J. OPERATOR THEORY
46(2001), 337–353

THE K-GROUPS OF $C(M) \times_\theta \mathbb{Z}_p$ FOR CERTAIN PAIRS (M, θ)

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Communicated by Norberto Salinas

Abstract. Let M be a connected, compact metric space with $\dim M \leqslant 2p-1$ ($p \geqslant 2$ is a prime) and let θ be a homeomorphism of M to itself with period p. Suppose that $\dim M_\theta \leqslant 2$, $H^2(M_\theta, \mathbb{Z}) \cong 0$ and that $\bigoplus\limits_{j=0}^{2p-1} H^j(M/\theta, \mathbb{Z})$ is finitely generated and torsion-free; $H^0(M_\theta, \mathbb{Z})$ is finitely generated. If θ is regular and $H^{2j+1}(M/\theta, \mathbb{Z}) \cong 0$, $1 \leqslant j \leqslant p-1$ or θ is strongly regular and M_θ is connected, then

$$\mathrm{K}_0(C(M) \times_\theta \mathbb{Z}_p) \cong \mathrm{K}^0(M/\theta) \oplus \bigoplus_{j=0}^{p-2} H^0(M_\theta, \mathbb{Z})$$

$$\mathrm{K}_1(C(M) \times_\theta \mathbb{Z}_p) \cong \mathrm{K}^{-1}(M/\theta) \oplus \bigoplus_{j=0}^{p-2} H^1(M_\theta, \mathbb{Z}).$$

The result leads us to compute some interesting examples when M is a sphere or a torus.

Keywords: *K-groups of C^*-algebras, crossed product of C^*-algebras, stable rank, Čech cohomology groups, regular self-homeomorphism of prime period.*

MSC (2000): 46L05.

0. INTRODUCTION

In many situations, we need to compute the K-groups of the crossed product $C(M) \times_\theta \mathbf{G}$, where M is a compact Hausdorff space and $\mathbf{G}$ is a locally compact group such that $g \to \theta_g$ is the action of $\mathbf{G}$ on M. So far, there are many results about the computation of $\mathrm{K}_i(C(M) \times_\theta \mathbf{G})$, $i = 0, 1$ such as P-V sequence for $C(M) \times_\theta \mathbb{Z}$ (cf. [1]), Phillips' Theorem for $C(M) \times_\theta \mathbf{G}$ here $\mathbf{G}$ is a finite group such that $\mathbf{G}$ acts on M freely (cf. [17]) and so on. In [15], Paschke considered the simplest crossed product $C(M) \times_\theta \mathbb{Z}_2$ and he established a simple exact sequence of $\mathrm{K}_i(C(M)\times_\theta \mathbb{Z}_2)$, $i=0,1$. Inspired by Paschke's work, we try to generalize it from $C(M)\times_\theta \mathbb{Z}_2$ to $C(M)\times_\theta \mathbb{Z}_p$, where $p\geqslant 2$ is a prime. But we find it is impossible to exert his idea in our framework. We have to find another way to compute the K-groups of $C(M)\times_\theta \mathbb{Z}_p$ while some restrictions on the pair (M,θ) are needed.

The paper consists of four sections. In Section 1, we will show when the natural homomorphism $i_{\mathcal{D}(\mathcal{M},\theta)}\colon U(\mathcal{D}(M,\theta))\to \mathrm{K}_1(\mathcal{D}(M,\theta))$ is isomorphic. We will be devoted to establish some exact sequences about U-groups and $\widehat{U}$-groups in Section 2. All these lead us to handle the K_1-group of $C(M)\times_\theta \mathbb{Z}_p$ for certain pairs (M,θ). We will compute the K_0-group of $C(M)\times_\theta\mathbb{Z}_p$ under some restrictions to the pair (M,θ) in Section 3. In the final section we will give some interesting examples of computing $\mathrm{K}_i(C(M)\times_\theta \mathbb{Z}_p)$, $i=0,1$ when M is a sphere or a torus.

Throughout the paper, we let $H^k(X,\mathbb{Z})$ denote the k^{th} Čech cohomology group of the compact Hausdorff space X and let $\mathrm{K}^{-i}(X)$ (respectively $\widetilde{\mathrm{K}}^{-i}(X)$) denote the (respectively reduced) K^{-i}-group of the compact Hausdorff space X $(i=0,1)$. It is well-known that if $\bigoplus\limits_{j\geqslant 0} H^j(X,\mathbb{Z})$ is torsion-free, then so is $\mathrm{K}^0(X)\oplus \mathrm{K}^{-1}(X)$ and $\mathrm{K}^0(X)\oplus \mathrm{K}^{-1}(X)\cong \bigoplus\limits_{j\geqslant 0} H^j(X,\mathbb{Z})$ (cf. [5]).

We write $\operatorname{Ker}\Phi$ (respectively $\operatorname{Im}\Phi$) to denote the kernel (respectively the range) of the homomorphism Φ between groups (or rings).

For convenience, we assume that throughout the paper the pair (M,θ) satisfies following conditions:

(1) M is a locally compact metric space with $\dim M\leqslant 2p-1$ (here $p\geqslant 2$ is a prime);

(2) θ is a self-homeomorphism of M with period p.

1. PRELIMINARIES

Let $\mathcal{A}$ be a C^*-algebra with unit 1. We denote by $\mathcal{U}(\mathcal{A})$ the group of unitary elements of $\mathcal{A}$ and $\mathcal{U}_0(\mathcal{A})$ the connected component of the unit 1 in $\mathcal{U}(\mathcal{A})$. The quotient group $U(\mathcal{A})=\mathcal{U}(\mathcal{A})/\mathcal{U}_0(\mathcal{A})$ whose multiplication is given by $[a][b]=[ab]$ is called the U-group, where $[a]$ stands for the equivalence class of a in $\mathcal{U}(\mathcal{A})$ about $\mathcal{U}_0(\mathcal{A})$. According to [1], there is a canonical homomorphism $i_{\mathcal{A}}\colon U(\mathcal{A})\to \mathrm{K}_1(\mathcal{A})$ given by $i_{\mathcal{A}}([a])=[\operatorname{diag}(a,1_n)]\in U(M_m(\mathcal{A}))$ for each $m>n$ where $\mathrm{M}_m(\mathcal{A})$ is the matrix algebra of order m over $\mathcal{A}$.

For the C^*-algebra $\mathcal{A}$ with unit 1, we regard $\operatorname{Lg}_n(\mathcal{A})$ as the set of all n-tuples $(a_1,\dots,a_n)$ from $\mathcal{A}^n$ which generate $\mathcal{A}$ as a left ideal. Set (cf. [18], [19])

$$
\begin{aligned}
&\mathrm{S}_n(\mathcal{A})=\Big\{(a_1,\dots,a_n)\,\Big|\,\sum_{k=1}^n a_k^*a_k=1\Big\}\\
&\operatorname{tsr}(\mathcal{A})=\min\{n\mid \operatorname{Lg}_n(\mathcal{A}) \text{ is dense in } \mathcal{A}^n\}\\
&\operatorname{csr}(\mathcal{A})=\min\{n\mid \mathcal{U}_0(M_n(\mathcal{A})) \text{ acts transitively on } \mathrm{S}_m(\mathcal{A}),\ \forall m\geqslant n\}\\
&\operatorname{gsr}(\mathcal{A})=\min\{n\mid \mathcal{U}(M_n(\mathcal{A})) \text{ acts transitively on } \mathrm{S}_m(\mathcal{A}),\ \forall m\geqslant n\}.
\end{aligned}
$$

If $\mathcal{A}$ has no unit, we put $U(\mathcal{A}) = U(\mathcal{A}^+)$, $\operatorname{tsr}(\mathcal{A}) = \operatorname{tsr}(\mathcal{A}^+)$, $\operatorname{csr}(\mathcal{A}) = \operatorname{csr}(\mathcal{A}^+)$ and $\operatorname{gsr}(\mathcal{A}) = \operatorname{gsr}(\mathcal{A}^+)$ where $\mathcal{A}^+$ is obtained from $\mathcal{A}$ by adjoining the unit 1.

The computation of the above functions of C^*-algebras is very interesting but sometimes is very difficult. However, if $\mathcal{A}$ is a purely infinite simple C^*-algebra, $\operatorname{tsr}(\mathcal{A}), \operatorname{csr}(\mathcal{A})$ and $\operatorname{gsr}(\mathcal{A})$ are completely determined by Rieffel and the author (cf. [18], [24], [25]). From [18], Theorem 2.9 from [19], [10] and Proposition 3.10 from [21] we have:

LEMMA 1.1. *Let $\mathcal{A}$ be a C^*-algebra and X be a compact Hausdorff space.*

(i) $\operatorname{gsr}(\mathcal{A}) \leqslant \operatorname{csr}(\mathcal{A}) \leqslant \operatorname{tsr}(\mathcal{A}) + 1$*;*
(ii) *if* $\operatorname{csr}(\mathcal{A}) \leqslant 2$*, then $i_{\mathcal{A}}$ is surjective;*
(iii) *if* $\operatorname{gsr}(C(\mathbf{S}^1) \otimes \mathcal{A}) \leqslant 2$*, then $i_{\mathcal{A}}$ is injective;*
(iv) $\operatorname{csr}(C(X)) = \min\{n \mid H^{2n-1}(X, \mathbb{Z}) \cong 0\}$*.*

REMARK 1.2. In fact, the author has found an equivalent condition that makes $i_{\mathcal{A}}$ injective (cf. Theorem 2.4 from [25]).

