\|. \quad\blacksquare\end{aligned}$$

The analogue of Lemma 2.7 is also true for the case of discrete operator semigroups with the obvious modifications which we leave to the reader.

3. THE HILBERT SPACE CASE

The operator semigroups in a Hilbert space H can be considered as a model for our goals in view of the explicit representation of the corresponding limit unitary groups by spectral integrals. The validity of the Plancherel theorem for $L_2(\mathbb{R}^+,H)$-functions allows to obtain a simple global stability criterion for a C_0-semigroup in H. The arguments of this type were used, in particular, in [11] for the study of operator groups similar to unitary groups.

THEOREM 3.1. *Let $(T(t))_{t\geqslant0}$ be a bounded C_0-semigroup in a Hilbert space H with generator A. Then $(T(t))_{t\geqslant0}$ is stable if and only if for every $x\in H$*

$$\lim_{\alpha\to0+}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=0. \tag{3.1}$$

REMARK 3.2. Observe, that if $(T(t))_{t\geqslant0}$ is a bounded C_0-semigroup in a Hilbert space H, then by the vector-valued Plancherel theorem,

$$\sup_{\alpha>0}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=\sup_{\alpha>0}2\pi\alpha\int_{0}^{\infty}\mathrm{e}^{-2\alpha t}\|T(t)x\|^2\,\mathrm{d}t\leqslant\pi C^2\|x\|^2, \tag{3.2}$$

where $C:=\sup\limits_{t\geqslant0}\|T(t)\|$. Hence the condition of Theorem 3.1 can be restated as follows: there exists a dense set $M\subset H$ such that for every $x\in M$

$$\lim_{\alpha\to0+}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=0.$$

Note that since $(T^*(t))_{t\geqslant0}$ is also a bounded C_0-semigroup in H, for every $x\in H$:

$$\sup_{\alpha>0}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A^*)x\|^2\,\mathrm{d}\beta\leqslant\pi C^2\|x\|^2. \tag{3.3}$$

Proof. NECESSITY. Suppose $(T(t))_{t\geqslant 0}$ is stable. By the vector-valued Plancherel theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=\int_0^{\infty}\mathrm{e}^{-2\alpha t}\|T(t)x\|^2\,\mathrm{d}t,\quad x\in H.$$

In view of the regularity of the Abel summation ([20], p. 505),

$$\lim_{\alpha\to 0+}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=2\pi\lim_{\alpha\to 0+}\alpha\int_0^{\infty}\mathrm{e}^{-2\alpha t}\|T(t)x\|^2\,\mathrm{d}t=0.$$

SUFFICIENCY. Suppose that for every $x\in H$

$$\lim_{\alpha\to 0+}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=0.$$

Define an equivalent norm $\|x\|_1:=\sup_{t\geqslant 0}\|T(t)x\|$ on H,

$$\|x\|_1\geqslant\|x\|\geqslant\frac{1}{C}\|x\|_1,$$

$C:=\sup_{t\geqslant 0}\|T(t)\|$. Since $(T(t))_{t\geqslant 0}$ is a contraction semigroup in the Banach space $X:=(H,\|\cdot\|_1)$ it follows that $\lim_{t\to\infty}\|T(t)x\|_1$ exists for every $x\in X$. Then for every $x\in H$

$$\begin{aligned}0&=\lim_{\alpha\to 0+}\alpha\int_{-\infty}^{\infty}\|R(\alpha+\mathrm{i}\beta,A)x\|^2\,\mathrm{d}\beta=2\pi\lim_{\alpha\to 0+}\alpha\int_0^{\infty}\mathrm{e}^{-2\alpha t}\|T(t)x\|^2\,\mathrm{d}t\\&\geqslant\frac{2\pi}{C^2}\limsup_{\alpha\to 0+}\alpha\int_0^{\infty}\mathrm{e}^{-2\alpha t}\|T(t)x\|_1^2\,\mathrm{d}t.\end{aligned}\tag{3.4}$$

In view of (3.4) and the regularity of the Abel summation,

$$\limsup_{t\to\infty}\|T(t)x\|\leqslant\lim_{t\to\infty}\|T(t)x\|_1=0.\quad\blacksquare$$

Thus, the above statement gives a global integrability criterion for the stability of C_0-semigroups in Hilbert spaces. The following statement is its “localized” version. We assume, in the sequel, that for a selfadjoint operator $\mathcal{C}$ with the corresponding spectral measure $E(\cdot)$, the resolution of the identity $E_t:=E((-\infty,t))$, is normalized such that

$$(E_tj(x),j(x))=\frac{(E_{t-0}j(x),j(x))+(E_{t+0}j(x),j(x))}{2},\quad t\in\mathbb{R}.$$

We will use freely basic properties of the spectral resolutions of selfadjoint (and also unitary) operators. These properties can be found, for example, in [9].

THEOREM 3.3. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Hilbert space H with generator A. Then $(T(t))_{t\geqslant 0}$ is stable if and only if for every $\beta_0 \in \mathbb{R}$ there exist $\varepsilon > 0$ and a dense set $M := M(\beta_0, \varepsilon)$ in H such that for every $x \in M$:*

$$\lim_{\alpha\to 0+} \alpha \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(\alpha + \mathrm{i}\beta, A)x\|^2 \,\mathrm{d}\beta = 0. \tag{3.5}$$

The necessity of (3.5) is proved in Theorem 3.1, so we only have to prove its sufficiency.

Proof. Let $(S(t))_{t\in\mathbb{R}}$ be the limit unitary C_0-group in a Hilbert space Y corresponding to $(T(t))_{t\geqslant 0}$, and let B be the generator of this group. By Stone's theorem, $B = \mathrm{i}\mathcal{C}$ for some selfadjoint operator $\mathcal{C}$. If $E(\cdot)$ is the spectral measure corresponding to the operator $\mathcal{C}$, then

$$\begin{aligned}\int_{\mathbb{R}} \frac{\alpha}{(\beta-t)^2+\alpha^2}\,\mathrm{d}(E_t\, j(x), j(x)) &= \alpha\|R(-\mathrm{i}\alpha+\beta, C)j(x)\|^2 = \alpha\|R(\alpha+\mathrm{i}\beta, B)j(x)\|^2 \\ &\leqslant \alpha\|j\|\,\|R(\alpha+\mathrm{i}\beta, A)x\|^2, \quad \alpha>0,\ x\in X.\end{aligned}$$

Let $\beta_0 \in \mathbb{R}$ be fixed. The condition

$$\lim_{\alpha\to 0+} \alpha \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(\alpha + \mathrm{i}\beta, A)x\|^2 \,\mathrm{d}\beta = 0, \quad x \in M,$$

implies that

$$\begin{aligned}0 &= \lim_{\alpha\to 0+} \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \int_{\mathbb{R}} \frac{\alpha}{(\beta-t)^2+\alpha^2}\,\mathrm{d}(E_t\, j(x), j(x))\,\mathrm{d}\beta \\ &= \lim_{\alpha\to 0+} \int_{\mathbb{R}} \mathrm{d}(E_t j(x), j(x)) \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \frac{\alpha}{(\beta-t)^2+\alpha^2}\,\mathrm{d}\beta \\ &= \lim_{\alpha\to 0+} \int_{\mathbb{R}} \mathrm{d}(E_t\, j(x), j(x)) \int_{\mathbb{R}} \frac{\alpha\, 1_{[\beta_0-\varepsilon,\beta_0+\varepsilon]}(\beta)}{(\beta-t)^2+\alpha^2}\,\mathrm{d}\beta \\ &= \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} f(t)\mathrm{d}(E_t\, j(x), j(x)),\end{aligned} \tag{3.6}$$

$$f(t) = \begin{cases} \pi, & t \in (\beta_0-\varepsilon, \beta_0+\varepsilon), \\ \frac{\pi}{2}, & t = \beta_0-\varepsilon \vee t = \beta_0+\varepsilon, \end{cases}$$

$$= \pi(E_{\beta_0+\varepsilon} j(x), j(x)) - (E_{\beta_0-\varepsilon} j(x), j(x)),$$

by the bounded convergence theorem and the radial limit properties of Poisson integrals. In particular, $E((\beta_0-\varepsilon, \beta_0+\varepsilon))j(x) = 0$, $x \in M$. As $\overline{M} = H$, the last equality holds for every $j(x)$, $x \in H$. Since the choice of $\beta_0 \in \mathbb{R}$ was arbitrary, we conclude by σ-semiadditivity of $E(\cdot)$ that

$$E(\mathbb{R})j(H) = 0,$$

or $j(H) = 0$, so the semigroup $(T(t))_{t\geqslant 0}$ is stable. ∎

Note that one can give the second proof of Theorem 3.3 using the distribution $Q(\cdot)$ and the inequalities (3.2), (3.3). The versions of these inequalities can be obtained easily for C_0-semigroups in Banach spaces with Fourier type greater than 1. (Recall that a Banach space X is a Hilbert space iff it has Fourier type equal to 2.) This fact can be used to generalize Theorem 3.3 for this class of C_0-semigroups when Stone's theorem is not available. We will not go into details here.

In contrast to Theorem 3.3, the following statement will make essential use of Stone's theorem.

THEOREM 3.4. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Hilbert space H with generator A. If for every $\beta_0 \in \mathbb{R}$ there exist $\varepsilon > 0$ and a dense set $M := M(\beta_0, \varepsilon)$ in H such that for every $x \in M$:*

$$\lim_{\alpha\to 0+} \sqrt{\alpha} R(\alpha + \mathrm{i}\beta, A)x = 0, \quad \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon), \tag{3.7}$$

then $(T(t))_{t\geqslant 0}$ is stable.

REMARK 3.5. Clearly, (3.7) is satisfied for $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \cap -\mathrm{i}\rho(A)$. On the other hand, the word "every" in the above statement is essential. For example, the resolvent $(\lambda - \mathrm{i}\beta_0)^{-1}$ of the one-dimensional semigroup $(\mathrm{e}^{\mathrm{i}\beta_0 t})_{t\geqslant 0}$ satisfies (3.7) at all points of $\mathrm{i}\mathbb{R}$ except $\mathrm{i}\beta_0$. Observe also that by the uniform boundedness principle the existence of the limit in (3.7) for *all* $x \in H$ and fixed $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \cap -\mathrm{i}\sigma(A)$ would contradict the spectral mapping theorem.

