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REFLEXIVITY OF FINITE DIMENSIONAL SUBSPACES OF OPERATORS

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ABSTRACT. We show that any n-dimensional subspace of $B(H)$ is $[\sqrt{2n}]$-reflexive, where $[t]$ denotes the largest integer that is less than or equal to $t \in \mathbb{R}$. As a corollary, we prove that if φ is an elementary operator on a C^*-algebra $\mathcal{A}$ with minimal length l, then φ is completely positive if and only if φ is $\max\{[\sqrt{2(l-1)}], 1\}$-positive.

KEYWORDS: *Reflexivity of subspace, separating vector, complete positivity.*

MSC (2000): Primary 47L30, 47L50; Secondary 47B47.

1. INTRODUCTION

Throughout this paper, let H be a complex separable Hilbert space, $B(H)$ the set of all bounded linear operators on H, $F(H)$ the set of finite rank operators on H, and $F_n(H)$ the set of operators with rank at most n. For $T \in B(H)$, let $R(T)$ denote the range of T. For any subspace $\mathcal{S} \subseteq B(H)$, define $\mathrm{ref}(\mathcal{S}) = \{T \in B(H) : Tx \in \mathrm{clin}(\mathcal{S}x)$, for any $x \in H\}$, where clin denotes norm closed linear span. $\mathcal{S}$ is called *reflexive* if $\mathrm{ref}(\mathcal{S}) = \mathcal{S}$. Define $\mathcal{S}^{(n)} = \{S^{(n)} \in B(H^{(n)}) : S \in \mathcal{S}\}$, where $H^{(n)}$ is the direct sum of n copies of H and $S^{(n)}$ is the direct sum of n copies of S acting on $H^{(n)}$. $\mathcal{S}$ is called *n-reflexive* if $\mathcal{S}^{(n)}$ is reflexive in $B(H^{(n)})$. A vector $x \in H$ is called a *separating vector* of $\mathcal{S}$ if the map $E_x : S \to Sx$, $S \in \mathcal{S}$ is injective. Let $\mathrm{sep}(\mathcal{S})$ denote the set of all separating vectors of $\mathcal{S}$ in H. The local dimension of $\mathcal{S}$, denoted by $k(\mathcal{S})$, is defined by $k(\mathcal{S}) = \max\limits_{x \in H}\{\dim \mathrm{clin}(\mathcal{S}x)\}$; clearly $k(\mathcal{S}) \leqslant \dim \mathcal{S}$. If $\dim \mathcal{S} < \infty$, it is not hard to see that $\mathrm{sep}(\mathcal{S}) \neq \emptyset$ if and only if $k(\mathcal{S}) = \dim \mathcal{S}$.

The notion of reflexivity was first introduced by Halmos ([7]) for subalgebras of algebra $B(H)$. Loginov and Shulman ([14]) extended reflexivity to subspaces of $B(H)$ which are not necessarily algebras. Reflexive subspaces have been useful in the analysis of operator algebras ([9], [10], [11]). A natural extension of the notion of reflexivity is n-reflexivity. It has been considered, for example, in [1], [10], [15]. In [12], Larson proved that if $\mathcal{S}$ is a finite dimensional subspace of $B(H)$, then $\mathrm{ref}(\mathcal{S}^{(n)}) = \mathcal{S}^{(n)} + \mathrm{ref}(\mathcal{S}^{(n)} \cap F(H^{(n)}))$. It follows immediately that $\mathcal{S}$ is n-reflexive if and only if $\mathcal{S} \cap F(H)$ is n-reflexive. Hence, we are only interested in which finite dimensional subspaces of $F(H)$ are n-reflexive.

In [15], Magajna stated the following question:

For each positive integer n, determine the smallest $k = k(n)$ such that all n-dimensional subspaces of $B(H)$ are k-reflexive.

In that paper, he proved $k(n) \leqslant n$. In [13], the first author improved the result and proved that if $\mathcal{S}$ is an n-dimensional subspace, then $\mathcal{S}$ is $([\frac{n}{2}] + 1)$-reflexive. Hence $k(n) \leqslant [\frac{n}{2}] + 1$. In this paper, our main result is Theorem 2.14. It states that if $\mathcal{S}$ is an n-dimensional subspace of $B(H)$, then $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive. Example 2.15 shows that $[\sqrt{2n}]$ is the smallest integer such that all n-dimensional subspaces of $B(H)$ are $[\sqrt{2n}]$-reflexive. Thus Theorem 2.14 and Example 2.15 provide the answer to Magajna's question. The proof of Theorem 2.14 will be prepared by a number of auxiliary steps, and we need to consider the local dimensions of subspaces. The method used in Theorem 2.14 can also be used to improve Theorem 3.6 in [2]. As an application of our main result, we prove that if φ is an elememtary operator on a C^*-algebra $\mathcal{A}$ with minimal legth l, then φ is completely positive if and only if φ is $\max\{[\sqrt{2(l-1)}], 1\}$-positive.