For the pair (M, θ), set $M_k = \{x \in M \mid \theta^k(x) = x\}$ and $M_\theta = M_1$. Clearly, $M_\theta \subset M_k$. On the other hand, $M_k \subset M_\theta$ when $p = 2$ or 3. If $3 \leqslant k \leqslant p-1$, there are two integers n_1, n_2 such that $n_1 p + n_2 k = 1$ (for $(p,k) = 1$)). Thus $\theta = \theta^{n_1 p}\theta^{n_2 k} = \theta^{n_2 k}$ and $M_k \subset M_\theta$. So $M_\theta = M_k$, $k = 2, \ldots, p-1$.

For the pair (M, θ) with M compact and $M_\theta \neq \varphi$, set $M_0 = M \backslash M_\theta$. Then $(M_0)_\theta = \varphi$ and the one-point compactification M_0^+ of M_0 is M/M_θ. Now let M/θ (or M_0/θ) denote the orbit space of θ and let P (respectively Q) be the canonical projective map of M or M_0 onto M/θ (or M_0/θ) (respectively of M onto M/M_θ). Identifing M_θ with the closed subset of M/θ, we have $P(M)/M_\theta \cong Q(M)/\widehat{\theta}$, where $\widehat{\theta}$ is a self-homeomorphism of $Q(M)$ with period p defined by $\widehat{\theta}(Q(x)) = Q(\theta(x))$, $\theta(*) = *$ $(* = Q(M_\theta))$ or $\widehat{\theta}(x) = \theta(x)$ for $x \in M_0$ and $\widehat{\theta}(+) = +$.

By Theorem 1.12.7, Theorem 1.7.7 from [6] and the proof of Proposition 2.1 from [23], we have the following:

LEMMA 1.3. *Let (M, θ) be the pair with M compact. Then* $\dim(M_0/\theta) = \dim M_0$, $\dim(M/\theta) = \dim M$ *and* $\dim(M/M_\theta) \leqslant \dim M$.

For the pair (M, θ), the dynamical system $(C_0(M), \theta, \mathbb{Z}_p)$ yields a crossed product C^*-algebra $C_0(M) \times_\theta \mathbb{Z}_p$. By 7.6.1 and 7.6.5 of [16], this algebra is $*$-isomorphic to the C^*-algebra

$$\mathcal{D}(M,\theta) = \left\{ \begin{bmatrix} f_0 & f_1 & \cdots & f_{p-1} \\ \theta(f_{p-1}) & \theta(f_0) & \cdots & \theta(f_{p-2}) \\ \cdots & \cdots & \cdots & \cdots \\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \cdots & \theta^{p-1}(f_0) \end{bmatrix} \;\middle|\; f_0, \ldots, f_{p-1} \in C_0(M) \right\}$$

contained in $\mathrm{M}_n(C_0(M))$, where $\theta(f)(x) = f(\theta(x))$, $\forall x \in M$, $f \in C_0(M)$.

LEMMA 1.4. *For the pair (M,θ) with M compact and $M_\theta \neq \varphi$, we have:*
(i) *every irreducible representation of $\mathcal{D}(M_0,\theta)$ is equivalent to the representation π_x, where $\pi_x(a) = a(x)$ for some $x \in M_0$ and $\forall a \in \mathcal{D}(M_0,\theta)$ and $P(x) \to [\pi_x]$ gives a homeomorphism of M_0/θ with $\widehat{\mathcal{D}(M_0,\theta)}$ — the spectrum of $\mathcal{D}(M_0,\theta)$ — where we identify $\mathcal{D}(M_0,\theta)$ with $\{a \in \mathcal{D}(M,\theta) \,|\, a|M_\theta = 0\}$;*
(ii) *$\mathcal{D}(M,\theta)$ is a p-homogeneous algebra which is $*$-isomorphic to $C_0(M_0/\theta, E)$, where E is a fiber bundle over M_0/θ with fiber $M_p(\mathbb{C})$.*

Proof. (i) is a combination of Lemma 16 from [8] and 7.7.1 from [16] comes from the combination of the proof of Theorem 14 from [8] and Theorem 3.2 from [7]. ∎

For the pair (M,θ), let $\mathrm{Det} : \mathrm{M}_n(\mathcal{D}(M,\theta)) \to C_0(M)$ denote the determinant as usual. Set $C_\theta(M) = \{f \in C_0(M) \mid \theta(f) = f\} \cong C_0(M/\theta)$. Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/p}$ and put

$$\Omega_p = \begin{bmatrix} \frac{1}{\sqrt{p}} & \frac{\omega}{\sqrt{p}} & \cdots & \frac{\omega^{p-1}}{\sqrt{p}} \\ \frac{1}{\sqrt{p}} & \frac{\omega^2}{\sqrt{p}} & \cdots & \frac{(\omega^2)^{p-1}}{\sqrt{p}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{p}} & \frac{\omega^{p-1}}{\sqrt{p}} & \cdots & \frac{(\omega^{p-1})^{p-1}}{\sqrt{p}} \\ \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} & \cdots & \frac{1}{\sqrt{p}} \end{bmatrix} \in \mathcal{U}(\mathrm{M}_n(\mathbb{C})).$$

Now define the map $\rho : C_\theta(M) \to \mathcal{D}(M,\theta)$ by $\rho(f) = \Omega_p^* \mathrm{diag}\,(f, 1_{p-1})\Omega_p$. In terms of some techniques from linear algebra (or Case 2 from [11]), we have

LEMMA 1.5. *Let (M,θ) be the pair with M compact. Then:*
(i) $\mathrm{Det}(a) \in C_\theta(M)$, $\forall a \in M_n(\mathcal{D}(M,\theta))$ *and* $\mathrm{Det} \circ \rho = \mathrm{id}$ *on* $C_\theta(M)$*;*
(ii) *ρ is a homomorphism of $\mathcal{U}(C_\theta(M))$ to $\mathcal{U}(\mathcal{D}(M,\theta))$;*
(iii) *for $f_0, \dots, f_{p-1} \in C_\theta(M)$,*

$$\Omega_p \begin{bmatrix} f_0 & f_1 & \cdots & f_{p-1} \\ f_{p-1} & f_0 & \cdots & f_{p-2} \\ \cdots & \cdots & \cdots & \cdots \\ f_1 & f_2 & \cdots & f_0 \end{bmatrix} \Omega_p^* = \mathrm{diag}\left(\sum_{j=0}^{p-1} \omega^{j(p-1)} f_j, \dots, \sum_{j=0}^{p-1} \omega^j f_j, \sum_{j=0}^{p-1} f_j \right).$$

COROLLARY 1.6. *Let (M,θ) be as above and $M_\theta \neq \varphi$. Then the homomorphism $\pi : \mathcal{D}(M,\theta) \to \bigoplus_{j=0}^{p-1} C(M_\theta)$ given by*

$$\pi\left(\begin{bmatrix} f_0 & f_1 & \cdots & f_{p-1} \\ \theta(f_{p-1}) & \theta(f_0) & \cdots & \theta(f_{p-2}) \\ \cdots & & & \\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \cdots & \theta^{p-1}(f_0) \end{bmatrix} \right)(x) = \left(\sum_{j=0}^{p-1} \omega^{j(p-1)} f_j(x), \dots, \sum_{j=0}^{p-1} f_j(x) \right),$$

induces the following exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{D}(M_0,\theta) \xrightarrow{l} \mathcal{D}(M,\theta) \xrightarrow{\pi} \bigoplus_{j=0}^{p-1} C(M_\theta) \longrightarrow 0, \tag{1.1}$$

where $x \in M_\theta$, $f_0, \dots, f_{p-1} \in C(M)$ and l is the inclusion map.

Obviously, replacing M by M_0^+ in (1.1), we have the following split exact sequence:

$$0 \longrightarrow \mathcal{D}(M_0,\theta) \xrightarrow{l} \mathcal{D}(M_0^+,\widehat{\theta}) \xrightarrow{\pi} \mathbb{C}^p \longrightarrow 0. \tag{1.2}$$

Here is the main result of the section.

PROPOSITION 1.7. *Let (M,θ) be a pair with M compact and $M_\theta \neq \varphi$, $\dim M_\theta \leqslant 2$. Then $i_{\mathcal{D}(M_0,\theta)}$ is isomorphic. In addition, if $H^2(M_\theta,\mathbb{Z}) \cong 0$, so is $i_{\mathcal{D}(M,\theta)}$.*

Proof. By Lemma 1.4 and Lemma 5 (b) from [14]

$$\operatorname{tsr}(\mathcal{D}(M_0,\theta)) = \left[\frac{\dim(M_0/\theta)-1}{2p}\right]+1;$$

here $[x]$ expresses the least integer $\geqslant x$. So by Lemma 2.4 in [13] and Lemma 1.3,

$$\begin{aligned}\operatorname{csr}(\mathcal{D}(M_0,\theta) &\leqslant \operatorname{tsr}(C([0,1]\otimes\mathcal{D}(M_0,\theta)) = \operatorname{tsr}(\mathcal{D}(M_0\times[0,1],\theta_1))\\ &= \left[\frac{\dim(M_0\times[0,1]/\theta_1)-1}{2p}\right]+1 \leqslant \left[\frac{\dim M}{2p}\right]+1\leqslant 2,\end{aligned}$$

where $\theta_1(x,t) = (\theta(x),t)$, $x\in M$, $t\in[0,1]$ and $(M\times[0,1])_{\theta_1} = M_\theta\times[0,1]$.

Using the same method as above, we can conclude that

$$\operatorname{gsr}(C(\mathbf{S}^1)\otimes\mathcal{D}(M_0,\theta)) \leqslant \operatorname{csr}(C(\mathbf{S}^1)\otimes\mathcal{D}(M_0,\theta)) \leqslant \left[\frac{\dim M\times\mathbf{S}^1}{2p}\right]+1\leqslant 2.$$

Thus $i_{\mathcal{D}(M_0,\theta)}$ is isomorphic by Lemma 1.1.

