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OPERATORS ON HILBERT H^*-MODULES

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Abstract. Let W be a Hilbert H^*-module over an H^*-algebra $\mathcal{E}$. We show that all bounded $\mathcal{E}$-linear operators on W are reduced by a suitable Hilbert space contained in W. This enables us to describe bounded $\mathcal{E}$-linear operators by lifting the appropriate results from Hilbert space theory. In particular, generalized compact $\mathcal{E}$-linear operators are characterized.

Keywords: *H^*-algebra, Hilbert H^*-module, bounded operator.*

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INTRODUCTION

A Hilbert H^*-module W over an H^*-algebra $\mathcal{E}$ is a left $\mathcal{E}$-module which possesses a $\tau(\mathcal{E})$-valued product. In the same time W is a Hilbert space with the inner product given by the action of the trace on the $\tau(\mathcal{E})$-valued product.

The notion is introduced by Saworotnow in [9] under the name of generalized Hilbert space. It has been studied by Smith, Molnár, Cabrera, Martínez and Rodríguez and others.

In contrast to the more general Hilbert modules over C^*-algebras two special features of Hilbert H^*-modules are of particular importance:

— each H^*-module is a Hilbert space by itself,

— each H^*-module has an $\mathcal{E}$-orthonormal basis.

The existence of an orthonormal basis in H^*-modules is proved in [4] (see also [7]). There is a simple but remarkable idea behind the construction: an orthonormal basis consists of so called basic vectors which are supported by minimal projections in $\mathcal{E}$. As an application, the theory of Hilbert-Schmidt operators on Hilbert H^*-modules is developed.

In the present paper we discuss bounded $\mathcal{E}$-linear operators on Hilbert H^*-modules in general.

According to the structural results of Smith our discussion can be essentially reduced to the case when the underlying H^*-algebra is topologically simple. The first important step is contained in our Lemma 1.4: each H^*-module over a simple

H^*-algebra has an orthonormal basis whose all members have the same supporting minimal projection. This leads to the discovery of a Hilbert space, denoted by W_e (see Lemmas 2.7 and 2.8 below) which is contained as a subspace in W. All bounded $\mathcal{E}$-linear operators are reduced by W_e. Remarkably, W_e contains all information about each bounded $\mathcal{E}$-linear operator on W. It turns out that each bounded $\mathcal{E}$-linear operator is, roughly speaking, the $\mathcal{E}$-homogenous extension of its restriction to W_e. More precisely, our Theorem 2.10 states that the algebra of all bounded $\mathcal{E}$-linear operators on W and the algebra of all bounded linear operators on W_e are naturally isomorphic as C^*-algebras.

This enables us to apply Hilbert space theory by lifting results from the Hilbert space W_e to the whole H^*-module W.

In Theorems 2.12 and 2.14 generalized compact $\mathcal{E}$-linear operators are described. Also, generalized Fredholm operators are introduced and an analogue of Atkinson's theorem is proved.

The paper is organized as follows:

In Section 1 our notation is fixed and some preliminary results are given. Also, the weak convergence in H^*-modules is briefly discussed.

In Section 2 the associated Hilbert space W_e is introduced and described. After that our results on bounded $\mathcal{E}$-linear and generalized compact operators are given.

1. BASIC NOTIONS AND PRELIMINARY RESULTS

An H^*-*algebra* $\mathcal{E}$ is a complex associative Banach algebra $\mathcal{E}$ with involution, whose underlying Banach space is a Hilbert space $(\mathcal{E}, \langle \cdot|\cdot \rangle)$ which has an involution $a \mapsto a^*$, such that $\langle ab|c\rangle = \langle b|a^*c\rangle = \langle a|cb^*\rangle$ for all $a, b, c \in \mathcal{E}$.

A *proper* H^*-algebra is an H^*-algebra with zero annihilator.

The trace-class in a proper H^*-algebra $\mathcal{E}$ is defined as the set $\tau(\mathcal{E}) = \{ab : a, b \in \mathcal{E}\}$. It is known that $\tau(\mathcal{E})$ is an ideal of $\mathcal{E}$ which is a Banach $*$-algebra under a suitable norm $\tau(\cdot)$. The norm τ is related to the given norm $\|\cdot\|$ on $\mathcal{E}$ by $\tau(a^*a) = \|a\|^2$ for all $a \in \mathcal{E}$. There exists a continuous linear form tr on $\tau(\mathcal{E})$ (called the *trace*) satisfying $\mathrm{tr}(ab) = \mathrm{tr}(ba) = \langle a|b^*\rangle$ for all $a, b \in \mathcal{E}$. In particular, $\mathrm{tr}(aa^*) = \mathrm{tr}(a^*a) = \langle a|a\rangle = \|a\|^2 = \tau(a^*a),\ \forall a \in \mathcal{E}$.

A projection is a selfadjoint idempotent e in $\mathcal{E}$; e is called *minimal* if $e \neq 0$ and $e\mathcal{E}e = \mathbb{C}e$.

Each simple H^*-algebra (that is, an H^*-algebra without nontrivial closed two-sided ideals) contains minimal projections. It is known that all minimal projections in a simple H^*-algebra have equal norms. Also, we shall need the following result ([1], Theorem 4.3):

Let $\mathcal{E}$ be a simple H^*-algebra and let (e_λ), $\lambda \in \Lambda$, be any family of minimal projections in $\mathcal{E}$. Then there exists a family of partial isometries $(g_{\lambda,\mu})$, $\lambda, \mu \in \Lambda$, with the following properties:

(p-i) $e_\lambda \mathcal{E} e_\mu = \mathbb{C} g_{\lambda,\mu},\ \forall \lambda, \mu \in \Lambda$;
(p-ii) $e_\lambda = g_{\lambda,\lambda},\ \forall \lambda \in \Lambda$;
(p-iii) $g_{\lambda,\mu} g_{\mu,\nu} = g_{\lambda,\nu},\ \forall \lambda, \mu, \nu \in \Lambda$;
(p-iv) $g_{\lambda,\mu} = g^*_{\mu,\lambda},\ \forall \lambda, \mu \in \Lambda$;
(p-v) $\|g_{\lambda,\mu}\| = \|g_{\upsilon,\nu}\|,\ \forall \lambda, \mu, \upsilon, \nu \in \Lambda$.

Actually, Theorem 4.3 from [1] states that each simple H^*-algebra $\mathcal{E}$ is the full algebra $\mathcal{HS}(H)^\alpha$ of Hilbert-Schmidt operators on some Hilbert space H of suitable dimension. Consequently, the set of all minimal projections in $\mathcal{E}$ coincides with the set of all orthogonal projections of rank 1 and all of them have the norm equals to α for some $\alpha \geqslant 1$.

If a proper H^*-algebra is not simple, then, by Theorem 4.2 from [1], it is the orthogonal sum $\mathcal{E} = \bigoplus_{i\in I} \mathcal{E}_i$ where each $\mathcal{E}_i$ is a simple H^*-algebra. By the preceding identification we have $\mathcal{E} = \bigoplus_{i\in I} \mathcal{HS}(H_i)^{\alpha_i}$.

Let us observe that the family (α_i) must be bounded. Indeed, if the family (α_i) contains an unbounded sequence (α_{i_k}) an easy argument from Hilbert spaces (see [6], Problem 41) gives a contradiction with the completeness of $\mathcal{E}$.

After all we conclude: the set $\mathcal{P}_m(\mathcal{E})$ of all minimal projections in a proper H^*-algebra $\mathcal{E}$ is bounded.