2. REFLEXIVITY OF FINITE DIMENSIONAL SUBSPACES

In the following, we always assume that $\mathcal{S}$ is a subspace of $B(H)$, $\dim \mathcal{S} < \infty$, and $\mathcal{S} \subseteq F(H)$ unless stated otherwise. Before we prove our main result, we need several lemmas and propositions.

LEMMA 2.1. ([4]) *The set* $\mathrm{sep}(\mathcal{S})$ *is an open subset of* H.

LEMMA 2.2. ([4]) *The set* $\mathrm{sep}(\mathcal{S})$ *is either empty or dense in* H.

Let M be a closed subspace of H and P be the orthogonal projection of H onto M. Define $\mathcal{S}_M = \{S \in \mathcal{S} : R(S) \subseteq M\}$. Let $\mathcal{S}_M^{\mathrm{c}}$ be any vector space complement of $\mathcal{S}_M$ in $\mathcal{S}$. Define $P^\perp \mathcal{S}_M^{\mathrm{c}} = \{P^\perp S : S \in \mathcal{S}_M^{\mathrm{c}}\}$.

PROPOSITION 2.3. $k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^{\mathrm{c}}) \leqslant k(\mathcal{S})$.

Proof. If $P^\perp \mathcal{S}_M^{\mathrm{c}} = 0$, it is obvious that $k(\mathcal{S}_M) \leqslant k(\mathcal{S})$. If $\mathcal{S}_M = 0$, it follows that $\mathcal{S}_M^{\mathrm{c}} = \mathcal{S}$ and

$$k(P^\perp \mathcal{S}_M^{\mathrm{c}}) = \max_{x \in H}\{\dim[P^\perp Sx : S \in \mathcal{S}_M^{\mathrm{c}}]\} \leqslant \max_{x \in H}\{\dim \mathrm{clin}(\mathcal{S}x)\} = k(\mathcal{S}).$$

Now suppose $k(\mathcal{S}_M) = m \neq 0$ and $k(P^\perp \mathcal{S}_M^{\mathrm{c}}) = l \neq 0$. Let $x_0 \in H$ be a separating vector of $\mathrm{span}\{S_1, \ldots, S_m\} \subseteq \mathcal{S}_M$. Similarly, there exist $P^\perp T_1, \ldots, P^\perp T_l \in \mathcal{S}_M^{\mathrm{c}}$ such that $\mathrm{span}\{P^\perp T_1, \ldots, P^\perp T_l\}$ has a separating vector. By Lemmas 2.1 and 2.2, we can choose $y \in H$ with $\|y\|$ small enough so that $x_0 + y$ is a separating vector for

$\mathrm{span}\{S_1,\dots,S_m\}$ and $\mathrm{span}\{P^\perp T_1,\dots,P^\perp T_l\}$. For any $\lambda_1,\dots,\lambda_m,\mu_1,\dots,\mu_l\in\mathbb{C}$, suppose

$$\lambda_1S_1(x_0+y)+\cdots+\lambda_mS_m(x_0+y)+\mu_1T_1(x_0+y)+\cdots+\mu_lT_l(x_0+y)=0. \tag{2.1}$$

Applying $P^\perp$ to both sides of (2.1), it follows

$$\mu_1P^\perp T_1(x_0+y)+\cdots+\mu_lP^\perp T_l(x_0+y)=0. \tag{2.2}$$

Since x_0+y is a separating vector of $\mathrm{span}\{P^\perp T_1,\dots,P^\perp T_l\}$, we have $\mu_1=\cdots=\mu_l=0$. Now (2.1) implies $\lambda_1=\cdots=\lambda_m=0$, since x_0+y is a separating vector of $\mathrm{span}\{S_1,\dots,S_m\}$. Hence $k(\mathcal{S})\geqslant k(\mathcal{S}_M)+k(P^\perp\mathcal{S}_M^{\mathrm{c}})$. ∎

PROPOSITION 2.4. *If $k(\mathcal{S}_M)=\dim M$, then $k(S_M)+k(P^\perp\mathcal{S}_M^{\mathrm{c}})=k(\mathcal{S})$.*

Proof. By Proposition 2.3, we only need to prove $k(\mathcal{S})\leqslant k(\mathcal{S}_M)+k(P^\perp\mathcal{S}_M^{\mathrm{c}})$.