Since $\dim M_\theta \leqslant 2$ and $H^3(M_\theta\times\mathbf{S}^1,\mathbb{Z}) \cong H^3(M_\theta,\mathbb{Z})\oplus H^2(M_\theta,\mathbb{Z})\cong 0$, we have $\operatorname{csr}(C(M_\theta))\leqslant 2$ and $\operatorname{csr}(C(M_\theta\times\mathbf{S}^1))\leqslant 2$ by Lemma 1.1. Thus by Corollary 1.6 and Lemma 2 from [12],

$$\operatorname{csr}(\mathcal{D}(M,\theta)) \leqslant \max\left\{\operatorname{csr}(\mathcal{D}(M_0,\theta)),\ \operatorname{csr}\left(\bigoplus_{j=0}^{p-1} C(M_\theta)\right)\right\}\leqslant 2$$

and also $\operatorname{gsr}(C(\mathbf{S}^1)\otimes\mathcal{D}(M,\theta)) \leqslant \operatorname{csr}(C(\mathbf{S}^1)\otimes\mathcal{D}(M,\theta))\leqslant 2$. Therefore $i_{\mathcal{D}(M,\theta)}$ is an isomorphism. ∎

2. THE COMPUTATION OF $K_1(\mathcal{D}(M,\theta))$ FOR CERTAIN (M,θ)

For the pair (M,θ) with M compact, set

$$\widehat{\mathcal{U}}(\mathcal{D}(M,\theta)) = \{u \in \mathcal{U}(\mathcal{D}(M,\theta)) \mid \mathrm{Det}(u)=1\},$$
$$\widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta)) = \{u \in \mathcal{U}((\mathcal{D}(M_0,\theta))^+) \mid \mathrm{Det}(u)=1\}$$

and let $\widehat{\mathcal{U}}_0(\mathcal{D}(M,\theta))$ (respectively $\widehat{\mathcal{U}}_0(\mathcal{D}(M_0,\theta))$) denote the connected component of 1 in $\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ (respectively $\widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta))$), where we identify $(\mathcal{D}(M_0,\theta))^+$ with the C^*-algebra $\{a \in \mathcal{D}(M,\theta) \mid a(x) \equiv \text{constant},\ x \in M_\theta\}$. Obviously, $\widehat{\mathcal{U}}_0(\mathcal{D}(M,\theta))$ (respectively $\widehat{\mathcal{U}}_0(\mathcal{D}(M_0,\theta))$) is a normal subgroup of $\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ (respectively $\widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta))$). Thus

$$\widehat{U}(\mathcal{D}(M,\theta)) = \widehat{\mathcal{U}}(\mathcal{D}(M,\theta))/\widehat{\mathcal{U}}_0(\mathcal{D}(M,\theta))$$
$$\widehat{U}(\mathcal{D}(M_0,\theta)) = \widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta))/\widehat{\mathcal{U}}_0(\mathcal{D}(M_0,\theta))$$

become groups under the multiplication $\langle uv\rangle = \langle u\rangle\langle v\rangle$, where $\langle u\rangle$ represents the equivalence class of u in $\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ about $\widehat{\mathcal{U}}_0(\mathcal{D}(M,\theta))$ (respectively $\widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta))$ about $\widehat{\mathcal{U}}_0(\mathcal{D}(M_0,\theta))$). Let $\langle 1\rangle$ denote the unit of $\widehat{U}(\mathcal{D}(M,\theta))$ or $\widehat{U}(\mathcal{D}(M_0,\theta))$.

LEMMA 2.1. *For the pair (M,θ) with M compact, the sequence of groups*

$$\langle 1\rangle \longrightarrow \widehat{U}(\mathcal{D}(M,\theta)) \xrightarrow{\ j\ } U(\mathcal{D}(M,\theta)) \longrightarrow \mathrm{Det}_* \longrightarrow U(C_\theta(M)) \longrightarrow 0 \tag{2.1}$$

is split exact; here $j(\langle a\rangle) = [a]$, $a \in \widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ and $\mathrm{Det}_*([u]) = [\mathrm{Det}(u)]$, $u \in \mathcal{U}(\mathcal{D}(M,\theta))$.

Proof. Since $\mathrm{Det}_*(j(\langle a\rangle)) = 0$ when $a \in \widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$, we get that $\mathrm{Im}\, j \subset \mathrm{Ker}\,\mathrm{Det}_*$. Let $[u] \in \mathrm{Ker}\,\mathrm{Det}_*$, i.e., $[\mathrm{Det}(u)] = 0$ in $U(C_\theta(M))$. Then there is a real continuous function h on M such that $\theta(h) = h$ and $\mathrm{Det}(u) = \mathrm{e}^{2\pi \mathrm{i} h}$. Put

$$v = u\,\mathrm{diag}\,(\mathrm{e}^{-2\pi\mathrm{i}h/p}, \ldots, \mathrm{e}^{-2\pi\mathrm{i}h/p}) \in \mathcal{U}(\mathcal{D}(M,\theta)).$$

Then $v \in \widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ and $j(\langle v\rangle) = [v] = [u]$ in $U(\mathcal{D}(M,\theta))$. So $\mathrm{Im}\, j = \mathrm{Ker}\,\mathrm{Det}_*$.

Now suppose that $u \in \widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ with $j(\langle u\rangle) = [1]$, where $[1]$ is the unit of $U(\mathcal{D}(M,\theta))$. Then $\mathrm{Det}(u) \equiv 1$ and there is a path u_t in $\mathcal{U}(\mathcal{D}(M,\theta))$ such that $u_0 = 1$ and $u_1 = u$. Put $v_t = u_t\rho(\mathrm{Det}(u_t^*))$, $0 \leqslant t \leqslant 1$. Then $\mathrm{Det}(v_t) \equiv 1, v_0 = 1$ and $v_1 = u$ by Lemma 1.3. Therefore $\langle u\rangle = \langle 1\rangle$, i.e., j is injective. From $\mathrm{Det}\circ\rho = \mathrm{id}$ on $\mathcal{U}(C_\theta(M))$, we have $\mathrm{Det}_* \circ \rho_* = \mathrm{id}$ on $U(C_\theta(M))$ where $\rho_*([f]) = [\rho(f)]$, $\forall f \in \mathcal{U}(C_\theta(M))$.

Thus we have proven that (2.1) is split exact. ∎

Assume that M is compact and $M_\theta \neq \varphi$. In terms of Corollary 1.6, we can define a homomorphism $\tau : \widehat{\mathcal{U}}(\mathcal{D}(M,\theta)) \to \bigoplus_{j=0}^{p-2} \mathcal{U}(C(M_\theta))$ for $x \in M_\theta$ by

$$\tau\left(\begin{bmatrix} f_0 & f_1 & \cdots & f_{p-1} \\ \theta(f_{p-1}) & \theta(f_0) & \cdots & \theta(f_{p-2}) \\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \cdots & \theta^{p-1}(f_0) \end{bmatrix}\right)(x) = \left(\sum_{j=0}^{p-1} \omega^{j(p-1)} f_j(x), \ldots, \sum_{j=0}^{p-1} \omega^{j} f_j(x)\right).$$

LEMMA 2.2. *Assume that M is compact and $M_\theta\neq\varphi$. The sequence of groups*

$$\widehat{U}(\mathcal{D}(M_0,\theta))\xrightarrow{l_*}\widehat{U}(\mathcal{D}(M,\theta))\xrightarrow{\tau_*}U\Big(\bigoplus_{j=0}^{p-2}C(M_\theta)\Big)\tag{2.2}$$

is exact in the middle, where $\tau_*(\langle u\rangle)=[\tau(u)]$, $u\in\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ and $l_*(\langle a\rangle)=\langle a\rangle$, $a\in\widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta))$.

Proof. Let $u\in\mathcal{U}((\mathcal{D}(M_0,\theta))^+)$ with $\mathrm{Det}(u)\equiv 1$. Then $u(x)\equiv\lambda 1$, $\forall x\in M_\theta$ and $\lambda^p=1$. It follows that $\tau(u)=(\lambda,\dots,\lambda)$. This implies that $\tau_*\circ l_*=\Big[\bigoplus_{j=0}^{p-2}1\Big]=0$, i.e., $\mathrm{Im}\,l_*\subset\mathrm{Ker}\,\tau_*$.

On the other hand, for v in $\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ with $\tau(v)\in\bigoplus_{j=0}^{p-2}\mathcal{U}_0(C(M_\theta))$, there are real functions $h_0,\dots,h_{p-2}$ in $C(M_\theta)$ such that $\tau(v)=(\mathrm{e}^{2\pi\mathrm{i}h_0},\dots,\mathrm{e}^{2\pi\mathrm{i}h_{p-2}})$. Pick real functions $\widetilde{h}_0,\dots,\widetilde{h}_{p-2}\in C_\theta(M)$ such that $\widetilde{h}_j|M_\theta=h_j$, $0\leqslant j\leqslant p-2$. Put

$$v_1=v\,\Omega_p^*\mathrm{diag}\left(\mathrm{e}^{-2\pi\mathrm{i}\widetilde{h}_0},\dots,\mathrm{e}^{-2\pi\mathrm{i}\widetilde{h}_{p-2}},\mathrm{e}^{2\pi\mathrm{i}\sum_{j=0}^{p-2}\widetilde{h}_j}\right)\Omega_p.$$

Then $v_1\in\widehat{\mathcal{U}}(\mathcal{D}(M_0,\theta))$ (i.e., $v_1\in\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ and $v_1(x)=1$, $x\in M_\theta$) and $l_*(\langle v_1\rangle)=\langle v_1\rangle=\langle v\rangle\in\mathrm{Ker}\,\tau_*$. ∎

In order to see when τ_* is surjective or l_* is injective in (2.2), we need to introduce the following:

DEFINITION 2.3. For the pair (M,θ) with M compact and $M_\theta\neq\varphi$, we say that θ is *strongly regular* if there is $h_\theta\in C(M)$ such that $\theta(h_\theta)=\omega h_\theta$ and $M_\theta=\{x\in M\mid h_\theta(x)=0\}$; θ is called to be *regular* if given $f_0,\dots,f_{p-1}\in\mathcal{U}(C(M_\theta))$ there are $F_0,\dots,F_{p-1}\in C_\theta(M)$ and $G_\theta\in C(M)$ such that $F_j|M_\theta=f_j$, $0\leqslant j\leqslant p-2$, $\theta(G_\theta)=\omega G_\theta$ and $\Big|\prod_{j=0}^{p-2}F_j(x)\Big|+|G_\theta(x)|\neq 0$, $\forall x\in M$.