Definition 1.1. A *Hilbert $\mathcal{E}$-module* is a left module W over a proper H^*-algebra $\mathcal{E}$ provided with a mapping $[\cdot|\cdot] : W \times W \to \tau(\mathcal{E})$ (called *$\tau(\mathcal{E})$-valued product*) which satisfies the following conditions:

(m-i) $[\alpha x|y] = \alpha[x|y], \ \forall \alpha \in \mathbb{C}, \ \forall x, y \in W$;
(m-ii) $[x + y|z] = [x|z] + [y|z], \ \forall x, y, z \in W$;
(m-iii) $[ax|y] = a[x|y], \ \forall a \in \mathcal{E}, \ \forall x, y \in W$;
(m-iv) $[x|y]^* = [y|x], \ \forall x, y \in W$;
(m-v) $\forall x \in W$, $x \neq 0$, $\exists a \in \mathcal{E}$, $a \neq 0$, such that $[x|x] = a^*a$;
(m-vi) W is a Hilbert space with the inner product $(x|y) = \operatorname{tr}\big([x|y]\big)$.

For the basic facts about Hilbert H^*-modules we refer to [4], [9] and [10]. In particular, we shall frequently use the following three immediate consequences of the above definition:

$$\begin{array}{ll} \|x\|^2 = \operatorname{tr}\big([x|x]\big) = \tau\big([x|x]\big) & \forall x \in W, \\ \big\|[x|y]\big\| \leqslant \tau\big([x|y]\big) \leqslant \|x\|\,\|y\|, & \forall x, y \in W, \\ \|ax\| \leqslant \|a\|\,\|x\|, & \forall a \in \mathcal{E}, \forall x \in W. \end{array}$$

Definition 1.2. An element $u \in W$ is said to be a *basic element* if there exists a minimal projection $e \in \mathcal{E}$ (called the *supporting projection*) such that $[u|u] = e$. An *orthonormal system in* W is a family of basic elements $(u_\lambda), \lambda \in \Lambda$ satisfying $[u_\lambda|u_\mu] = 0$ for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. An *orthonormal basis in* W is an orthonormal system generating a dense submodule of W.

We recall from [4] (and [7]) that each Hilbert H^*-module contains basic elements, orthonormal systems and orthonormal bases. Moreover, all orthonormal bases for W have the same cardinal number called the *hilbertian dimension* of W over $\mathcal{E}$ and denoted by $\mathcal{E}$-$\dim W$.

If (v_λ), $\lambda \in \Lambda$, is an orthonormal basis for W then the following two equivalent conditions are satisfied for each $x \in W$:

$$\begin{array}{ll} x = \sum_\lambda [x|v_\lambda] v_\lambda, & \text{(Fourier expansion)} \\ [x|x] = \sum_\lambda [x|v_\lambda]\,[x|v_\lambda]^*, & \text{(Parseval's identity).} \end{array}$$

In particular, since the trace is continuous, this implies $\|x\|^2 = \sum_\lambda \big\|[x|v_\lambda]\big\|^2$, $\forall x \in W$.

Let us observe that basic elements, in general, do not belong to the unit sphere because minimal projections may have norm greater then 1. However, the set of all basic elements in a Hilbert H^*-module must be bounded.

PROPOSITION 1.3 *Let W be a Hilbert module over an H^*-algebra $\mathcal{E}$. The set $\mathcal{M}(\mathcal{E})$ of all basic elements in W is bounded.*

Proof. Let us denote $\sup\{\|e\| : e \in \mathcal{P}_m(\mathcal{E})\} = M$ and take $v \in \mathcal{M}(\mathcal{E})$, $[v|v] = e \in \mathcal{P}_m(\mathcal{E})$. Then $\|v\|^2 = \operatorname{tr} e = \|e\|^2 \leqslant M^2$. ∎

Given any orthonormal basis (v_λ), $\lambda \in \Lambda$, the supporting projections $e_\lambda = [v_\lambda|v_\lambda]$ are, in general, different. The first remarkable fact we shall prove is that each Hilbert H^*-module over a simple H^*-algebra possesses an orthonormal basis $(w_\lambda), \lambda \in \Lambda$, such that all its members w_λ have the same supporting minimal projection. First, we need a lemma which slightly improves Lemma 1.3 from [4] (see also Lemma 1 in [7]).

LEMMA 1.4. *Let W be a Hilbert H^*-module over an arbitrary H^*-algebra $\mathcal{E}$, let $e \in \mathcal{E}$ be a projection (not necessarily minimal) and let $x \in W$ be such that $[x|x] = e$. Then $ex = x$.*

Proof. $[ex - x|ex - x] = e^3 - e^2 - e^2 + e = 0$ implies $x - ex = 0$. ∎

PROPOSITION 1.5. *Let W be a Hilbert H^*-module over a simple H^*-algebra $\mathcal{E}$ and let $e_0 \in \mathcal{E}$ be a minimal projection. Then there exists an orthonormal basis (w_λ), $\lambda \in \Lambda$ for W such that $[w_\lambda|w_\lambda] = e_0$ for all $\lambda \in \Lambda$.*

Proof. Let $(v_\lambda), \lambda \in \Lambda$, be an arbitrary orthonormal basis for W such that $[v_\lambda|v_\lambda] = e_\lambda$ for all $\lambda \in \Lambda$. Let $(g_{\lambda,\mu}), \lambda, \mu \in \Lambda' = \Lambda \cup \{0\}$, be the family of partial isometries satisfying conditions (p-i)–(p-v). Let us define $w_\lambda = g_{0,\lambda}v_\lambda$, $\forall \lambda \in \Lambda$. Obviously, $[w_\lambda|w_\lambda] = g_{0,\lambda}[v_\lambda|v_\lambda]g_{0,\lambda}^* = g_{0,\lambda}g_{\lambda,\lambda}g_{\lambda,0} = g_{0,0} = e_0$ $\forall \lambda \in \Lambda$ and $[w_\lambda|w_\mu] = 0$ for $\lambda \neq \mu$.

It remains to show that the orthonormal system (w_λ), $\lambda \in \Lambda$, generates a dense submodule of W. Let $x \in W$ satisfy $[x|w_\lambda] = 0$, $\forall \lambda \in \Lambda$. Then, for each $\lambda \in \Lambda$, we have $[x|g_{0,\lambda}v_\lambda] = 0 \Rightarrow [x|g_{\lambda,0}g_{0,\lambda}v_\lambda] = 0 \Rightarrow [x|e_\lambda v_\lambda] = 0$. By Lemma 1.4 this implies $[x|v_\lambda] = 0$, hence $x = 0$. ∎

REMARK 1.6. A Hilbert H^*-module over an H^*-algebra $\mathcal{E}$ is said to be *faithful* if it has zero annihilator in $\mathcal{E}$. Since the annihilator of a Hilbert H^*-module is a closed two-sided ideal in $\mathcal{E}$, one can always regard any Hilbert H^*-module as a faithful H^*-module over the orthogonal complement of its annihilator. Therefore, all Hilbert H^*-modules can be assumed faithful without loss of generality.

