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THE DRAZIN INVERSE FOR CLOSED LINEAR OPERATORS AND THE ASYMPTOTIC CONVERGENCE OF C_0-SEMIGROUPS

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ABSTRACT. The paper defines and studies the Drazin inverse for a closed linear operator A in a Banach space X in the case that 0 is an isolated spectral point of A. Results include an integral representation for the Drazin inverse of the infinitesimal generator of a C_0-semigroup and its application to a singular and singularly perturbed differential equation in a Banach space.

KEYWORDS: *Drazin inverse, closed linear operators, continuous semigroups, singular differential equation.*

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1. INTRODUCTION AND PRELIMINARIES

In 1958, Drazin ([4]) introduced a pseudoinverse in associative rings and semigroups that now carries his name. The inverse was extensively studied and applied in matrix setting (see the monograph [1] by Campbell and Meyer), as well as in the setting of bounded linear operators and elements of Banach algebras (see [2] and [11]). The conventional Drazin inverse was extended to closed linear operators by Nashed and Zhao in [12]; it exists if and only if 0 is at most a pole of the resolvent $R(\lambda; A)$ of the operator A.

The purpose of this paper is to introduce the Drazin inverse A^{D} of a closed linear operator A on a Banach space X which is defined if 0 is merely an isolated spectral point of A, and to investigate basic properties of A^{D}. For bounded linear operators and elements of a Banach algebra such inverse was introduced and studied by Koliha in [6], and further investigated in [3], [7] and [8].

A special attention is paid to the case when A is the infinitesimal generator of a C_0-semigroup. For this situation an integral representation of A^{D} is derived, which is then used to study the asymptotic behaviour of the solutions to a singular and singularly perturbed differential equation in a Banach space.

For basic concepts of operator theory of closed linear operators we rely on [15]. By $\mathcal{C}(X)$ we denote the space of all closed linear operators A with domain and range in a Banach space X; $\mathcal{D}(A), \mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the domain, nullspace and range of A, respectively. By $\mathcal{B}(X)$ we denote the space of all bounded linear operators defined on all of X. If $A \in \mathcal{C}(X)$, then $\rho(A)$ denotes the resolvent set of A and $\sigma(A)$ the spectrum of A. By $\operatorname{iso}\sigma(A)$ and $\operatorname{acc}\sigma(A)$ we define the set of all isolated and accumulation spectral points of A. The extended spectrum of $A \in \mathcal{C}(X)$ is denoted by $\sigma_{\rm e}(A)$; for $\lambda \in \rho(A)$, $R(\lambda; A)$ denotes the resolvent $(\lambda I - A)^{-1}$ of A.

Let $A \in \mathcal{C}(X)$ with $\sigma(A) \neq \mathbb{C}$. A subset σ of $\sigma_{\rm e}(A)$ is called a *spectral set* of A if it is both open and closed in the relative topology of $\sigma_{\rm e}(A)$ as a subset of $\mathbb{C} \cup \{\infty\}$. Let σ_1 be a bounded spectral set of A with the complement σ_2 in $\sigma_{\rm e}(A)$. By Theorem V.9.2 of [15], X is the direct sum $X = X_1 \oplus X_2$ of closed A-invariant subspaces, so that $A = A_1 \oplus A_2$ with respect to this sum, $\sigma(A_i) = \sigma_i$ for $i = 1, 2$, and A_1 is continuous. The projection $P \in \mathcal{B}(X)$ with $\mathcal{R}(P) = X_1$ and $\mathcal{N}(P) = X_2$ is called the *spectral projection* of A corresponding to σ_1.

A singleton $\{\mu\}$ is a spectral set of A if and only if μ is an isolated singularity of the resolvent $R(\lambda; A)$ of A. If $\mu \notin \operatorname{acc}\sigma(A)$, then either $\mu \in \operatorname{iso}\sigma(A)$ or μ is a resolvent point of A; we extend the concept of the spectral projection in the latter case by defining $P = 0$.

The following result from [9] is a generalization of Theorem 1.2 of [5] to closed operators. It will play an important role in our development of the Drazin inverse for closed operators.

Theorem (Theorem 1.4 of [9]) *Let A be a closed linear operator with domain $\mathcal{D}(A)$. The point 0 is an isolated spectral point of A if and only if there exists a nonzero projection P such that:*
(i) $\mathcal{R}(P) \subset \mathcal{D}(A)$;
(ii) $PAx = APx$ *for all* $x \in \mathcal{D}(A)$;
(iii) $\sigma(AP) = \{0\}$;
(iv) $A + \xi P$ *is invertible for some (in fact for all)* $\xi \neq 0$.
An operator P satisfying (i)–(iv) *is the spectral projection of A at* 0.

Many results of this paper involve interaction between closed and bounded operators. We set out the relevant properties in the following lemma for future reference. No proof is given as the arguments are fairly routine.

Lemma *Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{B}(X)$. Then the following are true:*
(i) $A + B \in \mathcal{C}(X)$ *and* $\mathcal{D}(A + B) = \mathcal{D}(A)$.
(ii) *If* $\mathcal{R}(B) \subset \mathcal{D}(A)$, *then* $AB \in \mathcal{B}(X)$.
(iii) *If* $\mathcal{R}(B) \subset \mathcal{D}(A)$, $ABx = BAx$ *for all* $x \in \mathcal{D}(A)$ *and A is invertible, then* $A^{-1}B = BA^{-1}$ *in* $\mathcal{B}(X)$.
(iv) *Let* $0 \in \operatorname{iso}\sigma(A)$ *and P be the spectral projection of A corresponding to* 0. *Then* $\mathcal{R}(P) \subset \mathcal{D}(A^n)$ *for all* $n \geqslant 1$. *If* $\mathcal{R}(B) \subset \mathcal{D}(A)$ *and* $ABx = BAx$ *for all* $x \in \mathcal{D}(A)$, *then* $BP = PB$.