Proof. Let $\beta_0 \in \mathbb{R}$ be fixed. Let $(S(t))_{t\in\mathbb{R}}$ be the limit unitary C_0-group in a Hilbert space Y corresponding to $(T(t))_{t\geqslant 0}$, and let B be the generator of this group. By Stone's theorem, $B = \mathrm{i}\mathcal{C}$ for some selfadjoint operator $\mathcal{C}$. If $E(\cdot)$ is the spectral measure corresponding to the operator $\mathcal{C}$, then as in the proof of Theorem 3.3 for $x \in H$

$$\int_{\mathbb{R}} \frac{\alpha}{(\beta - t)^2 + \alpha^2} \, \mathrm{d}(E_t j(x), j(x)) \leqslant \alpha \|j\| \, \|R(\alpha + \mathrm{i}\beta, A)x\|^2, \quad \alpha > 0. \tag{3.8}$$

For fixed $x \in M = M(\beta_0, \varepsilon)$ let $\mu := (E(\cdot)j(x), j(x))$. By assumption,

$$\lim_{\alpha\to 0+} \alpha \|R(\alpha + \mathrm{i}\beta, A)x\|^2 = 0, \quad \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon),\ x \in M,$$

so the Poisson integral in (3.8) has a radial limit at every point β from $(\beta_0 - \varepsilon, \beta_0 + \varepsilon)$. Hence, for every $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$, the symmetric derivative of the measure

μ, $(D\mu)(\beta)$, has the property

$$(D\mu)(\beta) = \lim_{\alpha\to 0+} \frac{\mu\big((\beta-\alpha,\beta+\alpha)\big)}{2\alpha} \tag{3.9}$$
$$\leqslant \lim_{\alpha\to 0+} \int\limits_{\beta-\alpha}^{\beta+\alpha} \frac{\alpha}{(\beta-t)^2+\alpha^2}\,\mathrm{d}(E_t\,j(x),j(x))$$
$$\leqslant \lim_{\alpha\to 0+} \int\limits_{\mathbb{R}} \frac{\alpha}{(\beta-t)^2+\alpha^2}\,\mathrm{d}(E_t\,j(x),j(x)) = 0.$$

Let $\mu := \frac{1}{\pi} f\,\mathrm{d}t + \mu_{\mathrm{sing}}$, $f \geqslant 0$, be the Lebesgue decomposition of μ. Suppose that the singular part μ_{sing} of μ satisfies $\mu_{\mathrm{sing}}((\beta_0-\varepsilon,\beta_0+\varepsilon)) \neq 0$. Since, according to [38], p. 169–170, $(D\mu_{\mathrm{sing}})(t) = +\infty$, μ_{sing}-a.e. on $\mathbb{R}$, there exists a point $t_0 \in (\beta_0-\varepsilon,\beta_0+\varepsilon)$ with $(D\mu_{\mathrm{sing}})(t_0) = +\infty$. Then $(D\mu)(t_0) = +\infty$, and we obtain a contradiction with (3.9). Hence, for the density $f \in L_1(\mathbb{R})$ of μ, we have $(D\mu)(t) = \frac{1}{\pi} f(t) = 0$, a.e. on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$ by Fatou's theorem. Therefore, for every $n \in \mathbb{N}$,

$$\left(E\left(\left[\beta_0 - \varepsilon\left(1-\frac{1}{n}\right), \beta_0 + \varepsilon\left(1-\frac{1}{n}\right)\right]\right) j(x), j(x)\right) = \frac{1}{\pi}\int\limits_{\beta_0-\varepsilon(1-\frac{1}{n})}^{\beta_0+\varepsilon(1-\frac{1}{n})} f(t)\,\mathrm{d}t = 0,$$

and then $E\big((\beta_0-\varepsilon,\beta_0+\varepsilon)\big)j(x) = 0$ by σ-semiadditivity of $E(\cdot)$. The denseness of M in H implies $j(H) \subset \overline{j(M)}$. Hence $E\big((\beta_0-\varepsilon,\beta_0+\varepsilon)\big)j(x) = 0$ for every $j(x) \in Y$. Since β_0 was arbitrary and $E(\cdot)$ is σ-semiadditive we have

$$\|E(\mathbb{R})j(x)\| = 0, \quad x \in H.$$

Thus, $(T(t))_{t\geqslant 0}$ is stable. ∎

Remark 3.6. In the proof we actually used a weak form of Loomis' tauberian theorem for positive harmonic functions ([29]).

Via the construction of the limit unitary group $(S(t))_{t\in\mathbb{R}}$, the Hilbert space stability conditions can be considered as "annihilation" conditions for the measures $(E(\cdot)j(x),j(x))$, $x \in H$, determined by $(S(t))_{t\in\mathbb{R}}$. Therefore, it is important to know criteria for the separate components in the Lebesgue decomposition of $(E(\cdot)j(x),j(x))$ to vanish. Along with the a priori information, such criteria could sharpen the stability characterization. Some of them are discussed below.

The reasoning from the proof of Theorem 3.4 shows that the condition

$$\lim_{\alpha\to 0+} \sqrt{\alpha}R(\alpha+\mathrm{i}\beta, A)x = 0 \quad \text{on } (\beta_0-\varepsilon,\beta_0+\varepsilon),\ \varepsilon>0,\ \text{a.e.,} \tag{3.10}$$

"annihilates" the absolutely continuous part of the restriction of $(E(\cdot)j(x),j(x))$ to $(\beta_0-\varepsilon,\beta_0+\varepsilon)$. Conversely, if the absolutely continuous part of this restriction is zero, then (3.10) holds by the Fatou theorem.

As follows from Theorem 3.3, the condition (3.10) is necessary for stability of a C_0-semigroup, and if the singular part of $(E(\cdot)j(x),j(x))$ is zero, it gives a simple stability criterion. In this connection, we recall the criterion for the stability of a C_0-semigroup $(T(t))_{t\geqslant 0}$ of completely nonunitary contractions in a Hilbert space. It was obtained in [10] by direct calculations involving the characteristic function of a contraction in the sense of [40].

Theorem 3.7. ([10]) *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup of completely nonunitary contractions in a Hilbert space H with generator A and cogenerator $G := (A+I)(A-I)^{-1}$. Then $(T(t))_{t\geqslant 0}$ is stable iff there is a set $E\subset\mathbb{R}$ of Lebesgue measure zero such that*

$$\lim_{\alpha\to 0+}\sqrt{\alpha}R(\alpha+\mathrm{i}\beta,A)x=0,\quad \beta\in\mathbb{R}\backslash E,$$

for every x from $\mathrm{Im}(I-GG^*)$. *(Note that* $\overline{\mathrm{Im}(I-GG^*)}=H$.*)*

To deduce our version of this statement, observe that if $(T(t))_{t\geqslant 0}$ is a C_0-semigroup of completely nonunitary contractions, then its minimal unitary dilation $(U(t))_{t\in\mathbb{R}}$ to a Hilbert space $K\supset H$ (in the sense of [40]) has absolutely continuous spectral measure (with respect to the Lebesgue measure) [32]. Define the subspace K_∞ of K as

$$K_\infty := \bigcap_{s\geqslant 0}\overline{\bigcup_{t>s}U(-t)H}.$$

Let $j_\infty : K\to K_\infty$ be the orthogonal projector on K_∞. Making use of results from [19], we infer that K_∞ is $U(t)$-invariant, $t\in\mathbb{R}$, and $(U(t))_{t\in\mathbb{R}}$ satisfies the properties (i)–(ii) of Theorem 2.2 in a Hilbert space K with intertwining operator $j=j_\infty|_H$. The property (ii) is mentioned explicitly in [19] and the property (i) follows from the representation

$$j_\infty x=\lim_{t\to\infty}U(-t)T(t)x,\quad x\in H,$$

indicated there. (The triple $((U(t))_{t\in\mathbb{R}},j,K)$ does not satisfy, in general, the property (iii) of Theorem 2.2). Hence, if $(T(t))_{t\geqslant 0}$ is a C_0-semigroup of completely nonunitary contractions, we can refine Theorem 3.4 by replacing (3.7) by (3.10). So, unitary dilation provides an alternative tool for proving stability of semigroups of contractions in Hilbert spaces.

Thus, for our purposes, it is important to know when the singular part of $(E(\cdot)j(x),j(x))$ is zero. We will discuss this in some more details.

To "annihilate" the singular part of $(E(\cdot)j(x),j(x))$, one can use conditions of other type than for the absolutely continuous part. Being combined with (3.10) they constitute Theorem 3.3.

Lemma 3.8. *Let $x\in H$, $\beta_0\in\mathbb{R}$, and $\varepsilon>0$ be fixed.*

(i) *If the restriction of the measure $(E(\cdot)j(x),j(x))$ to $(\beta_0-\varepsilon,\beta_0+\varepsilon)$ is absolutely continuous (with respect to the Lebesgue measure), then*

$$\lim_{\alpha\to 0+}\alpha\|R(\alpha+\mathrm{i}\cdot,B)j(x)\|^2 \tag{3.11}$$

exists in $L_1([\beta_0-\delta,\beta_0+\delta])$ for every $0<\delta<\varepsilon$. If the limit (3.11) *exists in $L_1([\beta_0-\varepsilon,\beta_0+\varepsilon])$, then $(E(\cdot)j(x),j(x))$ is absolutely continuous on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$.*

(ii) *The condition* (3.11) *is satisfied if the limit*

$$\lim_{\alpha\to 0+}\sqrt{\alpha}R(\alpha+\mathrm{i}\cdot,A)x \tag{3.12}$$

exists in $L_2([\beta_0-\varepsilon,\beta_0+\varepsilon],H)$.

Proof. (i) Let $0<\delta<\varepsilon$ be fixed. Let $j(x)=E(\mathbb{R}\setminus(\beta_0-\varepsilon,\beta_0+\varepsilon))j(x)$, $j(x)\in Y$. Since $\mathrm{dist}([\beta_0-\delta,\beta_0+\delta],\mathbb{R}\setminus(\beta_0-\varepsilon,\beta_0+\varepsilon))>0$, the limit in (3.11) exists in $L_1([\beta_0-\delta,\beta_0+\delta])$ and is equal to 0. So, we can assume that $j(x) = E((\beta_0-\varepsilon,\beta_0+\varepsilon))j(x)$. If the measure $\mu := (E(\cdot)j(x), j(x))$ is absolutely continuous on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$, then the symmetric derivative $(D\mu)(\cdot)$ belongs to $L_1(\mathbb{R})$. The limit in (3.11) exists in $L_1(\mathbb{R})$ as the $L_1(\mathbb{R})$-limit of the Poisson integral of $(D\mu)(t)$. Then the limit in (3.11) exists in $L_1([\beta_0-\delta,\beta_0+\delta])$.

Conversely, suppose that the limit in (3.11) exists and is equal to f. It is sufficient to consider $j(x)$ such that $j(x) = E([\beta_0-\varepsilon,\beta_0+\varepsilon])j(x)$. By Fatou's theorem,

$$(D\mu)(t) = \frac{1}{\pi} f(t) \quad \text{on } [\beta_0-\varepsilon,\beta_0+\varepsilon] \text{ a.e..}$$

Hence, $\mu = \frac{1}{\pi} f(t)\,\mathrm{d}t + \mu_{\mathrm{sing}}$, where μ_{sing} is the singular part of μ. According to equality (3.6) from the proof of Theorem 3.3 and (3.11),

$$\begin{aligned}\frac{1}{\pi}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} f(t)\,\mathrm{d}t + \mu_{\mathrm{sing}}(\beta_0-\varepsilon,\beta_0+\varepsilon) &\leqslant (E_{\beta_0+\varepsilon}\, j(x), j(x)) - (E_{\beta_0-\varepsilon}\, j(x), j(x)) \\ &= \lim_{\alpha\to 0+} \frac{\alpha}{\pi}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(\alpha+\mathrm{i}\beta, B)j(x)\|^2 = \frac{1}{\pi}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} f(t)\,\mathrm{d}t.\end{aligned}$$

Since μ is positive, $\mu_{\mathrm{sing}}((\beta_0-\varepsilon,\beta_0+\varepsilon)) = 0$.