We recall from [10] (see also [4]) that for each faithful Hilbert H^*-module over a proper H^*-algebra $\mathcal{E}$ there exists a family (W_i), $i \in I$, of Hilbert H^*-modules, where each W_i is a Hilbert H^*-module over a simple H^*-algebra $\mathcal{E}_i$ (contained as a minimal two-sided ideal in $\mathcal{E}$), such that W is equal to the mixed product of the family (W_i), i.e.

$$W = \mathop{\times}_{i\in I} W_i = \Big\{\{w_i\} \in \prod_{i\in I} W_i : \sum_{i\in I} \|w_i\|^2 < \infty\Big\}.$$

Now one can apply Proposition 1.5 in each W_i: there exists an orthonormal basis $(w_{\lambda,i})$, $\lambda \in \Lambda$, with the same supporting projections $e_i \in \mathcal{E}_i$.

In the last part of this section we briefly discuss the weak convergence in Hilbert H^*-modules.

Definition 1.7. A sequence (or net) (x_n) *converges weakly to* $x \in W$ in a Hilbert H^*-module W if $[x|y] = \lim_n [x_n|y]$, $\forall y \in W$. We denote weak convergence by $x_n \xrightarrow{[w]} x$.

Remark 1.8. Since the trace is continuous each weakly convergent sequence in a Hilbert H^*-module W also converges weakly (in the usual sense) in the Hilbert space $(W, (\,\cdot|\cdot\,))$, and therefore it must be bounded.

If $(v_\lambda), \lambda \in \Lambda$, is any orthonormal system in W then Parseval's identity applied in a closed submodule generated by all v_λ's forces $v_\lambda \xrightarrow{[w]} 0$ in the sense of "vanishing at infinity" (i.e. $\forall y, \varepsilon > 0$ the set $\{\lambda \in \Lambda : \|[v_\lambda|y]\| \geqslant \varepsilon\}$ is finite).

Proposition 1.9. *Let W be a Hilbert H^*-module. Each bounded sequence of mutually orthogonal elements in W weakly converges to* 0.

Proof. Let (x_n) be a sequence in W such that $[x_n|x_m] = 0$, $\forall n \neq m$ and $\|x_n\| \leqslant C$, $\forall n$, for some $C > 0$. Let W_n be the closed submodule generated by the element x_n and suppose $W = \bigoplus_{n=0}^{\infty} W_n$ with W_0 denoting the orthogonal complement of the closed submodule spanned by $\{x_n : n \in \mathbb{N}\}$. Now, for an arbitrary element $w = (w_n) \in W$, we have $\sum_{n=0}^{\infty} \|[w|x_n]\|^2 = \sum_{n=0}^{\infty} \|[w_n|x_n]\|^2 \leqslant \sum_{n=0}^{\infty} \|w_n\|^2 \|x_n\|^2 \leqslant C^2 \sum_{n=0}^{\infty} \|w_n\|^2 = C^2 \|w\|^2$. ∎

Remark 1.10. The proof of the above proposition cannot be based on the direct application of Parseval's identity to the sequence $(x_n/\|x_n\|)$. Namely, neither x_n's, nor their scalar multiples are (in general) basic elements. This becomes clear through the following phenomenon which serves as a remarkable difference between Hilbert H^*-modules and Hilbert spaces: the hilbertian dimension of a submodule W_x generated by an element x can be greater than 1 (even infinite). In fact, $\mathcal{E}$-$\dim W_x$ is equal to the cardinal number of the set $\mathcal{J}$ of indices such that $[x|x] = \sum_{\lambda \in \mathcal{J}} \alpha_\lambda e_\lambda$ where e_λ's are orthogonal minimal projections in $\mathcal{E}$ and $\alpha_\lambda > 0$. Namely, $x = \sum_{\lambda \in \mathcal{J}} \alpha_\lambda^{1/2} x_\lambda$ with respect to the orthonormal system $(x_\lambda)_{\lambda \in \mathcal{J}}$, $x_\lambda = \alpha_\lambda^{-1/2} e_\lambda x$.

2. Bounded operators on Hilbert H^*-modules

Definition 2.1. Let W be a Hilbert module over an H^*-algebra $\mathcal{E}$. An operator $A : W \to W$ is called *$\mathcal{E}$-linear* if it is linear and satisfies $A(ax) = aAx$, $\forall a \in \mathcal{E}$, $\forall x \in W$. The set of all bounded $\mathcal{E}$-linear operators on W is denoted by $\mathbf{B}_{\mathcal{E}}(W)$.

It is well known ([9]) that each $A \in \mathbf{B}_{\mathcal{E}}(W)$ has an adjoint $A^* \in \mathbf{B}_{\mathcal{E}}(W)$ in the sense $[Ax|y] = [x|A^*y]$, $\forall x, y \in W$. $\mathbf{B}_{\mathcal{E}}(W)$ is a C^*-algebra contained in the algebra $\mathbf{B}(W)$ of all bounded operators on W.

The strong (respectively weak) convergence of operators on Hilbert H^*-modules is defined in the standard way. It is easy to see that $\mathbf{B}_{\mathcal{E}}(W)$ is also closed in $\mathbf{B}(W)$ in the strong and weak operator topology, respectively.

Definition 2.2. Let W be a Hilbert H^*-module, let $v, w \in W$ be basic vectors and let the operator $F_{v,w} : W \to W$ be defined with $F_{v,w}(x) = [x|w]v$. The linear span of the set $\{F_{v,w} : v, w \in W\}$ is denoted by $\mathbf{F}_{\mathcal{E}}(W)$ and an operator A belonging to $\mathbf{F}_{\mathcal{E}}(W)$ is called a *generalized finite rank operator*.

Observe that $\mathbf{F}_{\mathcal{E}}(W) \subseteq \mathbf{B}_{\mathcal{E}}(W)$ and $F^*_{v,w} = F_{w,v}$, $AF_{v,w} = F_{Av,w}$, $F_{v,w}A = F_{v,A^*w}$, $\forall v, w \in W$, $\forall A \in \mathbf{B}_{\mathcal{E}}(W)$. Therefore $\mathbf{F}_{\mathcal{E}}(W)$ is a selfadjoint two-sided ideal in $\mathbf{B}_{\mathcal{E}}(W)$.

The above definition imitates the definition of Hilbert space finite rank operators, but, as we observed in Remark 1.10, the range of an operator $F_{x,y}$ may fail to have finite hilbertian dimension. However, if $x \in W$ is a basic element, the operator $F_{x,x}$ has a range whose hilbertian dimension is exactly 1. In fact, $F_{x,x}$ is the orthogonal projection to the closed submodule W_x generated by x. Of course, the range of $F_{x,x}$ regarded as a linear space may still be infinite dimensional. We shall illustrate this by the following example:

Example 2.3. Let $(H, (\cdot|\cdot))$ be an infinite dimensional Hilbert space and let $\mathcal{HS}(H)$ be the standard H^*-algebra of Hilbert-Schmidt operators on H. Let us denote by $\Theta_{x,y}$ the classical rank 1 operator on H: $\Theta_{x,y}(z) = (z|y)x$.

It is well known that H may be regarded as an H^*-module over $\mathcal{HS}(H)$. Given $x \in H$ and $T \in \mathcal{HS}(H)$, Tx is interpreted as the action of T. The $\mathcal{HS}(H)$-valued product on H is defined by $[x|y] = \Theta_{x,y}$. Since $\operatorname{tr}\Theta_{x,y} = (x|y)$ we conclude that the resulting norm on H coincides with the original one.

One can easily see that $\mathcal{HS}(H)$-$\dim H = 1$. Further, $F_{x,x} = I$, $\forall x \in H$, $\|x\| = 1$ (I denotes the identity operator on H) and $\mathbf{F}_{\mathcal{HS}(H)}(H) = \mathbf{B}_{\mathcal{HS}(H)}(H) = \{\alpha I : \alpha \in \mathbf{C}\}$.