Convention Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{B}(X)$ with $\mathcal{R}(B) \subset \mathcal{D}(A)$. For the sake of brevity we will write in the sequel

$$AB = BA \quad \text{to mean} \quad ABx = BAx \quad \text{for all } x \in \mathcal{D}(A).$$

2. THE DRAZIN INVERSE FOR CLOSED LINEAR OPERATORS

We start with a definition of the Drazin inverse of a closed operator that subsumes the conventional Drazin inverse defined by Nashed and Zhao (Definition 2.1, [12]).

DEFINITION Let $A \in \mathcal{C}(X)$. An operator $B \in \mathcal{B}(X)$ is called a *Drazin inverse* of A if $\mathcal{R}(B) \subset \mathcal{D}(A)$, $\mathcal{R}(I - AB) \subset \mathcal{D}(A)$, and

$$BAB = B, \quad AB = BA, \quad \sigma(A(I - AB)) = \{0\}. \tag{2.1}$$

The *Drazin index* $i(A)$ is defined to be $i(A) = 0$ if A is invertible, $i(A) = q$ if A is not invertible and $A(I - AB)$ is nilpotent of index q, and $i(A) = \infty$ otherwise. (The index is well defined since there is at most one operator $B \in \mathcal{B}(X)$ satisfying (2.1) — see (2.3).) An operator $A \in \mathcal{C}(X)$ that possesses a Drazin inverse is called *Drazin invertible*, and its Drazin inverse is denoted by A^{D}.

LEMMA *Let $A \in \mathcal{C}(X)$ be Drazin invertible with a Drazin inverse $B \in \mathcal{B}(X)$. Then the operator $P = I - AB$ is a continuous projection such that:*
(i) *$AP = PA$;*
(ii) *$\mathcal{R}(P) \subset \mathcal{D}(A^n)$ for all $n \geqslant 1$.*

Proof. From $BAB = B$ we obtain $(AB)^2 = ABAB = AB$, which implies $P^2 = P$.

If $y \in \mathcal{D}(A)$, then $ABy = y - Py \in \mathcal{D}(A)$; by the second condition in (2.1),

$$APy = A(y - ABy) = A(y - BAy) = (I - AB)Ay = PAy.$$

This proves (i).

Suppose that $y = Px \in \mathcal{D}(A^{n-1})$ for some $n \geqslant 2$. Then $Py = y$, and by (i),

$$Ay = APy = PAy \in \mathcal{D}(A),$$

which implies $y \in \mathcal{D}(A^n)$. The result follows by induction. ∎

We give some necessary and sufficient conditions for $A \in \mathcal{C}(X)$ to possess a Drazin inverse.

THEOREM *The following conditions on $A \in \mathcal{C}(X)$ are equivalent:*
(i) *$A \in \mathcal{C}(X)$ is Drazin invertible;*
(ii) *$0 \notin \operatorname{acc} \sigma(A)$;*
(iii) *$A = A_1 \oplus A_2$, where A_1 is bounded and quasinilpotent and A_2 is closed and invertible.*

Proof. Suppose that 0 is not an accumulation point of $\sigma(A)$. If $0 \in \rho(A)$, then A is invertible, and A^{-1} is a Drazin inverse of A. Suppose that $0 \in \sigma(A)$. Then $0 \in \operatorname{iso} \sigma(A)$, and the spectral projection P of A corresponding to 0 satisfies conditions (i)–(iv) of Theorem 1.1 with $\xi = 1$. Set $B = (A + P)^{-1}(I - P)$. Since $\mathcal{D}(A + P) = \mathcal{D}(A)$, $\mathcal{R}(B) = \mathcal{R}((A + P)^{-1}(I - P)) \subset \mathcal{D}(A)$. The commutativity condition of (2.1) is clear.

Further, $AB = A(I-P)(A+P)^{-1} = (A+P)(I-P)(A+P)^{-1} = I-P$, and $ABA = (A+P)^{-1}(I-P)^2 = (A+P)^{-1}(I-P) = B$. Finally, AP is quasinilpotent, that is, $\sigma(A(I - AB)) = \{0\}$. Hence B satisfies (2.1).

Conversely, suppose that B is a Drazin inverse of A, and set $P = I - AB$. By Lemma 2.2, the operator $P = I - AB$ is a continuous projection satisfying the commutativity condition of Theorem 1.1. Further, $AP = A(I - BA)$ is quasinilpotent. We need to verify that $A + P \in \mathcal{C}(X)$ is invertible. We have

$$(A+P)(B+P) = AB + AP + PB + P = I - P + AP + P = I + AP, \tag{2.2}$$

and $(B+P)(A+P)x = (I+AP)x$ for all $x \in \mathcal{D}(A)$. Since $I + AP$ is an invertible operator, so is $A + P$. By Theorem 1.1, $0 \notin \operatorname{acc}\sigma(A)$. We observe that, for any $\xi \neq 0$, $(A + \xi P)B = I - P$, which implies $B = (A + \xi P)^{-1}(I - P)$.