Obviously, if θ is strongly regular, then θ must be regular. Some conditions under which θ is regular or strongly regular will be given in Section 4.

PROPOSITION 2.4. *For the pair (M,θ) with M compact and $M_\theta\neq\varphi$, τ_* is surjective if θ is regular and l_* is injective if θ is strongly regular.*

Proof. Assume that θ is regular. Then for $f_0,\dots,f_{p-2}\in\mathcal{U}(C(M_\theta))$ there are $F_0,\dots,F_{p-2}\in C_\theta(M)$ and $G_\theta\in C(M)$ such that $F_j|M_\theta=f_j$, $0\leqslant j\leqslant p-2$,

$\theta(G_\theta) = \omega G_\theta$ and $\Big|\prod_{j=0}^{p-2} F(x)_j\Big| + |G_\theta(x)| \neq 0$, $\forall x \in M$. Set

$$A = \Omega_p^* \begin{bmatrix} F_0 & \omega G_\theta & & & \\ & \ddots & \ddots & & 0 \\ 0 & & \ddots & \ddots & \\ & & & F_{p-2} & \omega^{p-1} G_\theta \\ G_\theta^{*(p-1)} & 0 & \cdots & 0 & \prod_{j=0}^{p-2} F_j^* \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a \end{bmatrix} \Omega_p, \tag{2.3}$$

where $a(x) = \Big(\Big|\prod_{j=0}^{p-2} F_j(x)\Big|^2 + |G_\theta(x)|^{2(p-1)}\Big)^{-1}$, $x \in M$. Since $\theta(G_\theta) = \omega G_\theta$, $\theta(G_\theta^*) = \overline{\omega} G_\theta^*$, $\theta(F_j) = F_j$, $0 \leqslant j \leqslant p-2$, we conclude that $a \in C_\theta(M)$ and $A \in \mathcal{D}(M,\theta)$ with $\mathrm{Det}(A) \equiv 1$. Put $u = A(A^*A)^{-1/2}$. Then $u \in \widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ and $\tau(u) = (f_0, \ldots, f_{p-2})$ since $\theta(G_\theta) = \omega G_\theta$ indicates that $G_\theta | M_\theta = 0$.

Now suppose that θ is strongly regular, i.e., there is $h_\theta \in C_\theta(M)$ such that $M_\theta = \{x \in M \mid h_\theta(x) = 0\}$. Let $u \in \widehat{\mathcal{U}}((\mathcal{D}(M,\theta))^+)$ with $u(x) \equiv 1$, $\forall x \in M_\theta$ and $l(u) \in \widehat{\mathcal{U}}_0(\mathcal{D}(M,\theta))$ and let u_t be a path in $\widehat{\mathcal{U}}(\mathcal{D}(M,\theta))$ such that $u_0 = 1$ and $u_1 = u$ and let $\tau(u_t)(x) = (g_{0,t}(x), \ldots, g_{p-2,t}(x))$, $0 \leqslant t \leqslant 1$, $x \in M_\theta$. Then $g_{j,0}|M_\theta = g_{j,1}|M_\theta \equiv 1$, $g_{j,t} \in \mathcal{U}(C(M_\theta))$ and $t \to g_{j,t}$ is continuous, $0 \leqslant j \leqslant p-2$. Regarding $g_{j,t}$ as the functions in $C(M_\theta \times \mathbf{S}^1)$, $j = 0, \ldots, p-2$, we can pick $G_{j,t}$ in $C_\theta(M)$ such that $t \to G_{j,t}$ is continuous and $G_{j,t}|M_\theta = g_{j,t}$, $G_{j,0} = G_{j,1}$, $j = 0, \ldots, p-2$, $t \in [0,1]$. Put

$$A_t = \Omega_p^* \begin{bmatrix} G_{0,t} & \omega h_\theta & & & 0 \\ & 0 & & & \\ & & & G_{p-2,t} & \omega^{p-1} h_\theta \\ h_\theta^{*(p-1)} & 0 & \cdots & 0 & \prod_{j=0}^{p-2} G_{j,t}^* \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a_t \end{bmatrix} \Omega_p \tag{2.4}$$

and $B_t = A_0^{-1} A_t$, $0 \leqslant t \leqslant 1$, where $a_t(x) = \Big(\Big|\prod_{j=0}^{p-2} G_{j,t}(x)\Big|^2 + |h_\theta(x)|^{2(p-1)}\Big)^{-1}$, $x \in M$. Then $\mathrm{Det}(A_t) = \mathrm{Det}(B_t) \equiv 1$, $A_0 = A_1$, $A_t(x) = u_t(x)$, $x \in M_\theta$, $t \in [0,1]$. Set $c_t = B_t^{-1} u_t$ and $v_t = c_t(c_t^* c_t)^{-1/2}$, $0 \leqslant t \leqslant 1$. Then $v_0 = 1, v_1 = u, \mathrm{Det}(v_t) \equiv 1$ and $v_t(x) \equiv 1$, $x \in M_\theta$, $t \in [0,1]$. So $\langle u \rangle = \langle 1 \rangle$ in $\widehat{U}(\mathcal{D}(M_0,\theta))$, i.e., l_* is injective. ∎

COROLLARY 2.5. *For the pair (M,θ) with M compact and $M_\theta \neq \varphi$,*

$$\widehat{U}(\mathcal{D}(M_0^+, \widehat{\theta})) \cong \widehat{U}(\mathcal{D}(M_0,\theta)), \quad U(\mathcal{D}(M_0^+, \widehat{\theta})) \cong U(\mathcal{D}(M_0,\theta)).$$

Proof. It is easy to check that $U(\mathcal{D}(M_0^+, \widehat{\theta})) \cong U(\mathcal{D}(M_0,\theta))$ by (1.2) and the definition of U-group.

Since $(M/M_\theta)_{\widehat{\theta}} = \{*\}$, it follows from Lemma 2.2 that $l_* : \widehat{U}(\mathcal{D}(M_0,\theta)) \to \widehat{U}(\mathcal{D}(M_0^+, \widehat{\theta}))$ is surjective. Now in (2.4) we set $h_\theta = 0$ and $G_{j,t} = g_{j,t}(M_\theta)$, $0 \leqslant j \leqslant p-2$, $t \in [0,1]$. Then we can conclude that l_* is injective. ∎

THEOREM 2.6. *For the pair (M,θ) with M compact, $M_\theta \neq \varphi$, $\dim M_\theta \leqslant 2$ and $H^2(M_\theta,\mathbb{Z}) \cong 0$, if one of following conditions holds:*

(i) *θ is strongly regular and $\bigoplus_{j=0}^{p-1} H^{2j+1}(M/\theta,\mathbb{Z})$, $\mathrm{K}^{-1}(M/\theta)$ and $\mathrm{K}^{-1}(M_0^+/\widehat{\theta})$ are all finitely generated and torsion-free;*

(ii) *θ is regular and $H^{2j+1}(M/\theta,\mathbb{Z}) \cong 0$, $1 \leqslant j \leqslant p-1$ and $H^1(M/\theta,\mathbb{Z})$ is finitely generated, then*

$$\mathrm{K}_1(\mathcal{D}(M,\theta)) \cong \mathrm{K}^{-1}(M/\theta) \oplus \bigoplus_{j=0}^{p-2} H^1(M_\theta,\mathbb{Z}).$$

Proof. By Proposition 1.7 and Corollary 10.9.6 from [4],

$$\begin{aligned} U(\mathcal{D}(M_0,\theta)) &\cong \mathrm{K}_1(\mathcal{D}(M_0,\theta)) \cong \mathrm{K}_1(\mathcal{D}(M_0,\theta)\otimes\mathcal{K}) \\ &\cong \mathrm{K}_1(C_0(M_0/\theta)\otimes\mathcal{K}) \cong \mathrm{K}^{-1}(M_0^+/\widehat{\theta}), \end{aligned} \tag{2.5}$$

where $\mathcal{K}$ is the algebra of compact operators on l^2. Since $\dim M_\theta \leqslant 2$, it follows from the exact sequence of Čech cohomology groups (cf. [21])

$$\to H^{j-1}(M/\theta,\mathbb{Z}) \longrightarrow H^{j-1}(M_\theta,\mathbb{Z}) \longrightarrow H^j(M_0^+/\widehat{\theta},\mathbb{Z}) \longrightarrow H^j(M/\theta,\mathbb{Z}) \to \tag{2.6}$$

and $H^2(M_\theta,\mathbb{Z}) \cong 0$ that $H^{2j+1}(M_0^+/\widehat{\theta},\mathbb{Z}) \cong H^{2j+1}(M/\theta,\mathbb{Z})$, $1 \leqslant j \leqslant p-1$. Noting that $U(C_{\widehat{\theta}}(M_0^+)) \cong U(C(M_0^+/\widehat{\theta})) \cong H^1(M_0^+/\widehat{\theta},\mathbb{Z})$ by Proposition 3.10 from [21], we get that by Corollary 2.5, Lemma 2.1 and (2.5),

$$\begin{aligned} \widehat{U}(\mathcal{D}(M_0^+,\widehat{\theta})) \oplus U(C_{\widehat{\theta}}(M_0^+)) &\cong \widehat{U}(\mathcal{D}(M_0,\theta)) \oplus H^1(M_0^+/\widehat{\theta},\mathbb{Z}) \\ &\cong U(\mathcal{D}(M_0^+,\widehat{\theta})) \cong U(\mathcal{D}(M_0,\theta)) \cong \mathrm{K}^{-1}(M_0^+/\widehat{\theta}). \end{aligned} \tag{2.7}$$