Definition 2.4. An operator $A \in \mathbf{B}_{\mathcal{E}}(W)$ is said to be a *generalized compact* operator if there exists a sequence of generalized finite rank operators (F_n) such that $\lim\limits_n F_n = A$. The set of all generalized compact operators is denoted by $\mathbf{K}_{\mathcal{E}}(W)$.

By definition, $\mathbf{K}_{\mathcal{E}}(W) = \overline{\mathbf{F}_{\mathcal{E}}(W)}$ ($\overline{}$ denotes the topological closure), thus $\mathbf{K}_{\mathcal{E}}(W)$ is a closed selfadjoint two-sided ideal in $\mathbf{B}_{\mathcal{E}}(W)$.

Remark 2.5. Let W be a Hilbert H^*-module over an H^*-algebra $\mathcal{E}$. According to Remark 1.6, $W = \bigtimes_{i\in I} W_i$ where (W_i), $i \in I$, is a family of Hilbert H^*-modules, and each W_i is a module over a simple H^*-algebra $\mathcal{E}_i$.

Obviously, $W_i = \overline{\mathcal{E}_i W}$ ($\mathcal{E}_i$ being canonically embedded in $\mathcal{E} = \bigoplus_{i\in I} \mathcal{E}_i$), $\forall i \in I$. This implies $AW_i \subseteq W_i$, $\forall i \in I$, $\forall A \in \mathbf{B}_{\mathcal{E}}(W)$. Since the same holds true for A^* we conclude that each operator $A \in \mathbf{B}_{\mathcal{E}}(W)$ is reduced by all W_i's and therefore

$$\mathbf{B}_{\mathcal{E}}(W) = \bigoplus_{i\in I} \mathbf{B}_{\mathcal{E}_i}(W_i) = \Big\{\{A_i\} \in \prod_{i\in I} \mathbf{B}_{\mathcal{E}_i}(W_i) : \sup_{i\in I} \|A_i\| < \infty\Big\}.$$

The obtained decomposition of the C^*-algebra of all bounded $\mathcal{E}$-linear operators enables us to reduce our discussion to the case of Hilbert H^*-modules over simple H^*-algebras.

Definition 2.6. Let W be a Hilbert H^*-module over $\mathcal{E}$. For each $a \in \mathcal{E}$ we define the *left translation* $L_a : W \to W$ by $L_a x = ax$, $\forall\, x \in W$.

Obviously, each L_a is a bounded linear operator, but it is not $\mathcal{E}$-linear. As a bounded operator on the Hilbert space $(W, (\cdot|\cdot))$, L_a has the adjoint operator $L_a^{(*)}$; one has actually $L_a^{(*)} = L_{a^*}$.

Each of the operators L_a commutes with all $A \in \mathbf{B}_{\mathcal{E}}(W)$ (in fact, an operator A is $\mathcal{E}$-linear precisely when $AL_a = L_a A$ is satisfied for all $a \in \mathcal{E}$).

In particular, if $e \in \mathcal{E}$ is a projection then L_e is an orthogonal projection defined on the Hilbert space $(W, (\cdot|\cdot))$. Let us denote $W_e = L_e W$. The subspace W_e is a closed subspace of the Hilbert space $(W, (\cdot|\cdot))$ which reduces each operator $A \in \mathbf{B}_{\mathcal{E}}(W)$.

The special case when e is chosen to be a minimal projection will be of particular importance.

Lemma 2.7. *Let W be a Hilbert H^*-module over $\mathcal{E}$ and let e be a minimal projection in $\mathcal{E}$. The $W_e = \big\{x \in W : [x|x] = \lambda e,\ \lambda \geqslant 0\big\}$. If $\mathcal{E}$ is a simple H^*-algebra, then the subspace W_e generates a dense submodule in W.*

Proof. Let $x = ey$ be an arbitrary element in W_e. Then $[x|x] = e[y|y]e = \lambda e$ because e is a minimal projection. Conversely, if $x \in W$ and $[x|x] = \lambda e$, $\lambda > 0$, for $x' = \lambda^{-1/2}x$ we have $[x'|x'] = e$. By Lemma 1.4 $ex' = x'$ and this implies $x = \lambda^{1/2}x' = \lambda^{1/2}ex' = ex \in W_e$.

If $\mathcal{E}$ is a simple H^*-algebra we can choose an orthonormal basis $(w_\lambda)_{\lambda\in\Lambda}$ for W such that $[w_\lambda|w_\lambda] = e$, $\forall \lambda \in \Lambda$, hence $w_\lambda \in W_e$, $\forall \lambda \in \Lambda$ (Proposition 1.5). Now the conclusion follows by applying Fourier expansion with respect to (w_λ). ∎

Lemma 2.8. *Let W be a Hilbert H^*-module over a simple H^*-algebra $\mathcal{E}$ and let e be a minimal projection in $\mathcal{E}$. Then $[x|y] = \frac{1}{\|e\|^2}(x|y)e$ for all $x, y \in W_e$. An orthonormal system $(w_\lambda)_{\lambda\in\Lambda}$, $w_\lambda \in W_e$, $\lambda \in \Lambda$, is an orthonormal basis for W if and only if the system $(v_\lambda)_{\lambda\in\Lambda}$, $v_\lambda = \frac{w_\lambda}{\|e\|}$, $\lambda \in \Lambda$, is an orthonormal basis for the Hilbert space $(W_e, (\cdot|\cdot))$.*

Proof. Let $x, y \in W_e$, $[x|x] = \lambda e$, $[y|y] = \mu e$, $\lambda, \mu > 0$. If we denote $x' = \lambda^{-1/2}x$, $y' = \mu^{-1/2}y$ then $[x'|x'] = [y'|y'] = e$ and by Lemma 1.4 $x' = ex'$, $y' = ey'$. Now, because e is a minimal projection $[x'|y'] = [ex'|ey'] = e[x'|y']e = \alpha e$ for some

$\alpha \in \mathbb{C}$. This gives $[x|y] = \lambda^{-1/2}\mu^{-1/2}[x'|y'] = \lambda^{-1/2}\mu^{-1/2}\alpha e$ and $(x|y) = \operatorname{tr}[x|y] = \lambda^{-1/2}\mu^{-1/2}\alpha \operatorname{tr} e = \lambda^{-1/2}\mu^{-1/2}\alpha\|e\|^2$ wherefore $[x|y] = (x|y)e/\|e\|^2$.

Let us take an orthonormal basis $(w_\lambda)_{\lambda\in\Lambda}$ for W such that $[w_\lambda|w_\lambda] = e$, $\forall\lambda \in \Lambda$, and let $v_\lambda = w_\lambda/\|e\|$, $\lambda \in \Lambda$. Obviously, $(v_\lambda)_{\lambda\in\Lambda}$ is an orthonormal system in $(W_e, (\cdot|\cdot))$. Let $x \in W_e$ be a vector such that $(x|v_\lambda) = 0$, $\forall\lambda \in \Lambda$. In order to prove that $(v_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis for W_e it remains to show that $x = 0$. Indeed, we also have $(x|w_\lambda) = 0$, $\forall\lambda \in \Lambda$ and by the first statement of the lemma $[x|w_\lambda] = 0$, $\forall\lambda \in \Lambda$. Since $(w_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis for W we conclude $x = 0$.