Thus we have proved the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from Theorem V.9.2 of [15]. ∎

From the preceding proof we obtain a useful explicit formula for the Drazin inverse A^{D} in terms of the spectral projection P of A at 0, and a proof of uniqueness of A^{D}:

$$A^{\mathrm{D}} = (A + \xi P)^{-1}(I - P) \quad \text{for any } \xi \neq 0. \tag{2.3}$$

We also observe that $P = I - AA^{\mathrm{D}}$.

If $A = A_1 \oplus A_2$ is the decomposition of a Drazin invertible operator $A \in \mathcal{C}(X)$ described in the preceding theorem, then

$$A^{\mathrm{D}} = 0 \oplus A_2^{-1}. \tag{2.4}$$

Indeed,

$$A^{\mathrm{D}} = (A+P)^{-1}(I-P) = ((A_1+I)^{-1} \oplus A_2^{-1})(0 \oplus I) = 0 \oplus A_2^{-1}$$

by (2.3) with $\xi = 1$.

As a final result of this section we give a representation of the Drazin inverse in terms of the holomorphic calculus for a closed linear operator (see [15]) and a resulting expression for the spectrum of A^{D}.

Theorem *If $A \in \mathcal{C}(X)$ is Drazin invertible, then*

$$A^{\mathrm{D}} = f(A),$$

where f is a function holomorphic in an open neighborhood of $\sigma_{\mathrm{e}}(A)$ equal to 0 in an open neighborhood of 0 and at ∞, and to λ^{-1} for all λ in an open neighborhood of $\sigma(A)\backslash\{0\}$. If $i(A) > 0$, then

$$\sigma(A^{\mathrm{D}}) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(A) \setminus \{0\}\}.$$

Proof. We assume that $0 \in \operatorname{iso}\sigma(A)$. The spectral projection P of A corresponding to 0 can be expressed as $P = h(A)$, where h is a holomorphic function equal to 1 in an open neighborhood of 0 and to 0 in an open neighborhood of $\sigma_{\mathrm{e}}(A) \setminus \{0\}$. According to (2.3), the Drazin inverse is given by $A^{\mathrm{D}} = (A+P)^{-1}(I-P) = f(A)$, where $f(\lambda) = (\lambda + h(\lambda))^{-1}(1 - h(\lambda))$. From this expression for f we glean that $f(\lambda) = 0$ in an open neighborhood of 0 and $f(\lambda) = \lambda^{-1}$ in an open neighborhood of $\sigma_{\mathrm{e}}(A) \setminus \{0\}$.

By the spectral mapping theorem (Theorem V.9.5, [15]), $\sigma(f(A)) = f(\sigma_{\mathrm{e}}(A))$. If $\infty \in \sigma_{\mathrm{e}}(A)$, then $f(\infty) = 0$, otherwise $\sigma_{\mathrm{e}}(A) = \sigma(A)$. ∎

3. PROPERTIES OF THE DRAZIN INVERSE

This section studies properties of the Drazin inverse for closed linear operators. For the bounded case we recover many of the results of [6]. However, not all properties of the Drazin inverse for bounded linear operators find their counterpart in the closed operator theory, as witnessed by Theorem 3.4 and Examples 3.5 and 3.6. In [12], the Drazin inverse A^{d} of a closed linear operator A is defined for the case when A has a finite index $i(A)$, and several properties of A^{d} are stated without proof. Clearly, if A^{d} exists, then so does A^{D}, and $A^{\mathrm{d}} = A^{\mathrm{D}}$. Theorems 2.3 and 2.4 generalize Theorems 2.3 and 2.6 of [12], respectively. Further, Theorems 2.5 and 2.9 of [12] are recovered from Theorems 3.2 and 3.3, respectively. We give full proofs of these results.

We begin with the Laurent series expansion for the resolvent of A in a punctured neighborhood of an isolated spectral point of A.

Theorem *Let $A \in \mathcal{C}(X)$. If A is Drazin invertible, then there exists a punctured neighborhood Δ of 0 such that*

$$R(\lambda; A) = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n P - \sum_{n=0}^{\infty} \lambda^n (A^{\mathrm{D}})^{n+1}, \quad \lambda \in \Delta, \tag{3.1}$$

where $P = I - AA^{\mathrm{D}}$ is the spectral projection of A corresponding to 0.

Proof. Follows from Theorems 2.3, 1.1 and from the equation

$$(\lambda I - A)x = (\lambda I - AP)Px + (\lambda I - (A + P))(I - P)x$$

valid for all $x \in \mathcal{D}(A)$ and all λ in some punctured neighborhood Δ for which $\lambda I - (A + P)$ is invertible. Then

$$\begin{aligned} R(\lambda; A) &= (\lambda I - AP)^{-1} P + (\lambda I - (A + P))^{-1}(I - P) \\ &= \sum_{n=0}^{\infty} \lambda^{-n-1} A^n P - \sum_{n=0}^{\infty} \lambda^n ((A + P)^{-1}(I - P))^{n+1}. \end{aligned}$$

(Note that by Lemma 2.2, $A^n P$ is defined and bounded for all $n \geqslant 1$.) ∎

Theorem *Let $A \in \mathcal{C}(X)$ be Drazin invertible and let $B \in \mathcal{B}(X)$ be such that $\mathcal{R}(B) \subset \mathcal{D}(A)$ and $AB = BA$. Then $A^{\mathrm{D}}B = BA^{\mathrm{D}}$ in $\mathcal{B}(X)$.*