Assume that (i) is satisfied. Then by Theorem 1.6.6 from [1] and (2.7),

$$\widehat{U}(\mathcal{D}(M_0,\theta)) \cong \bigoplus_{j=1}^{p-1} H^{2j+1}(M_0^+/\widehat{\theta},\mathbb{Z}) \cong \bigoplus_{j=1}^{p-1} H^{2j+1}(M/\theta,\mathbb{Z})$$

and hence by Proposition 2.4,

$$\begin{aligned} \widehat{U}(\mathcal{D}(M,\theta)) &\cong \widehat{U}(\mathcal{D}(M_0,\theta)) \oplus U\Big(\bigoplus_{j=0}^{p-2} C(M_\theta)\Big) \\ &\cong \bigoplus_{j=1}^{p-1} H^{2j+1}(M/\theta,\mathbb{Z}) \oplus \bigoplus_{j=0}^{p-2} H^1(M_\theta,\mathbb{Z}). \end{aligned}$$

Furthermore, by Proposition 1.7 and Lemma 2.1,

$$\begin{aligned} \mathrm{K}_1(\mathcal{D}(M,\theta)) &\cong U(\mathcal{D}(M,\theta)) \cong \widehat{U}(\mathcal{D}(M,\theta)) \oplus U(C_\theta(M)) \\ &\cong \widehat{U}(\mathcal{D}(M,\theta)) \oplus H^1(M/\theta,\mathbb{Z}) \cong \mathrm{K}^{-1}(M/\theta) \oplus \bigoplus_{j=0}^{p-2} H^1(M_\theta,\mathbb{Z}). \end{aligned}$$

Suppose that (ii) is satisfied. Then $H^{2j+1}(M_0^+/\widehat{\theta},\mathbb{Z})\cong 0$, $1\leqslant j\leqslant p-1$ and $H^1(M_0^+/\widehat{\theta},\mathbb{Z})$ is finitely generated by (2.6) and $\mathrm{csr}(C(M_0^+/\widehat{\theta}))\leqslant 2$ by Lemma 1.1. So $\mathrm{K}^{-1}(M_0^+/\widehat{\theta})\cong U(C(M_0^+/\widehat{\theta}))\cong H^1(M_0^+/\widehat{\theta},\mathbb{Z})$ by Lemma 1.1 and Proposition 3.10 from [21]. Thus $\widehat{U}(\mathcal{D}(M_0,\theta))\cong 0$ by (2.7). Consequently, by Proposition 2.4, Lemma 2.1 and Proposition 1.7,

$$\begin{aligned}\mathrm{K}_1(\mathcal{D}(M,\theta))\cong U(\mathcal{D}(M,\theta))&\cong H^1(M/\theta,\mathbb{Z})\oplus\bigoplus_{j=0}^{p-2}H^1(M_\theta,\mathbb{Z})\\&\cong \mathrm{K}^{-1}(M/\theta)\oplus\bigoplus_{j=0}^{p-2}H^1(M_\theta,\mathbb{Z}).\qquad\blacksquare\end{aligned}$$

3. THE COMPUTATION OF $\mathrm{K}_0(\mathcal{D}(M,\theta))$ FOR CERTAIN (M,θ)

Let $\mathcal{A}$ be a unital C^*-algebra. For the projection $e\in\mathrm{M}_n(\mathcal{A})$, we write $[e]$ to denote the von Neumann-Murray equivalence class of $\mathrm{diag}\,(e,0)$ in $\mathrm{M}_m(\mathcal{A})$ for some $m\geqslant n$. Thus $\mathrm{K}_0(\mathcal{A})=\{[e]-[f]\mid e\in\mathrm{M}_n(\mathcal{A}),\ f\in\mathrm{M}_m(\mathcal{A})\text{ are projections}\}$. By Corollary 1.6, the pair (M,θ) with M compact and $M_\theta\neq\varphi$ raises following six-term exact sequence of K-groups (cf. Theorem 9.3.1 from [1])

$$\tag{3.1}\begin{array}{ccccc}\mathrm{K}_0(\mathcal{D}(M_0,\theta)) & \xrightarrow{\iota_*} & \mathrm{K}_0(\mathcal{D}(M,\theta)) & \xrightarrow{\pi_*} & \mathrm{K}_0\Big(\bigoplus\limits_{j=0}^{p-1}C(M_\theta)\Big)\\ \Big\uparrow{\scriptstyle\partial_2} & & & & \Big\downarrow{\scriptstyle\partial_1}\\ \mathrm{K}_1\Big(\bigoplus\limits_{j=0}^{p-1}C(M_\theta)\Big) & \xleftarrow{\pi_*} & \mathrm{K}_1(\mathcal{D}(M,\theta)) & \xleftarrow{\iota_*} & \mathrm{K}_1(\mathcal{D}(M_0,\theta)).\end{array}$$

Here ∂_1 is the exponential map and ∂_2 is the index map.

Let (M,θ) be the pair such that M is compact, $\dim M_\theta\leqslant 2$ and $H^2(M_\theta,\mathbb{Z})\cong 0$, $H^0(M_\theta,\mathbb{Z})\cong\mathbb{Z}^k$ $(1\leqslant k<\infty)$. Then $\mathrm{K}_0(C(M_\theta))\cong H^0(M_\theta,\mathbb{Z})$ by Theorem 1.2 from [22] and there exist k connected closed subsets $A_1,\dots,A_k$ in M_θ such that $M_\theta=\bigcup\limits_{j=1}^{k}A_i$ and $A_i\cap A_j=\varphi$, $i\neq j$. Set $h_j(x)=1$ when $x\in A_j$ and $h_j(x)=0$ when $x\in M_\theta/A_j$, $1\leqslant j\leqslant k$. Choose real functions $\widehat{h}_1,\dots,\widehat{h}_k$ in $C_\theta(M)$ such that $\sum\limits_{j=0}^{k}\widehat{h}_j=1$ and $\widehat{h}_j|M_\theta=h_j$, $1\leqslant j\leqslant k$. Put

$$e_{s,t}=[(\overbrace{0,\dots,0}^{s},h_t,\overbrace{0,\dots,0}^{p-1-s})]\in\mathrm{K}_0\Big(\bigoplus_{j=0}^{p-1}(C(M_\theta)\Big),\quad 0\leqslant s\leqslant p-1,\ 1\leqslant t\leqslant k.$$

Then $\{e_{s,t}\mid 0\leqslant s\leqslant p-1,\ 1\leqslant t\leqslant k\}$ forms a basis for $\mathrm{K}_0\Big(\bigoplus\limits_{j=0}^{p-1}C(M_\theta)\Big)$ and ∂_1 can be defined as

$$\tag{3.2}\partial_1(e_{s,t})=[\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_tP_s}]=[1-P_s+\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_s]$$

by 9.3.2 from [1], where $P_s = \Omega_p^* \operatorname{diag}(\overbrace{0,\dots,0}^{s},1,\overbrace{0,\dots,0}^{p-1-s})\Omega_p$ is a projection in $\mathcal{D}(M,\theta)$, $0 \leqslant s \leqslant p-1$, $t=1,\dots,k$. (Note that $P_sP_t=0$, $s\neq t$, $\sum_{s=0}^{p-1} P_s = 1$.)

LEMMA 3.1. *Suppose that M is compact and $M_\theta \neq \varphi$. Then for each $f \in \mathcal{U}(C(M_\theta))$, there is $F \in C_\theta(M)$ such that $F|M_\theta = f$ and $|F(x)| \leqslant 1$, $\forall x \in M$.*

Proof. Let $G \in C_\theta(M)$ such that $G|M_\theta = f$. Set $Z_G = \{x\in M \mid G(x)=0\}$. Since Z_G is closed in M and $Z_G \cap M_\theta = \varphi$, $\theta(Z_G)=Z_G$, it follows that there is a continuous function $h_0 : M \to [0,1]$ such that $h_0|Z_G = 1$ and $h_0|M_\theta = 0$.