Suppose that $k(\mathcal{S}_M)=m$ and $k(P^\perp\mathcal{S}_M^{\mathrm{c}})=l$. If $m+l=\dim\mathcal{S}$, it is obvious that $k(\mathcal{S})\leqslant k(\mathcal{S}_M)+k(P^\perp\mathcal{S}_M^{\mathrm{c}})$. If $m+l<\dim\mathcal{S}$, and $m+l<n\leqslant\dim\mathcal{S}$, we take n linearly independent operators from $\mathcal{S}$ in such a way that $S_1,\dots,S_{m_1}\in\mathcal{S}_M$, $T_1,\dots,T_{l_1}\in\mathcal{S}_M^{\mathrm{c}}$ and $m_1+l_1=n$. For any nonzero x_0 in H, we show that there are $\lambda_1,\dots,\lambda_{m_1},\mu_1,\dots,\mu_{l_1}$, not all zero, such that

$$\lambda_1S_1x_0+\cdots+\lambda_{m_1}S_{m_1}x_0+\mu_1T_1x_0+\cdots+\mu_{l_1}T_{l_1}x_0=0. \tag{2.3}$$

If $l_1\leqslant l$, then $m_1>m$, and choose $\mu_1=\cdots=\mu_{l_1}=0$. Since $k(\mathcal{S}_M)=m$, it follows that there are $\lambda_1,\dots,\lambda_{m_1}$, not all zero, such that $\lambda_1S_1x_0+\cdots+\lambda_{m_1}S_{m_1}x_0=0$. Suppose that $l_1>l$. If $\mathrm{span}\{P^\perp T_1x_0,\dots,P^\perp T_{l_1}x_0\}=(0)$, then $\mathrm{span}\{T_1x_0,\dots,T_{l_1}x_0\}\subseteq M$. Because $k(\mathcal{S}_M)=\dim M$, and $l_1+m_1=n>m+l$, it follows that there are $\lambda_1,\dots,\lambda_{m_1},\mu_1,\dots,\mu_{l_1}$, not all zero, satisfying (2.3). Without loss of generality, we may assume that $\{P^\perp T_1x_0,\dots,P^\perp T_tx_0\}, 1\leqslant t\leqslant l$ is linearly independent, and $P^\perp T_jx_0\in\mathrm{span}\{P^\perp T_1x_0,\dots,P^\perp T_tx_0\}$, $t+1\leqslant j\leqslant l_1$. Suppose that $P^\perp T_jx_0=\sum\limits_{i=1}^{t}a_{ij}P^\perp T_ix_0$, $t+1\leqslant j\leqslant l_1$. Let $B_j=T_j-\sum\limits_{i=1}^{t}a_{ij}T_i$. Then $B_jx_0\in M$, $t+1\leqslant j\leqslant l_1$. Since $S_ix_0\in M, 1\leqslant i\leqslant m_1$ and $\dim M=m<m_1+l_1-l\leqslant m_1+l_1-t$, we may choose $\lambda_1,\dots,\lambda_{m_1}$ and $\mu_{t+1},\dots,\mu_{l_1}$, not all zero, such that

$$\lambda_1S_1x_0+\cdots+\lambda_{m_1}S_{m_1}x_0+\mu_{t+1}B_{t+1}x_0+\cdots+\mu_{l_1}B_{l_1}x_0=0. \tag{2.4}$$

Hence

$$\begin{aligned}\lambda_1S_1x_0+\cdots+\lambda_{m_1}S_{m_1}x_0+\mu_{t+1}\Big(T_{l_1}-\sum_{i=1}^{t}a_{i\,t+1}T_i\Big)x_0+\cdots\\ +\mu_{l_1}\Big(T_{l_1}-\sum_{i=1}^{t}a_{i\,l_1}T_i\Big)x_0=0.\end{aligned} \tag{2.5}$$