Proof. Let P be the spectral projection of A at 0. By Lemma 1.2 (iv), $BP = PB$ in $\mathcal{B}(X)$. Hence

$$A^{\mathrm{D}}B = (A + P)^{-1}(I - P)B = B(A + P)^{-1}(I - P) = BA^{\mathrm{D}}$$

by (2.3) and Lemma 1.2 (iii). ∎

Theorem *Let $A \in \mathcal{C}(X)$ be Drazin invertible. Then, for each $n \geqslant 1$, A^n is Drazin invertible, and $(A^n)^{\mathrm{D}} = (A^{\mathrm{D}})^n$.*

Proof. Let $A = A_1 \oplus A_2$ with A_1 (bounded) quasinilpotent and A_2 (closed) invertible (see Theorem 2.3 (iii)). Then $A^n = A_1^n \oplus A_2^n$, for $n = 1, 2, \ldots$, where A_1^n is quasinilpotent and A_2^n invertible. Hence A^n is Drazin invertible by Theorem 2.3, and

$$(A^n)^{\mathrm{D}} = 0 \oplus (A_2^n)^{-1} = 0 \oplus (A_2^{-1})^n = (0 \oplus A_2^{-1})^n = (A^{\mathrm{D}})^n$$

for any $n \geqslant 1$. ∎

Theorem *Let $A \in \mathcal{C}(X)$ be Drazin invertible with the Drazin index $i(A) > 0$. Then A^{D} is Drazin invertible if and only if $\sigma(A)$ is bounded.*

Proof. By Theorem 2.4, $\sigma(A^{\mathrm{D}}) = \{0\} \cup \{\lambda^{-1} : \lambda \in \sigma(A) \setminus \{0\}\}$. According to Theorem 2.3, the operator A^{D} is Drazin invertible if and only if there exists a punctured neighborhood $\{\lambda : |\lambda| < r\}$ of 0 disjoint with $\sigma(A^{\mathrm{D}})$. This occurs if and only if $\sigma(A)$ is contained in $\{\lambda : |\lambda| \leqslant r^{-1}\}$. ∎

We give an example of an operator for which A^{D} is not Drazin invertible with $i(A) = \infty$ and with the "worst scenario" spectrum.

Example Consider the space ℓ^1 with a generic element $x = (\xi_1, \xi_2, \xi_3, \ldots)$. The operator A_1 on ℓ^1 defined by

$$A_1 x = \Big(0, \xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \ldots\Big)$$

belongs to $B(\ell^1)$, and is quasinilpotent but not nilpotent.

The right shift $T_2 x = (0, \xi_1, \xi_2, \xi_3, \ldots)$ on ℓ^1 is an injective bounded linear operator with $\sigma(T_2) = \{\lambda : |\lambda| \leqslant 1\}$. Its algebraic inverse A_2 is a closed linear operator with the domain $\mathcal{D}(A_2) = \{x \in \ell^1 : \xi_1 = 0\}$; A_2 is invertible in $C(\ell^1)$ with $A_2^{-1} = T_2$ and $\sigma(A_2) = \{\lambda : |\lambda| \geqslant 1\}$.

Define $A = A_1 \oplus A_2$ on $X = \ell^1 \oplus \ell^1$. Then $A \in \mathcal{C}(X)$, $\sigma(A) = \{0\} \cup \{\lambda : |\lambda| \geqslant 1\}$, A is Drazin invertible with $A^{\mathrm{D}} = 0 \oplus T_2$ and $i(A) = \infty$. But A^{D} is not Drazin invertible since $\sigma(A^{\mathrm{D}}) = \{\lambda : |\lambda| \leqslant 1\}$, and 0 is not an isolated spectral point of A.

In contrast with the unbounded case, the relation between a bounded operator and its Drazin inverse is more symmetrical. According to Theorem 5.2 of [6], if $A \in \mathcal{B}(X)$ is Drazin invertible with $i(A) > 0$, then A^{D} is also Drazin invertible, both operators have the same spectral projection corresponding to 0, and $i(A^{\mathrm{D}}) = 1$. In addition, $(A^{\mathrm{D}})^{\mathrm{D}} = A$ if and only if $i(A) = 1$ (Theorem 5.3 of [6]). For a Drazin invertible closed operator A with Drazin invertible A^{D}, the spectral projections of A and A^{D} at 0 need not be the same. This is demonstrated in the following example.

Example Let A_1 be as in Example 3.5, and let T_2 be defined on ℓ^1 by

$$T_2 x = \Big(\xi_1, 0, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \ldots\Big).$$

Then $T_2 \in B(\ell^1)$ and $\sigma(T_2) = \{0, 1\}$. The operator T_2 is injective, and its algebraic inverse A_2 is a closed linear operator with the domain $\mathcal{D}(A_2) = \mathcal{R}(T_2)$; A_2 is invertible in $C(\ell^1)$ with $A_2^{-1} = T_2$ and $\sigma(A_2) = \{1\}$.

Let $A = A_1 \oplus A_2$ on $X = \ell^1 \oplus \ell^1$. Then $A \in \mathcal{C}(X)$, $\sigma(A) = \{0, 1\}$ and A is Drazin invertible with $A^{\mathrm{D}} = 0 \oplus T_2$ and $i(A) = \infty$. Since $\sigma(A^{\mathrm{D}}) = \{0, 1\}$, A^{D} itself is Drazin invertible. Let $P = I \oplus 0$ be the spectral projection of A at 0. We observe that $A^{\mathrm{D}} + P = I \oplus T_2$ is not invertible in $\mathcal{B}(X)$ and, by Theorem 1.1, P is not the spectral projection of A^{D}.

