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PENTAGON SUBSPACE LATTICES ON BANACH SPACES

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ABSTRACT. If K, L and M are (closed) subspaces of a Banach space X satisfying $K \cap M = (0)$, $K \vee L = X$ and $L \subset M$, then $\mathcal{P} = \{(0), K, L, M, X\}$ is a *pentagon subspace lattice on* X. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are pentagons, every (algebraic) isomorphism $\varphi : \operatorname{Alg}\mathcal{P}_1 \to \operatorname{Alg}\mathcal{P}_2$ is quasi-spatial. The SOT-closure of the fin- ite rank subalgebra of $\operatorname{Alg}\mathcal{P}$ is $\{T \in \operatorname{Alg}\mathcal{P} : T(M) \subseteq L\}$. On separable Hilbert space H every positive, injective, non-invertible operator A and every non-zero subspace M satisfying $M \cap \operatorname{Ran}(A) = (0)$ give rise to a pentagon $\mathcal{P}(A; M)$. $\operatorname{Alg}\mathcal{P}(A; M)$ and $\operatorname{Alg}\mathcal{P}(B; N)$ are spatially isomorphic if and only if $T \operatorname{Ran}(A) = \operatorname{Ran}(B)$ and $T(M) = N$ for an invertible operator $T \in B(H)$. If $\mathcal{A}(A)$ is the set of operators leaving $\operatorname{Ran}(A)$ invariant, every isomorphism $\varphi : \mathcal{A}(A) \to \mathcal{A}(B)$ is implemented by an invertible operator T satisfying $T \operatorname{Ran}(A) = \operatorname{Ran}(B)$.

KEYWORDS: *algebraic isomorphism, invariant operator range, gap-dimension.*

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1. INTRODUCTION AND PRELIMINARIES

In abstract lattice theory two five-element lattices, the pentagon and the double triangle, play special roles. (A lattice is modular if and only if it has no pentagon sublattice; it is distributive if and only if it has neither a pentagon nor a double triangle sublattice. See [2].) Each of these lattices consists of a least element 0 and a greatest element 1 together with elements a, b and c satisfying

$$a \wedge b = b \wedge c = c \wedge a = 0 \quad \text{and} \quad a \vee b = b \vee c = c \vee a = 1$$

in the double triangle, and

$$a \wedge c = 0, \quad a \vee b = 1 \quad \text{and} \quad b < c$$

in the pentagon. Double triangle subspace lattices (on reflexive Banach space) were studied in [13]. Here we turn our attention to pentagons. More precisely, we study realizations of the pentagon as a lattice $\mathcal{P}$ of (closed) subspaces of complex

Banach space. The study of such realizations was begun by Halmos in 1971 ([6]). Apart from the facts that reflexive ([6]) and non-reflexive ([15], [19]) pentagon subspace lattices exist, and that $\operatorname{Alg}\mathcal{P}$ is always a semisimple Banach algebra ([8], [17]), not a lot is known about them. Below we present a more systematic study of them and, not surprisingly, several apparently difficult problems are isolated in the process. All of these problems concern various non-self-adjoint operator algebras and the possible existence of algebraic or spatial isomorphisms between them.

Throughout what follows X will denote a non-zero complex Banach space with (topological) dual X^*. Also, H will denote a non-zero complex Hilbert space on which the inner-product will be denoted by $(\cdot \mid \cdot)$. The notation "$\subset$" is reserved for *strict* set inclusion. The terms "operator" and "subspace" will mean "bounded linear transformation" and "norm closed linear manifold", respectively. As usual, if X_1 and X_2 are Banach spaces, $B(X_1, X_2)$ denotes the set of all operators $T : X_1 \to X_2$ and $B(X, X)$ is denoted simply by $B(X)$. If L and M are subspaces of X satisfying $L \subseteq M$, M/L denotes the quotient Banach space with norm $\|x + L\| = \inf\limits_{y\in L} \|x - y\|$ $(x \in M)$. If $n \in \mathbb{Z}^+$, $X^{(n)}$ denotes the Banach space consisting of the direct sum of n copies of X normed by $\|(x_i)_1^n\| = \Big(\sum\limits_{i=1}^{n} \|x_i\|^2\Big)^{1/2}$. The Banach space $X^{(\infty)}$ is defined as $\Big\{(x_i)_1^\infty : x_i \in X \ (i \in \mathbb{Z}^+) \text{ and } \sum\limits_{i=1}^{\infty} \|x_i\|^2 < \infty\Big\}$ with the usual operations and norm $\|x\| = \Big(\sum\limits_{i=1}^{\infty} \|x_i\|^2\Big)^{1/2}$. If $n \in \mathbb{Z}^+ \cup \{\infty\}$ and $A \in B(X)$, $A^{(n)}$ denotes the operator on $X^{(n)}$ defined by $A^{(n)}(x_i)_i^n = (Ax_i)_1^n$.

If $e, f, g, \ldots$ are vectors of X, then $\langle e, f, g, \ldots\rangle$ denotes their linear span. If $\{L_\gamma\}_\Gamma$ is a family of subspaces of X, $\bigvee\limits_\Gamma L_\gamma$ denotes the closed linear span of $\bigcup\limits_\Gamma L_\gamma$. For any vectors $f \in X$ and $e^* \in X^*$, $e^* \otimes f$ denotes the operator on X given by $(e^* \otimes f)x = e^*(x)f$ $(x \in X)$. If $T \in B(X)$, T^* denotes the adjoint of T, $\operatorname{Ran}(T)$ denotes the range of T and $G(T)$ denotes the graph of T, that is $G(T) = \{(x, Tx) : x \in X\}$. Also $T|K$ denotes the restriction of T to the subspace K. For any non-empty subset $Y \subseteq X$, $Y^\perp$ denotes its annihilator, that is, $Y^\perp = \{e^* \in X^* : e^*(y) = 0, \text{ for every } y \in Y\}$. For any non-empty subset $\mathcal{Z} \subseteq X^*, {}^\perp\mathcal{Z}$ denotes its pre-annihilator, that is, ${}^\perp\mathcal{Z} = \{f \in X : x^*(f) = 0, \text{ for every } x^* \in \mathcal{Z}\}$. Then $X^\perp = (0)$, $(0)^\perp = X^*$ and ${}^\perp(Y^\perp) = Y$, for every subspace $Y \subseteq X$.

A *subspace lattice on* X is a family $\mathcal{L}$ of subspaces of X satisfying

(i) $(0), X \in \mathcal{L}$ and
(ii) $\bigcap\limits_\Gamma L_\gamma \in \mathcal{L}$, $\bigvee\limits_\Gamma L_\gamma \in \mathcal{L}$, for every family $\{L_\gamma\}_\Gamma$ of elements of $\mathcal{L}$.

For any family $\mathcal{F}$ of subspaces of X we define $\operatorname{Alg}\mathcal{F}$, as usual, by

$$\operatorname{Alg}\mathcal{F} = \{T \in B(X) : T(L) \subseteq L, \text{ for every } L \in \mathcal{F}\}.$$

Then $\operatorname{Alg}\mathcal{F}$ is a unital algebra and is closed in the strong operator topology. For any subset $\mathcal{A} \subseteq B(X)$ we define $\operatorname{Lat}\mathcal{A}$, as usual, by

$$\operatorname{Lat}\mathcal{A} = \{L : L \text{ is a subspace of } X \text{ and } T(L) \subseteq L, \text{ for every } T \in \mathcal{A}\}.$$

Then $\operatorname{Lat}\mathcal{A}$ is a subspace lattice. Also, $\mathcal{F}\subseteq\operatorname{Lat}\operatorname{Alg}\mathcal{F}$, $\mathcal{A}\subseteq\operatorname{Alg}\operatorname{Lat}\mathcal{A}$, $\operatorname{Alg}\operatorname{Lat}\operatorname{Alg}\mathcal{F} = \operatorname{Alg}\mathcal{F}$ and $\operatorname{Lat}\operatorname{Alg}\operatorname{Lat}\mathcal{A} = \operatorname{Lat}\mathcal{A}$. A subspace lattice $\mathcal{L}$ on X is *reflexive* if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$, equivalently, if $\mathcal{L} = \operatorname{Lat}\mathcal{A}$ for some subset $\mathcal{A}\subseteq B(X)$.

If K and L are non-trivial subspaces of X that are (topologically) complementary, that is, $K\cap L = (0)$ and $K\vee L = X$, then $\mathcal{D} = \{(0), K, L, X\}$ is an *atomic Boolean subspace lattice on* X *with* 2 *atoms* (K and L); abbreviated 2-atom ABSL.

A subset $\mathcal{S}$ of H is an *operator range* if $\mathcal{S} = \operatorname{Ran}(T)$ for some operator $T\in B(H)$. Most of the basic results concerning operator ranges that we use are to be found in [3]. In particular, Douglas' theorem ([3], Theorem 2.1) states that, for every pair A, B of operators on H, $\operatorname{Ran}(A)\subseteq\operatorname{Ran}(B)$ if and only if $A = BC$ for some operator $C\in B(H)$. It easily follows that an operator $T\in B(H)$ maps $\operatorname{Ran}(A)$ into $\operatorname{Ran}(B)$ if and only if $TA = BS$ for some operator S. As usual, if M is a subspace of H, P_M denotes the orthogonal projection with range equal to M. We shall frequently use the following result due to Foiaş ([4]; see also [20], [21]).

Theorem 1.1. ([4]) *Let A be a positive operator on H. If the function $\psi : [0,\|A\|]\to\mathbb{R}$ is non-negative, non-decreasing, continuous and concave, then* $\operatorname{Ran}(\psi(A))$ *is invariant under every operator on H which leaves* $\operatorname{Ran}(A)$ *invariant.*

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras of operators on the Banach spaces X_1 and X_2, respectively, $\mathcal{A}_1$ and $\mathcal{A}_2$ are *algebraically isomorphic* if there exists a multiplicative, linear bijection $\varphi : \mathcal{A}_1\to\mathcal{A}_2$. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *spatially isomorphic* if there exists a bicontinuous bijection $T\in B(X_1, X_2)$ such that $T\mathcal{A}_1T^{-1} = \mathcal{A}_2$. Then the mapping $A\mapsto TAT^{-1}$ is a *spatial isomorphism* of $\mathcal{A}_1$ onto $\mathcal{A}_2$, *implemented* by T.

Let $\mathcal{P} = \{(0), K, L, M, X\}$ be a pentagon subspace lattice on X with $L\subset M$ (so $K\cap M = (0)$ and $K\vee L = X$). If $\dim M/L = n$ is a positive integer we say that $\mathcal{P}$ has gap-dimension n. Otherwise we say that $\mathcal{P}$ has infinite gap-dimension. In Section 2 we discuss how pentagon subspace lattices arise and show that, for every $n\in\mathbb{Z}^+\cup\{\infty\}$, $n>1$, there are both reflexive and non-reflexive pentagons with gap-dimension equal to n. All pentagons with gap-dimension one are reflexive ([6]; see also [16]). The closure of the finite rank subalgebra of $\operatorname{Alg}\mathcal{P}$ in the strong operator topology is shown to be $\{T\in\operatorname{Alg}\mathcal{P} : T(M)\subseteq L\}$.

For any pentagons $\mathcal{P}_1$ and $\mathcal{P}_2$ on Banach spaces X_1 and X_2, respectively, it is shown in Section 3 that every algebraic isomorphism $\varphi : \operatorname{Alg}\mathcal{P}_1\to\operatorname{Alg}\mathcal{P}_2$ is quasi-spatial (in a sense made precise below). A more general result, at least for reflexive Banach spaces, is given in [22], but our proof is direct and is unrestricted. It is not known whether such isomorphisms must be spatial, whether they must preserve gap-dimension, nor whether they can "switch the types of rank one operators".

Section 4 is devoted to a study of pentagons on separable Hilbert space H. More especially, the class of pentagons of the form $\mathcal{P}(A;M)$ on $H\oplus H$, which was introduced in [15], is further investigated. Here $A\in B(H)$ is a positive, injective, non-invertible operator and M is a non-zero subspace of H satisfying $M\cap\operatorname{Ran}(A) = (0)$; then $\mathcal{P}(A;M) = \{(0), G(-A), G(A), G(A) + (0)\oplus M, H\oplus H\}$ has gap-dimension equal to $\dim M$. Three operator algebras are associated with $\mathcal{P}(A;M)$, namely, $\operatorname{Alg}P(A;M)$, $\mathcal{A}(A;M) = \big\{T\in B(H) : T\operatorname{Ran}(A)\subseteq\operatorname{Ran}(A)$ and $T(M)\subseteq M\big\}$ and $\mathcal{A}_0(A;M)$, the finite rank subalgebra of $\mathcal{A}(A;M)$. For each pair of algebras of the same type, $\operatorname{Alg}\mathcal{P}(A;M)$, $\operatorname{Alg}\mathcal{P}(B;N)$, $\mathcal{A}(A;M)$, $\mathcal{A}(B;N)$ and $\mathcal{A}_0(A;M)$, $\mathcal{A}_0(B;N)$, they are shown to be spatially isomorphic if and only if there exists an invertible operator $T \in B(H)$ satisfying $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$ and $T(M) = N$. If only one of A, B is compact it is shown that $\mathrm{Alg}\,\mathcal{P}(A;M)$ and $\mathrm{Alg}\,\mathcal{P}(B;N)$ cannot even be algebraically isomorphic. An example is given to show that $\mathcal{A}_0(A;M)$ and $\mathcal{A}_0(B;N)$ can be algebraically isomorphic even if $\dim M \neq \dim N$. Whether $\mathcal{A}(A;M)$ and $\mathcal{A}(B;N)$ can be isomorphic in this case is not known. Taking $M = (0)$, $\mathcal{A}(A;M)$ becomes $\mathcal{A}(A)$. The latter type of operator algebras were introduced and extensively studied in [20]. Finally, we briefly consider the 2-atom ABSL $\mathcal{D}(A) = \{(0), G(-A), G(A), H \oplus H\}$ associated with the pentagon $\mathcal{P}(A;M)$. Here the algebra $\mathcal{A}(A)$ plays a role in determining $\mathcal{D}(A)$ analogous to that played by $\mathcal{A}(A;M)$ in determining $\mathcal{P}(A;M)$. We extend and unify earlier results in our last theorem. For example, we show that every algebraic isomorphism $\varphi : \mathcal{A}(A) \to \mathcal{A}(B)$ is spatially implemented by some invertible operator $T \in B(H)$ satisfying $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$.

2. GENERAL COMMENTS

How do pentagon subspace lattices arise? Since they are non-modular, they certainly do not arise on finite-dimensional spaces. If K and L are subspaces of a Banach space X satisfying $K \cap L = (0)$, $K \vee L = X$ and $K + L \neq X$, and M is a non-zero finite-dimensional subspace of X satisfying $M \cap (K + L) = (0)$, it is easily shown that $\mathcal{P} = \{(0), K, L, L \vee M, X\}$ is a pentagon on X with gap-dimension equal to $\dim M$. Conversely, and again easily proved, every pentagon subspace lattice on X with finite gap-dimension arises in this way.

With K and L as above, if M is an infinite-dimensional subspace satisfying $M \cap (K + L) = (0)$ with $L + M$ closed, then again $\mathcal{P} = \{(0), K, L, L \vee M, X\}$ is a pentagon, this time with infinite gap-dimension. The condition that $L + M$ be closed cannot be dropped in the preceding statement, as the following example shows (in it we even have $L \vee M = X$).