(ii) Suppose that there exists $f \in L_2([\beta_0-\varepsilon,\beta_0+\varepsilon], H)$ such that

$$\lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|\sqrt{\alpha}R(\alpha+\mathrm{i}\beta, A)x - f(\beta)\|^2\,\mathrm{d}\beta = 0.$$

Then for some $\alpha_k \to 0+$, $k\to\infty$, we have

$$\lim_{k\to\infty} \|\sqrt{\alpha_k}R(\alpha_k+\mathrm{i}\beta, A)x - f(\beta)\| = 0 \quad \text{on } [\beta_0-\varepsilon,\beta_0+\varepsilon] \text{ a.e..}$$

Since the operator $j: H\to Y$ transforms fundamental sequences in $L_2([\beta_0-\varepsilon,\beta_0+\varepsilon], H)$ into fundamental sequences in $L_2([\beta_0-\varepsilon,\beta_0+\varepsilon], Y)$, the limit

$$\lim_{\alpha\to 0+} \sqrt{\alpha}R(\alpha+\mathrm{i}\cdot, B)j(x)$$

exists in $L_2([\beta_0-\varepsilon,\beta_0+\varepsilon], Y)$. It is equal to $j\circ f$ almost everywhere on $[\beta_0-\varepsilon,\beta_0+\varepsilon]$ as the image of the pointwise limit of $\sqrt{\alpha_k}R(\alpha_k+\mathrm{i}\,\cdot, A)x$. In particular,

$$\lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \big|\sqrt{\alpha}\|R(\alpha+\mathrm{i}\beta, B)j(x)\| - \|j(f(\beta))\|\big|^2\,\mathrm{d}\beta = 0.$$

Since for almost every β from $[\beta_0-\varepsilon,\beta_0+\varepsilon]$,

$$\begin{aligned}&\big|\alpha\|R(\alpha+\mathrm{i}\beta, B)j(x)\|^2 - \|j(f(\beta))\|^2\big| \\ &\quad\leqslant \big|\sqrt{\alpha}\|R(\alpha+\mathrm{i}\beta, B)j(x)\| - \|j(f(\beta))\|\big|^2 \\ &\qquad + 2\big|\sqrt{\alpha}\|R(\alpha+\mathrm{i}\beta, B)j(x)\| - \|j(f(\beta))\|\big|\,\|j(f(\beta))\|,\end{aligned}$$

we obtain

$$\lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\big|\,\alpha\|R(\alpha+\mathrm{i}\beta,B)j(x)\|^2-\|j(f(\beta))\|^2\big|\,\mathrm{d}\beta=0.$$

The statement follows. ∎

Thus, (3.11) is necessary for the stability of C_0-semigroups in a Hilbert space on the one hand, and ensures that the singular part of $(E(\cdot)j(x),j(x))$ is zero on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$, on the other hand.

Observe that for every $x\in H$ and every $\alpha_n\to 0+$, $n\to\infty$, the sequence of measures $\{\alpha_n\|R(\alpha_n+\mathrm{i}\cdot,B)j(x)\|^2\mathrm{d}\beta : n\in\mathbb{N}\}$ converges to $(E(\cdot)j(x),j(x))$ in the w^*-topology of $(\mathrm{C}\big([\beta_0-\varepsilon,\beta_0+\varepsilon]\big))^*$ (see [27], Chapter 6, B). Hence, according to [27], p. 103, the integral condition

$$\sup_{\alpha>0}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} f(\alpha\|R(\alpha+\mathrm{i}\beta,B)j(x)\|^2)\,\mathrm{d}\beta<\infty, \tag{3.13}$$

where $f(x)$ is a nonnegative measurable function such that $\lim\limits_{x\to\infty}\frac{f(x)}{x}=\infty$, also implies the absolute continuity of $(E(\cdot)j(x),j(x))$ on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$. If, in addition, $f(x)$ is nondecreasing, then (3.13) follows from

$$\sup_{\alpha>0}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} f(\alpha\|j\|\,\|R(\alpha+\mathrm{i}\beta,A)x\|^2)\,\mathrm{d}\beta<\infty. \tag{3.14}$$

It is possible to give conditions of different type for the “annihilation” of the singular part of $(E(\cdot)j(x),j(x))$ on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$. For example, as follows from the proof of Theorem 3.4, the “pointwise” condition

$$\lim_{\alpha\to 0+}\sqrt{\alpha}\|R(\alpha+\mathrm{i}\beta,A)x\|<\infty, \tag{3.15}$$

for every β from $(\beta_0-\varepsilon,\beta_0+\varepsilon)$, could serve for this purpose.

Alternatively, one can use “weak type” inequalities. Remark that the resolvent majorization property (Lemma 2.7) implies for every $\alpha>0$

$$\{\beta\in\mathbb{R}:\sqrt{\alpha}\|R(\alpha+\mathrm{i}\beta,B)j(x)\|\geqslant 1\}\subset\{\beta\in\mathbb{R}:\sqrt{\alpha}\|j\|\,\|R(\alpha+\mathrm{i}\beta,A)x\|\geqslant 1\}.$$

According to [1], a positive singular Borel measure μ on $\mathbb{R}$ satisfies

$$\lim_{\alpha\to 0+}\sqrt{\alpha}\,\mathrm{mes}\Big\{\beta\in\mathbb{R}:\int\limits_{\mathbb{R}}\frac{\alpha}{(\beta-t)^2+\alpha^2}\,\mathrm{d}\mu(t)\geqslant 1\Big\}>0,$$

where mes denotes the Lebesgue measure.

Hence for fixed x from H such that $x=E((\beta_0-\varepsilon,\beta_0+\varepsilon))x$ the “global” condition

$$\lim_{\alpha\to 0+}\sqrt{\alpha}\,\mathrm{mes}\{\beta\in\mathbb{R}:\sqrt{\alpha}\|j\|\,\|R(\alpha+\mathrm{i}\beta,A)x\|\geqslant 1\}=0 \tag{3.16}$$

annihilates the singular part of $(E(\cdot)j(x),j(x))$ on $(\beta_0-\varepsilon,\beta_0+\varepsilon)$.

But it is not clear whether the conditions (3.14), (3.15), (3.16) are necessary for stability of a C_0-semigroup.

Summarizing the above discussion of the “annihilation” conditions, we can formulate the following statement.

PROPOSITION 3.9. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Hilbert space H with generator A. Let $(S(t))_{t\in\mathbb{R}}$ be the limit unitary C_0-group corresponding to $(T(t))_{t\geqslant 0}$, with generator B. Suppose $E(\cdot)$ is the spectral measure defined by the operator $\mathcal{C} = -\mathrm{i}B$, and let $x \in H$, $\beta_0 \in \mathbb{R}$ and $\varepsilon > 0$ be fixed.*

The conditions

(i) $\lim_{\alpha\to 0+} \sqrt{\alpha}\|R(\alpha+\mathrm{i}\beta, A)x\| = 0$, *for almost all β from $(\beta_0-\varepsilon, \beta_0+\varepsilon)$,*

(ii) *the limit* $\lim_{\alpha\to 0+} \sqrt{\alpha}R(\alpha+\mathrm{i}\cdot, A)x$ *exists in* $L_2([\beta_0-\varepsilon, \beta_0+\varepsilon], H)$,

are equivalent to the absence of the absolutely continuous and of the singular parts of $(E(\cdot)j(x), j(x))$ on $(\beta_0-\varepsilon, \beta_0+\varepsilon)$ respectively, and necessary for the stability of $(T(t))_{t\geqslant 0}$.

The "resolvent language" easily allows us to obtain the "individual" stability results.

COROLLARY 3.10. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A, and let $x \in X$ be fixed.*

(i) *Then* $\lim_{t\to\infty} \|T(t)x\| = 0$, *if and only if for every $\beta_0 \in \mathbb{R}$ there is $\varepsilon(\beta_0) > 0$ such that*

$$\lim_{\alpha\to 0+} \alpha \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(\alpha+\mathrm{i}\beta, A)x\|^2 \,\mathrm{d}\beta = 0.$$

(ii) *If for every $\beta \in \mathbb{R}$*

$$\lim_{\alpha\to 0+} \sqrt{\alpha}R(\alpha+\mathrm{i}\beta, A)x = 0,$$

then

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

Proof. The proof is straightforward. The restriction of the semigroup $(T(t))_{t\geqslant 0}$ to the closed linear span of $\{T(t)x : t \geqslant 0\}$ satisfies the conditions (3.5) or (3.7) for a dense set equal to the linear span of $\{T(t)x : t \geqslant 0\}$ (the same for all β_0). ∎

The analogous statements to Theorems 3.3, 3.4 are also true for discrete operator semigroups. The proofs follow the same lines as for the strongly continuous ones. The only differences are that in this case one deals with Parseval's equality instead of Plancherel's theorem, Abel summability of sequences instead of functions, and the unitary group $(S^n)_{n\in\mathbb{Z}}$ and the spectral representation of S.

THEOREM 3.11. *Let $(T^n)_{n\geqslant 0}$ be a bounded semigroup in a Hilbert space H. Then $(T^n)_{n\geqslant 0}$ is stable if and only if one of the following conditions holds:*

(i) (a) *For every x from a dense set M in H:*

$$\lim_{r\to 1+} (r-1) \int_0^{2\pi} \|R(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\|^2 \,\mathrm{d}\varphi = 0; \tag{3.17}$$

(b) *For every* $\varphi_0 \in \mathbb{R}$ *there are* $\varepsilon > 0$ *and a dense set* $M := M(\varphi_0, \varepsilon)$ *such that for every* $x \in M$*:*

$$\lim_{r\to 1+}(r-1)\int\limits_{\varphi_0-\varepsilon}^{\varphi_0+\varepsilon} \|R(re^{i\varphi},T)x\|^2\,\mathrm{d}\varphi = 0. \tag{3.18}$$

(ii) *If for every* $\varphi_0 \in \mathbb{R}$ *there are* $\varepsilon(\varphi_0) > 0$ *and a dense set* $M := M(\varphi_0, \varepsilon)$ *such that for every* $x \in M$*:*

$$\lim_{r\to 1+}(r-1)^{\frac{1}{2}}\|R(re^{\mathrm{i}\varphi},T)x\| = 0, \quad \varphi \in (\varphi_0-\varepsilon, \varphi_0+\varepsilon), \tag{3.19}$$

then $(T^n)_{n\geqslant 0}$ *is stable.*

We will give only the proofs of the first and of the third statement. The proof of the second statement is left to the reader.

Proof. (i) Suppose that for every $x \in H$, $\lim\limits_{n\to\infty}\|T^n x\| = 0$. Then by Parseval's identity,

$$(r-1)\frac{1}{2\pi}\int\limits_0^{2\pi}\|R(re^{\mathrm{i}\varphi},T)x\|^2\,\mathrm{d}\varphi = (1-t)\sum_{n=0}^{\infty}t^{2n+1}\|T^n x\|^2, \quad t = \frac{1}{r}.$$

Being the Abel limit of a sequence tending to zero, the right hand side of the above equality tends to zero as $r \to 1+$. Therefore,

$$\lim_{r\to 1+}(r-1)\frac{1}{2\pi}\int\limits_0^{2\pi}\|R(re^{\mathrm{i}\varphi},T)x\|^2\,\mathrm{d}\varphi = 0.$$

Conversely, suppose that (3.17) is true. Defining an equivalent norm on H:

$$\|x\|_1 := \sup_{n\geqslant 0}\|T^n x\|,$$

we have

$$\|x\| \leqslant \|x\|_1 \leqslant C\|x\|, \quad C := \sup_{n\geqslant 0}\|T^n\|.$$

The limit $\lim\limits_{n\to\infty}\|T^n x\|_1^2$ exists for every $x \in (H, \|\cdot\|_1)$. So,

$$\begin{aligned}0 &= \lim_{r\to 1+}(r-1)\frac{1}{2\pi}\int\limits_0^{2\pi}\|R(re^{\mathrm{i}\varphi},T)x\|^2\,\mathrm{d}\varphi = \lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty}t^{2n+1}\|T^n x\|^2\\ &\geqslant \limsup_{t\to 1-}(1-t)C^{-2}\sum_{n=0}^{\infty}t^{2n+1}\|T^n x\|_1^2.\end{aligned}$$

The sequence $\{\|T^n x\|_1^2 : n \geqslant 0\}$ is convergent and Abel convergent to zero, therefore its limit is zero. Hence the limit of $\{\|T^n x\| : n \geqslant 0\}$ is also zero.