Theorem *Let $T \in \mathcal{C}(X)$ and $S \in \mathcal{B}(X)$ be such that T is Drazin invertible, S quasinilpotent, $\mathcal{R}(S) \subset \mathcal{D}(T)$ and $TS = ST$. Then the operator $T + S \in \mathcal{C}(X)$ is Drazin invertible with*

$$(T + S)^{\mathrm{D}} = (T + S + P)^{-1}(I - P), \tag{3.2}$$

where P is the spectral projection of T corresponding to 0.

Proof. Let $T = T_1 \oplus T_2$ (relative to $X = X_1 \oplus X_2$) be the decomposition of a Drazin invertible operator T described in Theorem 2.3 (iii). By Lemma 1.2 (iv), $SP = PS$. Then $S = S_1 \oplus S_2$ relative to $X = X_1 \oplus X_2$ with S_i quasinilpotent and $T_iS_i = S_iT_i$ for $i = 1, 2$. Hence

$$T + S = (T_1 + S_1) \oplus (T_2 + S_2)$$

is Drazin invertible in view of Theorem 2.3 since $T_1 + S_1$ is quasinilpotent and $T_2 + S_2 = T_2(I + T_2^{-1}S_2)$ invertible. We note that P is the spectral projection of $T + S$ at 0, and hence $T + S + P = (T_1 + S_1 + I) \oplus (T_2 + S_2)$ is invertible. Finally,

$$(T + S + P)^{-1}(I - P) = 0 \oplus (T_2 + S_2)^{-1} = (T + S)^{\mathrm{D}}$$

as $I - P = 0 \oplus I$. ∎

The proofs of the following two theorems are omitted since they are similar to the proofs of Theorems 5.6 and 5.5 in [5] (plus some consideration of domains).

Theorem *Let $T \in \mathcal{C}(X)$, $S \in \mathcal{B}(X)$ be Drazin invertible operators such that $\mathcal{R}(S) \subset \mathcal{D}(T)$ and $TS = ST = 0$. Then $(T + S)^{\mathrm{D}}$ exists and*

$$(T + S)^{\mathrm{D}} = T^{\mathrm{D}} + S^{\mathrm{D}}. \tag{3.3}$$

Theorem *Let $T \in \mathcal{C}(X)$, $S \in \mathcal{B}(X)$ be Drazin invertible operators such that $\mathcal{R}(S) \subset \mathcal{D}(T)$ and $TSx = STx$ for all $x \in \mathcal{D}(T)$. Then the operator $TS \in \mathcal{B}(X)$ is Drazin invertible, and*

$$(TS)^{\mathrm{D}} = T^{\mathrm{D}}S^{\mathrm{D}}.$$

We close this section with an important decomposition of a Drazin invertible operator.

Theorem *An operator $A \in \mathcal{C}(X)$ is Drazin invertible if and only if there exist $C \in \mathcal{C}(X)$ and $Q \in \mathcal{B}(X)$ such that:*

(i) *$\mathcal{D}(C) = \mathcal{D}(A)$ and C is Drazin invertible with $i(C) \leqslant 1$;*
(ii) *$Q \in \mathcal{B}(X)$ is quasinilpotent and $\mathcal{R}(Q) \subset \mathcal{D}(A)$;*
(iii) *$A = C + Q$ and $CQ = QC = 0$.*

Then $C^{\mathrm{D}} = A^{\mathrm{D}}$ and such a decomposition is unique.

Proof. Suppose first that (i)–(iii) hold. Then Q is Drazin invertible with $Q^{\mathrm{D}} = 0$, and we can apply Theorem 3.8 to conclude that A is Drazin invertible with $A^{\mathrm{D}} = C^{\mathrm{D}} + Q^{\mathrm{D}} = C^{\mathrm{D}}$. If P is the spectral projection of A, then, according to Theorem 1.1, AP is quasinilpotent and $A + P$ invertible. Then $PC = CP = AP - QP$ is quasinilpotent (the sum of commuting quasinilpotent operators), and $C + P = A + P - Q = (A + P)(I - (A + P)^{-1}Q)$ is invertible; hence P is also the spectral projection of C by Theorem 1.1. According to Theorem 2.3 we can write $A = A_1 \oplus A_2$ relative to $X = X_1 \oplus X_2$; then $C = C_1 \oplus C_2$ relative to $X = X_1 \oplus X_2$ with C_2 invertible and with $C_1 = 0$ since $i(C) \leqslant 1$. From $QP = PQ$ we obtain

$Q = Q_1 \oplus Q_2$ relative to $X = X_1 \oplus X_2$. Then $0 = CQ = 0 \oplus C_2Q_2$, and the invertibility of C_2 implies $Q_2 = 0$. Consequently,

$$C = 0 \oplus A_2, \quad Q = A_1 \oplus 0,$$

which shows that the decomposition is unique.

Conversely, suppose that A is Drazin invertible. By Theorem 2.3, $A = A_1 \oplus A_2$, where A_1 is (bounded) quasinilpotent and A_2 (closed) invertible. Set $C = 0 \oplus A_2$ and $Q = A_1 \oplus 0$. The spectral projection of A at 0 is $P = I \oplus 0$. Then C is Drazin invertible and P is the spectral projection of C corresponding to 0.