Example 2.1. Let A be a positive, injective, non-invertible operator on a separable Hilbert space H. Put $K = G(-A)$ and $L = G(A)$. Then $K^\perp = \{(Ax, x) : x \in H\}$, $L^\perp = \{(-Ax, x) : x \in H\}$ and $K \cap L = (0)$, $K \vee L = H \oplus H$, $K + L \neq H \oplus H$. By a result of von Neumann (see [3], Theorem 3.6), there exists a unitary operator $U \in B(H)$ such that $\mathrm{Ran}(A) \cap \mathrm{Ran}(B) = (0)$ where $B = UAU^*$. Put $M = G(B)$. Then $M^\perp = \{(-Bx, x) : x \in H\}$ and $L \vee M = H \oplus H$. On the other hand, $M \cap (K + L) = (0)$, for $(x, Bx) = (y + z, -Ay + Az)$ gives $Bx = A(z - y) \in \mathrm{Ran}(A) \cap \mathrm{Ran}(B) = (0)$ so $Bx = 0$ and $x = 0$.

Such considerations as immediately above lead quite easily to examples of pentagon subspace lattices of arbitrary gap-dimension on separable Hilbert space. Indeed, let $A \in B(H)$ be as in the preceding example and let M be a non-zero subspace of H satisfying $M \cap \mathrm{Ran}(A) = (0)$. Since $G(A) + (0) \oplus H$ is closed and $G(A) \cap (0) \oplus H = (0)$, it follows that $G(A) + (0) \oplus M$ is closed. Clearly $\big((0) \oplus M\big) \cap \big(G(-A) + G(A)\big) = (0)$ so $\mathcal{P}(A;M) = \{(0), G(-A), G(A), G(A) + (0) \oplus M, H \oplus H\}$ is a pentagon subspace lattice. The gap-dimension of $\mathcal{P}(A;M)$ is $\dim M$ and, for any given A, it can be any positive integer or infinity. One way to see this is as follows. Firstly, there exists $B \in B(H)$ positive, injective, non-compact and non-invertible such that $\mathrm{Ran}(A) \subseteq \mathrm{Ran}(B)$. (If A is non-compact take $B = A$. Otherwise, if $(e_i)_1^\infty$ is an orthonormal basis of H that diagonalizes A, let B be the unique operator satisfying $Be_{2i-1} = Ae_{2i-1}$, $Be_{2i} = e_{2i}$, for every $i \in \mathbb{Z}^+$.) Next, by von Neumann's result, there exists a unitary operator $V \in B(H)$ such that $\mathrm{Ran}(B) \cap \mathrm{Ran}(VBV^*) = (0)$. Since VBV^* is not compact, there exists an infinite-dimensional subspace $M_\infty \subseteq \mathrm{Ran}(VBV^*)$ ([3], Theorem 2.5). For any non-zero subspace $M \subseteq M_\infty$ we have $M \cap \mathrm{Ran}(A) \subseteq M \cap \mathrm{Ran}(B) \subseteq \mathrm{Ran}(VBV^*) \cap \mathrm{Ran}(B) = (0)$, so $M \cap \mathrm{Ran}(A) = (0)$. Of course, the dimension of this M can be any positive integer or infinity.

Later, in Section 4, we will consider, in more detail, those pentagons of the form $\mathcal{P}(A; M)$ described above. This notation was introduced in [15]. It is easily shown that

$$\begin{aligned}\mathrm{Alg}\,\mathcal{P}(A; M) = \bigg\{ \begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix} &: X, Y, Z \in B(H), \\ &YA = AX \text{ and } Y(M) \subseteq M \bigg\}.\end{aligned}$$

If $\mathcal{P} = \{(0), K, L, M, X\}$ (where $L \subset M$) is a pentagon on a Banach space X then, of course, $\mathcal{P} \subseteq \mathrm{Lat}\,\mathrm{Alg}\,\mathcal{P}$. If $N \in \mathrm{Lat}\,\mathrm{Alg}\,\mathcal{P} \setminus \mathcal{P}$, then $L \subset N \subset M$ ([16]). Consequently, every pentagon with gap-dimension one is reflexive. The following shows how reflexive pentagons of arbitrary gap-dimension, even infinite, can be constructed.

Proposition 2.2. *Let $\mathcal{P} = \{(0), K, L, M, X\}$ be a pentagon subspace lattice (with $L \subset M$) on a Banach space X with gap-dimension equal to one. Then, for every $n \in \mathbb{Z}^+ \cup \{\infty\}$, $\mathcal{P}^{(n)} = \{(0), K^{(n)}, L^{(n)}, M^{(n)}, X^{(n)}\}$ is a reflexive pentagon on $X^{(n)}$ with gap-dimension equal to n.*

Proof. It is not difficult to verify that $\mathcal{P}^{(n)}$ is a pentagon on $X^{(n)}$ with gap-dimension n. Since $\mathcal{P}$ is reflexive, for every vector $y \in M \setminus L$, the linear manifold $(\mathrm{Alg}\,\mathcal{P})y$ is dense in M. For every i, j and every $A \in \mathrm{Alg}\,\mathcal{P}$ the mapping of $X^{(n)}$ into itself given by $(x_k)_1^n \mapsto (\delta_{ik} A x_j)_{k=1}^n$ defines an element of $\mathrm{Alg}\,\mathcal{P}^{(n)}$.

Suppose that $\mathcal{P}^{(n)}$ is not reflexive. Then there exists $N \in \mathrm{Lat}\,\mathrm{Alg}\,\mathcal{P}^{(n)}$ such that $L^{(n)} \subset N \subset M^{(n)}$. Let $(y_k)_1^n \in N \setminus L^{(n)}$. Then $y_j \notin L$, for some j. For every i and every $A \in \mathrm{Alg}\,\mathcal{P}$, $(\delta_{ik} A y_j)_{k=1}^n \in N$. Since $(\mathrm{Alg}\,\mathcal{P})y_j$ is dense in M, it follows that $M^{(n)} \subseteq N$. This is a contradiction. ∎

The first example of a non-reflexive pentagon (with gap-dimension 2) was given in [19]. Non-reflexive pentagons on separable Hilbert space of arbitrary finite gap-dimension are exhibited in [15] and, for each of them, its Lat Alg is fully determined. We now present a simple method by which examples of non-reflexive pentagons of arbitrary gap-dimension (even infinite) can be obtained.

EXAMPLE 2.3. Let A be a positive, injective, non-invertible operator on a separable Hilbert space H, and let N_∞ be an infinite-dimensional subspace satisfying $N_\infty \cap \operatorname{Ran}(A^{1/2}) = (0)$. Let N be a non-zero subspace satisfying $N \subseteq N_\infty$ and let $y \in \operatorname{Ran}(A^{1/2}) \setminus \operatorname{Ran}(A)$. Then $(N + \langle y \rangle) \cap \operatorname{Ran}(A) = (0)$. For if $x + \lambda y \in \operatorname{Ran}(A)$ with $x \in N$, $\lambda \in \mathbb{C}$, then $x \in N \cap \operatorname{Ran}(A^{1/2}) = (0)$ so $x = 0$. Then $\lambda y \in \operatorname{Ran}(A)$ gives $\lambda = 0$. Thus, using the notation mentioned earlier, $\mathcal{P}(A; N + \langle y \rangle)$ is a pentagon on $H \oplus H$ with gap-dimension equal to $1 + \dim N$ if $\dim N < \infty$ and infinity otherwise. We show that $\mathcal{P} = \mathcal{P}(A; N + \langle y \rangle)$ is non-reflexive by showing that $G(A) + (0) \oplus \langle y \rangle \in \operatorname{Lat} \operatorname{Alg} \mathcal{P}$.

Let $T \in \operatorname{Alg} \mathcal{P}$. Then $T = \begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix}$ for some operators X, Y, $Z \in B(H)$ satisfying $YA = AX$ and $Y(N + \langle y \rangle) \subseteq N + \langle y \rangle$. Since Y leaves $\operatorname{Ran}(A)$ invariant, it also leaves $\operatorname{Ran}(A^{1/2})$ invariant by Foiaş' theorem (as stated in Section 1). So $Yy \in \operatorname{Ran}(A^{1/2})$. Since Y also leaves $N + \langle y \rangle$ invariant, $Yy = x + \lambda y$, for some $x \in N$, $\lambda \in \mathbb{C}$. Thus $x = Yy - \lambda y \in N \cap \operatorname{Ran}(A^{1/2}) = (0)$, so $x = 0$. Hence $Y\langle y \rangle \subseteq \langle y \rangle$ and it easily follows that T leaves $G(A) + (0) \oplus \langle y \rangle$ invariant.

Let $\mathcal{P} = \{(0), K, L, M, X\}$ be a pentagon subspace lattice (as usual, with $L \subset M$) on a Banach space X. Let $\mathcal{R}$ denote the set of rank one operators of $\operatorname{Alg} \mathcal{P}$. By [16], Lemma 3.1 (see also [11]) the elements of $\mathcal{R}$ are of two distinct types. In fact, $\mathcal{R} = \mathcal{R}^- \cup \mathcal{R}^+$ where we define $\mathcal{R}^- = \{e^* \otimes f : 0 \neq f \in K,\, 0 \neq e^* \in M^\perp\}$ and $\mathcal{R}^+ = \{g^* \otimes h : 0 \neq h \in L,\, 0 \neq g^* \in K^\perp\}$. Note that $\mathcal{R}^- \cap \mathcal{R}^+ = \emptyset$. If R_1, $R_2 \in \mathcal{R}$ we shall say that R_1 and R_2 are of the *same type* if either both belong to $\mathcal{R}^-$ or both belong to $\mathcal{R}^+$. Otherwise we shall say that R_1 and R_2 are of *different types*.

THEOREM 2.4. *Let $\mathcal{P} = \{(0), K, L, M, X\}$ be a pentagon subspace lattice on a Banach space X (with $L \subset M$). Let $\mathcal{F}$, respectively $\mathcal{R}$, denote the set of finite rank, respectively rank one, operators of* $\operatorname{Alg} \mathcal{P}$. *Then:*

(i) *for every $n \in \mathbb{Z}^+$, every element of $\mathcal{F}$ of rank n is the sum of n elements of $\mathcal{R}$;*

(ii) *the closure of $\mathcal{F}$ in the strong operator topology is $\{T \in \operatorname{Alg} \mathcal{P} : T(M) \subseteq L\}$;*

(iii) $\operatorname{Lat} \mathcal{F} = \operatorname{Lat} \mathcal{R} = \mathcal{P} \cup \{N : N$ *a subspace of* X *and* $L \subset N \subset M\}$*;*

(iv) *the strongly closed algebra generated by $\mathcal{F}$ and I is* $\operatorname{Alg} \operatorname{Lat} \mathcal{R}$.

Proof. (i) Let $F \in \mathcal{F}$ have rank $n \in \mathbb{Z}^+$. Since $\operatorname{Ran}(F) = F(\overline{K + L}) \subseteq \overline{F(K + L)} = F(K) + F(L) \subseteq \operatorname{Ran}(F)$, $\operatorname{Ran}(F) = F(K) + F(L)$. Assume for the moment that $F(K) \neq (0)$ and $F(L) \neq (0)$ and let $\{x_1, x_2, \ldots, x_p\}$ and $\{y_1, y_2, \ldots, y_q\}$ be bases for $F(K)$ and $F(L)$, respectively. Then $\{x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q\}$ is a basis for $\operatorname{Ran}(F)$ and there exist $v_1^*, v_2^*, \ldots, v_p^*, w_1^*, w_2^*, \ldots, w_q^* \in X^*$ such that $F = \sum_{i=1}^{p} v_i^* \otimes x_i + \sum_{j=1}^{q} w_j^* \otimes y_j$. Suppose that $v_i^* \notin M^\perp$. Then $v_i^*(h) \neq 0$ for some vector $h \in M$. Now

$$Fh = \sum_{r=1}^{p} v_r^*(h) x_r + \sum_{j=1}^{q} w_j^*(h) y_j \in M,$$

so $\sum_{r=1}^{p} v_r^*(h)x_r \in K \cap M = (0)$ and so $v_i^*(h) = 0$. This contradiction shows that $v_i^* \otimes x_i \in \mathcal{R}$ $(i = 1, 2, \ldots, p)$.

Similarly, if we suppose that $w_j^* \notin K^\perp$, then $w_j^*(f) \neq 0$ for some $f \in K$. This leads to $\sum_{s=1}^{q} w_s^*(f)y_s \in K \cap L = (0)$, so $w_j^*(f) = 0$. Again a contradiction. Thus $w_j^* \otimes y_j \in \mathcal{R}$ $(j = 1, 2, \ldots, q)$. Since $p + q = n$, the assertion (i) follows.

Obvious modification to the above argument shows that (i) also holds when $F(K) = (0)$ or $F(L) = (0)$.

(ii) Let $\mathcal{E} = \{T \in \operatorname{Alg}\mathcal{P} : T(M) \subseteq L\}$. It is clear that $\mathcal{E}$ is a strongly closed subalgebra (even a 2-sided ideal) of $\operatorname{Alg}\mathcal{P}$ and, since $\mathcal{R} \subseteq \mathcal{E}$, we have $\mathcal{F} \subseteq \mathcal{E}$ by (i). Hence $\mathcal{E}$ contains the strong closure of $\mathcal{F}$. Now $\mathcal{D} = \{(0), K, M, X\}$ is a 2-atom ABSL on X, so by [1], Theorem 3.1 there exists a net $\{F_\alpha\}$ of finite rank operators of $\operatorname{Alg}\mathcal{D}$ such that $F_\alpha \to I$ (strongly). If $T \in \mathcal{E}$, then $TF_\alpha \to T$ (strongly) and $TF_\alpha \in \operatorname{Alg}\mathcal{P}$. Thus $TF_\alpha \in \mathcal{F}$ and T belongs to the strong closure of $\mathcal{F}$. This proves (ii).

(iii) This follows from (i) and [16], Proposition 3.2.

(iv) Let $\mathcal{A}$ denote the strongly closed algebra generated by $\mathcal{F}$ and I. Then $\operatorname{Lat}\mathcal{A} = \operatorname{Lat}\mathcal{F}$ so $\mathcal{A} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{F} = \operatorname{Alg}\operatorname{Lat}\mathcal{R}$. Let $A \in \operatorname{Alg}\operatorname{Lat}\mathcal{R}$. Then, by (iii), $A \in \operatorname{Alg}\mathcal{P}$ and $A(N) \subseteq N$ for every subspace N of X satisfying $L \subseteq N \subseteq M$. It follows that the operator $\widehat{A} : M/L \to M/L$ defined by $\widehat{A}(x + L) = Ax + L$ $(x \in M)$ leaves every subspace of M/L invariant. Thus $\widehat{A}$ is scalar, that is, there exists $\lambda \in \mathbb{C}$ such that $Ax - \lambda x \in L$, for every $x \in M$. Thus $A - \lambda \in \mathcal{E}$ so, since $\mathcal{E} \subseteq \mathcal{A}$, $A = (A - \lambda) + \lambda \in \mathcal{A}$. This proves (iv). ∎

Remark 2.5. With notation as in the preceding theorem, by (ii), $\mathcal{F}$ is not strongly dense in $\operatorname{Alg}\mathcal{P}$. (In the terminology of [1], p. 20, $\mathcal{P}$ does not have the strong rank one density property.) Even the strongly closed algebra $\mathcal{A}$ generated by $\mathcal{F}$ and I need not be $\operatorname{Alg}\mathcal{P}$. For, note that $\mathcal{A} = \operatorname{Alg}\mathcal{P}$ if and only if $\operatorname{Lat}\operatorname{Alg}\mathcal{P} = \operatorname{Lat}\mathcal{R}$. If the gap-dimension of $\mathcal{P}$ is greater than one and $\mathcal{P}$ is reflexive, then $\mathcal{P} = \operatorname{Lat}\operatorname{Alg}\mathcal{P} \neq \operatorname{Lat}\mathcal{R}$ so $\mathcal{A} \neq \operatorname{Alg}\mathcal{P}$. On the other hand, $\mathcal{A} = \operatorname{Alg}\mathcal{P}$ if $\mathcal{P}$ has gap-dimension one. Other examples in [15] show that it is possible to have $\mathcal{A} = \operatorname{Alg}\mathcal{P}$, with $\mathcal{P}$ having gap-dimension n, for any integer $n \geqslant 2$.