By (2.5), it follows that (2.3) is true. ∎

LEMMA 2.5. ([2]) *Let V be a vector space over a field $\mathbb{F}$ and let $L(V)$ be the set of all linear transformations on V. Suppose $\mathcal{S} \subseteq L(V)$ and* $\dim \mathcal{S}$ *is less than the cardinality of $\mathbb{F}$. Let x be a separating vector of $\mathcal{S}$ and W be a linear subspace of V satisfying $\mathcal{S}x \cap W = (0)$. Then for each vector $y \in V$, there is a scalar $\lambda \in \mathbb{F}$ so that $y + \lambda x$ separates $\mathcal{S}$ and $\mathcal{S}(y + \lambda x) \cap W = (0)$.*

LEMMA 2.6. *If $k(\mathcal{S}) = k$, then there exists an M with* $\dim M = k$ *and* $\dim \mathcal{S}_M^c \leqslant k$.

Proof. Since $k(\mathcal{S}) = k$, there exist $x_0 \in H$ and $A_1, \ldots, A_k \in \mathcal{S}$ such that $\max_{x \in H}\{\dim \operatorname{clin}(\mathcal{S}x)\} = \dim \operatorname{clin}(A_1x_0, \ldots, A_kx_0) = k$. Let $M = \operatorname{clin}(A_1x_0, \ldots, A_kx_0)$, $\widehat{\mathcal{S}} = \operatorname{span}\{A_1, \ldots, A_k\}$, and $\mathcal{S}_M = \{S \in \mathcal{S} : R(S) \subseteq M\}$. It is enough to prove $\mathcal{S} = \operatorname{span}\{\widehat{\mathcal{S}} \cup \mathcal{S}_M\}$. Since for any $S \in \mathcal{S}$, there exist $\lambda_1, \ldots, \lambda_k$ such that $Sx_0 = \sum_{i=1}^{k} \lambda_i A_i x_0$. Let $S_1 = S - \sum_{i=1}^{k} \lambda_i A_i$, then $S_1x_0 = 0$. If $S_1 = 0$, then $S \in \widehat{\mathcal{S}}$. If $S_1 \neq 0$, we show next that $S_1 \in \mathcal{S}_M$.

If $S_1 \notin \mathcal{S}_M$, there exists $y \in H$ such that $S_1y \notin M = \widehat{\mathcal{S}}x_0$. Let $W = \operatorname{clin}(S_1y)$. Then $\widehat{\mathcal{S}}x_0 \cap W = (0)$. By Lemma 2.5, there exists $\lambda \in \mathbb{C}$ such that $y + \lambda x_0$ separates $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}(y + \lambda x_0) \cap W = (0)$. Since $S_1 \neq 0$ and $S_1x_0 = 0$, it follows $\{A_1, \ldots, A_k, S_1\}$ is linearly independent. Let $\widetilde{\mathcal{S}} = \operatorname{span}\{A_1, \ldots, A_k, S_1\}$. Next we prove that $y + \lambda x_0$ separates $\widetilde{\mathcal{S}}$. For any $A \in \widehat{\mathcal{S}}$, $t \in \mathbb{C}$, if $(A + tS_1)(y + \lambda x_0) = 0$, then $A(y + \lambda x_0) = -tS_1y$. By $\widehat{\mathcal{S}}(y + \lambda x_0) \cap W = (0)$, it follows that $t = 0$ and $A(y + \lambda x_0) = 0$. Since $y + \lambda x_0$ is a separating vector of $\widehat{\mathcal{S}}$, we have $A = 0$. Hence $y + \lambda x_0$ separates $\widetilde{\mathcal{S}}$, which implies $k(\mathcal{S}) \geqslant k + 1$, a contradiction. ∎

DEFINITION 2.7. Suppose $\mathcal{S}$ is a subspace of $B(H)$. We say $\mathcal{S}$ has *property* A if for any subspace $\mathcal{S}_1$ of $\mathcal{S}$, we have $k(\mathcal{S}_1) \geqslant \{\sqrt{2 \dim \mathcal{S}_1} - 1/2\}$, where $\{t\}$ denotes the smallest integer that is greater than or equal to t.

We say $\mathcal{S}$ has *property* B if there exists a nonzero subspace M of H such that $k(\mathcal{S}_M) = \dim M$.

REMARK 2.8. Clearly if $\mathcal{S}$ has property A, then so does any subspace of $\mathcal{S}$. If $\mathcal{S}$ has property B, then so does any subspace of $B(H)$ containing $\mathcal{S}$.