(i) From $C = (I-P)A$ we have $\mathcal{D}(C) = \mathcal{D}(A)$. Further, $C^{\mathrm{D}} = 0 \oplus A_2^{-1} = A^{\mathrm{D}}$, and $i(C) = i(0 \oplus A_2) \leqslant 1$.

(ii) Since $Q = AP = PAP$, $\mathcal{R}(Q) \subset \mathcal{R}(P) \subset \mathcal{D}(A)$. Also, $\sigma(Q) = \sigma(A_1 \oplus 0) = \sigma(A_1) \cup \sigma(0) = \{0\}$.

(iii) Follows from the definition of C and Q. ∎

The operator C from the preceding theorem is called the *core part* of the Drazin invertible operator A. Its importance is seen from the following properties:

$$C^{\mathrm{D}} = A^{\mathrm{D}}, \quad \sigma(C) = \sigma(A), \quad R(C) = N(P), \quad N(C) = R(P), \tag{3.4}$$

where P is the spectral projection of A corresponding to 0. Only the spectral equality needs to be proved. We observe that, for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{D}(A)$,

$$(\lambda I - A)x = (\lambda I - C)(I - P)x + \lambda(I - Q)Px,$$
$$(\lambda I - C)x = (\lambda I - A)(I - P)x + \lambda Px.$$

Hence $\lambda I - A$ is invertible whenever $\lambda \in \rho(C) \setminus \{0\}$, and $\lambda I - C$ is invertible whenever $\lambda \in \rho(A) \setminus \{0\}$. Consequently, $\sigma(C) \cup \{0\} = \sigma(A) \cup \{0\}$. Considering separately the cases $0 \in \sigma(A)$ and $0 \notin \sigma(A)$, we conclude that $\sigma(C) = \sigma(A)$.

4. C_0-SEMIGROUPS AND THE DRAZIN INVERSE OF THE INFINITESIMAL GENERATOR

First we discuss some facts about C_0-semigroup that will be needed in the sequel.

Lemma *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A and let $P \in \mathcal{B}(X)$ be a projection satisfying the commutativity condition $T(t)P = PT(t)$ for all $t \geqslant 0$. Then*

$$S(t) := T(t)\exp(-tP) = \exp(-tP)T(t), \quad t \geqslant 0, \tag{4.1}$$

is a C_0-semigroup with the infinitesimal generator $A - P$.

Proof. The commutativity in (4.1) holds since $\exp(-tP) = I - P + \mathrm{e}^{-t}P$ for $t \geqslant 0$. A direct verification shows that $S(t)$ is a C_0-semigroup. Further, for each $x \in \mathcal{D}(A)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 S(t)x = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 T(t)\Big]\exp(0P)x + T(0)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \exp(-tP)x = Ax - Px,$$

which shows that $A - P$ is the infinitesimal generator of $S(t)$. ∎

We give a representation of the Drazin inverse of the infinitesimal generator. This result generalizes Theorem 6.3 of [6]. It will be applied to the study of the asymptotic behaviour of the solutions of a differential equation. In the following, the convergence of semigroups as $t \to \infty$ is understood in the operator norm topology.

Theorem *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A such that $T(t) \to P$ as $t \to \infty$. Then the following are true:*

(i) $0 \notin \operatorname{acc} \sigma(A)$ *and P is the spectral projection of A corresponding to* 0*;*

(ii) *there are constants $M > 0$ and $\mu > 0$ such that $\|T(t) - P\| \leqslant M\mathrm{e}^{-\mu t}$ for all $t \geqslant 0$;*

(iii) *for all $x \in X$ we have*

$$\int_0^\infty T(t)(I-P)x\,\mathrm{d}t = -A^{\mathrm{D}}x. \tag{4.2}$$

Proof. We observe that $P^2 = P$. For any $t > 0$, $T(t)P = \lim_{s\to\infty} T(t)T(s) = \lim_{s\to\infty} T(t+s) = P$; hence

$$T(t)P = P = PT(t) \quad \text{for all } t \geqslant 0. \tag{4.3}$$

Differentiating this equation at 0, we get $APx = PAx = 0$ for all $x \in \mathcal{D}(A)$. By the preceding lemma, $S(t) = T(t)\exp(-tP)$ is the C_0-semigroup generated by $A - P$. Expressing the exponential as a series, after a short calculation we obtain

$$T(t) = S(t) + (1 - \mathrm{e}^{-t})P \quad \text{and} \quad T(t)(I-P) = S(t)(I-P). \tag{4.4}$$

We observe that $S(t) \to 0$ as $t \to \infty$. From Proposition 1.2.2 of [13] we can deduce that there exist constants $K > 0$, $\mu > 0$ such that $\|S(t)\| \leqslant K\mathrm{e}^{-\mu t}$ for all $t \geqslant 0$. Hence $\|T(t) - P\| = \|T(t)(I-P)\| = \|S(t)(I-P)\| \leqslant \|I-P\|K\mathrm{e}^{-\mu t} \leqslant M\mathrm{e}^{-\mu t}$ for all $t \geqslant 0$.