Set $h(x) = \frac{1}{p}\sum_{j=0}^{p-1} h_0(\theta^j(x))$, $x\in M$. It is easy to check that $\theta(h)=h$, $0\leqslant h \leqslant 1$ and $h|Z_G = 1$, $h|M_\theta=0$. Therefore $F(x) = (|G(x)|+h(x))^{-1}G(x)$ verifies the assertion. ∎

Now let (M,θ) be a pair with M compact and $M_\theta\neq\varphi$, $\dim M_\theta \leqslant 2$. Then $U(C(M_\theta)) \cong \mathrm{K}_1(C(M_\theta))$ via $\mathrm{i}_{C(M_\theta)}$ by Lemma 1.1. Let $f_0,\dots,f_{p-1} \in \mathcal{U}\Big(\bigoplus_{j=0}^{p-1} C(M_\theta)\Big)$. Then there exist $F_0,\dots,F_{p-1}\in C_\theta(M)$ such that $F_j|M_\theta = f_j$ and $r_j(x) = |F_j(x)| \leqslant 1$, $j=0,\dots,p-1$, $x\in M$ by Lemma 3.1. Thus

$$w = \begin{bmatrix} \sum_{j=0}^{p-1} F_jP_j & \mathrm{i}\sum_{j=0}^{p-1}\sqrt{1-r_j^2}P_j \\ \mathrm{i}\sum_{j=0}^{p-1}\sqrt{1-r_j^2}P_j & \sum_{j=0}^{p-1} F_j^*P_j \end{bmatrix} \in \mathcal{U}(M_2(\mathcal{D}(M,\theta))$$

and $\pi_2(w) = \operatorname{diag}((f_0,\dots,f_{p-1}),(f_0^*,\dots,f_{p-1}^*))$. So by 8.3.1 from [1], ∂_2 can be expressed as

$$\partial_2([(f_0,\dots,f_{p-1})]) = \left[\begin{bmatrix} \sum_{j=0}^{p-1} r_j^2P_j & -\mathrm{i}\sum_{j=0}^{p-1}\sqrt{1-r_j^2}F_jP_j \\ \mathrm{i}\sum_{j=0}^{p-1}\sqrt{1-r_j^2}F_j^*P_j & \sum_{j=0}^{p-1}(1-r_j^2)P_j \end{bmatrix}\right] - [q_1]. \tag{3.3}$$

LEMMA 3.2. *Let the pair (M,θ) satisfy the conditions:*
(i) *M is connected and compact with $M_\theta\neq\varphi$, $\dim M_\theta\leqslant 2$ and $H^2(M_\theta,\mathbb{Z})\cong 0$;*
(ii) *$\widehat{U}(\mathcal{D}(M_0,\theta))\cong 0$ and $H^0(M_\theta,\mathbb{Z})\cong\mathbb{Z}^k$.*
Then $\operatorname{Ker}\partial_1 \cong \mathbb{Z}\oplus\bigoplus_{j=0}^{p-2} H^0(M_\theta,\mathbb{Z})$.

Proof. Since for $t=1,\dots,k$, $s_1\neq s_2$, $s_1,s_2 = 0,\dots,p-1$,

$$(\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_{s_1}+1-P_{s_1})(\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_{s_2}+1-P_{s_2})^* = 1-P_{s_1}-P_{s_2}+\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_{s_1}+\mathrm{e}^{-2\pi\mathrm{i}\widehat{h}_t}P_{s_2}$$

is in $\widehat{\mathcal{U}}((\mathcal{D}(M_0,\theta))^+)$, and $\widehat{U}(\mathcal{D}(M_0,\theta))\cong 0$, it follows that in $U(\mathcal{D}(M_0,\theta))$

$$[\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_{s_1}+1-P_{s_1}] = [\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_{s_2}+1-P_{s_2}],\quad s_1\neq s_2,\ t=1,\dots,k$$

and hence $\partial_1(e_{s,t})=[\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_0+1-P_0]$, $0\leqslant s\leqslant p-1$, $1\leqslant t\leqslant k$ by (3.2).

Let $a\in \mathrm{K}_0\Big(\bigoplus\limits_{j=0}^{p-1}C(M_\theta)\Big)$ such that $\partial_1(a)=0$. Since a can be expressed as $a=\sum\limits_{t=1}^{k}\sum\limits_{s=0}^{p-1}\lambda_{s,t}e_{s,t}$, $\lambda_{s,t}\in\mathbb{Z}$, we obtain that

$$0=\partial_1(a)=\sum_{t=1}^{k}\sum_{s=0}^{p-1}\lambda_{s,t}[\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}P_0+1-P_0]=\Big[\mathrm{e}^{2\pi\mathrm{i}\sum\limits_{t=1}^{k}\sum\limits_{s=0}^{p-1}\lambda_{s,t}\widehat{h}_t}P_0+1-P_0\Big]$$

in $U(\mathcal{D}(M_0,\theta))$. Consequently,

$$\mathrm{Det}\Big(\mathrm{e}^{2\pi\mathrm{i}\sum\limits_{t=1}^{k}\sum\limits_{s=0}^{p-1}\lambda_{s,t}\widehat{h}_t}P_0+1-P_0\Big)=\mathrm{e}^{2\pi\mathrm{i}\sum\limits_{t=1}^{k}\sum\limits_{s=0}^{p-1}\lambda_{s,t}\widehat{h}_t}\in\mathcal{U}_0((C_\theta(M_0))^+)$$

(here we identify $(C_\theta(M_0))^+$ with $\{f\in C_\theta(M)\mid f|M_\theta\equiv \text{constant}\}$) and there is a continuous function $h:M\to\mathbb{R}$ with $\theta(h)=h$ and $h|M_\theta\equiv k_0\in\mathbb{Z}$ such that $\mathrm{e}^{2\pi\mathrm{i}\sum\limits_{t=1}^{k}\Big(\sum\limits_{s=0}^{p-1}\lambda_{s,t}\Big)\widehat{h}_t}=\mathrm{e}^{2\pi\mathrm{i}h}$. Combining this identity with the assumption that M is connected and $h|M_\theta\equiv k_0$, $\widehat{h}_j|A_j=1$, $\widehat{h}_j|M_\theta\backslash A_j=0$, $j=1,\dots,k$, we have that there exists $n\in\mathbb{Z}$ such that $\sum\limits_{s=0}^{p-1}\lambda_{s,t}=n$, $t=1,\dots,k$. So

$$\begin{aligned}\mathrm{Ker}\,\partial_1&=\Big\{n_0\sum_{t=1}^{k}e_{0,t}+\sum_{t=1}^{k}\sum_{s=1}^{p-1}n_{s,t}(e_{s,t}-e_{0,t})\mid n_0,n_{s,t}\in\mathbb{Z}\Big\}\\&\cong\mathbb{Z}\oplus\bigoplus_{j=0}^{p-2}H^0(M_\theta,\mathbb{Z}).\quad\blacksquare\end{aligned}$$

Suppose that M is compact, $\dim M_\theta\leqslant 2$ and $H^2(M_\theta,\mathbb{Z})\cong 0$, $H^0(M_\theta,\mathbb{Z})\cong\mathbb{Z}^k$. Consider the six-term exact sequence of the triple $(C_0(M_0/\theta),C(M/\theta),C(M_\theta))$

$$\begin{array}{ccccc}\mathrm{K}_0(C_\theta(M_0)) & \xrightarrow{j_1} & \mathrm{K}_0(C_\theta(M)) & \xrightarrow{j_2} & \mathrm{K}_0(C(M_\theta))\\ \uparrow\partial_0 & & & & \downarrow\partial_0'\\ \mathrm{K}_1(C(M_\theta)) & \longleftarrow & \mathrm{K}_1(C_\theta(M)) & \longleftarrow & \mathrm{K}_1(C_\theta(M_0))\end{array}\tag{3.4}$$

where ∂_0 is the index map given by

$$\partial_0([f])=\left[\begin{bmatrix} r^2 & -\mathrm{i}\sqrt{1-r^2}F\\ \mathrm{i}\sqrt{1-r^2}F^* & 1-r^2\end{bmatrix}\right]-[q_1]\in\mathrm{K}_0(C_0(M_0/\theta)),$$

$f\in\mathcal{U}(C(M_\theta))$, $F\in C_\theta(M)$ with $F|M_\theta=f$ and $0\leqslant r(x)=|F(x)|\leqslant 1$; ∂_0' is given by $\partial_0'([h_t])=[\mathrm{e}^{2\pi\mathrm{i}\widehat{h}_t}]$, $t=1,\dots,k$ (see (3.1) and (3.2)).

Analogous to the proof of last paragraph of Lemma 3.2, we have:

LEMMA 3.3. *Let* (M,θ) *be a pair satisfying the following conditions:*
(i) M *is a connected, compact space with* $M_\theta \neq \varphi$;
(ii) $\dim M_\theta \leqslant 2$, $H^2(M_\theta,\mathbb{Z}) \cong 0$ *and* $H^0(M_\theta,\mathbb{Z}) \cong \mathbb{Z}^k$.
Then $\operatorname{Im} j_2 = \operatorname{Ker} \partial_0' = \{n[1] \mid n \in \mathbb{Z}\}$.

For the pair (M,θ) with M compact and $M_\theta \neq \varphi$, define the $*$-homomorphism Ψ of $C_\theta(M_0)$ to $\mathcal{D}(M_0,\theta)$ by $\Psi(f) = fP_{p-1}$. Then Ψ can be extended to the homomorphism of $(C_\theta(M_0))^+$ to $(\mathcal{D}(M_0,\theta))^+$ by $\Psi(f) = fP_{p-1}+f(M_\theta)(1-P_{p-1})$.

LEMMA 3.4. *The induced homomorphism* $\Psi_*: \mathrm{K}_0(C_\theta(M_0)) \to \mathrm{K}_0(\mathcal{D}(M_0,\theta))$ *is isomorphic.*

Proof. Simple computation shows that $P_{p-1}\mathcal{D}(M_0,\theta)P_{p-1} = \{fP_{p-1} \mid f \in C_\theta(M_0)\}$. This means that Ψ is an isomorphism of $C_\theta(M_0)$ onto $P_{p-1}\mathcal{D}(M_0,\theta)P_{p-1}$. So, in order to show that Ψ_* is isomorphic, we need only to prove that the induced homomorphism k_* of the inclusion map $k: P_{p-1}\mathcal{D}(M_0,\theta)P_{p-1} \to \mathcal{D}(M_0,\theta)$ is an isomorphism of $\mathrm{K}_0(P_{p-1}\mathcal{D}(M_0,\theta)P_{p-1})$ to $\mathrm{K}_0(\mathcal{D}(M_0,\theta))$.