3. QUASI-SPATIALITY OF ISOMORPHISMS

If X_1 and X_2 are Banach spaces and $\mathcal{A}_1, \mathcal{A}_2$ are subalgebras of $B(X_1)$ and $B(X_2)$, respectively, an algebraic isomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is called *quasi-spatial* if there exists a closed, densely defined, injective linear transformation $T : \operatorname{Dom}(T) \subseteq X_1 \to X_2$ with dense range, and with its domain $\operatorname{Dom}(T)$ invariant under every element of $\mathcal{A}_1$, such that $\varphi(A)Tx = TAx$, for every $x \in \operatorname{Dom}(T)$ and every $A \in \mathcal{A}_1$. The notion of quasi-spatiality was introduced in [10], where it is shown that every algebraic isomorphism $\varphi : \operatorname{Alg}\mathcal{L}_1 \to \operatorname{Alg}\mathcal{L}_2$ is quasi-spatial for any pair $\mathcal{L}_1, \mathcal{L}_2$ of ABSL's on Banach space. (For a definition of "ABSL" see [1], p. 5.) It is also shown in [10] (see also [9]) that such a mapping φ need not be spatial.

The techniques of [10] are used in [22] to show the necessity of quasi-spatiality of algebraic isomorphisms between the Alg's of a broader class of subspace lattices, containing all ABSL's and all pentagons, at least if the underlying spaces are reflexive ([22], Theorem 3.3.5, p. 126). We give a more direct proof of this for pentagons below, with no restrictions. The proof is included also for the convenience of the reader, and because we wish to use aspects of it in the next section.

Recall that an element s of an abstract algebra $\mathcal{A}$ is called a *single element* of $\mathcal{A}$ if $asb = 0$ and $a, b \in \mathcal{A}$ implies that either $as = 0$ or $sb = 0$. Every rank one operator is a single element of every operator algebra containing it. (On the other hand, there are operator algebras containing single elements of rank more than one, even infinity [5], [12], [18].)

Lemma 3.1. *Let $\mathcal{P} = \{(0), K, L, M, X\}$ be a pentagon subspace lattice (with $L \subset M$) on a Banach space X. Every non-zero single element of* $\operatorname{Alg}\mathcal{P}$ *has rank one.*

Proof. Let $S \in \operatorname{Alg}\mathcal{P}$ be non-zero and single. Note that $R_1AR_2 = 0$, for every $A \in \operatorname{Alg}\mathcal{P}$ and every pair R_1, R_2 of rank one operators of $\operatorname{Alg}\mathcal{P}$ of different types (that is, either $R_1 \in \mathcal{R}^-$, $R_2 \in \mathcal{R}^+$ or vice-versa). Suppose that $S(K) \neq (0)$. Then $SR_- \neq 0$, for some $R_- \in \mathcal{R}^-$ so $R_+S = 0$, for every $R_+ \in \mathcal{R}^+$. Thus $\operatorname{Ran}(S) \subseteq K$ so $\operatorname{Ran}(S) \not\subseteq M$. Hence $R'_-S \neq 0$, for some $R'_- \in \mathcal{R}^-$, so $SR_+ = 0$, for every $R_+ \in \mathcal{R}^+$. Thus $S(L) = (0)$. This shows that $S(K) = (0)$ or $S(L) = (0)$.

Now, since $\operatorname{Ran}(S)$ is not contained in $K \cap M$, there exists $R \in \mathcal{R}^- \cup \mathcal{R}^+$ such that $RS \neq 0$. Since RS has rank one, for every $f_1, f_2 \in K$ there exist scalars α_1, α_2, not both zero, such that $RS(\alpha_1 f_1 + \alpha_2 f_2) = 0$. If $0 \neq e^* \in M^\perp$, $RS(e^* \otimes (\alpha_1 f_1 + \alpha_2 f_2)) = 0$ so $S(\alpha_1 f_1 + \alpha_2 f_2) = 0$, since S is single. From this it follows that $\dim S(K) \leqslant 1$. Similarly $\dim S(L) \leqslant 1$.

Since $\operatorname{Ran}(S) = \overline{S(K+L)} \subseteq \overline{S(K) + S(L)}$, it follows that S has rank one. ∎

Remark 3.2. In the proof of the preceding lemma we noted that, given the rank one operators R_1 and R_2 of $\operatorname{Alg}\mathcal{P}$, if they are of different types then $R_1AR_2 = 0$, for every $A \in \operatorname{Alg}\mathcal{P}$. The converse is also true. Indeed, suppose that $R_1, R_2 \in \mathcal{R}^-$. Then $R_i = e_i^* \otimes f_i$ with $f_i \in K$, $e_i^* \in M^\perp$ $(i = 1, 2)$. Since $f_2 \notin M$ there exists $e^* \in M^\perp$ such that $e^*(f_2) = 1$. Also, $e_1^* \notin K^\perp$ so there exists $f \in K$ such that $e_1^*(f) = 1$. Then $A_1 = e^* \otimes f \in \operatorname{Alg}\mathcal{P}$ and $R_1A_1R_2 = e_2^* \otimes f_1 \neq 0$. Similarly, there exists $A_2 \in \operatorname{Alg}\mathcal{P}$ such that $R_1A_2R_2 \neq 0$ if $R_1, R_2 \in \mathcal{R}^+$.

Let $\mathcal{P}_i = \{(0), K_i, L_i, M_i, X_i\}$ be a pentagon subspace lattice (with $L_i \subset M_i$) on a Banach space X_i $(i = 1, 2)$. Let $\varphi : \operatorname{Alg}\mathcal{P}_1 \to \operatorname{Alg}\mathcal{P}_2$ be an algebraic isomorphism. The property of being single is purely algebraic so is preserved by algebraic isomorphisms. But, using Lemma 3.1, the set of non-zero single elements of $\operatorname{Alg}\mathcal{P}_i$ is just $\mathcal{R}_i$, the set of rank one operators of $\operatorname{Alg}\mathcal{P}_i$ $(i = 1, 2)$. Hence $\varphi(\mathcal{R}_1) = \mathcal{R}_2$. More can be deduced. Using Theorem 2.4 (i) and Lemma 3.1 it follows that φ is rank-preserving, that is, $\operatorname{rank}\varphi(F) = \operatorname{rank} F$, for every $F \in \operatorname{Alg}\mathcal{P}_1$. Also, since the characterization of rank one operators being of different types, given in the preceding remark, is also purely algebraic, it follows that either φ preserves the types of rank one operators in the sense that $\varphi(\mathcal{R}_1^-) = \mathcal{R}_2^-$ and $\varphi(\mathcal{R}_1^+) = \mathcal{R}_2^+$, or φ switches their types in the sense that $\varphi(\mathcal{R}_1^-) = \mathcal{R}_2^+$ and $\varphi(\mathcal{R}_1^+) = \mathcal{R}_2^-$.

QUESTION 3.3. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are pentagon subspace lattices on Banach spaces can an algebraic isomorphism $\varphi : \operatorname{Alg} \mathcal{P}_1 \to \operatorname{Alg} \mathcal{P}_2$ switch the types of rank one operators?

THEOREM 3.4. *Let $\mathcal{P}_i = \{(0), K_i, L_i, M_i, X_i\}$ be a pentagon subspace lattice (with $L_i \subset M_i$) on a Banach space X_i $(i = 1, 2)$. Every algebraic isomorphism $\varphi : \operatorname{Alg} \mathcal{P}_1 \to \operatorname{Alg} \mathcal{P}_2$ is quasi-spatial.*

Proof. Let $\varphi : \operatorname{Alg} \mathcal{P}_1 \to \operatorname{Alg} \mathcal{P}_2$ be an algebraic isomorphism. Choose $f_0 \in K_1$, $e_0^* \in M_1^\perp$ such that $e_0^*(f_0) = 1$. Then $e_0^* \otimes f_0 \in \mathcal{R}_1$ and $\varphi(e_0^* \otimes f_0) \in \mathcal{R}_2$. If $0 \neq e^* \in M_1^\perp$ then $(e_0^* \otimes f_0)(e^* \otimes f_0) = e^* \otimes f_0$ so $\varphi(e_0^* \otimes f_0)\varphi(e^* \otimes f_0) = \varphi(e^* \otimes f_0)$. It follows that $\varphi(e^* \otimes f_0)$ and $\varphi(e_0^* \otimes f_0)$ have the same range. Let b_0 span the range of $\varphi(e_0^* \otimes f_0)$. Then, for every $e^* \in M_1^\perp$ there exists a unique vector $a^* \in X_2^*$ such that $\varphi(e^* \otimes f_0) = a^* \otimes b_0$. This defines a linear mapping $S_0 : M_1^\perp \to X_2^*$ given by $S_0 e^* = a^*$. Thus $\varphi(e^* \otimes f_0) = S_0 e^* \otimes b_0$ for every $e^* \in M_1^\perp$.

Similarly, choosing $h_0 \in L_1$, $g_0 \in K_1^\perp$ such that $g_0^*(h_0) = 1$ and letting d_0 span the range of $\varphi(g_0^* \otimes h_0)$, there exists a linear mapping $S_0 : K_1^\perp \to X_2^*$ such that $\varphi(g^* \otimes h_0) = S_0 g^* \otimes d_0$, for every $g^* \in K_1^\perp$. Extend the definition of S_0 to $K_1^\perp + M_1^\perp$ by linearity. Either $S_0(M_1^\perp) \subseteq M_2^\perp$ and $S_0(K_1^\perp) \subseteq K_2^\perp$ if φ preserves the types of rank one operators, or $S_0(M_1^\perp) \subseteq K_2^\perp$ and $S_0(K_1^\perp) \subseteq M_2^\perp$ if it switches them.

A similar argument shows that there exists a linear mapping $T_0 : K_1 + L_1 \to K_2 + L_2$ such that $\varphi(e_0^* \otimes f) = S_0 e_0^* \otimes T_0 f$ and $\varphi(g_0^* \otimes h) = S_0 g_0^* \otimes T_0 h$ for every $f \in K_1$, $h \in L_1$. Also, $T_0(K_1) \subseteq K_2$ and $T_0(L_1) \subseteq L_2$ if φ preserves types and $T_0(K_1) \subseteq L_2$, $T_0(L_1) \subseteq K_2$ if it switches them.

Since $e_0^* \otimes f_0$ is non-zero and idempotent, so is $\varphi(e_0^* \otimes f_0)$. But $\varphi(e_0^* \otimes f_0) = S_0 e_0^* \otimes T_0 f_0$ (since $b_0 = T_0 f_0$). Hence $S_0 e_0^*(T_0 f_0) = 1$. Then, for every $e^* \in M_1^\perp$ and $f \in K_1$, we have

$$\begin{aligned}\varphi(e^* \otimes f) &= \varphi\big((e_0^* \otimes f)(e^* \otimes f_0)\big) = \varphi(e_0^* \otimes f)\varphi(e^* \otimes f_0)\\ &= (S_0 e_0^* \otimes T_0 f)(S_0 e^* \otimes T_0 f_0) = S_0 e^* \otimes T_0 f.\end{aligned}$$

Similarly, $\varphi(g^* \otimes h) = S_0 g^* \otimes T_0 h$, for every $g^* \in K_1^\perp$ and $h \in L_1$.

Let $v^* \otimes \omega \in \operatorname{Alg} \mathcal{P}_1$. Then $\varphi(v^* \otimes \omega) = S_0 v^* \otimes T_0 \omega$ and $(v^* \otimes \omega)^2 = v^*(\omega)(v^* \otimes \omega)$ give $\big(\varphi(v^* \otimes \omega)\big)^2 = S_0 v^*(T_0 \omega)\varphi(v^* \otimes \omega) = v^*(\omega)\varphi(v^* \otimes \omega)$, so $S_0 v^*(T_0 \omega) = v^*(\omega)$. It readily follows that $S_0 v^*(T_0 \omega) = v^*(\omega)$, for every $v^* \in K_1^\perp + M_1^\perp$ and $\omega \in K_1 + L_1$. From this, it is clear that both S_0 and T_0 are injective.

Next, we show that $\varphi(A) T_0 \omega = T_0 A \omega$, for every $A \in \operatorname{Alg} \mathcal{P}_1$ and every $\omega \in K_1 + L_1$. Suppose first that $0 \neq f \in K_1$. Choose $e^* \in M_1^\perp$ such that $e^*(f) = 1$. Then $S_0 e^*(T_0 f) = e^*(f) = 1$ and

$$\begin{aligned}\varphi(A) T_0 f &= \varphi(A)(S_0 e^* \otimes T_0 f) T_0 f = \varphi(A)\varphi(e^* \otimes f) T_0 f\\ &= \varphi(e^* \otimes A f) T_0 f = (S_0 e^* \otimes T_0 A f) T_0 f = T_0 A f.\end{aligned}$$

Similarly, $\varphi(A) T_0 h = T_0 A h$, for every $0 \neq h \in L_1$, and combining these two cases gives the desired result.

Each of the linear mappings $T_0|K_1$, $T_0|L_1$, $S_0|M_1^\perp$ and $S_0|K_1^\perp$ is continuous. Indeed, by [8] and [17], $\operatorname{Alg} \mathcal{P}_2$ is semisimple so by Johnson's theorem ([7]), φ is automatically continuous in the operator norm. For example, if $f_n \to f$ with f_n, $f \in K_1$, then $e^* \otimes f_n \to e^* \otimes f$ in operator norm for every $0 \neq e^* \in M_1^\perp$. Then $\varphi(e^* \otimes f_n) \to \varphi(e^* \otimes f)$, that is, $S_0 e^* \otimes T_0 f_n \to S_0 e^* \otimes T_0 f$. Hence $T_0 f_n \to T_0 f$ and $T_0|K_1$ is continuous. Similar arguments apply to $T_0|L_1$, $S_0|M_1^\perp$ and $S_0|K_1^\perp$. Also, if φ preserves types, we have $T_0(K_1) = K_2$, $T_0(L_1) = L_2$, $S_0(M_1^\perp) = M_2^\perp$ and $S_0(K_1^\perp) = K_2^\perp$. For, let $0 \neq b \in K_2$. Then $S_0 e_0^* \otimes b = \varphi(e_1^* \otimes f_1)$ for some non-zero vectors $e_1^* \in M_1^\perp$, $f_1 \in K_1$. Thus $S_0 e_0^* \otimes b = S_0 e_1^* \otimes T_0 f_1$ so $b \in \langle T_0 f_1 \rangle$ and $b \in T_0(K_1)$. The other range equalities are proved similarly. On the other hand, if φ switches types, we have $T_0(K_1) = L_2$, $T_0(L_1) = K_2$, $S_0(M_1^\perp) = K_2^\perp$ and $S_0(K_1^\perp) = M_2^\perp$, by a similar argument.