(ii) Let $(S^n)_{n\in\mathbb{Z}}$ be the limit unitary group corresponding to the semigroup $(T^n)_{n\geqslant 0}$. Suppose $E^{\mathrm{d}}(\cdot)$ is the spectral measure on $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ corresponding to S, and $E^d_\xi := E^{\mathrm{d}}(\delta_\xi)$, $\delta_\xi = \{e^{i\varphi} : \varphi \in [0,\xi)\}$, $\xi \in [0,2\pi)$ is the resolution of the identity associated with $E^{\mathrm{d}}(\cdot)$. Suppose further that φ_0 and the corresponding $\varepsilon = \varepsilon(\varphi_0) > 0$ are fixed. According to our assumption, for every $x \in M(\varphi_0,\varepsilon)$:

$$\begin{aligned}
&\lim_{r\to 1-} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\varphi-\xi)+r^2}\,\mathrm{d}(E^d_\xi j(x), j(x)) \\
&\quad = \lim_{r\to 1-} (1-r^2)\|(I-re^{-i\varphi}S)^{-1}j(x)\|^2 \\
&\quad = \lim_{r\to 1-} 2\left(\frac{1}{r}-1\right)\left\|R\left(\frac{e^{i\varphi}}{r},S\right)j(x)\right\|^2 \\
&\quad \leqslant \lim_{r\to 1-} 2\left(\frac{1}{r}-1\right)\|j\|\left\|R\left(\frac{e^{i\varphi}}{r},T\right)x\right\|^2 = 0.
\end{aligned} \tag{3.20}$$

Let $\mu := (E^{\mathrm{d}}(\cdot)j(x), j(x))$ and let $\mu = \frac{1}{2\pi} f\,\mathrm{d}\xi + \mu_{\mathrm{sing}}$, $f \geqslant 0$, be the Lebesgue decomposition of the measure μ. Observe that for fixed $\varphi \in (\varphi_0-\varepsilon, \varphi_0+\varepsilon)$ and $0 \leqslant r < 1$,

$$\lim_{\xi\to\varphi} \frac{1-r^2}{1-2r\cos(\varphi-\xi)+r^2} = \frac{1+r}{1-r}.$$

Hence, there is $\delta := \delta(\varphi)$, $\varepsilon > \delta > 0$ such that

$$\frac{1-r^2}{1-2r\cos(\varphi-\xi)+r^2} \geqslant \frac{1+r}{2(1-r)}, \quad \xi \in (\varphi-\delta, \varphi+\delta).$$

By (3.19) and (3.20) the symmetric derivative $D\mu(\varphi)$ of μ satisfies the condition

$$\begin{aligned}
(D\mu)(\varphi) &= \lim_{r\to 1-} \frac{\mu\big((\varphi-(1-r), \varphi+(1-r))\big)}{2(1-r)} \\
&\leqslant \lim_{r\to 1-} \int_{\varphi-\delta}^{\varphi+\delta} \frac{1-r^2}{(1+r)(1-2r\cos(\varphi-\xi)+r^2)}\,\mathrm{d}(E^d_\xi j(x), j(x)) \\
&\leqslant \lim_{r\to 1-} \int_0^{2\pi} \frac{1-r^2}{(1+r)(1-2r\cos(\varphi-\xi)+r^2)}\,\mathrm{d}(E^d_\xi j(x), j(x)) \\
&\leqslant \lim_{r\to 1-} \left(\frac{1}{r}-1\right)\|j\|\left\|R\left(\frac{e^{i\varphi}}{r},T\right)x\right\|^2 = 0.
\end{aligned} \tag{3.21}$$

If $\mu_{\mathrm{sing}}\big((\varphi_0-\varepsilon, \varphi_0+\varepsilon)\big) \neq 0$, then as in the proof of Theorem 3.4, there exists $\varphi \in (\varphi_0-\varepsilon, \varphi_0+\varepsilon)$ with $(D\mu)(\varphi) = +\infty$, which is impossible in view of (3.21). Therefore,

$$(D\mu)(\xi) = \frac{1}{2\pi} f(\xi) = 0 \quad \text{on } (\varphi_0-\varepsilon, \varphi_0+\varepsilon) \quad \text{a.e.},$$

and

$$(E^{\mathrm{d}}\big((\varphi_0-\varepsilon,\varphi_0+\varepsilon)\big)j(x),j(x))\leqslant \frac{1}{2\pi}\sum_{n=1}^{\infty}\int\limits_{\varphi-\varepsilon(1-\frac{1}{n})}^{\varphi_0+\varepsilon(1-\frac{1}{n})} f(\xi)\,\mathrm{d}\xi=0.$$

Hence, $E^{\mathrm{d}}\big((\varphi_0-\varepsilon,\varphi_0+\varepsilon)\big)j(x)=0$ by the denseness of $M(\varphi_0,\varepsilon)$ in H. As the choice of φ_0 was arbitrary,

$$\|E^{\mathrm{d}}(\mathbb{R})j(x)\|=0,\quad x\in H.$$

Hence $(T^n)_{n\geqslant 0}$ is stable. ∎

4. THE BANACH SPACE CASE

In the "absence" of Stone's theorem for semigroups on Banach spaces we have to invoke substitutes for it as mentioned in the Introduction. The following statement illustrates this approach.

THEOREM 4.1. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X with generator A. If for every $\beta_0\in\mathbb{R}$ there exist $\varepsilon>0$ and a dense set $M:=M(\beta_0,\varepsilon)$ in X such that for every $x\in M$:*

$$\lim_{\alpha\to 0+}\alpha\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\|R^2(\alpha+\mathrm{i}\beta,A)x\|\,\mathrm{d}\beta=0, \tag{4.1}$$

then the C_0-semigroup $(T(t))_{t\geqslant 0}$ is stable.

Proof. Assume that (4.1) holds. Let $(S(t))_{t\in\mathbb{R}}$ be the limit isometric C_0-group corresponding to $(T(t))_{t\geqslant 0}$ with generator B. Let $\beta_0\in\mathbb{R}$ and $g\in\mathcal{D}\big((\beta_0-\varepsilon,\beta_0+\varepsilon)\big)$, $\varepsilon>0$, be fixed. Define a linear operator $Q^g_{\beta_0}$ on the Banach space Y by the equality

$$Q^g_{\beta_0}y:=\lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}(R(\alpha+\mathrm{i}\beta,B)-R(-\alpha+\mathrm{i}\beta,B))y\,g(\beta)\,\mathrm{d}\beta,\quad y\in Y. \tag{4.2}$$

This definition is correct in view of Theorem 2.4 and the remark after it. Moreover, $Q^g_{\beta_0}$ is a bounded operator by the closed graph theorem. If $x\in M$, then by Lemma 2.7,

$$\begin{aligned}\|Q^g_{\beta_0}j(x)\|&\leqslant \lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\|\big(R(\alpha+\mathrm{i}\beta,B)-R(-\alpha+\mathrm{i}\beta,B)\big)j(x)\|\,|g(\beta)|\,\mathrm{d}\beta,\\ &\leqslant \lim_{\alpha\to 0+}6\|j\|\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\alpha\|R^2(\alpha+\mathrm{i}\beta,A)x\|\,\mathrm{d}\beta\max_{[\beta_0-\varepsilon,\beta_0+\varepsilon]}|g(\beta)|=0.\end{aligned}$$

Since $j(X)\subset\overline{j(M)}$, and since $Q^g_{\beta_0}$is continuous, it follows that $Q^g_{\beta_0}j(x)=0$ for every $x\in X$. As $\beta_0\in\mathbb{R}$ and $g\in\mathcal{D}\big((\beta_0-\varepsilon,\beta_0+\varepsilon)\big)$ were arbitrary, we obtain $\operatorname{supp}Q(\cdot)j(x)=\emptyset$, where $Q(\cdot)$ is defined by (2.4). According to Lemma 2.5, this is impossible, unless $j(x)=0$. Therefore, $j(X)=\{0\}$, so $(T(t))_{t\geqslant 0}$ is stable. ∎

The next corollary clarifies somewhat the condition of integrability of the local resolvents (4.1).

COROLLARY 4.2. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space X. If for every $\beta_0\in\mathbb{R}$ there are $\varepsilon>0$ and a dense set $M:=M(\beta_0,\varepsilon)\subset X$ such that for every $x\in M$ and some $f_x\in L_1([\beta_0-\varepsilon,\beta_0+\varepsilon],X)$:*

$$\lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\|R(\alpha+\mathrm{i}\beta,A)x-f_x(\beta)\|\,\mathrm{d}\beta=0, \tag{4.3}$$

then $(T(t))_{t\geqslant 0}$ is stable.

(Compare with [12], Theorem 3.4.)

Proof. If $x\in M$, then by Lemma 2.7 and (4.3),

$$\lim_{\alpha\to 0+}\alpha\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\|R^2(\alpha+\mathrm{i}\beta,A)x\|\,\mathrm{d}\beta\leqslant 2\lim_{\alpha\to 0+}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\|(R(\alpha+\mathrm{i}\beta,A)-R(2\alpha+\mathrm{i}\beta,A))x\|\,\mathrm{d}\beta=0.$$

Therefore, the desired statement follows from Theorem 4.1. ∎

REMARK 4.3. If a Banach space X has the analytic Radon-Nikodym property (see [12], [21] for a discussion of the role of this property in stability theory), then the convergence condition (4.3) can be replaced by the condition

$$\sup_{\alpha>0}\int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon}\|R(\alpha+\mathrm{i}\beta,A)x\|\,\mathrm{d}\beta<\infty. \tag{4.4}$$

In this case, as proved in [12], (4.4) implies the existence of the limit in (4.3). Obviously, (4.4) is weaker than the "local" boundedness of $R(\alpha+\mathrm{i}\beta,A)x$:

$$\sup_{\alpha>0}\ \sup_{\beta\in(\beta_0-\varepsilon,\beta_0+\varepsilon)}\|R(\alpha+\mathrm{i}\beta,A)x\|<\infty \tag{4.5}$$

for every $\beta_0\in\mathbb{R}$ and corresponding $\varepsilon(\beta_0)>0$. However, the conditions (4.4), (4.5) are not sufficient in Banach spaces without the analytic Radon-Nikodym property as Example 1.7 in [21] shows.

Now, we are to obtain the Banach space counterpart of Theorem 3.4. When X was a Hilbert space, we were using essentially the representation of the difference

$$\big(R(\alpha+\mathrm{i}\beta,B)-R(-\alpha+\mathrm{i}\beta,B)\big)y,\quad y\in Y,$$

as the Poisson integral of an operator-valued measure. On the other hand, as was indicated in [24], p. 52, there is a Banach space and a C_0-group in it, such that the representation of the above difference as the Poisson integral of a Y-valued measure does not exist.

To overcome this difficulty, one can apply theorems of the Phragmen-Lindelöf type.

Theorem 4.4. *Let f^+, f^- be a pair of functions analytic in the rectangles $P^+ = (0,1) \times (a,b)$, $P^- = (-1,0) \times (a,b)$, $\{a,b\} \subset \mathbb{R}$, respectively, and let $f(\alpha + \mathrm{i}\beta) := f^+(\alpha + \mathrm{i}\beta) - f^-(-\alpha + \mathrm{i}\beta)$, $(\alpha, \beta) \in P^+$. Suppose that:*

(i) *for $(\alpha + \mathrm{i}\beta) \in P^+$:*

$$\sup_{\beta \in (a,b)} |f(\alpha + \mathrm{i}\beta)| = \mathrm{O}\left(\frac{1}{\alpha}\right), \quad \alpha \to 0+, \tag{4.6}$$

(ii) *for every $\beta \in (a,b)$:*

$$\lim_{\alpha \to 0+} |f(\alpha + \mathrm{i}\beta)| = 0. \tag{4.7}$$

Then there exists a function F analytic in $P := (-1,1) \times (a,b)$, such that $F|P^+ = f^+$, $F|P^- = f^-$.