Conversely, let us take an orthonormal basis $(v_\lambda)_{\lambda\in\Lambda}$ for the Hilbert space $(W_e, (\cdot|\cdot))$, and define $w_\lambda = \|e\|v_\lambda$, $\forall\lambda \in \Lambda$. It follows immediately that $(w_\lambda)_{\lambda\in\Lambda}$ is an orthonormal system in W. Let W' be the closed submodule of W generated by all w_λ, $\lambda \in \Lambda$. We have to show that $W' = W$. Let us suppose $W' \neq W$. Then $W'^\perp \neq \{0\}$ and by Proposition 1.7 from [4] there exists a basic element $v_0 \in W'^\perp$ with supporting projection $e_0 \in \mathcal{E}$. As in the proof of Proposition 1.5 we can find a partial isometry $g_{0,e}$ such that for $w_0 = g_{0,e}v_0 \neq 0$ we have $[w_0|w_0] = e$. This implies $w_0 \in W_e$ and, on the other hand, $[w_0|w_\lambda] = g_{0,e}[v_0|w_\lambda] = 0$, $\forall\lambda \in \Lambda$. This gives $(w_0|w_\lambda) = (w_0|v_\lambda) = 0$ for all $\lambda \in \Lambda$, which is impossible because $(v_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis for W_e and w_0, being a basic element, is not 0. ∎

COROLLARY 2.9. *Let W be a Hilbert H^*-module over a simple H^*-algebra $\mathcal{E}$ and let e be an arbitrary minimal projection in $\mathcal{E}$. Then $\mathcal{E}\text{-}\dim W = \dim W_e$.*

As we mentioned before, the Hilbert space W_e reduces all operators $A \in \mathbf{B}_\mathcal{E}(W)$. In this way, for any $A \in \mathbf{B}_\mathcal{E}(W)$, the induced operator $\widehat{A} \in \mathbf{B}(W_e)$, $\widehat{A} = A|W_e : W_e \to W_e$ is well defined.

THEOREM 2.10. *Let W be a Hilbert H^*-module over an H^*-algebra $\mathcal{E}$.*

(i) *If $\mathcal{E}$ is simple and $e \in \mathcal{P}_m(\mathcal{E})$, then the map $\varphi : \mathbf{B}_\mathcal{E}(W) \to \mathbf{B}(W_e)$, $\varphi(A) = \widehat{A}$ is an isomorphism of C^*-algebras. The map φ also preserves the strong and weak convergence of operators, respectively.*

(ii) *If $\mathcal{E}$ is not a simple H^*-algebra then there exists a family (W_{e_i}), $i \in I$, of Hilbert spaces such that $\mathbf{B}_\mathcal{E}(W)$ and $\bigoplus_{i\in I} \mathbf{B}(W_{e_i})$ are isomorphic C^*-algebras.*

Proof. (i) Obviously, the map φ is a morphism of C^*-algebras. It is injective because $W = \overline{\operatorname{span}\{\mathcal{E}W_e\}}$. Therefore, φ is an isometry. By definition, φ preserves strong convergence. The weak convergence of operators is preserved under φ because the trace is continuous.

To prove the surjectivity let us fix an orthonormal basis $(w_\lambda)_{\lambda\in\Lambda}$ for W such that $[w_\lambda|w_\lambda] = e$, $\forall\lambda \in \Lambda$. According to Lemma 2.8 the system $(v_\lambda)_{\lambda\in\Lambda}$, $v_\lambda = w_\lambda/\|e\|$, $\lambda \in \Lambda$, is an orthonormal basis for W_e. Let $T \in \mathbf{B}(W_e)$ be arbitrarily chosen. For $x \in W_e$ we have $x = \sum_{\lambda\in\Lambda}(x|v_\lambda)v_\lambda$ and $Tx = \sum_{\lambda\in\Lambda}(x|v_\lambda)Tv_\lambda$. For each finite set of indices $S \subset \Lambda$ let us define $T_S \in \mathbf{B}(W_e)$, $T_S x = \sum_{\lambda\in S}(x|v_\lambda)Tv_\lambda$, and $A_S \in \mathbf{F}_\mathcal{E}(W) \subseteq \mathbf{B}_\mathcal{E}(W)$, $A_S w = \sum_{\lambda\in S}[w|w_\lambda]Tw_\lambda$. Then

$$A_S v_\mu = \begin{cases} 0, & \text{if } \mu \notin S; \\ \sum_{\lambda\in S}[v_\mu|w_\lambda]Tw_\lambda = [v_\mu|w_\mu]Tw_\mu = eTv_\mu = Tv_\mu, & \text{if } \mu \in S; \end{cases}$$

(for the last equality we used the fact that $Tv_\mu \in W_e$, so $Tv_\mu = eTv_\mu$). This shows $\varphi(A_S) = \widehat{A}_S = T_S$ for each finite set of indices $S \subset \Lambda$.

Now, let us observe that the net (T_S) converges strongly to T in $\mathbf{B}(W_e)$. Therefore it remains to prove the existence of a strong limit of the net (A_S) in $\mathbf{B}_{\mathcal{E}}(W)$. Then the conclusion will follow from the uniqueness of the strong limit since the strong convergence is preserved under the action of φ.

It is clear that $\sum\limits_{\lambda\in\Lambda} [w|w_\lambda]Tw_\lambda$ exists for all $w \in W_e$. The limit also exists for all elements of the form aw, $a \in \mathcal{E}$, $w \in W_e$ (since $a \mapsto aw$ is a continuous map) and for all their finite sums.

Now we can apply the standard argument: the net (A_S) converges strongly on a dense subset $\operatorname{span}\{\mathcal{E}W_e\}$ of W and $\|A_S\| = \|\varphi(A_S)\| = \|T_S\| \leqslant \|T\|$, for all finite sets $S \subset \Lambda$. This ensures that there exists $A \in \mathbf{B}_{\mathcal{E}}(W)$ such that $Ax = \sum\limits_{\lambda\in\Lambda} [x|w_\lambda]Tw_\lambda$ for all $x \in W$.

(ii) By Remark 2.5, $\mathbf{B}_{\mathcal{E}}(W) = \bigoplus\limits_{i\in I} \mathbf{B}_{\mathcal{E}_i}(W_i)$ where W_i is a Hilbert H^*-module over a simple H^*-algebra $\mathcal{E}_i$, for each $i \in I$. In each $\mathcal{E}_i$ one can chose a minimal projection e_i. Let W_{e_i} be the Hilbert space associated to each W_i. Now, it is clear that

$$\varphi : \bigoplus_{i\in I} \mathbf{B}_{\mathcal{E}_i}(W) \to \bigoplus_{i\in I} \mathbf{B}(W_{e_i}), \qquad \varphi(\{A_i\}) = \{\varphi_i(A_i)\}$$

(here φ_i denotes the isomorphism from the first part of the proof) is an isomorphism. ∎

REMARK 2.11. Let us observe that part (i) from the preceding proof actually shows that φ is a homeomorphism with respect to the strong operator topologies. In particular, a net (A_j) converges strongly to A in $\mathbf{B}_{\mathcal{E}}(W)$ if and only if $Ax = \lim\limits_j A_j x,\ \forall x \in W_e$.

We also note the following consequence of the above theorem: if $\mathcal{E}$ is a simple H^*-algebra then $\varphi(\mathbf{F}_{\mathcal{E}}(W)) = \mathbf{F}(W_e)$ and $\varphi(\mathbf{K}_{\mathcal{E}}(W)) = \mathbf{K}(W_e)$ where $\mathbf{F}(W_e)$ and $\mathbf{K}(W_e)$ denote the ideal of finite rank operators on W_e and the ideal of compact operators on W_e, respectively.