By the spectral inclusion from Theorem 2.2.3 of [14], $\sigma(A-P)$ lies in the open left half plane, which shows that $A - P$ is invertible. The conditions of Theorem 1.1 are satisfied, $0 \notin \operatorname{acc}\sigma(A)$, and P is the spectral projection of A corresponding to 0. According to Theorem 4.2.4 (b) of [14], for all $x \in X$ we have

$$\int_0^\infty S(t)x\,\mathrm{d}t = -(A-P)^{-1}x \quad \text{for all } x \in X, \tag{4.5}$$

and, using (2.3) with $\xi = -1$,

$$\int_0^\infty T(t)(I-P)x\,\mathrm{d}t = \int_0^\infty S(t)(I-P)x\,\mathrm{d}t = -(A-P)^{-1}(I-P)x = -A^{\mathrm{D}}x. \tag{2.5}$$ ∎

5. APPLICATIONS TO DIFFERENTIAL EQUATIONS

In this section we consider the abstract Cauchy problem for the infinitesimal generator of a C_0-semigroup and its singular perturbation. All functions f are defined on $[0,\infty)$ with values in X. We apply the results of previous sections to obtain a generalization of asymptotic theorems of Pazy (Chapter 4 of [14]). In particular, our first theorem generalizes Theorem 4.4.4 of [14].

Theorem *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A such that $T(t)\to P$. Let f be bounded and Lebesgue measurable on $[0,\infty)$, and let Pf be integrable on $[0,\infty)$. If $\lim\limits_{t\to\infty} f(t)=f_0$, then the mild solution $u(t)$ of the differential problem*

$$\frac{\mathrm{d}u}{\mathrm{d}t}=Au(t)+f(t), \quad u(0)=x, \tag{5.1}$$

satisfies

$$\lim_{t\to\infty} u(t)=Px-A^{\mathrm{D}}f_0+\int_0^\infty Pf(s)\,\mathrm{d}s. \tag{5.2}$$

Proof. The mild solution to the problem is given by

$$u(t)=T(t)x+\int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

By (4.3), $T(t)P=P=PT(t)$ for all $t\geqslant 0$. Let

$$u_1(t)=P\int_0^t T(t-s)f(s)\,\mathrm{d}s \quad\text{and}\quad u_2(t)=(I-P)\int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

Then

$$u_1(t)=\int_0^t PT(t-s)f(s)\,\mathrm{d}s=\int_0^t Pf(s)\,\mathrm{d}s.$$

Since Pf is integrable on $[0,\infty)$,

$$\lim_{t\to\infty} u_1(t)=\lim_{t\to\infty}\int_0^t Pf(s)\,\mathrm{d}s=\int_0^\infty Pf(s)\,\mathrm{d}s.$$

Further,

$$\begin{aligned}u_2(t)&=\int_0^t T(t-s)(I-P)(f(s)-f_0)\,\mathrm{d}s+\int_0^t T(t-s)(I-P)f_0\,\mathrm{d}s\\&=v_1(t)+v_2(t).\end{aligned}$$

By Theorem 4.2 there exist positive constants M,μ such that

$$\|T(t)(I-P)\|=\|T(t)-P\|\leqslant M\mathrm{e}^{-\mu t}\quad\text{for all } t\geqslant 0.$$

Write $\|f\|_\infty = \sup_{t\geqslant 0} \|f(t)\|$. Let $\eta > 0$ and let t_0 be such that $\|f(s) - f_0\| < \eta$ if $s \geqslant t_0$. Then

$$\begin{aligned}\|v_1(t)\| &\leqslant \int_0^t \|T(t-s) - P\|\,\|f(s) - f_0\|\,\mathrm{d}s \leqslant \int_0^t M\mathrm{e}^{-\mu(t-s)}\|f(s) - f_0\|\,\mathrm{d}s \\ &\leqslant \int_0^{t_0} M\mathrm{e}^{-\mu(t-s)} 2\|f\|_\infty\,\mathrm{d}s + \int_{t_0}^t M\mathrm{e}^{-\mu(t-s)}\eta\,\mathrm{d}s \\ &\leqslant 2M\|f\|_\infty\,\mu^{-1}\big(\mathrm{e}^{-\mu(t-t_0)} - \mathrm{e}^{-\mu t}\big) + \eta M\mu^{-1}\big(1 - \mathrm{e}^{-\mu(t-t_0)}\big),\end{aligned}$$

and $\limsup_{t\to\infty} \|v_1(t)\| \leqslant \eta M\mu^{-1}$. Since $\eta > 0$ was arbitrary, $\lim_{t\to\infty} \|v_1(t)\| = 0$.

By Theorem 4.2,

$$\begin{aligned}\lim_{t\to\infty} v_2(t) &= \int_0^t T(t-s)(I-P)f_0\,\mathrm{d}s = \lim_{t\to\infty} \int_0^t T(\tau)(I-P)f_0\,\mathrm{d}\tau \\ &= \int_0^\infty T(\tau)(I-P)f_0\,\mathrm{d}\tau = -A^{\mathrm{D}} f_0.\end{aligned}$$

Finally,

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} T(t)x + \lim_{t\to\infty} \int_0^t T(t-s)f(s)\,\mathrm{d}s = Px - A^{\mathrm{D}} f_0 + \int_0^\infty Pf(s)\,\mathrm{d}s. \quad ∎$$

Next, we derive conditions under which the mild solution of a singularly perturbed problem has limit as $\varepsilon \to 0+$. This result generalizes Theorem 4.4.5 of [14]; see also [10].