It is necessary to make some comments concerning Theorem 4.4. It was claimed in [43] that the conditions:

(i′) for $(\alpha, \beta) \in P^+$:

$$\sup_{\beta \in (a,b)} |f(\alpha + \mathrm{i}\beta)| = \mathrm{o}\left(\frac{1}{\alpha^2}\right), \quad \alpha \to 0+,$$

(ii′) for every $\beta \in (a,b)$ except a countable set:

$$\lim_{\alpha \to 0+} |f(\alpha + \mathrm{i}\beta)| = 0,$$

imply the same conclusion as in Theorem 4.4. However, the function $\left(\lambda - \frac{\mathrm{i}(a+b)}{2}\right)^{-1}$ satisfies the conditions (i′)′, (ii′) and has a pole at $\frac{\mathrm{i}(a+b)}{2}$. The arguments and, clearly, the conditions (i′)′, (ii′) from [43] require corrections. In view of this and the importance of Theorem 4.4 for further reasonings, we give a proof with our changes. The present form of Theorem 4.4 is not the most general, but is sufficient for our purposes. For results related to Theorem 4.4 for the case of unit disc see [3], [13], [42], where somewhat different ideas are used.

Let $B(\xi, r) := \{\lambda \in \mathbb{C} : |\xi - \lambda| < r\}$, $B^0(\xi, r) := \{\lambda \in \mathbb{C} : 0 < |\xi - \lambda| < r\}$. Here and in the sequel we will use the following estimate for subharmonic functions.

Lemma 4.5. *Let $u(\xi)$ be a nonnegative subharmonic function on $B^0(\mathrm{i}\beta_0, R)$ such that*

$$u(\xi) \leqslant \frac{1}{|\operatorname{Re} \xi|}, \quad \xi \in B^0(\mathrm{i}\beta_0, R), \operatorname{Re} \xi \neq 0. \tag{4.8}$$

Then there exists $C > 0$ such that

$$u(\xi) \leqslant \frac{C}{|\mathrm{i}\beta_0 - \xi|}, \quad \xi \in B^0\left(\mathrm{i}\beta_0, \frac{R}{2}\right). \tag{4.9}$$

Lemma 4.5 is implicit in [14], p. 53–54. Its counterparts for the case of unit disc can be found in [41], Lemma 5.8 and [36], Lemma 23. Since we do not know the precise reference for our version of Lemma 4.5 and since we will use it essentially in further reasonings, we will give the proof modifying arguments from [14].

Proof. Choose $\delta > 0$ so that

$$\frac{1}{\mathrm{e}} + \frac{4\mathrm{e}^2}{\pi\delta} < 1$$

and $v_0 > 0$ so that

$$\delta\frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-v_0} < \frac{R}{3}.$$

For fixed $v \geqslant v_0$ set $r = \delta\mathrm{e}^{-v}$.

Consider $\xi_0 \in B^0(\mathrm{i}\beta_0, \frac{R}{2})$ such that $u(\xi_0) = \mathrm{e}^v$ for some $v \geqslant v_0$. Note that $B(\xi_0, r) \subset B(\mathrm{i}\beta_0, R)$. First, we prove the alternative: either the disc $B(\xi_0, r)$ contains $\mathrm{i}\beta_0$ or there exists $\xi_1 \in B(\xi_0, r)$ such that $u(\xi_1) > \mathrm{e}^{v+1}$. Suppose, on the contrary, that $B(\xi_0, r)$ does not contain $\mathrm{i}\beta_0$ and $u \leqslant \mathrm{e}^{v+1}$ in $B(\xi_0, r)$. Then by subharmonicity of u in $B(\xi_0, r)$ and (4.8) we have

$$\begin{aligned}
\mathrm{e}^v \leqslant u(\xi_0) &\leqslant \frac{1}{\pi r^2}\iint\limits_{B(\xi_0,r)} u(x+\mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \mathrm{e}^{v-1}\frac{1}{\pi r^2}\mathrm{mes}\{\eta : u(\eta) \leqslant \mathrm{e}^{v-1},\ \eta \in B(\xi_0, r)\} \\
&\quad + \mathrm{e}^{v+1}\frac{1}{\pi r^2}\mathrm{mes}\{\eta : u(\eta) > \mathrm{e}^{v-1},\ \eta \in B(\xi_0, r)\} \\
&\leqslant \mathrm{e}^{v-1} + \mathrm{e}^{v+1}\frac{1}{\pi r^2}\mathrm{mes}\Big\{\eta : \frac{1}{|\mathrm{Re}\,\eta|} > \mathrm{e}^{v-1},\ \eta \in B(\xi_0, r)\Big\} \\
&\leqslant \mathrm{e}^{v-1} + \mathrm{e}^{v+1}\frac{1}{\pi r^2}2\mathrm{e}^{-v+1}2r = \mathrm{e}^{v-1} + \frac{4\mathrm{e}^2}{\pi r}.
\end{aligned}$$

Thus

$$\mathrm{e}^v \leqslant \mathrm{e}^v\left(\frac{1}{\mathrm{e}} + \frac{4\mathrm{e}^2}{\pi}\frac{\mathrm{e}^{-v}}{r}\right) = \mathrm{e}^v\left(\frac{1}{\mathrm{e}} + \frac{4\mathrm{e}^2}{\pi\delta}\right) < \mathrm{e}^v.$$

The obtained contradiction proves the alternative.

Further, we inductively construct a set $\{\xi_n\}$ as follows. If for $n \in \mathbb{N} \cup \{0\}$ the disc $B(\xi_{n_0}, \delta\mathrm{e}^{-v-n_0})$ contains $\mathrm{i}\beta_0$, then

$$|\xi_0 - \mathrm{i}\beta_0| \leqslant \sum_{k=0}^{n}|\xi_k - \xi_{k+1}| \leqslant \delta\frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-v} = \delta\frac{\mathrm{e}}{\mathrm{e}-1}(u(\xi_0))^{-1}, \quad \xi_{n+1} = \mathrm{i}\beta_0.$$

Therefore, u satisfies the estimate (4.9) at ξ_0 with $C = \delta\frac{\mathrm{e}}{\mathrm{e}-1}$, and we finish the construction of $\{\xi_n\}$.

If for $n \in \mathbb{N}\cup\{0\}$ the disc $B(\xi_n, \delta\mathrm{e}^{-v-n})$ does not contain $\mathrm{i}\beta_0$, then, according to the proved alternative, there is $\xi^* \in B(\xi_n, \delta\mathrm{e}^{-v-n})$ such that

$$u(\xi^*) \geqslant \mathrm{e}^{v+n+1}, \quad |\xi_n - \xi^*| \leqslant \delta\mathrm{e}^{-v-n}.$$

Put in this case $\xi_{n+1} = \xi^*$. Note that on every step we have $\xi_n \in B(\mathrm{i}\beta, R)$ since

$$\begin{aligned}
|\xi_n - \mathrm{i}\beta_0| &\leqslant \sum_{k=1}^{n}|\xi_k - \xi_{k-1}| + |\xi_0 - \mathrm{i}\beta_0| \\
&\leqslant \frac{R}{2} + \sum_{k=1}^{\infty}|\xi_k - \xi_{k-1}| \leqslant \frac{R}{2} + \delta\frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-v} < R.
\end{aligned}$$

If each of the constructed discs $B(\xi_n, \delta \mathrm{e}^{-v-n})$, $n \geqslant 0$ does not contain $\mathrm{i}\beta_0$, then $\lim\limits_{n\to\infty} u(\xi_n) = +\infty$. So $\lim\limits_{n\to\infty} \xi_n = \mathrm{i}\beta_0$ and

$$|\xi_0 - \mathrm{i}\beta_0| \leqslant \sum_{n=0}^{\infty} |\xi_n - \xi_{n+1}| \leqslant \delta \frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-v}.$$

Thus, for every $\xi \in B^0(\mathrm{i}\beta_0, \frac{R}{2})$ such that $u(\xi) \geqslant \mathrm{e}^{v_0}$, the inequality (4.9) holds with $C = \frac{\mathrm{e}\delta}{\mathrm{e}-1}$.

Observe, that if $u(\xi) = \mathrm{e}^v$, $v < v_0$, $\xi \in B^0(\mathrm{i}\beta_0, \frac{R}{2})$, then

$$u(\xi) < \mathrm{e}^{v_0} < \mathrm{e}^{v_0}\frac{R}{|\xi - \mathrm{i}\beta_0|} = \frac{\mathrm{e}^{v_0}R}{|\xi - \mathrm{i}\beta_0|}.$$

Setting finally $C := \max(\frac{\mathrm{e}\delta}{\mathrm{e}-1}, \mathrm{e}^{v_0}R)$ in (4.9), we obtain the required estimate. ∎

Proof of Theorem 4.4. We will say that a point $\mathrm{i}\beta \in (\mathrm{i}a, \mathrm{i}b)$ is singular for the pair (f^+, f^-) if there is no neighbourhood N_β of $\mathrm{i}\beta$ and a function F_β analytic in N_β such that

$$F_\beta(\lambda) = \begin{cases} f^+(\lambda), & \lambda \in N_\beta \cap P^+, \\ f^-(\lambda), & \lambda \in N_\beta \cap P^- . \end{cases}$$

Let S be the set of singular points of (f^+, f^-) in $(\mathrm{i}a, \mathrm{i}b)$. If $S = \emptyset$, then $F_\beta(\lambda)$ does not depend on β and the conclusion of the theorem holds. Suppose that S is nonempty.

Fix an arbitrary segment $[\mathrm{i}c, \mathrm{i}d] \subset (\mathrm{i}a, \mathrm{i}b)$, and assume that $S_{c,d} := S \cap [\mathrm{i}c, \mathrm{i}d]$ is nonempty. The set $S_{c,d}$ is closed by the definition of S. For every $n \in \mathbb{N}$, the set

$$\begin{aligned}\{\mathrm{i}\beta \in [\mathrm{i}c, \mathrm{i}d] &: |f(\alpha + \mathrm{i}\beta)| \leqslant n,\ \alpha \in (0,1]\} \\ &= \bigcap_{k=1}^{\infty} \Big\{\mathrm{i}\beta \in [\mathrm{i}c, \mathrm{i}d] : |f(\alpha + \mathrm{i}\beta)| \leqslant n,\ \alpha \in \Big[\frac{1}{k}, 1\Big]\Big\}\end{aligned}$$

is also closed by continuity of f. According to (4.7),

$$S_{c,d} = \bigcup_{n=1}^{\infty} (\{\mathrm{i}\beta \in [\mathrm{i}c, \mathrm{i}d] : |f(\alpha + \mathrm{i}\beta)| \leqslant n,\ \alpha \in (0,1]\} \cap S) = \bigcup_{n=1}^{\infty} S^n_{c,d},$$

and by the Baire category theorem, there is an interval $(\mathrm{i}e, \mathrm{i}f) \subset [\mathrm{i}c, \mathrm{i}d]$ and $n_0 \in \mathbb{N}$ such that $(\mathrm{i}e, \mathrm{i}f) \cap S_{c,d} \neq \emptyset$ and $(\mathrm{i}e, \mathrm{i}f) \cap S_{c,d} \subset S^{n_0}_{c,d}$.