Finally, we show that the closure $\overline{G(T_0)}$ of the graph of T_0 is the graph of a linear transformation T which quasi-spatially implements φ. Let $\mathcal{D} = \big\{x \in X_1 : (x, y) \in \overline{G(T_0)},\ \text{for some } y \in X_2\big\}$. Then $\mathcal{D}$ is a linear manifold and it is dense in X_1 since $K_1 + L_1 \subseteq \mathcal{D}$. Let $x \in \mathcal{D}$ and suppose that $(x, y_1), (x, y_2) \in \overline{G(T_0)}$ with y_1, $y_2 \in X_2$. Then, $(0, y_1 - y_2) \in \overline{G(T_0)}$ so there exists a sequence $(\omega_n)_1^\infty$ of elements of $K_1 + L_1$ such that $\omega_n \to 0$ and $T_0 \omega_n \to y_1 - y_2$. Now, for every $v^* \in K_1^\perp + M_1^\perp$, $S_0 v^*(T_0 \omega_n) = v^*(\omega_n) \to 0$ so $S_0 v^*(y_1 - y_2) = 0$. Since $S_0(K_1^\perp + M_1^\perp) = K_2^\perp + M_2^\perp$ it follows that $y_1 = y_2$. Thus $\overline{G(T_0)} = G(T)$ for some linear transformation $T : \mathcal{D} \to X_2$. Since T extends T_0 and the latter has dense range, T has dense range. Also, T is injective. For, suppose that $Tx = 0$ with $x \in \mathcal{D}$. Then $(x, 0) \in \overline{G(T_0)}$, so there exists a sequence $(\omega_n')_1^\infty$ of elements of $K_1 + L_1$ such that $\omega_n' \to x$ and $T_0 \omega_n' \to 0$. Then, for every $v^* \in K_1^\perp + M_1^\perp$, $S_0 v^*(T_0 \omega_n') = v^*(\omega_n') \to 0$ so $v^*(x) = 0$. It follows that $x = 0$.

The proof is completed by showing that $\mathcal{D}$ is invariant under every element A of $\operatorname{Alg}\mathcal{P}_1$ and $\varphi(A)Tx = TAx$, for every $x \in \mathcal{D}$. Given A and x, there exists a sequence $(\omega_n'')_1^\infty$ of elements of $K_1 + L_1$ such that $\omega_n'' \to x$ and $T_0 \omega_n'' \to Tx$. Then $A\omega_n'' \to Ax$ and $T_0 A \omega_n'' = \varphi(A) T_0 \omega_n'' \to \varphi(A)Tx$. Thus $(A\omega_n'', T_0 A \omega_n'') \to (Ax, \varphi(A)Tx)$. Hence $(Ax, \varphi(A)Tx) \in G(T)$ and so $Ax \in \mathcal{D}$ and $\varphi(A)Tx = TAx$. ∎

Remarks 3.5. (1) It is perhaps worthwhile to comment briefly on the set of linear transformations that quasi-spatially implement a given algebraic isomorphism φ between $\operatorname{Alg}\mathcal{P}_1$ and $\operatorname{Alg}\mathcal{P}_2$ (with $\mathcal{P}_1, \mathcal{P}_2$ and φ as in the statement of the preceding theorem). Let S_0 and T_0 be as in the proof of Theorem 3.4. In the proof, the vectors b_0 and d_0 are determined only up to non-zero scalar multiples; so the same is true for S_0 on $M_1^\perp$ and S_0 on $K_1^\perp$. This means that T_0 on K_1 and T_0 on L_1 are determined only up to non-zero multiples as well. Hence, for all scalars $\lambda, \mu \neq 0$, if $T_0' : K_1 + L_1 \to X_2$ is the linear transformation defined by $T_0'(x + y) = \lambda T_0 x + \mu T_0 y$ then T_0' has a closed extension which quasi-spatially implements φ.

On the other hand, suppose that $T' : \mathcal{D}' \to X_2$ quasi-spatially implements φ. Since $\mathcal{D}'$ is dense in X_1, for every $0 \neq e^* \in M_1^\perp$ there exists $z \in \mathcal{D}'$ such that $e^*(z) = 1$. Then, since $\mathcal{D}'$ is invariant under every element of $\operatorname{Alg}\mathcal{P}_1$, $(e^* \otimes f)z \in \mathcal{D}'$, for every $f \in K_1$, so $K_1 \subseteq \mathcal{D}'$. Similarly $L_1 \subseteq \mathcal{D}'$, so $K_1 + L_1 \subseteq \mathcal{D}'$. Let $f_1 \in K_1$ and $e_1^* \in M_1^\perp$ satisfy $e_1^*(f_1) = 1$. Then, for every $x \in K_1$, we have

$$T'x = T'(e_1^* \otimes x) f_1 = \varphi(e_1^* \otimes x) T' f_1 = (S_0 e_1^* \otimes T_0 x) T' f_1 = S_0 e_1^*(T' f_1) T_0 x.$$

Thus $T'|K_1 = \lambda T_0|K_1$ (with $\lambda = S_0 e_1^*(T'f_1)$). Similarly $T'|L_1 = \mu T_0|L_1$ for some scalar μ. Hence T' is a closed extension of a transformation of the type T_0' described above.

(2) If X_1 and X_2 are reflexive Banach spaces, then $\mathcal{P}_i^\perp$ defined by $\mathcal{P}_i^\perp = \{(0), K_i^\perp, M_i^\perp, L_i^\perp, X_i^*\}$ is a pentagon subspace lattice (with $M_i^\perp \subset L_i^\perp$) on X_i^* ($i = 1, 2$). With φ and S_0 as in the proof of Theorem 3.4, the mapping $\psi : \operatorname{Alg}\mathcal{P}_1^\perp \to \operatorname{Alg}\mathcal{P}_2^\perp$ defined by $\psi(A) = (\varphi(\tau^{-1}A^*\tau))^*$, where $\tau : X_1 \to X_1^{**}$ is the canonical embedding, is an algebraic isomorphism. This isomorphism is quasi-spatially implemented by S where $\overline{G(S_0)} = G(S)$. We leave the verification of these facts to the reader.

(3) Let X_1 and X_2 be Banach spaces and let $T \in B(X_1, X_2)$ be a bicontinuous bijection. For every subset $\mathcal{A} \subseteq B(X_1)$, $\operatorname{Lat}(T\mathcal{A}T^{-1}) = T(\operatorname{Lat}\mathcal{A})$. Also, for every family $\mathcal{F}$ of subspaces of X_1, $T(\operatorname{Alg}\mathcal{F})T^{-1} = \operatorname{Alg} T(\mathcal{F})$. It follows that, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices on X_1 and X_2, respectively, then $T(\operatorname{Alg}\mathcal{L}_1)T^{-1} = \operatorname{Alg}\mathcal{L}_2$ if and only if $T(\operatorname{Lat}\operatorname{Alg}\mathcal{L}_1) = \operatorname{Lat}\operatorname{Alg}\mathcal{L}_2$. So clearly $T(\operatorname{Alg}\mathcal{L}_1)T^{-1} = \operatorname{Alg}\mathcal{L}_2$ need not imply that $T(\mathcal{L}_1) = \mathcal{L}_2$ (for example, let $\mathcal{L}_1$ be a non-reflexive subspace lattice, let $\mathcal{L}_2 = \operatorname{Lat}\operatorname{Alg}\mathcal{L}_1$ and $T = I$). However, for pentagons this implication is valid.

PROPOSITION 3.6. *Let $\mathcal{P}_i = \{(0), K_i, L_i, M_i, X_i\}$ be a pentagon subspace lattice (with $L_i \subset M_i$) on a Banach space X_i ($i = 1, 2$). Let $T \in B(X_1, X_2)$ be a bicontinuous bijection. Then $T(\operatorname{Alg}\mathcal{P}_1)T^{-1} = \operatorname{Alg}\mathcal{P}_2$ if and only if $T(K_1) = K_2$, $T(L_1) = L_2$ and $T(M_1) = M_2$.*

Proof. By the remark immediately above we must show that $T(\operatorname{Lat}\operatorname{Alg}\mathcal{P}_1) = \operatorname{Lat}\operatorname{Alg}\mathcal{P}_2$ if and only if $T(K_1) = K_2$, $T(L_1) = L_2$ and $T(M_1) = M_2$. Clearly, if the latter condition is satisfied, $T(\mathcal{P}_1) = \mathcal{P}_2$ so $\operatorname{Lat}\operatorname{Alg}\mathcal{P}_2 = \operatorname{Lat}\operatorname{Alg} T(\mathcal{P}_1) = \operatorname{Lat} T(\operatorname{Alg}\mathcal{P}_1)T^{-1} = T(\operatorname{Lat}\operatorname{Alg}\mathcal{P}_1)$.

Conversely, suppose that $T(\operatorname{Lat}\operatorname{Alg}\mathcal{P}_1) = \operatorname{Lat}\operatorname{Alg}\mathcal{P}_2$. As noted earlier, if $N \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}_i$ then either $N \in \mathcal{P}_i$ or $L_i \subset N \subset M_i$. Since $T(L_1) \subset T(M_1)$ and each is non-trivial, we must have $L_2 \subseteq T(L_1) \subset T(M_1) \subseteq M_2$. Since $T(K_1) \cap T(L_1) = (0)$ and $L_2 \subseteq T(L_1)$, it follows that $T(K_1) \cap L_2 = (0)$. Thus $L_2 \not\subseteq T(K_1)$ so $T(K_1) = K_2$. If $T(L_1) \neq L_2$ then we would have $T^{-1}(L_2) \subset L_1$ so, since $T^{-1}(L_1) \in \operatorname{Lat}\operatorname{Alg}\mathcal{P}_1$, $T^{-1}(L_2) = (0)$. Hence $T(L_1) = L_2$. Similarly $T(M_1) = M_2$. ∎

It follows from the preceding proposition that, if there exists a spatial isomorphism $\varphi : \operatorname{Alg}\mathcal{P}_1 \to \operatorname{Alg}\mathcal{P}_2$ then the pentagon $\mathcal{P}_1$ is reflexive if and only if $\mathcal{P}_2$ is. For if $T \in B(X_1, X_2)$ implements φ then $T(\mathcal{P}_1) = \mathcal{P}_2$ and $T(\operatorname{Lat}\operatorname{Alg}\mathcal{P}_1) = \operatorname{Lat}\operatorname{Alg}\mathcal{P}_2$. It is clear that such a (spatial) isomorphism preserves the types of rank one operators and that its existence implies that $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same gap-dimensions. In general, can there exist an algebraic isomorphism between $\operatorname{Alg}\mathcal{P}_1$ and $\operatorname{Alg}\mathcal{P}_2$ if one of the pentagons is reflexive and the other not?

Question 3.7. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pentagon subspace lattices on Banach spaces.

(i) Must every algebraic isomorphism between $\operatorname{Alg}\mathcal{P}_1$ and $\operatorname{Alg}\mathcal{P}_2$ be spatial?

(ii) If $\operatorname{Alg}\mathcal{P}_1$ and $\operatorname{Alg}\mathcal{P}_2$ are algebraically isomorphic must they be spatially isomorphic?

(iii) If $\operatorname{Alg}\mathcal{P}_1$ and $\operatorname{Alg}\mathcal{P}_2$ are algebraically isomorphic must $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same gap-dimensions?

We present some results relating to these questions in the next section.

4. PENTAGONS ON HILBERT SPACES

Throughout this section H will denote a separable infinite-dimensional Hilbert space. We shall consider in more detail pentagons on $H \oplus H$ of the form $\mathcal{P}(A;M) = \{(0), G(-A), G(A), G(A) + (0) \oplus M, H \oplus H\}$ where $A \in B(H)$ is an arbitrary positive, injective, non-invertible operator and M is a non-zero subspace (possibly infinite-dimensional) of H satisfying $M \cap \operatorname{Ran}(A) = (0)$. A description of $\operatorname{Alg}\mathcal{P}(A;M)$ was given earlier. Let us now write it as

$$\operatorname{Alg}\mathcal{P}(A;M) = \left\{ \begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix} : X, Y, Z \in B(H),\ Y \in \mathcal{A}(A;M) \text{ and } YA = AX \right\}$$

where, by definition,

$$\mathcal{A}(A;M) = \{T \in B(H) : T\operatorname{Ran}(A) \subseteq \operatorname{Ran}(A) \text{ and } T(M) \subseteq M\}.$$

Much of this section will be concerned with the structure of the unital algebras of the form $\mathcal{A}(A;M)$. (This notation was introduced in [15].) They largely determine the structures of the Alg's and the Lat Alg's of the corresponding pentagons and are interesting in their own right. For example, if $2 \leqslant \dim M = n < \infty$, it is possible to have $\mathcal{A}(A;M)|M$ as (i) the algebra of all upper triangular $n \times n$ matrices or (ii) the algebra of all diagonal $n \times n$ matrices or (iii) the algebra $\mathbb{C}I_n$, relative to appropriate bases of M ([15]). We let $\mathcal{A}_0(A;M)$ denote the algebra of finite rank operators of $\mathcal{A}(A;M)$.

Proposition 4.1. *For every operator algebra on H of the form $\mathcal{A}(A;M)$ described above:*

(i) $\mathcal{A}_0(A;M) = \{AX(1 - P_M) : X \in B(H)$ *and* X *has finite rank*$\}$;

(ii) *for every pair e, f of non-zero vectors of H, $e \otimes f \in \mathcal{A}(A;M)$ if and only if $f \in \operatorname{Ran}(A)$ and $e \in M^{\perp}$;*

(iii) *for every $n \in \mathbb{Z}^+$, every element of $\mathcal{A}(A;M)$ of rank n is the sum of n rank one elements of $\mathcal{A}(A;M)$;*

(iv) *every non-zero single element of $\mathcal{A}(A;M)$ has rank one.*

Proof. (i) Let $F \in \mathcal{A}_0(A;M)$. Then $\operatorname{Ran}(F) = F\overline{\operatorname{Ran}(A)} \subseteq \overline{F\operatorname{Ran}(A)} = F\operatorname{Ran}(A) \subseteq \operatorname{Ran}(A)$. Thus $F = AX$ for some operator $X \in B(H)$. Since A is injective, X has finite rank. Since $F(M) \subseteq M \cap \operatorname{Ran}(A) = (0)$, $F(M) = (0)$.

Thus $X(M) = (0)$ so $X(1 - P_M) = X$. Hence $F = AX(1 - P_M)$ and $\mathcal{A}_0(A; M) \subseteq \{AX(1 - P_M) : X \in B(H)$ and X has finite rank$\}$. The reverse inclusion is obvious.

(ii) This follows immediately from (i).

(iii) Let $n > 1$ and let $F \in \mathcal{A}(A; M)$ have rank n. Then if $\{f_i\}_1^n$ is a basis for $\mathrm{Ran}(F)$, $f_i \in \mathrm{Ran}(A)$ for every i and $F = \sum_{i=1}^{n} e_i \otimes f_i$ for some non-zero vectors $e_i \in H$ $(i = 1, 2, \ldots, n)$. Since $F(M) = (0)$, $e_i \in M^\perp$ so $e_i \otimes f_i \in \mathcal{A}(A; M)$ by (ii), for every $i = 1, 2, \ldots, n$.