Let $\mathcal{A}$ be the C^*-subalgebra of $\mathcal{D}(M_0,\theta)$ generated by $\mathcal{D}(M_0,\theta)P_{p-1}\mathcal{D}(M_0,\theta)$. Since $\pi_x(\mathcal{D}(M_0,\theta)) = \mathrm{M}_p(\mathbb{C})$ by Lemma 1.4 and $\mathrm{M}_p(\mathbb{C})P_{p-1}\mathrm{M}_p(\mathbb{C}) = \mathrm{M}_p(\mathbb{C})$ ($\mathrm{M}_p(\mathbb{C})$ is a simple C^*-algebra), it follows that $\pi_x|\mathcal{A}$ is irreducible for every $x \in M_0$ and $\pi_{x_1}|\mathcal{A}$ is not equivalent to $\pi_{x_2}|\mathcal{A}$ when $P(x_1) \neq P(x_2)$ in M_0/θ. So $\mathcal{A} = \mathcal{D}(M_0,\theta)$ by Lemma 11.1.4 from [4], that is, P_{p-1} is a full projection in the sense of [3]. Therefore k_* is an isomorphism by Corollary 2.6 from [3]. ∎

LEMMA 3.5. *Consider a pair with* M *compact and* θ *regular, and* $\dim M_\theta \leqslant 2$. *Then* $\operatorname{Im}\partial_2 \subset \operatorname{Im}\Psi_*$ *and the diagram*

$$\begin{array}{ccc} \mathrm{K}_1(C(M_\theta)) & \xrightarrow{\partial_0} & \mathrm{K}_0(C_\theta(M_0)) \\ \Big\downarrow K & & \Psi_* \Big\downarrow \\ \mathrm{K}_1\Big(\bigoplus_{j=0}^{p-1} C(M_\theta)\Big) & \xrightarrow{\partial_2} & \mathrm{K}_0(\mathcal{D}(M_0,\theta)) \end{array} \tag{3.5}$$

is commutative, where $K([f]) = [(1,\ldots,1,f)]$, $f \in \mathcal{U}(C(M_\theta))$.

Proof. The second assertion comes from the definition of K, Ψ_*, ∂_2 and ∂_0.

Suppose that $a = \partial_2([f_0,\ldots,f_{p-1}])$ for some $f_0,\ldots,f_{p-1} \in \mathcal{U}(C(M_\theta))$. Since θ is regular, it follows from (2.3) that there are $u_0,\ldots,u_{p-2} \in \mathcal{U}(\mathcal{D}(M,\theta))$ such that $\pi(u_j) = (\overbrace{1,\ldots,1}^{j}, f_j, \overbrace{1,\ldots,1}^{p-2-j}, f_j^*) \in \bigoplus_{j=0}^{p-1} \mathcal{U}(C(M_\theta))$, $0 \leqslant j \leqslant p-2$. So

$$\begin{aligned} a &= \partial_2([(f_0,1,\ldots,1,f^*)]) + \cdots + \partial_2([(1,\ldots,1,f_{p-2},f_{p-2}^*)]) \\ &\quad + \partial_2 \circ \mathrm{K}([f_0\cdots f_{p-1}]) \\ &= \partial_2 \circ \mathrm{K}([f_0\cdots f_{p-1}]) + \partial_2 \circ \pi_*([u_0]) + \cdots + \partial_2 \circ \pi_*([u_{p-2}]) \\ &= \partial_2 \circ \mathrm{K}([f_0\cdots f_{p-1}]) = \Psi_* \circ \partial_0([f_0\cdots f_{p-1}]). \end{aligned} \tag{3.6}$$

(for $\partial_2 \circ \pi_* = 0$ by (3.1)). ∎

The following theorem demonstrates what $\mathrm{K}_0(\mathcal{D}(M,\theta))$ is.

THEOREM 3.6. *Let (M,θ) satisfy the following conditions:*
(i) *M is connected, compact with $M_\theta \neq \varphi$, $\dim M_\theta \leqslant 2$, $H^2(M_\theta,\mathbb{Z}) \cong 0$;*
(ii) *$\mathrm{K}_0(C(M/\theta))$ and $H^0(M_\theta,\mathbb{Z})$ are all finitely generated and θ is regular;*
(iii) *M_θ is connected or $H^{2j+1}(M/\theta,\mathbb{Z}) \cong 0$, $1 \leqslant j \leqslant p-1$.*
Then $\mathrm{K}_0(\mathcal{D}(M,\theta)) \cong \mathrm{K}^0(M/\theta) \oplus \bigoplus_{j=0}^{p-2} H^0(M_\theta,\mathbb{Z})$.

Proof. Let $\{e_j\}_1^N$ be the sequence of generators in $\mathrm{K}_0(C_\theta(M))$ other than $[1]$ such that the set $\{[1], e_1, \dots, e_t\}$ is independent and $\{e_j\}_{j=t+1}^N$ is the set of all torsion elements in $\{e_j\}_1^N$. Thus

$$\mathrm{K}_0(C_\theta(M)) = \Big\{ n[1] + \sum_{j=1}^N \lambda_j e_j \mid n, \lambda_j \in \mathbb{Z} \Big\}. \tag{3.7}$$

The proof of the assertion consists of the following steps:

STEP 1. We claim that

$$\operatorname{Im} j_1 = \operatorname{Ker} j_2 = \Big\{ \sum_{j=1}^t \lambda_j (e_j - n_j[1]) + \sum_{j=t+1}^N \lambda_j e_j \mid \lambda_j \in \mathbb{Z} \Big\}, \tag{3.8}$$

where $n_j[1] = j_2(e_j)$, $j = 1, \dots, t$. Since $\mathrm{K}_0(C(M_\theta)) \cong H^0(M_\theta,\mathbb{Z})$ is torsion-free, $j_2(e_j) = 0$, $t+1 \leqslant j \leqslant N$ by (3.4). Now, by Lemma 3.3, we can choose $n_j \in \mathbb{Z}$ such that $j_2(e_j) = n_j[1]$, $1 \leqslant j \leqslant t$. Noting that $j_2([1]) = [1]$, we have

$$\Big\{ \lambda_1(e_1 - n_1[1]) + \cdots + \lambda_t(e_t - n_t[1]) + \sum_{j=t+1}^N \lambda_j e_j \mid \lambda_j \in \mathbb{Z} \Big\} \subset \operatorname{Ker} j_2.$$

On the other hand, let $a = n[1] + \sum_{j=1}^N \lambda_j e_j \in \operatorname{Ker} j_2$. Then $n = -\sum_{j=1}^t \lambda_j n_j$. Thus a can be written as

$$a = \sum_{j=1}^t \lambda_j (e_j - n_j[1]) + \sum_{j=t+1}^N \lambda_j.$$

Equation (3.7) is proven.

STEP 2. We have that $\mathrm{K}_0(\mathcal{D}(M,\theta))/\operatorname{Im} \partial_2 \cong \widetilde{\mathrm{K}}^0(M/\theta)$. To do this, we take $\eta_j \in \mathrm{K}_0(C_\theta(M))$ such that

$$j_1(\eta_j) = e_j - n_j[1],\ 1 \leqslant j \leqslant t \quad \text{and} \quad j_1(\eta_j) = e_j,\ t+1 \leqslant j \leqslant N. \tag{3.9}$$

Put $\xi_j = \Psi_*(\eta_j)$, $j = 1, \dots, N$. Then we can conclude from the identity $\operatorname{Im} \partial_0 = \operatorname{Ker} j_1$, Lemma 3.4 and Lemma 3.5 that

(A) $\lambda \xi_j \notin \operatorname{Im} \partial_2$, $\forall \lambda \in \mathbb{Z} \backslash \{0\}$, $1 \leqslant j \leqslant t$;
(B) $\xi_j \notin \operatorname{Im} \partial_2$ and $k_j \xi_j \in \operatorname{Im} \partial_2$ iff $k_j e_j = 0$, $k_j \in \mathbb{Z}$, $t+1 \leqslant j \leqslant N$ and
(C) if there exist $\lambda_1, \dots, \lambda_t \in \mathbb{Z}$ such that $\sum_{j=1}^t \lambda_j \xi_j \in \operatorname{Im} \partial_2$, then $\lambda_j = 0$.

Now, for each $a \in \operatorname{Im}\Psi_*$, there is by Lemma 3.4 and Lemma 3.5 a unique $b \in \mathrm{K}_0(C_0(M_0/\theta))$ such that $a = \Psi_*(b)$, since

$$j_1(b) = \sum_{j=1}^{t} \lambda_j(e_j - n_j[1]) + \sum_{j=t+1}^{N} \lambda_j e_j$$

for some $\lambda_1, \dots, \lambda_N \in \mathbb{Z}$ by (3.8). Therefore there exists $c \in \mathrm{K}_1(C(M_\theta))$ such that $b - \sum_{j=1}^{N} \lambda_j\eta_j = \partial_0(c)$ by (3.4) and (3.9) and hence $a = \sum_{j=1}^{N} \lambda_j\xi_j + \partial_2(K(c))$ by Lemma 3.5. So from (A), (B), (C) and Lemma 3.4, we obtain that

$$\mathrm{K}_0(\mathcal{D}(M,\theta))/\operatorname{Im}\partial_2 \cong \widetilde{\mathrm{K}}_0(C_\theta(M)) \cong \widetilde{\mathrm{K}}^0(M/\theta).$$

STEP 3. By (3.1), we have

$$\operatorname{Ker}\pi_* = \operatorname{Im} l_* \cong \mathrm{K}_0(\mathcal{D}(M_0,\theta))/\operatorname{Ker} l_* \cong \widetilde{\mathrm{K}}_0(M/\theta).$$

So if M_θ is connected, $\partial_1 = 0$ by (3.2) and furthermore

$$\mathrm{K}_0(\mathcal{D}(M,\theta)) \cong \operatorname{Ker}\pi_* \oplus \operatorname{Im}\pi_* \cong \mathrm{K}^0(M/\theta) \oplus \mathbb{Z}^{p-1};$$

if $H^{2j+1}(M/\theta,\mathbb{Z}) \cong 0$, $1 \leqslant j \leqslant p-1$, then by the proof of Theorem 2.6, $U(\mathcal{D}(M_0,\theta)) \cong 0$ and hence by Lemma 3.2,

$$\mathrm{K}_0(\mathcal{D}(M,\theta)) \cong \operatorname{Ker}\pi_* \oplus \operatorname{Im}\pi_* \cong \mathrm{K}^0(M/\theta) \oplus \bigoplus_{j=0}^{p-2} H^0(M_\theta,\mathbb{Z}). \quad ∎$$

4. EXAMPLES

We realize that the notions "regular" or "strongly regular" self-homeomorphism play a very important role in the computation of $\mathrm{K}_i(\mathcal{D}(M,\theta))$, $i = 0,1$. The following proposition shows when θ is regular or strongly regular.