The set $(\mathrm{i}e, \mathrm{i}f) \setminus S_{c,d}$ is open, so $(\mathrm{i}e, \mathrm{i}f) \setminus S_{c,d} = \bigcup\limits_{n=1}^{\infty} (\mathrm{i}a_n, \mathrm{i}b_n)$, $n \in \mathbb{N}$. Fix $n_1 \in \mathbb{N}$. Consider the rectangle $P_{n_1} = (0,1) \times (a_{n_1}, b_{n_1})$, and its half, $P_{n_1,h} = (0,1) \times (a_{n_1}, \frac{1}{2}(a_{n_1} + b_{n_1}))$. By assumption, $f|\partial P_{n_1,h}$ extends continuously to the whole $\partial P_{n_1,h}$. Let g be the solution of the Dirichlet problem in $P_{n_1,h}$ with these boundary conditions and let $u = f - g$. Then g is a bounded harmonic function in $P_{n_1,h}$ and u is a harmonic function in $P_{n_1,h}$ such that u is zero on $(\mathrm{i}a_{n_1}, \frac{1}{2}(\mathrm{i}a_{n_1} + \mathrm{i}b_{n_1}))$ and on $(\mathrm{i}a_{n_1}, \mathrm{i}a_{n_1} + 1)$. Moreover, u is continuous on $P_{n_1,h} \setminus \{\mathrm{i}a_{n_1}\}$. Observe that the function u can be extended harmonically into the region

$$\mathcal{R} = \Big\{\alpha + \mathrm{i}\beta : (\alpha, \beta) \in (-1,1) \times \Big(\frac{3a_{n_1} - b_{n_1}}{2}, \frac{a_{n_1} + b_{n_1}}{2}\Big)\Big\} \setminus \{\mathrm{i}a_{n_1}\}.$$

By a double application of the Schwartz reflection principle ([8], p. 405), first on the interval $(\mathrm{i}a_{n_1}, \frac{1}{2}(\mathrm{i}a_{n_1} + \mathrm{i}b_{n_1}))$, and then on the intervals $(\mathrm{i}a_{n_1} - 1, \mathrm{i}a_{n_1})$ and $(\mathrm{i}a_{n_1}, \mathrm{i}a_{n_1} + 1)$, it extends to the harmonic function u_c in $\mathcal{R}\backslash\big(\frac{3\mathrm{i}a_{n_1} - \mathrm{i}b_{n_1}}{2}, \mathrm{i}a_{n_1}\big)$. Then setting

$$u_c(\mathrm{i}a_{n_1} - \mathrm{i}x) = -\overline{u_c(\mathrm{i}a_{n_1} + \mathrm{i}x)}, \quad x \in \Big(0, \frac{b_{n_1} - a_{n_1}}{2}\Big),$$

we obtain the required extension in view of harmonicity of u_c in $\big((-1,1)\times (a_{n_1}, \frac{a_{n_1}+b_{n_1}}{2})\big)$. Denote the obtained extension again by u_c.

By construction, $\mathrm{i}a_{n_1}$ is an isolated singular point of u_c (in the usual sense). Therefore,

$$u_c(\lambda) = u_0 \log|\lambda - \mathrm{i}a_{n_1}| + \sum_{k=-\infty}^{\infty} (u_k^1 \cos k\varphi + u_k^2 \sin k\varphi)|\lambda - \mathrm{i}a_{n_1}|^k, \tag{4.10}$$

where $\varphi \in [0, 2\pi)$ and the series converges in some $B^0(\mathrm{i}a_{n_1}, r) = \{\lambda \in \mathbb{C} : 0 < |\lambda - \mathrm{i}a_{n_1}| < r\} \subset \mathcal{R}$ (see, for example, [8], p. 316, Example 11). Since g is bounded in $P_{n_1,h}$ and (i) holds, the nonnegative subharmonic function $|u_c(\lambda)|$ satisfies the estimate (4.8) in $B^0(\mathrm{i}\beta_0, r)$. Hence it also satisfies (4.9) in $B^0(\mathrm{i}\beta_0, \frac{r}{2})$. Hence in (4.10): $u_k^1 = u_k^2 = 0$, $k \leqslant -2$. Along the ray $l_0 := \{\mathrm{i}a_{n_1} + t : t > 0\}$:

$$\lim_{\lambda \to \mathrm{i}a_{n_1}} |u_c(\lambda)| = 0, \quad \lambda \in l_0.$$

Therefore, $u_0 = u_{-1}^1 = 0$.

Along the ray $l_{\frac{\pi}{2}} := \{\mathrm{i}a_{n_1} + \mathrm{i}t : t > 0\}$:

$$\lim_{\lambda \to \mathrm{i}a_{n_1}} |u_c(\lambda)| = 0, \quad \lambda \in l_{\frac{\pi}{2}}.$$

So $u_{-1}^2 = 0$, and u_c is thus harmonic in $\mathcal{R} \cup \{\mathrm{i}a_{n_1}\}$. Since the solution of the considered Dirichlet problem in $P_{n_1,h}$ is unique, $u = 0$, and $f = g$ is bounded in $P_{n_1,h}$. As $(\mathrm{i}e, \mathrm{i}f) \cap S_{c,d} \subset S_{c,d}^{n_0}$,

$$\sup_{P_{n_1,h}} |f(\alpha + \mathrm{i}\beta)| \leqslant \max(n_0, m), \quad m := \max_{[1+\mathrm{i}a_n, 1+\mathrm{i}b_n]} |f(\alpha + \mathrm{i}\beta)|,$$

by the maximum principle for harmonic functions. Repeating the same reasonings for the other half of the rectangle P_{n_1}, and then for all the rectangles P_n, $n \geqslant 1$, we obtain

$$\sup\{|f(\alpha + \mathrm{i}\beta)| : (\alpha, \beta) \in (0,1) \times (e, f)\} \leqslant \max(n_0, m).$$

Under the last condition, Theorem 2.4 implies that there is a function F analytic in $(-1,1) \times (e, f)$ such that

$$F(\alpha + \mathrm{i}\beta) = \begin{cases} f^+(\alpha + \mathrm{i}\beta), & (\alpha, \beta) \in (0,1) \times (e, f); \\ f^-(\alpha + \mathrm{i}\beta), & (\alpha, \beta) \in (-1,0) \times (e, f). \end{cases}$$

On the other hand, $(\mathrm{i}e, \mathrm{i}f) \cap S \neq \emptyset$ by our construction; a contradiction. Thus, for arbitrary $[\mathrm{i}c, \mathrm{i}d] \subset (\mathrm{i}a, \mathrm{i}b)$, we obtain $S \cap [\mathrm{i}c, \mathrm{i}d] = \emptyset$. Therefore, $S = \emptyset$. ∎

REMARK 4.6. Observe that Theorem 3.4 can be deduced from Theorem 4.4. However, we prefered a more explicit proof of Theorem 3.4 since it allows us to formulate various Hilbert space stability conditions depending on the concrete situation.

To obtain the Banach space analogue of Theorem 3.4, we need to combine both Theorem 4.4 and Lemma 2.5.

THEOREM 4.7. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X with generator A. If for every $\beta_0 \in \mathbb{R}$ there exist $\varepsilon > 0$ and a dense set $M := M(\beta_0, \varepsilon)$ in X such that for every $x \in M$:*

$$\lim_{\alpha\to 0+} \alpha R^2(\alpha + \mathrm{i}\beta, A)x = 0, \quad \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon), \tag{4.11}$$

then $(T(t))_{t\geqslant 0}$ is stable.

(As in Theorem 3.4, the existence of the limit in (4.11) for all $x \in X$ and fixed $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \cap -\mathrm{i}\sigma(A)$ would contradict the spectral mapping theorem.)

Proof. Let $\beta_0 \in \mathbb{R}$ be fixed, and let $(S(t))_{t\in\mathbb{R}}$ be the limit isometric C_0-group corresponding to $(T(t))_{t\geqslant 0}$ with generator B. For any $x \in M$, $y^* \in Y^*$ consider the analytic functions

$$f^{\pm}(\pm\alpha + \mathrm{i}\beta) = y^*\big(R(\pm\alpha + \mathrm{i}\beta, B)j(x)\big)$$

in the rectangles $P^+_{\beta_0} = (0,1) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$, $P^-_{\beta_0} = (-1,0) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ respectively. By Lemma 2.7,

$$\begin{aligned}|f^+(\alpha + \mathrm{i}\beta) - f^-(-\alpha + \mathrm{i}\beta)| &\leqslant \|R(\alpha + \mathrm{i}\beta, B)j(x) - R(-\alpha + \mathrm{i}\beta, B)j(x)\|\,\|y^*\| \\ &\leqslant 6\alpha\|j\|\,\|R^2(\alpha + \mathrm{i}\beta, A)x\|\,\|y^*\|, \quad (\alpha, \beta) \in P^+_{\beta_0}.\end{aligned}$$

Hence, by the assumption,

$$\lim_{\alpha\to 0+} |f^+(\alpha + \mathrm{i}\beta) - f^-(-\alpha + \mathrm{i}\beta)| = 0, \quad \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon).$$

Moreover, since the C_0-group $(S(t))_{t\in\mathbb{R}}$ is isometric,

$$|f^{\pm}(\pm\alpha + \mathrm{i}\beta)| \leqslant \frac{\|y^*\|\,\|j(x)\|}{\alpha}, \quad (\alpha, \beta) \in P^+_{\beta_0}.$$

Thus, $f(\alpha + \mathrm{i}\beta) := f^+(\alpha + \mathrm{i}\beta) - f^-(-\alpha + \mathrm{i}\beta)$ satisfies the conditions of Theorem 4.4 in $P^+_{\beta_0}$.

Fix $g \in \mathcal{D}\big((\beta_0 - \varepsilon, \beta_0 + \varepsilon)\big)$. If $Q^g_{\beta_0}$ is the linear bounded operator on Y defined by (4.2), then $y^*(Q^g_{\beta_0} j(x)) = 0$, $y^* \in Y^*$, $x \in M$, in view of Theorem 4.4. The denseness of M in X and the Hahn-Banach theorem imply that $Q^g_{\beta_0} j(x) = 0$, $x \in X$. Since $\beta_0 \in \mathbb{R}$ and $g \in \mathcal{D}\big((\beta_0 - \varepsilon, \beta_0 + \varepsilon)\big)$ were arbitrary, we obtain $\operatorname{supp} Q(\cdot)\, j(x) = \emptyset$, $x \in X$, were $Q(\cdot)$ is defined by (2.4). Refering as before to Lemma 2.5, we conclude that $(T(t))_{t\geqslant 0}$ is stable. ∎

REMARK 4.8. The proof of Theorem 4.7 shows that Lemma 4.4 is true also for X-valued functions $f^{\pm}$. Note, in connection with Theorem 4.7, that if A is the generator of a bounded C_0-semigroup in X, and for fixed $x \in X$, $\beta_0 \in \mathbb{R}$:

$$\lim_{\alpha\to 0+} \alpha R^2(\alpha + \mathrm{i}\beta_0, A)x = 0, \tag{4.12}$$

then the limit in (4.12) exists uniformly in every sector $S_\xi := \{\mathrm{i}\beta_0 + t\mathrm{e}^{\mathrm{i}\varphi} : t \geqslant 0, |\varphi| \leqslant \xi < \frac{\pi}{2}\}$. This can be shown directly by means of the first resolvent identity and the estimate $\alpha\|R(\alpha + \mathrm{i}\beta, A)\| \leqslant C$, $\alpha > 0$, $C > 0$, with arguments similar to those of Lemma 2.7.

The conditions for the stability of individual orbits of Banach space semigroups are direct consequences of Theorems 4.1, 4.7. They can be obtained analogously to Corollary 3.10.

COROLLARY 4.9. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X with generator A, and let $x \in X$ be fixed. Suppose that one of the following two conditions holds:*

(i) *for every $\beta_0 \in \mathbb{R}$ there is $\varepsilon(\beta_0) > 0$ such that*

$$\lim_{\alpha\to 0+} \alpha \int_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R^2(\alpha + \mathrm{i}\beta, A)x\| \,\mathrm{d}\beta = 0;$$

(ii) *for every $\beta \in \mathbb{R}$,*

$$\lim_{\alpha\to 0+} \alpha R^2(\alpha + \mathrm{i}\beta, A)x = 0.$$

Then

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

The next theorem consists of the discrete counterparts of Theorems 4.1 and 4.7.