(iv) Let $S \in \mathcal{A}(A; M)$ be a non-zero single element. Let $f \in \mathrm{Ran}(A)$ such that $Sf \neq 0$. If $e \in M^\perp$ and $(Sf|e) = 0$, then $S^*e = 0$. For, if $e \neq 0$ we have $(e \otimes f)S(e \otimes f) = (Sf|e)(e \otimes f) = 0$ so, since S is single, $S^*e = 0$. Thus $M^\perp \cap \langle Sf\rangle^\perp \subseteq \ker S^*$ so $\overline{\mathrm{Ran}(S)} \subseteq \overline{M + \langle Sf\rangle}$. Now $S\,\mathrm{Ran}(A) \subseteq \mathrm{Ran}(A) \cap (M + \langle Sf\rangle) = \langle Sf\rangle$ and $\mathrm{Ran}(S) = S\overline{\mathrm{Ran}(A)} \subseteq \overline{S\,\mathrm{Ran}(A)} \subseteq \langle Sf\rangle$, so S has rank one. ∎

THEOREM 4.2. *Let $A, B \in B(H)$ be positive, injective, non-invertible operators and let M, N be non-zero subspaces of H satisfying $M \cap \mathrm{Ran}(A) = N \cap \mathrm{Ran}(B) = (0)$. Let $\mathcal{P}(A; M), \mathcal{P}(B; N)$ be the corresponding pentagons on $H \oplus H$ (as described above). The following are equivalent:*

(i) $\mathrm{Alg}\,\mathcal{P}(A; M)$ *and* $\mathrm{Alg}\,\mathcal{P}(B; N)$ *are spatially isomorphic;*

(ii) $\mathcal{A}(A; M)$ *and* $\mathcal{A}(B; N)$ *are spatially isomorphic;*

(iii) $\mathcal{A}_0(A; M)$ *and* $\mathcal{A}_0(B; N)$ *are spatially isomorphic;*

(iv) *there exists an invertible operator $T \in B(H)$ such that $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$ and $T(M) = N$.*

Proof. We prove that (i) ⇒ (iv) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (iv). Suppose that there exists an invertible operator $T_0 \in B(H \oplus H)$ such that $T_0 \,\mathrm{Alg}\,\mathcal{P}(A; M)\, T_0^{-1} = \mathrm{Alg}\,\mathcal{P}(B; N)$. By Proposition 3.6, T_0 maps $G(-A), G(A)$ and $G(A) + (0) \oplus M$ onto, respectively, $G(-B), G(B)$ and $G(B) + (0) \oplus N$. It follows that T_0 has the form $T_0 = \begin{pmatrix} S + UA & U \\ BUA & T + BU \end{pmatrix}$, where $S, T, U \in B(H)$ satisfy $TA = BS$, $T(M) \subseteq N$. Similarly, T_0^{-1} has the form $T_0^{-1} = \begin{pmatrix} \widetilde{S} + \widetilde{U}B & \widetilde{U} \\ A\widetilde{U}B & \widetilde{T} + A\widetilde{U} \end{pmatrix}$, where $\widetilde{T}B = A\widetilde{S}$ and $\widetilde{T}(N) \subseteq M$.

For every $x \in H$,

$$\begin{pmatrix} x \\ -Ax \end{pmatrix} = T_0^{-1}T_0 \begin{pmatrix} x \\ -Ax \end{pmatrix} = T_0^{-1} \begin{pmatrix} Sx \\ -BSx \end{pmatrix} = \begin{pmatrix} \widetilde{S}Sx \\ -A\widetilde{S}Sx \end{pmatrix},$$

so $\widetilde{S}S = 1$. For every $y \in H$,

$$\begin{pmatrix} y \\ -By \end{pmatrix} = T_0T_0^{-1} \begin{pmatrix} y \\ -By \end{pmatrix} = T_0 \begin{pmatrix} \widetilde{S}y \\ -A\widetilde{S}y \end{pmatrix} = \begin{pmatrix} S\widetilde{S}y \\ -BS\widetilde{S}y \end{pmatrix},$$

so $S\widetilde{S} = 1$. Thus S is invertible and $\widetilde{T}TA = \widetilde{T}BS = A\widetilde{S}S = A$ implies that $\widetilde{T}T = 1$, and $T\widetilde{T}B = TA\widetilde{S} = BS\widetilde{S} = B$ implies that $T\widetilde{T} = 1$. Thus T is also invertible. Also, $\widetilde{T}(N) \subseteq M$ gives $N \subseteq T(M)$, so $T(M) = N$. Since $TA = BS$, $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$.

(iv) $\Rightarrow$ (ii). It is easy to verify that $T\mathcal{A}(A;M)T^{-1} = \mathcal{A}(B;N)$ if $T \in B(H)$ has properties as described in (iv).

(ii) $\Rightarrow$ (iii). Obviously, if $T \in B(H)$ is invertible and $T\mathcal{A}(A;M)T^{-1} = \mathcal{A}(B;N)$ then $T\mathcal{A}_0(A;M)T^{-1} = \mathcal{A}_0(B;N)$.

(iii) $\Rightarrow$ (i). Let $T \in B(H)$ be an invertible operator satisfying $T\mathcal{A}_0(A;M)T^{-1} = \mathcal{A}_0(B;N)$. Let $0 \neq f \in \operatorname{Ran}(A)$ and $0 \neq e \in M^\perp$. Then, by Proposition 4.1 (ii), $e \otimes f \in \mathcal{A}_0(A;M)$ so $T(e \otimes f)T^{-1} = T^{*-1}e \otimes Tf \in \mathcal{A}_0(B;N)$. By the same proposition, $Tf \in \operatorname{Ran}(B)$ and $T^{*-1}e \in N^\perp$. Thus $T\operatorname{Ran}(A) \subseteq \operatorname{Ran}(B)$ and $T^{*-1}M^\perp = (TM)^\perp \subseteq N^\perp$ so $N \subseteq T(M)$. Similarly, $T^{-1}\operatorname{Ran}(B) \subseteq \operatorname{Ran}(A)$ and $M \subseteq T^{-1}(N)$. Hence $T\operatorname{Ran}(A) = \operatorname{Ran}(B)$ and $T(M) = N$. From the former $TA = BS$ for a unique invertible operator S. The operator $T_0 = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ is invertible and maps $G(-A)$, $G(A)$ and $G(A) + (0) \oplus M$ onto, respectively, $G(-B)$, $G(B)$ and $G(B) + (0) \oplus N$. By Proposition 3.6, $T_0 \operatorname{Alg}\mathcal{P}(A;M)T_0^{-1} = \operatorname{Alg}\mathcal{P}(B;N)$. ∎

On the Hilbert space H no invertible operator can map the range of a compact operator onto the range of an operator which is not compact. Hence, using the preceding theorem, to show that, given $n \in \mathbb{Z}^+ \cup \{\infty\}$ and a specified reflexivity condition (that is, either reflexive or non-reflexive) there exist pentagons $\mathcal{P}(A;M)$ and $\mathcal{P}(B;N)$ on separable Hilbert space, each having gap-dimension n and each satisfying the specified reflexivity condition, yet whose Alg's (so whose associated algebras $\mathcal{A}(A;M)$ and $\mathcal{A}(B;N)$) are not spatially isomorphic, it is enough to provide such examples with A compact and B non-compact.

In Example 2.3 the operator A can be either compact or non-compact and the dimension of the subspace M can be any positive integer or infinity. In any case the resulting pentagon is non-reflexive.

Proposition 2.2 provides examples for the reflexive case with one exception. Indeed, if A is a positive, injective, non-invertible (compact or non-compact) operator on H and $e \in H$ is a vector satisfying $e \notin \operatorname{Ran}(A)$, then $\mathcal{P} = \mathcal{P}(A;\langle e\rangle)$ is a reflexive pentagon on $H \oplus H$. Thus, by Proposition 2.2, for any $n \in \mathbb{Z}^+ \cup \{\infty\}$, $\mathcal{P}(A^{(n)};\langle e\rangle^{(n)})$ (which can be identified with $\mathcal{P}^{(n)}$) is a reflexive pentagon with gap-dimension n. For finite n, the operator $A^{(n)}$ is compact if and only if A is. On the other hand, $A^{(\infty)}$ will always be non-compact. An example supplying the last remaining case immediately follows.

EXAMPLE 4.3. We show that, on a separable Hilbert space, there exists a reflexive pentagon $\mathcal{P}(B;N)$ with B compact and N infinite-dimensional.

On $\mathcal{H} = \ell^2(\mathbb{Z}^+)$ let S^* be the backward shift operator. Let $0 < a < 1$ and let A be the positive, injective, compact operator given by $A = \operatorname{diag}(1, a, a^2, a^3, \ldots)$ (with respect to the usual basis). Also, let $e \in \mathcal{H}$ be the vector $e = (1, a, a^2, a^3, \ldots)$. Then $e \notin \operatorname{Ran}(A)$ and $S^*A = aAS^*$, $S^*e = ae$.

For every $j \in \mathbb{Z}^+$, $\mathcal{P}(a^{j-1}A;\langle e\rangle)$ is a pentagon on $\mathcal{H} \oplus \mathcal{H}$. For every $i, j \in \mathbb{Z}^+$ let R_{ij} be the operator on $\mathcal{H} \oplus \mathcal{H}$ given by $R_{ij} = \begin{pmatrix} aS^* & 0 \\ 0 & a^{i-j}S^* \end{pmatrix}$. Then R_{ij} maps $G(-a^{j-1}A)$, $G(a^{j-1}A)$ and $G(a^{j-1}A) + (0) \oplus \langle e\rangle$ onto, respectively, $G(-a^{i-1}A)$, $G(a^{i-1}A)$ and $G(a^{i-1}A) + (0) \oplus \langle e\rangle$.

On $H = \mathcal{H}^{(\infty)}$, let B be the operator $B = \operatorname{diag}(A, aA, a^2A, a^3A, \ldots)$. Then B is positive and injective. Also, B is compact. If $F \in B(\mathcal{H})$ is a finite rank operator then, for every $m \in \mathbb{Z}^+$,

$$\|B - \operatorname{diag}(F, aF, a^2F, \ldots, a^{m-1}F, 0, 0, \ldots)\| \leqslant \|A - F\| + a^m$$

and $\operatorname{diag}(F, aF, a^2F, \ldots, a^{m-1}F, 0, 0, \ldots)$ has finite rank. Let N be the subspace of H given by $N = \langle e \rangle^{(\infty)} = \{(\lambda_k e)_1^\infty : (\lambda_k)_1^\infty \in l^2(\mathbb{Z}^+)\}$. Then $N \cap \operatorname{Ran}(B) = (0)$, and so $\mathcal{P}(B; N)$ is a pentagon on $H \oplus H$ with infinite gap-dimension.

We show that $\mathcal{P} = \mathcal{P}(B; N)$ is reflexive. Suppose that it is not. Let $L \in \operatorname{Lat}\operatorname{Alg}\mathcal{P} \setminus \mathcal{P}$. Then $G(B) \subset L \subset G(B) + (0) \oplus N$. Let $y \in L$, $y \notin G(B)$. Then $y = (x_k)_1^\infty \oplus (a^{k-1}Ax_k + \lambda_k e)_1^\infty$ where, because $y \notin G(B)$, we have $\lambda_j \neq 0$ for some $j \geqslant 1$. For this j and for every $i \geqslant 1$, the operator $\widetilde{R}_{ij} : H \to H$ which sends $(u_k)_1^\infty \oplus (v_k)_1^\infty \in H \oplus H$ to $(\delta_{ik} a S^* u_j)_{k=1}^\infty \oplus (\delta_{ik} a^{i-j} S^* v_j)_{k=1}^\infty$, belongs to $\operatorname{Alg}\mathcal{P}$. Hence applying these operators (as i varies) to y and using the fact that $G(B) \subseteq L$, we obtain $0 \oplus (\delta_{ik} e)_{k=1}^\infty \in L$, for every $i \geqslant 1$. Thus $(0) \oplus N \subseteq L$ so $L = G(B) + (0) \oplus N$. This is a contradiction.

Actually, if in the pentagons $\mathcal{P}(A; M)$ and $\mathcal{P}(B; N)$ one of the operators A, B is compact and the other is non-compact, then $\operatorname{Alg}\mathcal{P}(A; M)$ and $\operatorname{Alg}\mathcal{P}(B; N)$ cannot even be algebraically isomorphic as the following shows.

THEOREM 4.4. *Let $A, B \in B(H)$ be positive, injective, non-invertible operators and let M, N be non-zero subspaces of H satisfying $M \cap \operatorname{Ran}(A) = N \cap \operatorname{Ran}(B) = (0)$. Let $\mathcal{P}(A; M), \mathcal{P}(B; N)$ be the corresponding pentagons on $H \oplus H$ and let $\varphi : \operatorname{Alg}\mathcal{P}(A; M) \to \operatorname{Alg}\mathcal{P}(B; N)$ be an algebraic isomorphism. Then:*

(i) *if φ preserves the types of rank one operators, there exists an invertible operator $T \in B(H)$ such that $T \operatorname{Ran}(A) = \operatorname{Ran}(B)$;*

(ii) *if φ switches the types of rank one operators, there exist operators $V, W \in B(H)$, injective on $\operatorname{Ran}(A)$ and $\operatorname{Ran}(B)$, respectively, such that $V \operatorname{Ran}(A) = \operatorname{Ran}(B)$ and $W \operatorname{Ran}(B) = \operatorname{Ran}(A)$.*

Proof. By Theorem 3.4, φ is quasi-spatial. As in the proof of that theorem (particularized to the present situation) there exists a linear transformation T_0 mapping $G(-A) + G(A)$ injectively onto $G(-B) + G(B)$ with both $T_0|G(-A)$ and $T_0|G(A)$ continuous. Put $T_0|G(A) = T_+$ and $T_0|G(-A) = T_-$. Noting that $G(\pm A) + (0) \oplus H = H \oplus H$, $T_\pm$ can be extended to be operators on $H \oplus H$ by defining each of them to be zero on $(0) \oplus H$. Now $T_\pm \in B(H \oplus H)$ must have the forms $T_- = \begin{pmatrix} P & 0 \\ R & 0 \end{pmatrix}$ and $T_+ = \begin{pmatrix} Q & 0 \\ S & 0 \end{pmatrix}$. For every $x, y \in H$,

$$T_0 \begin{pmatrix} x \\ -Ax \end{pmatrix} = T_- \begin{pmatrix} x \\ -Ax \end{pmatrix} = \begin{pmatrix} P & 0 \\ R & 0 \end{pmatrix} \begin{pmatrix} x \\ -Ax \end{pmatrix} = \begin{pmatrix} Px \\ Rx \end{pmatrix},$$

and

$$T_0 \begin{pmatrix} y \\ Ay \end{pmatrix} = T_+ \begin{pmatrix} y \\ Ay \end{pmatrix} = \begin{pmatrix} Q & 0 \\ S & 0 \end{pmatrix} \begin{pmatrix} y \\ Ay \end{pmatrix} = \begin{pmatrix} Qy \\ Sy \end{pmatrix},$$

so

$$T_0 \begin{pmatrix} x + y \\ -Ax + Ay \end{pmatrix} = \begin{pmatrix} Px + Qy \\ Rx + Sy \end{pmatrix}.$$

Now $\begin{pmatrix} A & 1 \\ A^2 & A \end{pmatrix} \in \operatorname{Alg}\mathcal{P}(A;M)$ and $\begin{pmatrix} B & 1 \\ B^2 & B \end{pmatrix} \in \operatorname{Alg}\mathcal{P}(B;N)$. Let

$$\varphi\begin{pmatrix} A & 1 \\ A^2 & A \end{pmatrix} = \begin{pmatrix} X'+Z'B & Z' \\ BZ'B & Y'+BZ' \end{pmatrix},$$

where $X', Y', Z' \in B(H)$, $Y'B = BX'$ and $Y'(N) \subseteq N$. Let

$$\begin{pmatrix} B & 1 \\ B^2 & B \end{pmatrix} = \varphi\begin{pmatrix} X+ZA & Z \\ AZA & Y+AZ \end{pmatrix},$$

where $X, Y, Z \in B(H)$, $YA = AX$ and $Y(M) \subseteq M$. Then, for every $x, y \in H$,

$$\begin{pmatrix} X'+Z'B & Z' \\ BZ'B & Y'+BZ' \end{pmatrix} T_0 \begin{pmatrix} x+y \\ -Ax+Ay \end{pmatrix} = T_0 \begin{pmatrix} A & 1 \\ A^2 & A \end{pmatrix} \begin{pmatrix} x+y \\ -Ax+Ay \end{pmatrix},$$

and

$$\begin{pmatrix} B & 1 \\ B^2 & B \end{pmatrix} T_0 \begin{pmatrix} x+y \\ -Ax+Ay \end{pmatrix} = T_0 \begin{pmatrix} X+ZA & Z \\ AZA & Y+AZ \end{pmatrix} \begin{pmatrix} x+y \\ -Ax+Ay \end{pmatrix}.$$

Case (i). Suppose that φ preserves the types of rank one operators. Then $T_0G(-A) = G(-B)$ and $T_0G(A) = G(B)$ so $R = -BP$ and $S = BQ$. Clearly P and Q are invertible operators. The two equations above become, respectively,

$$\begin{pmatrix} X'Px + (X'+2Z'B)Qy \\ -BX'Px + B(X'+2Z'B)Qy \end{pmatrix} = \begin{pmatrix} 2QAy \\ 2BQAy \end{pmatrix}$$

and

$$\begin{pmatrix} 2BQy \\ 2B^2Qy \end{pmatrix} = \begin{pmatrix} PXx + Q(X+2ZA)y \\ -BPXx + BQ(X+2ZA)y \end{pmatrix}.$$

Since these are true for every $x, y \in H$, then $X' = 0$, $Z'BQ = QA$, $X = 0$ and $QZA = BQ$. Thus $AQ^{-1} = WB$ and $BQ = VA$ where $W = Q^{-1}Z'$ and $V = QZ$. Then $WBQ = WVA = A$ so $WV = 1$, and $VAQ^{-1} = VWB = B$ so $VW = 1$. Hence $T = V$ is invertible and $T\operatorname{Ran}(A) = \operatorname{Ran}(B)$.