Theorem 2.10 enables us to investigate $\mathcal{E}$-linear operators by applying the standard Hilbert space theory. The procedure is simple and efficient: for an operator A in $\mathbf{B}_{\mathcal{E}}(W)$ one should take $\varphi(A)$, then derive the suitable result and finally lift the obtained conclusion back to $\mathbf{B}_{\mathcal{E}}(W)$.

As an example, we briefly comment the Hilbert-Schmidt class in $\mathbf{B}_{\mathcal{E}}(W)$. Hilbert-Schmidt operators on H^*-modules are introduced in [4]. An operator $A \in \mathbf{B}_{\mathcal{E}}(W)$ is said to be Hilbert-Schmidt if there is an orthonormal basis $(w_\lambda)_{\lambda\in\Lambda}$ for W such that $\sum\limits_{\lambda\in\Lambda} \|Aw_\lambda\|^2 = \|A\|_2^2$ is finite. It is shown in [4] that the number $\|A\|_2^2$ is independent of the particular choice of the basis $(w_\lambda)_{\lambda\in\Lambda}$. Further, the class of all Hilbert-Schmidt operators on W denoted by $\mathcal{HS}_{\mathcal{E}}(W)$ serves as a selfadjoint two-sided ideal in $\mathbf{B}_{\mathcal{E}}(W)$ which is an H^*-algebra with the inner product $(A|B) = \sum\limits_{\lambda\in\Lambda} (Aw_\lambda|Bw_\lambda)$. Finally, $\forall A \in \mathcal{HS}_{\mathcal{E}}(W)$, $\|A\| \leqslant \|A\|_2$, $\mathbf{F}_{\mathcal{E}}(W) \subseteq \mathcal{HS}_{\mathcal{E}}(W)$, and each A in $\mathcal{HS}_{\mathcal{E}}(W)$ can be obtained as the limit of a sequence of generalized finite rank operators in the norm $\|\cdot\|_2$.

Now, Theorem 2.10 (together with Lemma 2.8) implies that $A \in \mathcal{HS}_{\mathcal{E}}(W)$ if and only if the operator $\varphi(A)$ belongs to the standard Hilbert-Schmidt class on the Hilbert space W_e (or $\bigoplus_{i\in I} \mathbf{B}(W_{e_i})$).

Also, if $\mathcal{E}$ is simple we get $\|A\|_2 = \|e\|\,\|\varphi(A)\|_2$. It should be observed that this equality is independent on the particular choice of minimal projections in $\mathcal{E}$ because they all have the same norm. The presence of the factor $\|e\|$ in the above equality comes from the fact that the basic vectors in W do not generally belong to the unit sphere in W: if $[w_\lambda|w_\lambda] = e$ then $\|w_\lambda\| = \|e\|$.

If the underlying H^*-algebra $\mathcal{E}$ is not simple, then using part (ii) of Theorem 2.10, we conclude $\|A\|_2 \leqslant \Big(\sup_{e\in\mathcal{P}_m(\mathcal{E})} \|e\|\Big)\|\varphi(A)\|_2$.

Finally, we note that $\|A\| \leqslant \|A\|_2$ for all $A \in \mathcal{HS}_{\mathcal{E}}(W)$ implies $\mathbf{F}_{\mathcal{E}}(W) \subseteq \mathcal{HS}_{\mathcal{E}}(W) \subseteq \mathbf{K}_{\mathcal{E}}(W)$.

We are going to describe generalized compact operators in some more details. First we note (immediately from Theorem 2.10 and Corollary 2.9) that, if $\mathcal{E}\text{-}\dim W < \infty$, then $\mathbf{F}_{\mathcal{E}}(W) = \mathbf{K}_{\mathcal{E}}(W) = \mathbf{B}_{\mathcal{E}}(W)$. In the following we shall therefore assume $\mathcal{E}\text{-}\dim W = \infty$.

It is well known that an operator A on a Hilbert space is compact if and only if, for each sequence (x_n) weakly converging to 0, $\lim_n Ax_n = 0$ is satisfied. Moreover, A is compact if and only if it satisfies $\lim_n Av_n = 0$ for each orthonormal sequence (v_n) ([5] or [2]).

One may ask if generalized compact operators are also characterized by these properties.

Let (x_n) be an arbitrary sequence in a Hilbert H^*-module such that $x_n \xrightarrow{[w]} 0$. If F is an arbitrary operator in $\mathbf{F}_{\mathcal{E}}(W)$ then obviously $\lim_n Fx_n = 0$. Consequently, each generalized compact operator A, being a limit of a sequence of generalized operators of finite rank, also satisfies $\lim_n Ax_n = 0$.

To prove the converse we shall use Theorem 2.10. We also need the following fact from Hilbert space theory:

Let $\{H_i\}$, $i \in I$, be a family of Hilbert spaces. An operator $\{A_i\} \in \bigoplus_{i\in I} \mathbf{B}(H_i) \subseteq \mathbf{B}\Big(\bigoplus_{i\in I} H_i\Big)$ is compact if and only if the following two conditions are satisfied:

$$\begin{aligned} &A_i \in \mathbf{K}(H_i), \quad \forall i \in I,\\ &\lim_i \|A_i\| = 0 \quad \text{(i.e. the family } (\|A_i\|) \text{ vanishes at infinity).} \end{aligned}$$

THEOREM 2.12. *Let W be a Hilbert H^*-module and $A \in \mathbf{B}_{\mathcal{E}}(W)$. The following conditions are mutually equivalent:*

(i) $A \in \mathbf{K}_{\mathcal{E}}(W)$.

(ii) $\lim_n Ax_n = 0$ *for each sequence (x_n) in W such that $x_n \xrightarrow{[w]} 0$.*

(iii) $\lim_n Ax_n = 0$ *for each bounded sequence (x_n) of mutually orthogonal elements in W.*

(iv) $\lim_n Ax_n = 0$ *for each orthonormal sequence (x_n) in W.*

Proof. (i) $\Rightarrow$ (ii) is already noted in the preceding discussion. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial (see Propositions 1.3 and 1.9).

For the proof of (iv) $\Rightarrow$ (i) suppose first that the underlying H^*-algebra is simple. Let A satisfy (iv). Using Theorem 2.10 (i) we see that the same is true for $\varphi(A) \in \mathbf{B}(W_e)$. Therefore $\varphi(A)$ is compact, and by Remark 2.11, $A \in \mathbf{K}_{\mathcal{E}}(W)$.

For the proof of (iv) $\Rightarrow$ (i) in the general case let

$$\varphi : \bigoplus_{i\in I} \mathbf{B}_{\mathcal{E}_i}(W_i) \to \bigoplus_{i\in I} \mathbf{B}(W_{e_i}), \quad \varphi(\{A_i\}) = \{\varphi_i(A_i)\}$$

be the isomorphism from Theorem 2.10 (ii) and let $A = \{A_i\}$ be given. For an index $j \in I$ the operator $A_j \in \mathbf{B}_{\mathcal{E}_j}(W_j)$ also satisfies (iv), so by the first part of the proof $A_j \in \mathbf{K}_{\mathcal{E}_j}(W_j)$. Consequently, $\varphi_j(A_j) \in \mathbf{K}(W_{e_j})$. We also claim that $\lim_i \|\varphi_i(A_i)\| = 0$.