THEOREM 4.10. *Let $(T^n)_{n\geqslant 0}$ be a bounded semigroup in a Banach space X. Then the semigroup $(T^n)_{n\geqslant 0}$ is stable if one of the following conditions holds:*

(i) *for every $\varphi_0 \in \mathbb{R}$ there exist $\varepsilon > 0$ and a dense set $M := M(\varphi_0, \varepsilon)$ such that for every $x \in M$:*

$$\lim_{r\to 1+} (r-1) \int_{\varphi_0-\varepsilon}^{\varphi_0+\varepsilon} \|R^2(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\| \,\mathrm{d}\varphi = 0; \tag{4.13}$$

(ii) *for every $\varphi_0 \in \mathbb{R}$ there exist $\varepsilon(\varphi_0) > 0$ and a dense set $M := M(\varphi_0, \varepsilon)$ in a Banach space X such that for every $x \in M$:*

$$\lim_{r\to 1+} (r-1) R^2(r\mathrm{e}^{\mathrm{i}\varphi}, T)x = 0, \quad \varphi \in (\varphi_0 - \varepsilon, \varphi_0 + \varepsilon). \tag{4.14}$$

We will give the proof only for the first statement. The proof of the second statement can be done in a similar way.

Proof. Suppose $(S^n)_{n\in\mathbb{Z}}$ is the limit isometric C_0-group corresponding to $(T^n)_{n\geqslant 0}$. Observe that there exists $\delta > 0$ such that for every $\varphi_0 \in \mathbb{R}$ the mapping $\lambda \to \mathrm{e}^{\lambda}$ is conformal in

$$\mathcal{T}_\delta := \{\lambda = r\mathrm{e}^{\mathrm{i}\varphi} : \mathrm{e}^{-\delta} \leqslant r \leqslant \mathrm{e}^{\delta},\, \varphi_0 - \delta \leqslant \varphi \leqslant \varphi_0 + \delta\}.$$

Set $\varepsilon_0 = \min(\varepsilon, \delta)$, where ε is given by (4.13). Then the functions

$$F^{\pm}(\pm\alpha + \mathrm{i}\beta)y = R(\mathrm{e}^{\pm\alpha+\mathrm{i}\beta}, S)y, \quad y \in Y$$

are analytic in the open sets $P^+_{\varphi_0} = (0, \varepsilon_0) \times (\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0)$, $P^-_{\varphi_0} = (-\varepsilon_0, 0) \times (\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0)$ respectively, and satisfy the conditions

$$\begin{aligned} \|\big(F^+(\alpha + \mathrm{i}\beta) - F^-(-\alpha + \mathrm{i}\beta)\big)j(x)\| &\leqslant 6(\mathrm{e}^{\alpha} - 1)\|R^2(\mathrm{e}^{\alpha+\mathrm{i}\beta}, S)j(x)\| \\ &\leqslant 6(\mathrm{e}^{\alpha} - 1)\|j\|\,\|R^2(\mathrm{e}^{\alpha}\mathrm{e}^{\mathrm{i}\beta}, T)x\|, \quad (\alpha, \beta) \in P^+_{\varphi_0},\, x \in X, \end{aligned} \tag{4.15}$$

$$\|F^{\pm}(\pm\alpha + \mathrm{i}\beta)y\| \leqslant \frac{\|y\|}{\alpha}, \quad (\alpha, \beta) \in P^+_{\varphi_0},\, y \in Y. \tag{4.16}$$

Suppose (4.13) holds. Fix $\varphi_0 \in \mathbb{R}$ and $g \in \mathcal{D}\big((\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0)\big)$. Define a linear operator $Q^g_{\varphi_0} : Y \to Y$ by the equality

$$Q^g_{\varphi_0} y := \lim_{\alpha\to 0+} \int_{\varphi_0-\varepsilon_0}^{\varphi_0+\varepsilon_0} \big(F^+(\alpha + \mathrm{i}\beta) - F^-(-\alpha + \mathrm{i}\beta)\big)y\, g(\beta)\,\mathrm{d}\beta, \quad y \in Y.$$

From the proof of Theorem 2.4 and (4.16) it follows that $Q^g_{\varphi_0}$ is bounded. Taking into account $\overline{M} = X$, (4.15) and (4.13), we obtain as in the proof of Theorem 4.1 $Q^g_{\varphi_0} j(x) = 0$, $x \in X$. Since $g \in \mathcal{D}\big((\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0)\big)$ was arbitrary,

$$\operatorname{supp} Q(\cdot)j(x) \cap (\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0) = \emptyset, \quad x \in X, \tag{4.17}$$

where $Q(\cdot) : \mathcal{D}\big((\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0)\big) \to L(Y)$ is defined by (2.4). By Theorem 2.4, there is a function F analytic in $P_{\varphi_0} := (-\varepsilon_0, \varepsilon_0) \times (\varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0)$ such that

$$F(\alpha + \mathrm{i}\beta) = \begin{cases} F^+(\alpha + \mathrm{i}\beta), & (\alpha, \beta) \in P^+_{\varphi_0}, \\ F^-(\alpha + \mathrm{i}\beta), & (\alpha, \beta) \in P^-_{\varphi_0}. \end{cases}$$

By the choice of ε_0, $R(\lambda, S)j(x)$ is analytic in $\mathcal{T}_{\varepsilon_0}$. Now, as $\varphi_0 \in [0, 2\pi)$ was arbitrary, we conclude that $R(\lambda, S)j(x)$ is a bounded entire function, so $j(x) = 0$, $x \in X$. The stability of $(T^n)_{n\geqslant 0}$ is proved. ∎

REMARK 4.11. It is natural to compare the obtained stability conditions in a Banach space with the corresponding Hilbert space stability conditions.

Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X with the generator A. Then for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$ the operator $A - \lambda I$ is the generator of a bounded C_0-semigroup. By the Hardy-Landau inequality for such operators ([23]),

$$\|(A - \lambda I)x\|^2 \leqslant 2C(C + 1)\|x\|\,\|(A - \lambda I)^2 x\|, \tag{4.18}$$

where x belongs to the domain $D(A^2)$ of A^2 and $C = \sup_{t\geqslant 0} \|T(t)\|$. (The inequality (4.18) was proved in [23] for generators of contraction semigroups. For our case, the proof remains essentially the same. Alternatively, see the next paragraph.) If $\lambda \in \rho(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant 0\}$, then

$$\|R(\lambda, A)x\|^2 \leqslant 2C(C+1)\|R^2(\lambda, A)x\| \, \|x\| \tag{4.19}$$

for $x \in D(A^2)$, and by continuity for $x \in X$. Hence, in the case X is a Hilbert space, the stability conditions (4.1), (4.11) are not weaker, than the corresponding conditions (3.5), (3.7).

Similarly, let T be a linear bounded operator in X such that $\sup\limits_{n\geqslant 0}\|T^n\| \leqslant C$. Then in the equivalent norm $\|x\|_1 := \sup\limits_{n\geqslant 0}\|T^n x\|$, we have $\|T\|_1 \leqslant 1$. So $G = \frac{\mathrm{e}^{-\mathrm{i}\varphi}}{r}T - I$, $r > 1$, $\varphi \in [0, 2\pi)$, is the generator of a contraction C_0-semigroup $(\mathrm{e}^{Gt})_{t\geqslant 0}$. By the inequality (4.18),

$$\left\|\left(\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{r}T - I\right)x\right\|_1^2 \leqslant 4\left\|\left(\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{r}T - I\right)^2 x\right\|_1 \|x\|_1, \quad x \in X,$$

hence

$$\|R(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\|_1^2 \leqslant 4\|R^2(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\|_1 \, \|x\|_1.$$

Thus,

$$\begin{aligned}\|R(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\|^2 &\leqslant \|R(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\|_1^2 \leqslant 4\|R^2(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\|_1 \, \|x\|_1,\\ &\leqslant 4C^2\|R^2(r\mathrm{e}^{\mathrm{i}\varphi}, T)x\| \, \|x\|, \quad r > 1, \ \varphi \in [0, 2\pi).\end{aligned}$$

Therefore, if X is a Hilbert space, then the stability conditions for the discrete operator semigroups (4.13), (4.14) are not weaker than (3.18), (3.19) respectively.

It is well-known, that the condition

$$\sigma_{\mathrm{p}}(A^*) \cap \mathrm{i}\mathbb{R} = \emptyset \tag{4.20}$$

is necessary for the stability of a C_0-semigroup $(T(t))_{t\geqslant 0}$ (see, for example, [2] where this fact is proved by means of Banach limits). Since a stable C_0-semigroup is necessarily bounded, (4.20) is equivalent to

$$\lim_{\alpha\to 0+} \alpha R(\alpha + \mathrm{i}\beta, A)x = 0, \quad x \in X, \tag{4.21}$$

by the Abel mean ergodic theorem [20], p. 520. Moreover, if a C_0-semigroup $(T(t))_{t\geqslant 0}$ is stable, then in view of

$$\alpha \, \|R(\alpha + \mathrm{i}\beta, A)x\| \leqslant \alpha \int_0^\infty \mathrm{e}^{-\alpha t}\|T(t)x\| \, \mathrm{d}t, \quad x \in X, \ \alpha > 0,$$

and the regularity of Abel summation, (4.21) is satisfied *uniformly* in β from $\mathbb{R}$. Hence this property is also necessary for the stability of $(T(t))_{t\geqslant 0}$ (but it is not sufficient as Example 1.7 from [21] shows). The next proposition shows that the stability conditions (3.5), (3.7), (4.1), (4.11) ensure (4.21) uniformly in β on compacts from $\mathbb{R}$. In view of Remark 4.11, it is sufficient to consider only (3.5), (3.7). By Heine-Borel arguments it is sufficient to prove the assertion only for segments.

PROPOSITION 4.12. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X, with generator A. Let $x \in X$, $\beta \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. Suppose that one of the following two conditions holds:*

$$\lim_{\alpha\to 0+} \alpha \int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(\alpha+\mathrm{i}\beta, A)x\|^2 \,\mathrm{d}\beta = 0. \tag{4.22}$$

$$\lim_{\alpha\to 0+} \sqrt{\alpha} R(\alpha+\mathrm{i}\beta, A)x = 0, \quad \beta \in [\beta_0-\varepsilon, \beta_0+\varepsilon]. \tag{4.23}$$

Then (4.21) *holds with fixed x uniformly in $\beta \in [\beta_0-\varepsilon, \beta_0+\varepsilon]$.*

Proof. Let $\beta_0 \in \mathbb{R}$ and $x \in X$ be fixed.

1) Suppose (4.22) is true. Consider the rectangle with vertexes in $(\frac{\alpha}{2}+\mathrm{i}(\beta_0-\varepsilon))$, $(\frac{\alpha}{2}+\mathrm{i}(\beta_0+\varepsilon))$, $(\frac{3\alpha}{2}+\mathrm{i}(\beta_0-\varepsilon))$, $(\frac{3\alpha}{2}+\mathrm{i}(\beta_0+\varepsilon))$, where $\alpha < 2\varepsilon$.

By subharmonicity of the function $\|R(\alpha+\mathrm{i}\,\cdot, A)x\|^2$ in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ we obtain

$$\begin{aligned}
\frac{\pi\alpha^2}{4}\|R(\alpha+\mathrm{i}\beta, A)x\|^2 &\leqslant \int\limits_{\frac{\alpha}{2}}^{\frac{3\alpha}{2}} \int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(a+\mathrm{i}b, A)x\|^2 \,\mathrm{d}b\,\mathrm{d}a \\
&\leqslant \alpha \sup_{a\in[\frac{\alpha}{2},\frac{3\alpha}{2}]} \int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(a+\mathrm{i}b, A)x\|^2 \,\mathrm{d}b \\
&= \alpha \int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(a^*(\alpha)+\mathrm{i}b, A)x\|^2 \,\mathrm{d}b \\
&\leqslant 2a^*(\alpha) \int\limits_{\beta_0-\varepsilon}^{\beta_0+\varepsilon} \|R(a^*(\alpha)+\mathrm{i}b, A)x\|^2 \,\mathrm{d}b
\end{aligned} \tag{4.24}$$

for some $a^* \in \left[\frac{\alpha}{2}, \frac{3\alpha}{2}\right]$. If (4.22) holds, then the last expression in (4.24) tends to 0 as $\alpha \to 0+$. Therefore, $\alpha\, R(\alpha+\mathrm{i}\beta, A)x \to 0$, $\alpha \to 0+$ uniformly in $\beta \in [\beta_0-\varepsilon, \beta_0+\varepsilon]$.