Case (ii). Suppose that φ switches the types of rank one operators. Then $T_0G(-A) = G(B)$ and $T_0G(A) = G(-B)$. In this case $R = BP$ and $S = -BQ$, and P, Q are again invertible. We get

$$\begin{pmatrix} (X'+2Z'B)Px + X'Qy \\ B(X'+2Z'B)Px - BX'Qy \end{pmatrix} = \begin{pmatrix} 2QAy \\ -2BQAy \end{pmatrix}$$

and

$$\begin{pmatrix} 2BPx \\ 2B^2Px \end{pmatrix} = \begin{pmatrix} PXx + Q(X+2ZA)y \\ BPXx - BQ(X+2ZA)y \end{pmatrix}.$$

It follows that $X' = -2Z'B$, $X'Q = 2QA$, $X = -2ZA$ and $PX = 2BP$. Hence $VA = BP$ and $WB = AQ^{-1}$ with $V = -PZ$ and $W = -Q^{-1}Z'$. Clearly V and W are injective on $\operatorname{Ran}(A)$ and $\operatorname{Ran}(B)$, respectively, and $V\operatorname{Ran}(A) = \operatorname{Ran}(B)$, $W\operatorname{Ran}(B) = \operatorname{Ran}(A)$. ∎

REMARKS 4.5. (1) With A, B as in the statement of the preceding theorem, if $V \in B(H)$ is an operator, injective on $\operatorname{Ran}(A)$ and satisfying $V \operatorname{Ran}(A) = \operatorname{Ran}(B)$, it need not be true that V is invertible. In fact, V need not even be injective (on H). An example of this is given in the proof of Proposition 4.11.

(2) Let $\mathcal{P}(A; M)$ and $\mathcal{P}(A; N)$ be pentagons on $H \oplus H$ (as described at the beginning of this section). Suppose that there exists an operator range, $\mathcal{S}$ say, which is invariant under every operator which leaves $\operatorname{Ran}(A)$ invariant, and which satisfies $M \subseteq \mathcal{S}$, $N \not\subseteq \mathcal{S}$. Then $\operatorname{Alg} \mathcal{P}(A; M)$ and $\operatorname{Alg} \mathcal{P}(A; N)$ (and so $\mathcal{A}(A; M)$ and $\mathcal{A}(A; N)$) cannot be spatially isomorphic. If they were, then by Theorem 4.2, there would exist an invertible operator $T \in B(H)$ such that $T \operatorname{Ran}(A) = \operatorname{Ran}(A)$ and $T(M) = N$. But since this T leaves $\operatorname{Ran}(A)$ invariant, $N = T(M) \subseteq T(\mathcal{S}) \subseteq \mathcal{S}$, so $N \subseteq \mathcal{S}$. The next two propositions follow from this observation and Example 2.3.

PROPOSITION 4.6. *For every positive, injective, non-invertible operator $A \in B(H)$ and every integer $n \geqslant 2$, there exists a sequence $(\mathcal{P}(A; M_k))_1^\infty$ of non-reflexive pentagons, each having gap-dimension n, no two of whose Alg's are spatially isomorphic.*

Proof. Let A and $n \geqslant 2$ be given. Let $y \in \operatorname{Ran}(A^{1/2}) \setminus \operatorname{Ran}(A)$. Let $(N_k)_1^\infty$ be a sequence of $(n-1)$-dimensional subspaces satisfying $N_k \cap \operatorname{Ran}(A^{2^{-k}}) = (0)$ and $N_k \subseteq \operatorname{Ran}(A^{2^{-k-1}})$ for every $k \geqslant 1$. Let $M_k = N_k + \langle y \rangle$ $(k \geqslant 1)$. Then, for every $k \geqslant 1$, as in Example 2.3, $\mathcal{P}(A; M_k)$ is a non-reflexive pentagon on $H \oplus H$ with gap-dimension n. Let $k > \ell \geqslant 1$. Then $\operatorname{Ran}(A^{2^{-\ell-1}}) \subseteq \operatorname{Ran}(A^{2^{-k}})$ so, although $M_\ell \subseteq \operatorname{Ran}(A^{2^{-\ell-1}})$, $M_k \not\subseteq \operatorname{Ran}(A^{2^{-\ell-1}})$. (In fact $M_k \cap \operatorname{Ran}(A^{2^{-\ell-1}}) \subseteq M_k \cap \operatorname{Ran}(A^{2^{-k}}) = \langle y \rangle$.) Moreover, by Foiaş's theorem, $\operatorname{Ran}(A^{2^{-\ell-1}})$ is invariant under every operator which leaves $\operatorname{Ran}(A)$ invariant. Hence $\operatorname{Alg} \mathcal{P}(A; M_\ell)$ and $\operatorname{Alg} \mathcal{P}(A; M_k)$ are not spatially isomorphic by the preceding remark. ∎

PROPOSITION 4.7. *For every positive, injective, non-invertible operator $A \in B(H)$ there exists a sequence $(\mathcal{P}(A; K_k))_1^\infty$ of pentagons each having gap-dimension one (so reflexive), no two of whose Alg's are spatially isomorphic.*

Proof. Let A be given. For each $k \geqslant 1$ choose a vector $e_k \in H$ satisfying $e_k \notin \operatorname{Ran}(A^{2^{-k+1}})$, $e_k \in \operatorname{Ran}(A^{2^{-k}})$. Put $K_k = \langle e_k \rangle$ $(k \geqslant 1)$. Then, for every $k \geqslant 1$, $\mathcal{P}(A; K_k)$ is a pentagon on $H \oplus H$ with gap-dimension one. If $k > \ell \geqslant 1$, then $e_\ell \in \operatorname{Ran}(A^{2^{-\ell}})$ and $e_k \notin \operatorname{Ran}(A^{2^{-\ell}})$. By Foiaş's theorem and our earlier remark, $\operatorname{Alg} \mathcal{P}(A; K_\ell)$ and $\operatorname{Alg} \mathcal{P}(A; K_k)$ are not spatially isomorphic. ∎

On a more positive note, however, we have the following.

PROPOSITION 4.8. *Let $A \in B(H)$ be a positive, injective, non-invertible operator. Let $e, f \in H$ satisfy $e \notin \operatorname{Ran}(A)$, $f \notin \operatorname{Ran}(A)$ and let $\mathcal{P}(A; \langle e \rangle)$ and $\mathcal{P}(A; \langle f \rangle)$ be the corresponding pentagons. If e and f are linearly dependent modulo* $\operatorname{Ran}(A)$, *then* $\operatorname{Alg} \mathcal{P}(A; \langle e \rangle)$ *and* $\operatorname{Alg} \mathcal{P}(A; \langle f \rangle)$ *are spatially isomorphic.*

Proof. Let e, f be linearly dependent modulo $\operatorname{Ran}(A)$. By Theorem 4.2 it is enough to show that there exists an invertible operator $T \in B(H)$ such that $T \operatorname{Ran}(A) = \operatorname{Ran}(A)$ and $T\langle e \rangle = \langle f \rangle$. If e and f are linearly dependent we can take $T = I$. Suppose that e and f are linearly independent. There exist non-zero scalars α, β such that $\alpha e + \beta f \in \operatorname{Ran}(A)$. We can assume that $\alpha = 1$. Let $e + \beta f = Ah$.

Now $(1-E)f \neq 0$ where $E = P_{\langle e\rangle}$. Put $g = (ae + b(1-E)f)/(a\|e\|^2)$ where $a = \overline{\beta}\|(1-E)f\|^2$ and $b = -(\|e\|^2 + \overline{\beta}(e|f))$. Put $R = g \otimes h$. Note that

$$\begin{aligned}(Ah|ae + b(1-E)f) &= (e + \beta f|ae + b(1-E)f) \\ &= \overline{a}\|e\|^2 + \beta\overline{a}(f|e) + \beta\overline{b}\|(1-E)f\|^2 \\ &= -\overline{a}\overline{b} + \beta\overline{b}\|(1-E)f\|^2 \\ &= -\overline{b}(\overline{a} - \beta\|(1-E)f\|^2) = 0.\end{aligned}$$

Thus $(Ah|g) = 0$ so $RAR = 0$. Put $T = 1 - AR$ and $S = 1 - RA$. Then T and S are invertible with $T^{-1} = 1 + AR$ and $S^{-1} = 1 + RA$. Also $TA = AS$ so $T\,\mathrm{Ran}(A) = \mathrm{Ran}(A)$. Since $Te = e - ARe = e - (e|g)Ah = e - Ah = \beta f$, $T\langle e\rangle = \langle f\rangle$. ∎

What precisely is the analogue of Theorem 4.2 for algebraic, as opposed to spatial, isomorphisms is not yet clear. However, we do have the following.

THEOREM 4.9. *Let $A, B \in B(H)$ be positive, injective, non-invertible operators and let M, N be non-zero subspaces of H satisfying $M \cap \mathrm{Ran}(A) = N \cap \mathrm{Ran}(B) = (0)$. Let $\mathcal{P}(A;M), \mathcal{P}(B;N)$ be the corresponding pentagons on $H \oplus H$. Concerning the following three statements,* (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii)*.*

(i) $\mathrm{Alg}\,\mathcal{P}(A;M)$ *and* $\mathrm{Alg}\,\mathcal{P}(B;N)$ *are algebraically isomorphic by an isomorphism that preserves the types of rank one operators.*

(ii) $\mathcal{A}(A;M)$ *and* $\mathcal{A}(B;N)$ *are algebraically isomorphic.*

(iii) $\mathcal{A}_0(A;M)$ *and* $\mathcal{A}_0(B;N)$ *are algebraically isomorphic.*

Proof. (i) $\Rightarrow$ (ii). Suppose that there exists an algebraic isomorphism $\varphi : \mathrm{Alg}\,\mathcal{P}(A;M) \to \mathrm{Alg}\,\mathcal{P}(B;N)$ that preserves the types of rank one operators. Note that, for the pentagon $\mathcal{P}(A;M)$ and with the obvious notation, $\mathcal{R}_A^- = \left\{\begin{pmatrix} -RA & R \\ ARA & -AR\end{pmatrix} : R = e \otimes f \text{ with } 0 \neq f \in H \text{ and } 0 \neq e \in M^\perp\right\}$. Also note that $\{T \in \mathrm{Alg}\,\mathcal{P}(A;M) : TF = 0, \text{ for every } F \in \mathcal{R}_A^-\}$ is the set $\left\{\begin{pmatrix} ZA & Z \\ AZA & AZ\end{pmatrix} : Z \in B(H)\right\}$. Similar notes apply to $\mathcal{P}(B;N)$. Since $\varphi(\mathcal{R}_A^-) = \mathcal{R}_B^-$, it follows that φ maps the set $\left\{\begin{pmatrix} ZA & Z \\ AZA & AZ\end{pmatrix} : Z \in B(H)\right\}$ onto the set $\left\{\begin{pmatrix} Z'B & Z' \\ BZ'B & BZ'\end{pmatrix} : Z' \in B(H)\right\}$.

Let $Y \in \mathcal{A}(A;M)$. Then $YA = AX$ for a unique operator $X \in B(H)$, and $\begin{pmatrix} X & 0 \\ 0 & Y\end{pmatrix} \in \mathrm{Alg}\,\mathcal{P}(A;M)$. Thus, there exist unique operators $X', Y', Z' \in B(H)$ with $Y' \in \mathcal{A}(B;N)$ and $Y'B = BX'$, such that

$$\varphi\begin{pmatrix} X & 0 \\ 0 & Y\end{pmatrix} = \begin{pmatrix} X' + Z'B & Z' \\ BZ'B & Y' + BZ'\end{pmatrix}.$$

This defines a mapping $\theta : \mathcal{A}(A;M) \to \mathcal{A}(B;N)$ given by $\theta(Y) = Y'$. This mapping is clearly linear. We show that it is an algebraic isomorphism.

θ is injective: Let $\theta(Y) = 0$. Then $\varphi\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} Z'B & Z' \\ BZ'B & BZ' \end{pmatrix}$, for some $Z' \in B(H)$. But $\begin{pmatrix} Z'B & Z' \\ BZ'B & BZ' \end{pmatrix} = \varphi\begin{pmatrix} ZA & Z \\ AZA & AZ \end{pmatrix}$, for some $Z \in B(H)$. Hence, since φ is injective, $Z = 0$ and $Y = AZ = 0$.

θ is multiplicative: Let $Y_1, Y_2 \in \mathcal{A}(A; M)$. Then

$$\varphi\left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}\begin{pmatrix} X_2 & 0 \\ 0 & Y_2 \end{pmatrix}\right) = \varphi\begin{pmatrix} X_1X_2 & 0 \\ 0 & Y_1Y_2 \end{pmatrix} = \begin{pmatrix} * & Z''' \\ * & \theta(Y_1Y_2) + BZ''' \end{pmatrix}$$

and

$$\varphi\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}\varphi\begin{pmatrix} X_2 & 0 \\ 0 & Y_2 \end{pmatrix} = \begin{pmatrix} X' + Z'B & Z' \\ BZ'B & \theta(Y_1) + BZ' \end{pmatrix}\begin{pmatrix} * & Z'' \\ * & \theta(Y_2) + BZ'' \end{pmatrix}$$
$$= \begin{pmatrix} * & X'Z'' + 2Z'BZ'' + Z'\theta(Y_2) \\ * & \theta(Y_1)\theta(Y_2) + 2BZ'BZ'' + \theta(Y_1)BZ'' + BZ'\theta(Y_2) \end{pmatrix}.$$

Thus $Z''' = X'Z'' + 2Z'BZ'' + Z'\theta(Y_2)$ and $\theta(Y_1Y_2) = \theta(Y_1)\theta(Y_2) - B(X'Z'' + 2Z'BZ'' + Z'\theta(Y_2)) + 2BZ'BZ'' + \theta(Y_1)BZ'' + BZ'\theta(Y_2) = \theta(Y_1)\theta(Y_2) - BX'Z'' + \theta(Y_1)BZ'' = \theta(Y_1)\theta(Y_2)$, since $\theta(Y_1)B = BX'$.