To prove this, let us suppose the opposite: there exist $\varepsilon > 0$ and a sequence of indices (j_n) such that $\|\varphi_{j_n}(A_{j_n})\| \geqslant \varepsilon$. In each $W_{e_{j_n}}$ we can choose a unit vector v_{j_n} such that $\|\varphi_{j_n}(A_{j_n})v_{j_n}\| > \frac{\varepsilon}{2}$. Now $w_{j_n} = \|e_{j_n}\|v_{j_n}$ are basic elements in W_{j_n} and $\|A_{j_n}w_{j_n}\| > \|e_{j_n}\|\frac{\varepsilon}{2} \geqslant \frac{\varepsilon}{2}$ (because each projection in an H^*-algebra has norm not smaller then 1). The obtained sequence (w_{j_n}) contradicts the assumption (iv).

From what we just proved follows that $\varphi(A) = \{\varphi_i(A_i)\}$ is a compact operator on $\bigoplus_{i\in I} W_{e_i}$. It remains to apply once again an easy argument from unordered summation.

Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_i \|\varphi_i(A_i)\| = 0$ the set of indices $\{i \in I : \|\varphi_i(A_i)\| \geqslant \varepsilon\}$ is finite; let it be $\{i_1, \ldots, i_n\} \subset I$. Let us define $\{T_i\} \in \bigoplus_{i\in I} \mathbf{B}(W_{e_i})$ with

$$T_i = \begin{cases} \varphi_i(A_i), & \text{if } i \in \{i_1, \ldots, i_n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\|\{\varphi_i(A_i)\} - \{T_i\}\| = \sup_{i\neq i_1,\ldots,i_n} \|\varphi_i(A_i)\| \leqslant \varepsilon$.

Since $\varphi_i(A_i)$ are compact operators and $\varphi_i(\mathbf{F}_{\mathcal{E}_i}(W_i)) = \mathbf{F}(W_{e_i})$ there exist $F_i \in \mathbf{F}_{\mathcal{E}_i}(W_i)$ such that $\|\varphi_i(A_i) - \varphi_i(F_i)\| < \varepsilon$, $\forall i = i_1, \ldots, i_n$. If we define $\{F_i\} \in \bigoplus_{i\in I} \mathbf{B}(W_{e_i})$ with

$$F_i = \begin{cases} F_{i_k}, & \text{if } i = i_k \in \{i_1, \ldots, i_n\}; \\ 0, & \text{otherwise.} \end{cases}$$

then $\|\{A_i\} - \{F_i\}\| = \|\{\varphi_i(A_i)\} - \{\varphi_i(F_i)\}\| \leqslant 2\varepsilon$. Since $\{F_i\}$ is a finite orthogonal sum of generalized finite rank operators, we conclude that $\{F_i\} \in \mathbf{F}_{\mathcal{E}}(W)$ and, by definition, $A = \{A_i\} \in \mathbf{K}_{\mathcal{E}}(W)$. ∎

REMARK 2.13. The proof of the above theorem also shows that in the case of a simple H^*-algebra $\mathcal{E}$ an operator A is a generalized compact operator if and only if the following, weaker condition is satisfied:

(iv′) There exists a minimal projection e in $\mathcal{E}$ such that $\lim_n Aw_n = 0$ for each orthonormal sequence (w_n) with the common supporting projection e.

The next variant of Theorem 2.10 can be proved in the same way, so we omit the proof. The corresponding statement for Hilbert space operators is proved in [5].

Theorem 2.14. *Let W be a Hilbert H^*-module and $A \in \mathbf{B}_{\mathcal{E}}(W)$. The following conditions are mutually equivalent:*

(i) $A \in \mathbf{K}_{\mathcal{E}}(W)$.

(ii) $\lim_n [Ax_n|x_n] = 0$ *for each sequence* (x_n) *in* W *such that* $x_n \overset{[w]}{\to} 0$.

(iii) $\lim_n [Ax_n|x_n] = 0$ *for each bounded sequence* (x_n) *of mutually orthogonal elements in* W.

(iv) $\lim_n [Ax_n|x_n] = 0$ *for each orthonormal sequence* (x_n) *in* W.

The preceding two theorems show that generalized compact operators are characterized in the standard way. Moreover, they share numerous properties with the compact operators on Hilbert spaces. As an illustration we will state the following diagonalization theorem:

Theorem 2.15. *Let W be a Hilbert H^*-module and let $A \in \mathbf{K}_{\mathcal{E}}(W)$ be a positive operator. Then there exist an orthonormal sequence (w_n) in W and a sequence (λ_n) of nonnegative real numbers converging to* 0 *such that*

$$Ax = \sum_{n=1}^{\infty} \lambda_n [x|w_n] w_n, \quad \forall x \in W.$$

The proof is an easy application of Theorem 2.10 and hence omitted.

A Hilbert space has a countable orthogonal dimension if and only if it is separable. The same is not true for Hilbert H^*-modules. The hilbertian dimension of a non separable Hilbert H^*-module W may be countable, even finite (Example 2.3, provided that the initial Hilbert space is non separable). This enables us to transfer typical "separable" theorems from Hilbert space theory to (possibly) non separable Hilbert H^*-modules. Indeed, the necessary condition $\mathcal{E}\text{-}\dim W \leqslant \aleph_0$ must be satisfied. As an example we include the Weyl-von Neumann-Berg theorem.

Definition 2.16. Let W be a Hilbert H^*-module over an H^*-algebra $\mathcal{E}$ such that $\mathcal{E}\text{-}\dim W = \aleph_0$. An operator $D \in \mathbf{B}_{\mathcal{E}}(W)$ is called *diagonal* if there exist an orthonormal basis (w_n) for W and a sequence of complex numbers (λ_n) such that $Dw_n = \lambda_n w_n$, $\forall n$.

Theorem 2.17. *Let W be a Hilbert H^*-module over an H^*-algebra $\mathcal{E}$ such that $\mathcal{E}\text{-}\dim W = \aleph_0$. Let $A \in \mathbf{B}_{\mathcal{E}}(W)$ be a normal operator and $\varepsilon > 0$. Then there exist a diagonal operator $D \in \mathbf{B}_{\mathcal{E}}(W)$ and $T \in \mathbf{K}_{\mathcal{E}}(W)$ such that $A = D + T$ and $\|T\| \leqslant \varepsilon$.*

Proof. If the underlying H^*-algebra is simple the proof is a direct application of Berg's theorem ([3]), Theorem 2.10 (i) and Lemma 2.8.

In the general case we apply the isomorphism from Theorem 2.8 (ii)

$$\varphi : \bigoplus_{n \in \mathbf{N}} \mathbf{B}_{\mathcal{E}_n}(W_n) \to \bigoplus_{n \in \mathbf{N}} \mathbf{B}(W_{e_n}), \quad \varphi(\{A_n\}) = \{\varphi_n(A_n)\}.$$

Now by the first part of the proof for each n we can find D_n and T_n with $\|T_n\| \leqslant \frac{\varepsilon}{n}$. ∎

Von Neumann's theorem asserts that for a selfadjoint operator A one can find a Hilbert-Schmidt perturbation T such that $\|T\|_2 < \varepsilon$. The same is true for $\mathcal{E}$-linear selfadjoint operators. This is proved by the same argument as above provided that the operators T_n satisfy $\|T_n\|_2 \leqslant \frac{\varepsilon}{2^n}$.