PROPOSITION 4.1. *Let (M,θ) be a pair with M compact and $M_\theta \neq \varphi$. If (M,θ) satisfies* (i) *or* (ii)*, then θ is regular and if (M,θ) satisfies* (iii)*, then θ is strongly regular:*

(i) *M is a 2-dimensional manifold and θ is self-differomorphic;*

(ii) *$i^* : H^1(M/\theta,\mathbb{Z}) \to H^1(M_\theta,\mathbb{Z})$ is surjective, where i^* is the induced homomorphism of the inclusion map $i : M_\theta \to M/\theta$;*

(iii) *$M \subset \mathbb{C}$ and the zero-points of $h_\theta(x) = \sum_{j=0}^{p-1} \omega^{p-1-j}\theta^j(x)$ is the set M_θ.*

Proof. Assume that (i) holds. Let $f_0, \dots, f_{p-2} \in \mathcal{U}(C(M_\theta))$. Then there exist $H_0, \dots, H_{p-2} \in C(M)$ such that $H_j|M_\theta = f_j$, $0 \leqslant j \leqslant p-2$. Set $\widehat{H}_j = \frac{1}{p}\sum_{k=0}^{p-1} \theta^k(H_j)$, $0 \leqslant j \leqslant p-2$. Then $\widehat{H}_j \in C_\theta(M)$ and $\widehat{H}_j|M_\theta = f_j$, $0 \leqslant j \leqslant p-2$. Since M is a compact manifold, we can find differentiable functions $\widetilde{H}_0, \dots, \widetilde{H}_{p-2} \in C_\theta(M)$ such that $\|\widehat{H}_j - \widetilde{H}_j\| < 1/2$, $0 \leqslant j \leqslant p-2$ (cf. Theorem 2.3.3 from [9]).

Now by Sard's Theorem (6.1 from [2]), we can choose a regular value a_j of $\widetilde{H}_j : M \to \mathbb{C}$ such that $|a_j| < 1/2$, $0 \leqslant j \leqslant p-2$. Set $G_j(x) = \widetilde{H}_j(x) - a_j$, $0 \leqslant j \leqslant p-2$, $x \in M$. Then $\|H_j - G_j\| < 1$ and $G_j^{-1}(0)$ is either empty or finite (by Lemma 5.9 from [2]), $0 \leqslant j \leqslant p-2$, for $\dim M = \dim \mathbb{C} = 2$.

Set $G(x) = \prod_{j=0}^{p-2} G_j(x)$, $x \in M$. Then $G^{-1}(0)$ is either empty or finite and $\theta(G^{-1}(0)) = G^{-1}(0)$, $G^{-1}(0) \cap M_\theta = \varphi$. If $G^{-1}(0) = \varphi$, we take $G_\theta(x) = 0$, $\forall x \in M$; if $G^{-1}(0)$ finite, we can pick a function K_0 on $G^{-1}(0)$ such that $\sum_{j=0}^{p-1} \omega^{p-1-j} K_0(\theta^j(x)) \neq 0$, $\forall x \in G^{-1}(0)$. Let $\widetilde{K} \in C(M)$ such that $\widetilde{K}|G^{-1}(0) = K_0$ and set $G_\theta(x) = \sum_{j=0}^{p-1} \omega^{p-1-j} \widetilde{K}(\theta^j(x))$, $x \in M$. Then $G_\theta|G^{-1}(0) = K_0 \neq 0$ and $\theta(G_\theta) = \omega G_\theta$.

Note that $\|H_j - G_j\| < 1$ implies $\|f_j - G_j|M_\theta\| < 1$, $0 \leqslant j \leqslant p-2$. Thus there is $h_j \in C(M_\theta)$ such that $f_j = \mathrm{e}^{h_j} G_j|M_\theta$, $0 \leqslant j \leqslant p-2$. Let $\widetilde{h}_j \in C_\theta(M)$ such that $\widetilde{H}_j|M_\theta = h_j$ and set $F_j = \mathrm{e}^{\widetilde{h}_j} G_j$, $0 \leqslant j \leqslant p-2$. Then by the above argument, $F_j|M_\theta = f_j$, $0 \leqslant j \leqslant p-2$ and $\theta(G_\theta) = \omega G_\theta$, $\left|\prod_{j=0}^{p-2} F_j(x)\right| + |G_\theta(x)| \neq 0$, $\forall x \in M$, i.e., θ is regular.

By Corollary VIII. 2 from [10], condition (ii) is equivalent to the statement "Every $f \in \mathcal{U}(C(M_\theta))$ has a continuous extension $F : M \to \mathbf{S}^1$ with $\theta(F) = F$". Take $G_\theta = 0$ in Definition 2.3. We see that θ is regular.

Let h_θ be as in condition (iii). Since $M_\theta = \{x \in M \mid h_\theta(x) = 0\}$ and $\theta(h_\theta) = \omega h_\theta$, it follows that θ is strongly regular. ∎

REMARK 4.2. It is easy to verify that if $p = 2$ and $M \subset \mathbb{C}$, then condition (iii) of Proposition 4.1 is satisfied. We see that if $\dim M \leqslant 1$, then condition (ii) of Proposition 4.1 is also satisfied by Lemma 1.3 and Theorem 3.2.10 from [6].

EXAMPLE 4.3. Let $M = \mathbf{S}^1 \times \mathbf{S}^1 = \{(z_1, z_2) \,\big|\, |z_1| = |z_2| = 1\}$ and $\theta(z_1, z_2) = (z_2, z_1)$. Then $M_\theta = \{(z, z) \mid \forall z \in \mathbf{S}^1\} \cong \mathbf{S}^1$ and θ is regular by Proposition 4.2 (i). We will show that $M/\theta \cong \mathbf{S}^1 \times [0, 1]$.

Set $S = \{(z_1 z_2, z_1 + z_2) \mid z_1, z_2 \in \mathbf{S}^1\}$. Then it is easy to check that $M/\theta \cong S$ by the homeomorphic map $\beta(\langle z_1, z_2\rangle) = (z_1 z_2, z_1 + z_2)$, where $\langle z_1, z_2\rangle = P(z_1, z_2)$.

Define the continuous map $\Gamma : \mathbf{S}^1 \times [0, 1] \to S$ by $\Gamma(z, t) = (z^2, 2zt)$. (Here $z_1 = (t + \mathrm{i}\sqrt{1 - t^2})z$, $z_2 = (t - \mathrm{i}\sqrt{1 - t^2})z$.) Obviously, Γ is injective. Now, for $z_1, z_2 \in \mathbf{S}^1$ there is $z \in \mathbf{S}^1$ such that $z^2 = z_1 z_2$. Thus

$$z_1 + z_2 = z_1 + \overline{z}_1 z^2 = \begin{cases} z(\overline{z} z_1 + z\overline{z}_1) & \text{if } \overline{z} z_1 + z\overline{z}_1 \geqslant 0, \\ -z(-\overline{z} z_1 - z\overline{z}_1) & \text{if } \overline{z} z_1 + z\overline{z}_1 < 0. \end{cases}$$

This implies that Γ is also surjective.

Finally, from Theorem 2.6 and Theorem 3.6, we get that

$$\mathrm{K}_0(\mathcal{D}(M, \theta)) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad \mathrm{K}_1(\mathcal{D}(M, \theta)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

EXAMPLE 4.4. Let $M=\mathbf{S}^2=\{(x,z)\in[-1,1]\times\mathbb{C}\mid x^2+|z|^2=1\}$ and $\theta(x,z)=(x,\mathrm{e}^{2\pi\mathrm{i}/3}z)$. Then $M_\theta=\{(-1,0),(1,0)\}$ and θ is regular by Proposition 4.1 (ii). Define the homeomorphic map $\beta: M_0/\theta\to(-1,1)\times\mathbf{S}^1$ by

$$\beta(\langle x,z\rangle)=(x,(1-x^2)^{-\frac{3}{2}}z^3)$$

where $\langle x,z\rangle=P(x,z)$, $(x,z)\in\mathbf{S}^2$. So $M_0^+/\widehat{\theta}\cong((-1,1)\times\mathbf{S}^1)^+\cong(\mathbf{S}^1\times\mathbb{R}^1)^+$. Since $H^1(\mathbf{S}^2,\mathbb{Z})\cong0$, we have $H^1(M/\theta,\mathbb{Z})\cong0$ so that $\mathrm{K}^{-1}(M/\theta)\cong0$. Therefore $\mathrm{K}^0(M/\theta)\cong\mathbb{Z}^2$ by (3.4). Finally, by Theorem 2.6 and Theorem 3.6,

$$\mathrm{K}_0(\mathcal{D}(M,\theta))\cong\mathbb{Z}^6,\quad \mathrm{K}_1(\mathcal{D}(M,\theta))\cong0.$$

Acknowledgements. The author wishes to thank the referee for his or her helpful suggestions. The author also wishes to thank Professor Zongyao Wang for his great help in revising the paper.

The work was supported by the National Sciences Foundation, P.R. China.

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Received February 10, 1999; revised May 14, 1999.