θ maps onto $\mathcal{A}(B; N)$: Let $Y' \in \mathcal{A}(B; N)$. Then $Y'B = BX'$ for a unique operator $X' \in B(H)$, and there exist operators $X, Y, Z \in B(H)$ with $Y \in \mathcal{A}(A; M)$ and $YA = AX$ such that $\varphi\begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix} = \begin{pmatrix} X' & 0 \\ 0 & Y' \end{pmatrix}$. Now

$$\varphi\begin{pmatrix} X + ZA & Z \\ AZA & Y + AZ \end{pmatrix} = \varphi\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} + \varphi\begin{pmatrix} ZA & Z \\ AZA & AZ \end{pmatrix}$$
$$= \begin{pmatrix} * & Z'' \\ * & \theta(Y) + BZ'' \end{pmatrix} + \begin{pmatrix} * & Z''' \\ 0 & BZ''' \end{pmatrix}$$
$$= \begin{pmatrix} * & Z'' + Z''' \\ * & \theta(Y) + B(Z'' + Z''') \end{pmatrix},$$

so $Z'' + Z''' = 0$ and $\theta(Y) = Y'$.

Thus $\theta : \mathcal{A}(A; M) \to \mathcal{A}(B; N)$ is an algebraic isomorphism.

(ii) $\Rightarrow$ (iii). Let $\theta : \mathcal{A}(A; M) \to \mathcal{A}(B; N)$ be an algebraic isomorphism. By Proposition 4.1 (iii) and (iv), the algebra $\mathcal{A}_0(A; M)$ has a purely algebraic description as a subset of $\mathcal{A}(A; M)$, namely, $\mathcal{A}_0(A; M) = \{F \in \mathcal{A}(A; M) : F$ is a finite sum of single elements$\}$. A similar comment applies to $\mathcal{A}_0(B; N)$ as a subset of $\mathcal{A}(B; N)$. Hence $\theta(\mathcal{A}_0(A; M)) = \mathcal{A}_0(B; N)$. ∎

REMARKS 4.10. (1) With (i), (ii) and (iii) the statements in the preceding theorem, it is not known whether the implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are always valid. Neither is it known, despite Theorem 3.4 and Proposition 4.1 (iii) and (iv), whether every algebraic isomorphism $\theta : \mathcal{A}(A; M) \to \mathcal{A}(B; N)$ must be quasi-spatial.

(2) In connection with Question 3.7 raised earlier, and with the valid implication (i) $\Rightarrow$ (ii) of the preceding theorem in mind, it would be interesting to know whether $\dim M$ must equal $\dim N$ if $\mathcal{A}(A; M)$ and $\mathcal{A}(B; N)$ are algebraically isomorphic; they need not be if only their finite rank subalgebras are isomorphic.

Proposition 4.11. *There exist positive, injective, non-invertible operators $A, B \in B(H)$ and subspaces M, N of H with $\dim M = 2$ and $\dim N = 1$ such that $M \cap \operatorname{Ran}(A) = N \cap \operatorname{Ran}(B) = (0)$ with the algebras $\mathcal{A}_0(A; M)$ and $\mathcal{A}_0(B; N)$ algebraically isomorphic.*

Proof. Let $(e_n)_1^\infty$ be an arbitrary fixed orthonormal basis for H. Let S be the forward shift operator with respect to this basis. Let $\mathcal{S}_1$ be a proper dense operator range not containing e_1. Then $\mathcal{S}_1 = \operatorname{Ran}(A)$ for some positive, injective, non-invertible operator $A \in B(H)$. Now $S^*\mathcal{S}_1 = \operatorname{Ran}(S^*A)$ is proper and dense. For, AS is injective and if $\operatorname{Ran}(S^*A) = H$, then $SS^*\mathcal{S}_1 = SH = \langle e_1\rangle^\perp \subseteq \mathcal{S}_1 + \langle e_1\rangle$ (using $SS^* = 1 - P_{\langle e_1\rangle}$), so $H = \mathcal{S}_1 + \langle e_1\rangle$. This would contradict the fact that $\mathcal{S}_1$ is of infinite codimension in H ([3], p. 262). Thus $S^* \operatorname{Ran}(A) = \operatorname{Ran}(B)$ for some positive, injective, non-invertible operator $B \in B(H)$. Of course $\ker S^* = \langle e_1\rangle$. Let M be any 2-dimensional subspace satisfying $e_1 \in M, M \cap \operatorname{Ran}(A) = (0)$. Put $N = S^*(M)$. Then $\dim N = 1$. Also $N \cap \operatorname{Ran}(B) = (0)$. For, let $x \in N \cap \operatorname{Ran}(B)$. Then $x = S^*y = S^*z$ with $y \in M$ and $z \in \operatorname{Ran}(A)$. Then $z - y \in \ker S^* = \langle e_1\rangle \subseteq M$ so $z \in M \cap \operatorname{Ran}(A) = (0)$ and $z = 0$. Then $x = S^*z = 0$.

Now consider the mapping $\theta : \mathcal{A}_0(A; M) \to B(H)$ given by $\theta(X) = S^*XS$. This mapping is clearly linear. Note that, for every $F \in \mathcal{A}_0(A; M)$, $Fe_1 = 0$ and $\operatorname{Ran}(F) \subseteq \operatorname{Ran}(A)$.

θ is multiplicative: Let $X, Y \in \mathcal{A}_0(A; M)$. Then $\theta(X)\theta(Y) = S^*XSS^*YS = S^*X(1 - P_{\langle e_1\rangle})YS = S^*XYS = \theta(XY)$.

θ is injective: Let $Z \in \mathcal{A}_0(A; M)$ and suppose that $\theta(Z) = 0$. Then $S^*ZS = 0$ so $(1 - P_{\langle e_1\rangle})Z(1 - P_{\langle e_1\rangle}) = 0 = (1 - P_{\langle e_1\rangle})Z$. Hence $\operatorname{Ran}(Z) \subseteq \langle e_1\rangle$. But $\operatorname{Ran}(Z) \subseteq \operatorname{Ran}(A)$ as well, so $Z = 0$.

θ maps $\mathcal{A}_0(A; M)$ into $\mathcal{A}_0(B; N)$: Let $X \in \mathcal{A}_0(A; M)$. Then $\operatorname{Ran}(\theta(X)) = \operatorname{Ran}(S^*XS) \subseteq S^* \operatorname{Ran}(X) \subseteq S^* \operatorname{Ran}(A) = \operatorname{Ran}(B)$. Also,

$$\begin{aligned}\theta(X)(N) &= S^*XS(N) = S^*XSS^*(M) = S^*X(1 - P_{\langle e_1\rangle})(M)\\ &= S^*X(M) = S^*(0) = (0).\end{aligned}$$

Since $\theta(X)$ clearly has finite rank, it follows by Proposition 4.1 that $\theta(X) \in \mathcal{A}_0(B; N)$.

θ maps $\mathcal{A}_0(A; M)$ onto $\mathcal{A}_0(B; N)$: Let $Y' \in \mathcal{A}_0(B; N)$. Then $Y' = BZ'$ for some finite rank operator $Z' \in B(H)$ satisfying $Z'(N) = (0)$. Let Q be the unique operator satisfying $S^*A = BQ$. If $Qx = 0$, then $S^*Ax = 0$ so $Ax \in \ker S^* = \langle e_1\rangle$. But $Ax \in \operatorname{Ran}(A)$ as well, so $Ax = 0$ and $x = 0$. Thus Q is injective. Since it is easily shown that $\operatorname{Ran}(Q) = H$, Q is invertible. Then $AQ^{-1}Z'S^*$ has finite rank, has range contained in $\operatorname{Ran}(A)$ and $AQ^{-1}Z'S^*(M) = AQ^{-1}Z'(N) = (0)$. Thus $AQ^{-1}Z'S^* \in \mathcal{A}_0(A; M)$. Also, $\theta(AQ^{-1}Z'S^*) = S^*AQ^{-1}Z'S^*S = S^*AQ^{-1}Z' = BZ' = Y'$.

Thus $\theta : \mathcal{A}_0(A; M) \to \mathcal{A}_0(B; N)$ is an algebraic isomorphism. ∎

We conclude by briefly considering the 2-atom ABSL $\mathcal{D}(A)$ associated with the pentagon $\mathcal{P}(A;M)$. By definition

$$\mathcal{D}(A) = \{(0), G(-A), G(A), H \oplus H\}\,.$$

It is easily shown that

$$\operatorname{Alg}\mathcal{D}(A) = \left\{\begin{pmatrix} X+ZA & Z \\ AZA & Y+AZ \end{pmatrix} : X, Y, Z \in B(H),\ Y \in \mathcal{A}(A) \text{ and } YA = AX\right\},$$

where the unital algebra $\mathcal{A}(A)$ is defined by

$$\mathcal{A}(A) = \{T \in B(H) : T\operatorname{Ran}(A) \subseteq \operatorname{Ran}(A)\}\,.$$

Analogously, we let $\mathcal{A}_0(A)$ denote the algebra of finite rank operators of $\mathcal{A}(A)$. The operator algebras of the form $\mathcal{A}(A)$ (with $A \in B(H)$ a positive, injective and non-invertible operator) were introduced, and extensively studied, in [20]. For Boolean algebras of the form $\mathcal{D}(A)$ the theory is much closer to being complete. For example, every algebraic isomorphism $\varphi : \operatorname{Alg}\mathcal{D}(A) \to \operatorname{Alg}\mathcal{D}(B)$ is spatial ([14]). (This result has been extended to arbitrary 2-atom ABSL's on separable Hilbert spaces ([22]).) The results presented in the theorem below should be compared with the analogous results that we have obtained for pentagons above.

PROPOSITION 4.12. *For every operator algebra on H of the form $\mathcal{A}(A)$ described above:*

(i) $\mathcal{A}_0(A) = \{AX : X \in B(H)$ *and X has finite rank*$\}$*;*

(ii) *for every pair e, f of non-zero vectors of H, $e \otimes f \in \mathcal{A}(A)$ if and only if $f \in \operatorname{Ran}(A)$;*

(iii) *for every $n \in \mathbb{Z}^+$, every element of $\mathcal{A}(A)$ of rank n is the sum of n rank one elements of $\mathcal{A}(A)$;*

(iv) *every non-zero single element of $\mathcal{A}(A)$ has rank one.*

A proof of this proposition follows immediately by taking $M = (0)$ in the proof of Proposition 4.1.

We thank H. Radjavi for assisting in observing statement (i)(b) in the following theorem. The theorem is a significant strengthening of the main result of [23].

THEOREM 4.13. *Let A, B be positive, injective, non-invertible operators on the separable Hilbert space H. Let $\mathcal{D}(A), \mathcal{D}(B)$ be the corresponding 2-atom Boolean algebras on $H \oplus H$, and let $\mathcal{A}(A), \mathcal{A}(B)$ and $\mathcal{A}_0(A), \mathcal{A}_0(B)$ be the corresponding operator algebras on H, as described above. Then:*

(i) *for each pair of algebras*

(a) $\operatorname{Alg}\mathcal{D}(A)$, $\operatorname{Alg}\mathcal{D}(B)$,

(b) $\mathcal{A}(A)$, $\mathcal{A}(B)$,

(c) $\mathcal{A}_0(A)$, $\mathcal{A}_0(B)$,

every algebraic isomorphism between them is spatial;

(ii) *for each pairs described in* (i)*, the algebras are algebraically isomorphic if and only if there exists an invertible operator $T \in B(H)$ such that $T\operatorname{Ran}(A) = \operatorname{Ran}(B)$;*

(iii) *for each pairs* (b) *and* (c) *described in* (i)*, every invertible operator $T \in B(H)$ satisfying $T\operatorname{Ran}(A) = \operatorname{Ran}(B)$ spatially implements an algebraic isomorphism from the first onto the second. Conversely, every algebraic isomorphism* *from the first onto the second is spatially implemented by such an invertible operator T;*

(iv) *if $S, T, U \in B(H)$ are operators with S, T and $S + 2UA$ invertible and with $TA = BS$, then each of the operators $T_0^+, T_0^- \in B(H \oplus H)$ given by*

$$T_0^{\pm} = \begin{pmatrix} \pm(S+UA) & U \\ \pm BUA & T+BU \end{pmatrix}$$

is an invertible operator which spatially implements an algebraic isomorphism from $\operatorname{Alg}\mathcal{D}(A)$ *onto* $\operatorname{Alg}\mathcal{D}(B)$. *Conversely, every algebraic isomorphism from* $\operatorname{Alg}\mathcal{D}(A)$ *onto* $\operatorname{Alg}\mathcal{D}(B)$ *is spatially implemented by an operator of the form T_0^+ or T_0^-.*

Proof. (i) (a) Every algebraic isomorphism between $\operatorname{Alg}\mathcal{D}(A)$ and $\operatorname{Alg}\mathcal{D}(B)$ is spatial by [14], Theorem 2.

(i) (b) Let $\theta : \mathcal{A}(A) \to \mathcal{A}(B)$ be an algebraic isomorphism. Our proof that θ is spatial is similar to the proof of Theorem 3.4. By Proposition 4.12 (iv), θ maps the set of rank one operators of $\mathcal{A}(A)$ onto the set of rank one operators of $\mathcal{A}(B)$. Choose $f_0 \in \operatorname{Ran}(A)$, $e_0 \in H$ such that $(f_0|e_0) = 1$. Then $e_0 \otimes f_0 \in \mathcal{A}(A)$ so $\theta(e_0 \otimes f_0) = g_0 \otimes h_0$ for some vectors $h_0 \in \operatorname{Ran}(B)$, $g_0 \in H$ satisfying $(h_0|g_0) = 1$ (since $(e_0 \otimes f_0)^2 = e_0 \otimes f_0$ implies that $(g_0 \otimes h_0)^2 = g_0 \otimes h_0$). For every $0 \neq e \in H$, $(e_0 \otimes f_0)(e \otimes f_0) = e \otimes f_0$ so $\theta(e_0 \otimes f_0)\theta(e \otimes f_0) = \theta(e \otimes f_0)$ and $\operatorname{Ran}(\theta(e \otimes f_0)) = \langle h_0 \rangle$. It follows that there exists an injective linear mapping $S : H \to H$ such that $\theta(e \otimes f_0) = Se \otimes h_0$, for every $e \in H$. Similarly, for every $0 \neq f \in \operatorname{Ran}(A)$, $\ker\theta(e_0 \otimes f_0) = \ker\theta(e_0 \otimes f) = \langle g_0 \rangle^{\perp}$ (since $(e_0 \otimes f)(e_0 \otimes f_0) = e_0 \otimes f$) and it follows that there exists an injective linear mapping $T : \operatorname{Ran}(A) \to \operatorname{Ran}(B)$ such that $\theta(e_0 \otimes f) = g_0 \otimes Tf$, for every $f \in \operatorname{Ran}(A)$.

For every pair of vectors $e \in H$, $f \in \operatorname{Ran}(A)$ we have $(e_0 \otimes f)(e \otimes f_0) = e \otimes f$ so, applying θ, $\theta(e \otimes f) = (g_0 \otimes Tf)(Se \otimes h_0) = Se \otimes Tf$. Since $(e \otimes f)^2 = (f|e)(e \otimes f)$, $(Tf|Se)(Se \otimes Tf) = (f|e)(Se \otimes Tf)$ so $(Tf|Se) = (f|e)$.

The mapping S maps onto H. For, let $0 \neq g \in H$. Then $g \otimes h_0 \in \mathcal{A}(B)$ so $\theta(e \otimes f) = g \otimes h_0$ for some $e \in H$, $f \in \operatorname{Ran}(A)$. Hence $Se \otimes Tf = g \otimes h_0$, so $(g_0|Tf)Se = g$. Similarly, T maps $\operatorname{Ran}(A)$ onto $\operatorname{Ran}(B)$.

The mapping S is continuous. For this, it is enough to show that S is closed. Let $e_n \to e$ and $Se_n \to g$. Then, for every $f \in \operatorname{Ran}(A)$, $(Tf|Se_n) = (f|e_n) \to (f|e)$ and $(Tf|Se_n) \to (Tf|g)$. Thus $(Tf|g) = (f|e)$. Now $g = Se'$ for some vector e' so $(Tf|Se') = (f|e') = (f|e)$ and since this is true for every $f \in \operatorname{Ran}(A)$, $e = e'$.