It should be emphasized that in spite of the above theorems (and other analogies) generalized compact operators cannot map bounded sets into relatively compact ones. As shown by Example 2.3, this is impossible even in the case of finite $\mathcal{E}$-dimensional Hilbert H^*-module over a simple H^*-algebra.

In the rest of the paper we briefly discuss generalized Fredholm operators.

Definition 2.18. Let W be a Hilbert H^*-module. The *generalized Calkin algebra* $\mathbf{C}_{\mathcal{E}}(W)$ is defined as a quotient C^*-algebra $\mathbf{B}_{\mathcal{E}}(W)/\mathbf{K}_{\mathcal{E}}(W)$. The quotient map is denoted by π.

Definition 2.19. Let W be a Hilbert C^*-module. An operator $A \in \mathbf{B}_{\mathcal{E}}(W)$ is said to be a *generalized Fredholm operator* if $\pi(A)$ is an invertible element in $\mathbf{C}_{\mathcal{E}}(W)$.

We are going to prove Atkinson's theorem for generalized Fredholm operators. In the proof we shall need the following well known fact for classical Fredholm operators:

Let H_1, H_2 be a Hilbert spaces and $A = A_1 \oplus A_2 \in \mathbf{B}(H_1) \oplus \mathbf{B}(H_2) \subset \mathbf{B}(H_1 \oplus H_2)$. Then A is a Fredholm operator if and only if A_1 and A_2 are Fredholm operators. Moreover, $\operatorname{Im} A = \operatorname{Im} A_1 \oplus \operatorname{Im} A_2$, $\ker A = \ker A_1 \oplus \ker A_2$ and $\operatorname{ind} A = \operatorname{ind} A_1 + \operatorname{ind} A_2$.

We also need the following simple lemma.

Lemma 2.20. *Let W be a Hilbert H^*-module over a simple H^*-algebra $\mathcal{E}$ and $A \in \mathbf{B}_{\mathcal{E}}(W)$. Then* $\operatorname{Im} \varphi(A) = e \operatorname{Im} A$ *and* $\ker \varphi(A) = e \ker A$ *(with e and φ having the same meaning as in Theorem* 2.10 (i)*).*

The proof is straightforward and hence omitted.

Remark 2.21. A similar result can be obtained in the general case. Also, one can prove

$$\overline{\operatorname{Im} A} = \overline{\operatorname{span}\{\mathcal{E} \operatorname{Im} \varphi(A)\}}, \qquad \ker A = \overline{\operatorname{span}\{\mathcal{E} \ker \varphi(A)\}}.$$

Theorem 2.22. *Let W be a Hilbert H^*-module and $A \in \mathbf{B}_{\mathcal{E}}(W)$. Then A is a generalized Fredholm operator if and only if the image of A is a closed submodule and both* $\mathcal{E}$-$\dim \ker A$ *and* $\mathcal{E}$-$\dim \ker A^*$ *are finite.*

Proof. Suppose first that the underlying H^*-algebra $\mathcal{E}$ is simple. Let e be a minimal projection in $\mathcal{E}$ and let $\varphi : \mathbf{B}_{\mathcal{E}}(W) \to \mathbf{B}(W_e)$ be the isomorphism from Theorem 2.10.

Let A be a generalized Fredholm operator. Then there exist an operator $B \in \mathbf{B}_{\mathcal{E}}(W)$ and generalized compact operators T_1, T_2 such that $AB - T_1 = I$ and $BA - T_2 = I$. Since $\varphi(\mathbf{K}_{\mathcal{E}}(W)) = \mathbf{K}(W_e)$ the operator $\varphi(A)$ is Fredholm. In particular, $\varphi(A)$ has closed range and finite dimensional kernel and cokernel. Applying Lemmas 1.4 and 2.20 and Corollary 2.9 to the Hilbert H^*-modules $\ker A$ and $\ker A^*$, respectively, we get $\mathcal{E}$-$\dim \ker A$, $\mathcal{E}$-$\dim \ker A^* < \infty$.

To prove that $\operatorname{Im} A$ is closed it is enough to show that $A|(\ker A)^{\perp}$ is bounded from below. (This is the standard argument for Hilbert space operators and it can be easily seen that it extends to bounded $\mathcal{E}$-linear operators. Also, the converse holds true.)

Suppose the opposite. Then there exists a sequence of unit elements $(x_n) \in (\ker A)^{\perp}$ such that $\lim_n Ax_n = 0$. This implies $\lim_n A(ex_n) = 0$ and, since $(ex_n) \in (\ker \varphi(A))^{\perp}$ and $\operatorname{Im} \varphi(A)$ is closed we obtained a contradiction.

The converse is proved in exactly the same way.

The general case reduces to the previous one by using the isomorphism φ from Theorem 2.10 (ii). It should be noted that an operator $\{\varphi_i(A_i)\} \in \bigoplus_i \mathbf{B}(W_{e_i})$ is Fredholm if and only if each of the operators $\varphi_i(A_i)$ is a Fredholm operator and $\varphi_i(A_i)$ are invertible operators on W_{e_i} for all but finitely many indices $i \in I$. ∎

DEFINITION 2.23. Let W be a Hilbert H^*-module and $A \in \mathbf{B}_{\mathcal{E}}(W)$ be a Fredholm operator. The *generalized Fredholm index of* A is the integer defined by

$$\mathcal{E}\text{-ind}(A) = (\mathcal{E}\text{- dim} \ker A) \; - \; (\mathcal{E}\text{- dim} \ker A^*).$$

CONCLUDING REMARKS. (1) We restricted ourselves to present only a few basic results on $\mathcal{E}$-linear operators. Clearly, using the same technique (basically Theorem 2.10), various results from Hilbert spaces can also be extended onto Hilbert H^*-modules.

As an example let us briefly discuss the spectrum of a bounded $\mathcal{E}$-linear operator. The spectrum $\sigma(A)$ of an operator $A \in \mathbf{B}_{\mathcal{E}}(W)$ is defined in the standard way. It is equal to the spectrum of A regarded as a bounded operator on the Hilbert space $(W, (\cdot|\cdot))$ because the spectrum of an element of a C^*-algebra remains unchanged when it is calculated in some C^*-subalgebra. Since the map φ from Theorem 2.10 is an isomorphism we also have $\sigma(A) = \sigma(\varphi(A))$. Finally, from Lemma 2.20 and Corollary 2.9 immediately follows that the same formulae hold for the point, approximative and continuous spectra, respectively.

(2) However, there are some remarkable differences between the operators on Hilbert H^*-modules and Hilbert spaces. For example (in contrast to Hilbert spaces), if w and z are two basic elements in a Hilbert H^*-module W with different supporting projections then there does not exist an operator $U \in \mathbf{B}_{\mathcal{E}}(W)$ such that $Uw = z$. Consequently, given two orthonormal bases, in general there does not exists a unitary $\mathcal{E}$-linear operator which maps one of them into another.

(3) The technique used in this paper can also be extended to closed densely defined $\mathcal{E}$-linear operators on Hilbert H^*-modules. With some additional technical difficulties similar results are obtained. This will be presented elsewhere.

(4) Altough the results obtained in this paper depend on the special structure of the underlying H^*-algebra, our main results (Theorems 2.10, 2.12, 2.14, 2.15, 2.17 and 2.22) can be extended to Hilbert C^*-modules over C^*-algebras of compact operators. This is the contents of our subsequent paper. Moreover, there is some evidence that these results can be applied in more general Hilbert C^*-modules using the recent results on extensions of Hilbert C^*-modules.

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