2) We can assume without loss of generality that $(T(t))_{t\geqslant 0}$ is a contraction semigroup, so that $\alpha\|R(\alpha+\mathrm{i}\beta, A)\| \leqslant 1$. If (4.23) holds, then $\alpha\, R(\alpha+\mathrm{i}\beta, A)x \to 0$, $\alpha \to 0+$, for every $\beta \in [\beta_0-\varepsilon, \beta_0+\varepsilon]$. Observe that

$$\begin{aligned}
&\frac{1}{2^{n+1}}\, R\bigg(\frac{1}{2^{n+1}}+\mathrm{i}\beta, A\bigg)x \\
&\quad= \frac{1}{2^{n+1}}\, R\bigg(\frac{1}{2^n}+\mathrm{i}\beta, A\bigg)x + \frac{1}{2^{2n+2}}\, R\bigg(\frac{1}{2^n}+\mathrm{i}\beta, A\bigg) R\bigg(\frac{1}{2^{n+1}}+\mathrm{i}\beta, A\bigg)x \\
&\quad= \bigg(\frac{1}{2}I + \frac{1}{2}\bigg(\frac{1}{2^{n+1}}\, R\bigg(\frac{1}{2^{n+1}}+\mathrm{i}\beta, A\bigg)\bigg)\bigg)\frac{1}{2^n}\, R\bigg(\frac{1}{2^n}+\mathrm{i}\beta, A\bigg)x, \quad n \in \mathbb{N}.
\end{aligned}$$

Therefore,

$$\frac{1}{2^{n+1}}\left\|R\left(\frac{1}{2^{n+1}}+\mathrm{i}\beta, A\right)x\right\| \leqslant \frac{1}{2^n}\left\|R\left(\frac{1}{2^n}+\mathrm{i}\beta, A\right)x\right\|, \quad n \in \mathbb{N}.$$

Hence the decreasing sequence $\left\{\frac{1}{2^n}\left\|R\left(\frac{1}{2^n}+\mathrm{i}\cdot, A\right)x\right\| : n \in \mathbb{N}\right\}$ of continuous functions on the compact set $[\beta_0-\varepsilon, \beta_0+\varepsilon]$ converges to zero. By Dini's theorem, this convergence is uniform on $[\beta_0-\varepsilon, \beta_0+\varepsilon]$. Since for $\alpha \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$:

$$\begin{aligned}
&\left\|\alpha R(\alpha+\mathrm{i}\beta, A)x - \frac{1}{2^n}R\left(\frac{1}{2^n}+\mathrm{i}\beta, A\right)x\right\| \\
&\leqslant \left(\frac{1}{2^n}-\alpha\right)\left\|R\left(\frac{1}{2^n}+\mathrm{i}\beta, A\right)x\right\| \|\alpha R(\alpha+\mathrm{i}\beta, A)\| + \left(\frac{1}{2^n}-\alpha\right)\left\|R\left(\frac{1}{2^n}+\mathrm{i}\beta, A\right)x\right\| \\
&\leqslant \frac{1}{2^n}\left\|R\left(\frac{1}{2^n}+\mathrm{i}\beta, A\right)x\right\|,
\end{aligned}$$

the statement follows.

Note that we used the boundedness of $(T(t))_{t\geqslant 0}$ only for the proof of the second part of the proposition. ∎

If $(T(t))_{t\geqslant 0}$ is a stable C_0-semigroup in a Hilbert space, then, using Jensen's inequality with the probability measure $\alpha \mathrm{e}^{-\alpha t}\,\mathrm{d}t$, $\alpha > 0$, we obtain

$$\begin{aligned}
(\alpha\,\|R(\alpha+\mathrm{i}\beta, A)x\|)^2 &\leqslant \left(\int_0^\infty \alpha \mathrm{e}^{-\alpha t}\|T(t)x\|\,\mathrm{d}t\right)^2 \leqslant \int_0^\infty \alpha \mathrm{e}^{-\alpha t}\|T(t)x\|^2\,\mathrm{d}t \\
&= \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} \left\|R\left(\frac{\alpha}{2}+\mathrm{i}\beta, A\right)x\right\|^2 \mathrm{d}\beta.
\end{aligned}$$

Hence the global condition (3.1) implies that (4.21) is satisfied uniformly in $\beta \in \mathbb{R}$. However, it is not clear how to obtain the property (4.21) "uniform" in all $\beta \in \mathbb{R}$ from the local "integrability" conditions (3.5), (4.1) or from the "pointwise convergence" conditions (3.7), (4.11). It would be interesting to know whether there is a type of asymptotic behaviour of $(T(t))_{t\geqslant 0}$, equivalent to the "uniform" condition (4.21).

5. CONCLUDING REMARKS

We start with an indication of some possible refinements of the statements obtained in Sections 2 and 3. We will consider only C_0-semigroups, although the remarks (A), (B) below are valid with evident changes for the discrete operator semigroups.

(A) The conditions of all our stability results for a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ could be formulated with the exception of closed countable sets $S := S(\varepsilon, \beta_0) \subset (\beta_0-\varepsilon, \beta_0+\varepsilon)$ (depending on ε, β_0) on which we could only require:

(5.1) $$\lim_{\alpha\to 0+} \alpha R(\alpha+\mathrm{i}\beta, A)x = 0, \quad x \in X\ (x \in H),\ \beta \in S.$$

For example, the corresponding version of Theorem 4.7 can be stated as follows.

THEOREM 5.1. *Let $(T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup in a Banach space X with generator A. If for every $\beta_0 \in \mathbb{R}$ there exist $\varepsilon > 0$, a closed countable set $S := S(\varepsilon, \beta_0) \subset (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ and a dense set $M := M(\beta_0, \varepsilon) \subset X$ such that:*

(i) $\lim\limits_{\alpha\to 0+} \alpha R^2(\alpha + \mathrm{i}\beta, A)x = 0$, *$\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \setminus S$, $x \in M$;*

(ii) $\lim\limits_{\alpha\to 0+} \alpha R(\alpha + \mathrm{i}\beta, A)x = 0$, *$\beta \in S$, $x \in X$;*

then $(T(t))_{t\geqslant 0}$ is stable.

In this case, for fixed $\beta_0 \in \mathbb{R}$, $x \in X$ and corresponding $\varepsilon := \varepsilon(\beta_0)$, the reasoning similar to the proof of Theorem 4.7 shows that

$$S_{\beta_0} := \operatorname{supp} Q(\cdot)j(x) \cap (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \quad \text{is countable,}$$

where $Q(\cdot)$ is defined by (2.4). Hence, if S_{β_0} is nonempty, then it contains an isolated point $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$. The local resolvent $R(\lambda, B)j(x)$ extends analytically into some punctured disc $B^0(\mathrm{i}\beta, r)$. Then by Lemma 4.5, there exists $C > 0$ such that

$$\|R(\lambda, B)j(x)\| \leqslant \frac{C}{|\lambda - \mathrm{i}\beta|}, \quad \lambda \in B^0\Big(\mathrm{i}\beta, \frac{r}{2}\Big).$$

Hence from the Laurent expansion of $R(\lambda, B)j(x)$ in $B^0(\mathrm{i}\beta, \frac{r}{2})$ it follows that

$$\lim_{\alpha\to 0} \alpha\|R(\alpha + \mathrm{i}\beta, B)j(x)\| = \lim_{\lambda\to \mathrm{i}\beta} |\lambda - \mathrm{i}\beta|\, \|R(\lambda, B)j(x)\| > 0.$$

On the other hand, by resolvent majorization,

$$\alpha\|R(\alpha + \mathrm{i}\beta, B)j(x)\| \leqslant \alpha\|j\|\, \|R(\alpha + \mathrm{i}\beta, A)x\| \to 0, \quad \alpha \to 0+,$$

which contradicts the previous inequality, and the observation follows.

Thus, our stability results with the above formulation (A) generalize Theorem 1.2, Theorem 1.3 and Corollary 1.4 from the Introduction. Indeed, for every closed countable set $E \subset \mathbb{R}$ and every $\beta_0 \in \mathbb{R}$ there is δ, $0 < \delta < \varepsilon$, such that $E_{\beta_0} := (\beta_0 - \delta, \beta_0 + \delta) \cap E$ is closed and countable. However, the proofs then are reduced to the study of two parts of $\operatorname{supp} Qj(x)$: $E_{\beta_0} \setminus \operatorname{supp} Qj(x)$ and $E_{\beta_0} \cap \operatorname{supp} Qj(x)$. The study of the second part subsumes implicitly refering to the previous spectral stability criterias. Is the "distinguishing" of countable exceptional sets essential and do our integrability conditions (3.5), (4.1) include Theorem 1.3 or Theorem 1.2, as they are stated in the paper ? This question remained unclear. Despite the fact that Theorem 3.3 gives necessary and sufficient conditions for the stability of a bounded C_0-semigroup in a Hilbert space, we do not know how to deduce Theorem 1.2 from it directly.

In connection with Corollary 1.4 from the Introduction note the following.

(B) Any conditions for the stability of a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ can be given in the form:

"There exists a dense set $M \in X$ such that for every $x \in M$ these conditions hold for the restriction of $(T(t))_{t\geqslant 0}$ to the closed linear span, X_x, of $\{T(t)x : t \geqslant 0\}$."

Formally, the form (B) is more general than the initial, but it is not clear whether these two forms are equivalent.

Further, as an illustration, we will show how to obtain Theorem 1.2 from Theorems 3.4 and 4.7 in their original formulation.

Suppose that a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ satisfies the conditions of Theorem 1.2, so $\sigma(A)\cap \mathrm{i}\mathbb{R}$ is countable, and $\sigma_{\mathrm{p}}(A^*)\cap \mathrm{i}\mathbb{R}=\emptyset$. By a variant of the Mittag-Leffler theorem ([16]),

$$M := \bigcap_{\beta\in\mathbb{R}} \mathrm{Im}(A-\mathrm{i}\beta I) \quad \text{is dense in } X.$$

Since $\lim\limits_{\alpha\to 0+} \alpha R(\alpha+\mathrm{i}\beta, A)x = 0$, $\beta\in\mathbb{R}$, $x\in M$, and $\sup\limits_{\alpha>0} \alpha\|R(\alpha+\mathrm{i}\beta, A)\|<\infty$, we have

$$\lim_{\alpha\to 0+} \alpha R(\alpha+\mathrm{i}\beta, A)x = 0, \quad \beta\in\mathbb{R},$$

for every $x\in X$. It then follows from [28],

$$M = \Big\{x\in X : \text{ for every } \beta\in\mathbb{R},\ \exists \lim_{\alpha\to 0+} R(\alpha+\mathrm{i}\beta, A)x\Big\}.$$

By Lemma 2.7, for fixed $\beta\in\mathbb{R}$ and $x\in X$, the property

$$\exists \lim_{\alpha\to 0+} R(\alpha+\mathrm{i}\beta, A)x \tag{5.2}$$

implies

$$\lim_{\alpha\to 0+} \alpha R^2(\alpha+\mathrm{i}\beta, A)x = 0. \tag{5.3}$$

Thus,

$$\overline{\{x\in X : \text{ for every } \beta\in\mathbb{R}\ \lim_{\alpha\to 0+} \alpha R^2(\alpha+\mathrm{i}\beta, A)x = 0\}} = \overline{M} = X.$$

Thus, Theorem 1.2 can be deduced from Theorem 4.7. Taking into account Remark 4.11, we see that Theorem 1.2 also follows from Theorem 3.4 in the Hilbert space case.

REMARK 5.2. R. Chill had a conjecture that (5.2) satisfied for all $\beta\in\mathbb{R}$ and x from a dense set in X implies the stability of a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$.

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