For every $f \in \operatorname{Ran}(A)$ and every $e \in H$, $(S^{*-1}f - Tf|Se) = (S^{*-1}f|Se) - (Tf|Se) = (f|e) - (f|e) = 0$. Hence S^{*-1} is an extension of T to H. Denote this extension by T so that $T = S^{*-1}$.

Let $0 \neq f \in \operatorname{Ran}(A)$ and $X \in \mathcal{A}(A)$. Choose $e \in H$ such that $(f|e) = 1$. Then

$$\begin{aligned}\theta(X)Tf &= \theta(X)(Se \otimes Tf)Tf = \theta(X)\theta(e \otimes f)Tf = \theta(X(e \otimes f))Tf \\ &= \theta(e \otimes Xf)Tf = (Se \otimes TXf)Tf = TXf.\end{aligned}$$

Hence, since $\operatorname{Ran}(A)$ is dense in H, $\theta(X) = TXT^{-1}$.

(i) (c) Let $\theta : \mathcal{A}_0(A) \to \mathcal{A}_0(B)$ be an algebraic isomorphism. In $\mathcal{A}_0(A)$ and $\mathcal{A}_0(B)$ every non-zero single element has rank one (again, simply take $M = (0)$ in the proof of Proposition 4.1). Hence θ maps the set of rank one operators of $\mathcal{A}_0(A)$ onto the set of rank one operators of $\mathcal{A}_0(B)$. As in the proof of (i) (b), there exists an invertible operator $T \in B(H)$ such that $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$ and $\theta(X) = TXT^{-1}$, for every $X \in \mathcal{A}_0(A)$.

(ii) The proofs given above show that, for each of the pairs (i) (b) and (i) (c), if the algebras are algebraically isomorphic then there exists an invertible operator $T \in B(H)$ such that $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$.

Let $\varphi : \mathrm{Alg}\,\mathcal{D}(A) \to \mathrm{Alg}\,\mathcal{D}(B)$ be an algebraic isomorphism. We show that, once again, such an invertible operator T exists (this is proved in [23] but we shall need the details later). Now φ is spatial so, for some invertible operator $T_0 \in B(H \oplus H)$, $\varphi(X) = T_0XT_0^{-1}$, for every $X \in \mathrm{Alg}\,\mathcal{D}(A)$. Since $\mathrm{Alg}\,\mathcal{D}(B) = T_0\,\mathrm{Alg}\,\mathcal{D}(A)T_0^{-1} = \mathrm{Alg}(T_0\mathcal{D}(A))$ and every ABSL is reflexive ([6]; see also [16]), $T_0\mathcal{D}(A) = \mathcal{D}(B)$. Hence either

$$T_0G(-A) = G(-B) \quad \text{and} \quad T_0G(A) = G(B), \text{ or} \tag{1}$$

$$T_0G(-A) = G(B) \quad \text{and} \quad T_0G(A) = G(-B). \tag{2}$$

It readily follows that in case (1) T_0 has the form $T_0 = \begin{pmatrix} S+UA & U \\ BUA & T+BU \end{pmatrix}$ or, in case (2), the form $T_0 = \begin{pmatrix} -(S+UA) & U \\ -BUA & T+BU \end{pmatrix}$, where, in each case, $S, T, U \in B(H)$ and $TA = BS$.

In case (1) the proof that T and S are invertible is exactly the proof of the first part of Theorem 4.2 taking $M = N = (0)$. Hence $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$. Notice also that $S+2UA$ is invertible since $T_0G(A) = G(B)$ and, for every $x \in H$,

$$T_0 \begin{pmatrix} x \\ Ax \end{pmatrix} = \begin{pmatrix} (S+2UA)x \\ B(S+2UA)x \end{pmatrix}.$$

In case (2) we proceed similarly. The operator T_0^{-1} has the form $T_0^{-1} = \begin{pmatrix} -(\widetilde{S}+\widetilde{U}B) & \widetilde{U} \\ -A\widetilde{U}B & \widetilde{T}+A\widetilde{U} \end{pmatrix}$, where $\widetilde{S}, \widetilde{T}, \widetilde{U} \in B(H)$ and $\widetilde{T}B = A\widetilde{S}$. For every $x \in H$,

$$\begin{pmatrix} x \\ Ax \end{pmatrix} = T_0^{-1}T_0 \begin{pmatrix} x \\ Ax \end{pmatrix} = T_0^{-1} \begin{pmatrix} -Sx \\ BSx \end{pmatrix} = \begin{pmatrix} (\widetilde{S}+2\widetilde{U}B)Sx \\ A(\widetilde{S}+2\widetilde{U}B)Sx \end{pmatrix},$$

and

$$\begin{pmatrix} x \\ -Ax \end{pmatrix} = T_0^{-1}T_0 \begin{pmatrix} x \\ -Ax \end{pmatrix} = T_0^{-1} \begin{pmatrix} -(S+2UA)x \\ -B(S+2UA)x \end{pmatrix} = \begin{pmatrix} \widetilde{S}(S+2UA)x \\ -A\widetilde{S}(S+2UA)x \end{pmatrix},$$

so $(\widetilde{S}+2\widetilde{U}B)S = \widetilde{S}(S+2UA) = 1$. Also, for every $y \in H$,

$$\begin{pmatrix} y \\ -By \end{pmatrix} = T_0T_0^{-1} \begin{pmatrix} y \\ -By \end{pmatrix} = T_0 \begin{pmatrix} -(\widetilde{S}+2\widetilde{U}B)y \\ -A(\widetilde{S}+2\widetilde{U}B)y \end{pmatrix} = \begin{pmatrix} S(\widetilde{S}+2\widetilde{U}B)y \\ -BS(\widetilde{S}+2\widetilde{U}B)y \end{pmatrix},$$

and

$$\begin{pmatrix} y \\ By \end{pmatrix} = T_0T_0^{-1} \begin{pmatrix} y \\ By \end{pmatrix} = T_0 \begin{pmatrix} -\widetilde{S}y \\ A\widetilde{S}y \end{pmatrix} = \begin{pmatrix} (S+2UA)\widetilde{S}y \\ B(S+2UA)\widetilde{S}y \end{pmatrix},$$

so $S(\widetilde{S}+2\widetilde{U}B) = (S+2UA)\widetilde{S} = 1$. Thus again S and $S+2UA$ are invertible. Also, $(\widetilde{T}+2A\widetilde{U})TA = (\widetilde{T}+2A\widetilde{U})BS = A(\widetilde{S}+2\widetilde{U}B)S = A$ implies that $(\widetilde{T}+2A\widetilde{U})T = 1$, and $T(\widetilde{T} + 2A\widetilde{U})B = TA(\widetilde{S} + 2\widetilde{U}B) = BS(\widetilde{S} + 2\widetilde{U}B) = B$ implies that $T(\widetilde{T} + 2A\widetilde{U}) = 1$. Hence T is also invertible and since $TA = BS$, $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$.

Conversely, suppose that there exists an invertible operator $T \in B(H)$ such that $T\,\mathrm{Ran}(A) = \mathrm{Ran}(B)$. Then $TA = BS$ for a unique invertible operator S. It is easy to verify that

$$T\mathcal{A}(A)T^{-1} = \mathcal{A}(B), \quad T\mathcal{A}_0(A)T^{-1} = \mathcal{A}_0(B)$$

and

$$\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \mathrm{Alg}\,\mathcal{D}(A) \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}^{-1} = \mathrm{Alg}\,\mathcal{D}(B)$$

(the latter is equivalent to $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \mathcal{D}(A) = \mathcal{D}(B)$).

(iii) This has already been proved above.

(iv) Because of the proof of (ii) above, it only remains to show that if $S, T, U \in B(H)$ are operators with S, T and $S + 2UA$ invertible and with $TA = BS$, then

$$T_0^{\pm} = \begin{pmatrix} \pm(S + 2UA) & U \\ \pm BUA & T + BU \end{pmatrix}$$

are invertible operators which spatially implement algebraic isomorphisms from $\mathrm{Alg}\,\mathcal{D}(A)$ onto $\mathrm{Alg}\,\mathcal{D}(B)$. Assume, for a moment, that they are invertible. Then they implement isomorphisms if and only if $T_0^{\pm}\mathcal{D}(A) = \mathcal{D}(B)$ and the latter readily follows from the facts that, for every $x \in H$,

$$\begin{pmatrix} S + UA & U \\ BUA & T + BU \end{pmatrix} \begin{pmatrix} x \\ -Ax \end{pmatrix} = \begin{pmatrix} Sx \\ -BSx \end{pmatrix},$$

$$\begin{pmatrix} S + UA & U \\ BUA & T + BU \end{pmatrix} \begin{pmatrix} x \\ Ax \end{pmatrix} = \begin{pmatrix} (S + 2UA)x \\ B(S + 2UA)x \end{pmatrix}$$

and

$$\begin{pmatrix} -(S + UA) & U \\ -BUA & T + BU \end{pmatrix} \begin{pmatrix} x \\ -Ax \end{pmatrix} = \begin{pmatrix} -(S + 2UA)x \\ -B(S + 2UA)x \end{pmatrix},$$

$$\begin{pmatrix} -(S + UA) & U \\ -BUA & T + BU \end{pmatrix} \begin{pmatrix} x \\ Ax \end{pmatrix} = \begin{pmatrix} -Sx \\ BSx \end{pmatrix}.$$

Finally, we prove the invertibility of T_0^+ and T_0^-.

For T_0^+, define $\widetilde{U} = -(S + 2UA)^{-1}UT^{-1}$, $\widetilde{S} = S^{-1}$ and $\widetilde{T} = T^{-1}$. Then $\widetilde{T}B = A\widetilde{S}$ and $(S + 2UA)\widetilde{U}B = -UA\widetilde{S}$. Also $\widetilde{S} + 2\widetilde{U}B = (S + 2UA)^{-1}$, for $(S + 2UA)(\widetilde{S} + 2\widetilde{U}B) = (S + 2UA)\widetilde{S} - 2UA\widetilde{S} = 1 + 2UA\widetilde{S} - 2UA\widetilde{S} = 1$. It is now easy to verify that $T_0^+S_0^+ = S_0^+T_0^+ = 1$, where

$$S_0^+ = \begin{pmatrix} \widetilde{S} + \widetilde{U}B & \widetilde{U} \\ A\widetilde{U}B & \widetilde{T} + A\widetilde{U} \end{pmatrix}.$$

For T_0^-, define $\widetilde{U} = (S+2UA)^{-1}UT^{-1}$, $\widetilde{S} = S^{-1} - 2\widetilde{U}B$ and $\widetilde{T} = T^{-1} - 2A\widetilde{U}$. Then $\widetilde{T}B = A\widetilde{S}$ for, $\widetilde{T}B = T^{-1}B - 2A\widetilde{U}B = AS^{-1} - 2A\widetilde{U}B = A(S^{-1} - 2\widetilde{U}B) = A\widetilde{S}$. Clearly $(S+2UA)\widetilde{U}B = UAS^{-1}$. Also, $\widetilde{S} = (S+2UA)^{-1}$ since $(S+2UA)\widetilde{S} =$

$(S+2UA)(S^{-1}-2\widetilde{U}B)=1+2UAS^{-1}-2(S+2UA)\widetilde{U}B=1+2UAS^{-1}-2UAS^{-1}=1$. Since $UA\widetilde{S}-S\widetilde{U}B=UAS^{-1}-2UA\widetilde{U}B-S\widetilde{U}B=UAS^{-1}-(S+2UA)\widetilde{U}B=UAS^{-1}-UAS^{-1}=0$, $UA\widetilde{S}=S\widetilde{U}B$. Also, $\widetilde{U}T=\widetilde{S}U$ for $\widetilde{U}T=(S+2UA)^{-1}U=\widetilde{S}U$. It is now easy to verify that $T_0^-S_0^-=S_0^-T_0^-=1$, where

$$S_0^-=\begin{pmatrix}-(\widetilde{S}+\widetilde{U}B) & \widetilde{U}\\ -A\widetilde{U}B & \widetilde{T}+A\widetilde{U}\end{pmatrix}. \quad ∎$$

REFERENCES

1. S. Argyros, M. Lambrou, W.E. Longstaff, Atomic Boolean subspace lattices and applications to the theory of bases, *Mem. Amer. Math. Soc.* **445**(1991).
2. T. Donnellan, *Lattice Theory*, Pergamon Press, Oxford–New York 1968.
3. P.A. Fillmore, J.P. Williams, On operator ranges, *Adv. in Math.* **7**(1971), 254–281.
4. C. Foiaş, Invariant para-closed subspaces, *Indiana Math. J.* **21**(1972), 881–907.
5. F. Gilfeather, R.L. Moore, Isomorphisms of certain CSL algebras, *J. Funct. Anal.* **67**(1986), 264–291.
6. P.R. Halmos, Reflexive lattices of subspaces, *J. London Math. Soc.* **4**(1971), 257–263.
7. B.E. Johnson, The uniqueness of the complete norm topology, *Bull. Amer. Math. Soc.* **73**(1967), 537–539.
8. A. Katavolos, E. Katsoulis, Semisimplicity in operator algebras and subspace lattices, *J. London Math. Soc. (2)* **42**(1990), 365–372.
9. A. Katavolos, M.S. Lambrou, M. Papadakis, On some algebras diagonalized by M-bases of l^2, *Integral Equations Operator Theory* **17**(1993), 68–94.
10. M.S. Lambrou, On some reflexive lattices and related algebras, Ph.D. Dissertation, University of London, Kings College, 1977.
11. M.S. Lambrou, Approximants, commutants and double commutants in normed algebras, *J. London Math. Soc. (2)* **25**(1982), 499–512.
12. M.S. Lambrou, On the ranks of operators in reflexive algebras, *Linear Algebra Appl.* **142**(1990), 211–235.
13. M.S. Lambrou, W.E. Longstaff, Finite rank operators leaving double triangles invariant, *J. London Math. Soc. (2)* **45**(1992), 153–168.
14. M.S. Lambrou, W.E. Longstaff, Spatiality of isomorphisms between certain reflexive algebras, *Proc. Amer. Math. Soc.* **122**(1994), 1065–1073.
15. M.S. Lambrou, W.E. Longstaff, Non-reflexive pentagon subspace lattices, *Studia Math. (2)* **125**(1997), 187–199.
16. W.E. Longstaff, Strongly reflexive lattices, *J. London Math. Soc. (2)* **11**(1975), 491–498.
17. W.E. Longstaff, Remarks on semisimple reflexive algebras, "Conference on Automatic Continuity and Banach Algebras", Canberra, 1989, pp. 273–287, *Proc. Centre Math. Anal. Austral. Nat. Univ.*, vol. 21, Austral Nat. Univ., Canberra, 1989.
18. W.E. Longstaff, O. Panaia, On the ranks of single elements of reflexive operator algebras, *Proc. Amer. Math. Soc.* **125**(1997), 2875–2882.
19. W.E. Longstaff, P. Rosenthal, On two questions of Halmos concerning subspace lattices, *Proc. Amer. Math. Soc.* **75**(1979), 85–86.
20. E. Nordgren, M. Radjabalipour, H. Radjavi, P. Rosenthal, On invariant operator ranges, *Trans. Amer. Math. Soc.* **251**(1979), 389–398.

21. S.-C. Ong, Converse of a theorem of Foiaş and reflexive lattices of operator ranges, *Indiana Math. J.* **30**(1981), 57–63.
22. O. Panaia, Quasi-spatiality of isomorphisms for certain classes of operator algebras, Ph.D. Dissertation, University of Western Australia, 1995.
23. M. Papadakis, On isomorphisms between certain non-CSL algebras, *Proc. Amer. Math. Soc.* **119**(1993), 1157–1164.

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