Theorem *Let $T(t)$ be a C_0-semigroup with the infinitesimal generator A satisfying $T(t) \to P$ as $t \to \infty$. Let f be continuous and bounded on $[0,\infty)$. The limit as $\varepsilon \to 0+$ of the mild solution $u_\varepsilon(t)$ of the singularly perturbed problem*

$$\varepsilon \frac{\mathrm{d}u_\varepsilon(t)}{\mathrm{d}t} = Au_\varepsilon(t) + f(t), \quad u_\varepsilon(0) = x, \quad \varepsilon > 0, \tag{5.3}$$

exists if and only if $Pf(t) = 0$ for all $t \geqslant 0$. If this is the case, then

$$u(t) := \lim_{\varepsilon\to 0+} u_\varepsilon(t) = Px - A^{\mathrm{D}} f(t), \tag{5.4}$$

where A^{D} is the Drazin inverse of A. The limit u is a solution of the reduced equation

$$0 = Au(t) + f(t), \quad u(0) = Px - A^{\mathrm{D}} f(0). \tag{5.5}$$

Proof. The mild solution to (5.3) is given by

$$u_\varepsilon(t) = T_\varepsilon(t)x + \varepsilon^{-1}\int_0^t T_\varepsilon(t-s)f(s)\,\mathrm{d}s,$$

where $T_\varepsilon(t) = T(t/\varepsilon)$. From (4.3) we deduce that $T_\varepsilon(t)P = P = PT_\varepsilon(t)$ for all $t \geqslant 0$. Let

$$u_{1\varepsilon}(t) = \varepsilon^{-1}P\int_0^t T_\varepsilon(t-s)f(s)\,\mathrm{d}s, \quad u_{2\varepsilon}(t) = \varepsilon^{-1}(I-P)\int_0^t T_\varepsilon(t-s)f(s)\,\mathrm{d}s.$$

Then $u_{1\varepsilon}(t) = \varepsilon^{-1}\int_0^t Pf(s)\,\mathrm{d}s$, and $\lim_{\varepsilon\to 0+} u_{1\varepsilon}(t)$ exists pointwise for $t \geqslant 0$ if and only if $\int_0^t Pf(s)\,\mathrm{d}s = 0$ for all $t \geqslant 0$; this occurs if and only if $Pf(t) = 0$ for all $t \geqslant 0$. Write

$$\begin{aligned} u_{2\varepsilon}(t) &= \varepsilon^{-1}\int_0^t T_\varepsilon(t-s)(I-P)(f(s)-f(t))\,\mathrm{d}s + \varepsilon^{-1}\int_0^t T_\varepsilon(t-s)(I-P)f(t)\,\mathrm{d}s \\ &= v_{1\varepsilon}(t) + v_{2\varepsilon}(t). \end{aligned}$$

By Theorem 4.2, there are constants $M > 0$, $\mu > 0$ such that

$$\|T_\varepsilon(t)(I-P)\| = \|T_\varepsilon(t) - P\| \leqslant M\mathrm{e}^{-\mu t/\varepsilon} \quad \text{for all } t \geqslant 0.$$

Keep $t > 0$ fixed. If $\eta > 0$, choose $t_0 \in (0,t)$ such that $\|f(s) - f(t)\| < \eta$ if $t_0 \leqslant s \leqslant t$. Similarly as in the preceding proof,

$$\begin{aligned} \|v_{1\varepsilon}(t)\| &\leqslant \varepsilon^{-1}\int_0^t M\mathrm{e}^{-\mu(t-s)/\varepsilon}\|f(s)-f(t)\|\,\mathrm{d}s \\ &= \varepsilon^{-1}\int_0^{t_0} M\mathrm{e}^{-\mu(t-s)/\varepsilon}2\|f\|_\infty\,\mathrm{d}s + \varepsilon^{-1}\int_{t_0}^t M\mathrm{e}^{-\mu(t-s)/\varepsilon}\eta\,\mathrm{d}s \\ &\leqslant 2M\mu^{-1}\|f\|_\infty\big(\mathrm{e}^{-\mu(t-t_0)/\varepsilon} - \mathrm{e}^{-\mu t/\varepsilon}\big) + \eta M\mu^{-1}(1-\mathrm{e}^{-\mu(t-t_0)/\varepsilon}). \end{aligned}$$

Hence $\limsup_{\varepsilon\to 0+}\|v_{1\varepsilon}(t)\| \leqslant \eta M\mu^{-1}$. Since $\eta > 0$ was arbitrary, we conclude that $\lim_{\varepsilon\to 0+}\|v_{1\varepsilon}(t)\| = 0$.

Therefore,

$$\lim_{\varepsilon\to 0+} v_{2\varepsilon}(t) = \lim_{\varepsilon\to 0+} \varepsilon^{-1}\int_0^t T_\varepsilon(t-s)(I-P)f(t)\,\mathrm{d}s$$
$$= \lim_{\varepsilon\to 0+}\int_0^{t/\varepsilon} T(\tau)(I-P)f(t)\,\mathrm{d}\tau$$
$$= \int_0^\infty T(\tau)(I-P)f(t)\,\mathrm{d}\tau = -A^{\mathrm{D}}f(t)$$

by Theorem 4.2. Therefore we conclude that

$$\lim_{\varepsilon\to 0+} u_\varepsilon(t) = Px - A^{\mathrm{D}}f(t)$$

if and only if $Pf(t) = 0$ for all $t \geqslant 0$. Under this assumption we have

$$Au(t) + f(t) = A(Px - A^{\mathrm{D}}f(t)) + f(t) = Pf(t) = 0,$$

and $u(0) = Px - A^{\mathrm{D}}f(0)$, which completes the proof. ∎

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