For $x, y \in H$, let $x \otimes y$ denote the rank-one operator $u \to (u, x)y$.

LEMMA 2.9. ([8]) *Let $A, B \in B(H)$ and $\mathcal{S} = \operatorname{span}\{A, B\}$. Then $k(\mathcal{S}) = 1$ if and only if one of the following holds:*

(i) $\dim \mathcal{S} = 1$;

(ii) *there exist $x_0, x_1, x_2 \in H$ such that $A = x_1 \otimes x_0$, $B = x_2 \otimes x_0$.*

LEMMA 2.10. *Suppose* $\dim \mathcal{S} = n \geqslant 2$. *If $k(\mathcal{S}) < \{\sqrt{2n} - 1/2\}$, then $\mathcal{S}$ has property* B.

Proof. If $n = 2$, then $k(\mathcal{S}) = 1$. Lemma 2.9 now implies that $\mathcal{S}$ has property B.

Suppose the statement is true for all $\mathcal{S}$ with $2 \leqslant \dim \mathcal{S} \leqslant n - 1$, $n \geqslant 3$. For any $\mathcal{S}$ with $\dim \mathcal{S} = n$, let $k(\mathcal{S}) = k$. By Lemma 2.6, there exists a subspace M of H such that $\dim M = k$ and $\dim \mathcal{S}_M^c \leqslant k$.

If $\mathcal{S}_M = \mathcal{S}$, clearly $k(\mathcal{S}_M) = k(\mathcal{S}) = \dim M$.

If $\mathcal{S}_M \nsubseteq \mathcal{S}$, then let P be the orthogonal projection of H onto M. We have, for any $\mathcal{S}_M^{\mathrm{c}}, P^{\perp}\mathcal{S}_M^{\mathrm{c}} \neq (0)$, so $k(P^{\perp}\mathcal{S}_M^{\mathrm{c}}) \geqslant 1$. Hence $k(\mathcal{S}_M) \leqslant k-1$, by Proposition 2.3. Since $k < \{\sqrt{2n}-1/2\}$, we have $\{\sqrt{2n}-1/2\}-1 \leqslant \{\sqrt{2(n-k)}-1/2\}$. So $k-1 < \{\sqrt{2n}-1/2\}-1 \leqslant \{\sqrt{2(n-k)}-1/2\}$. Hence $k(\mathcal{S}_M) < \{\sqrt{2(n-k)}-1/2\} \leqslant \{\sqrt{2\dim\mathcal{S}_M}-1/2\}$. (Since $\dim\mathcal{S}_M+\dim\mathcal{S}_M^{\mathrm{c}}=n$, it follows that $\dim\mathcal{S}_M = n-\dim\mathcal{S}_M^{\mathrm{c}}$. Since $\dim\mathcal{S}_M^{\mathrm{c}} \leqslant k$, it follows $\dim\mathcal{S}_M \geqslant n-k$.) By the induction hypothesis, $\mathcal{S}_M$ has property B. It follows that $\mathcal{S}$ has property B. ∎

Lemma 2.11. *If* $\dim\mathcal{S}=n$ *and* $\mathcal{S}$ *has property* A *then* $\mathcal{S}$ *is* $[\sqrt{2n}]$*-reflexive, where* $[t]$ *denotes the largest integer that is less than or equal to* t.

Proof. If $n=1$, Lemma 10 from [9] implies that $\mathcal{S}$ is reflexive.

Suppose the statement is true for all $\mathcal{S}$ with property A and $\dim\mathcal{S}\leqslant n-1$, $n\geqslant 2$. Suppose $\dim\mathcal{S}=n$, $\mathcal{S}$ has property A, and $k(\mathcal{S})=k$. Since $\mathcal{S}$ has property A, $k\geqslant\{\sqrt{2n}-1/2\}$. If $k=n$, then $\mathcal{S}$ has a separating vector, so $\mathcal{S}$ is 2-reflexive. Hence $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive, since $n\geqslant 2$ and $[\sqrt{2n}]\geqslant 2$.

Suppose that $\{\sqrt{2n}-1/2\}\leqslant k\leqslant n-1$. Let $m=[\sqrt{2n}]$. Since $k(\mathcal{S})=k$, there exist $x_1\in H$ and $\{A_1,\dots,A_k\}\subseteq\mathcal{S}$ such that $\{A_ix_1\}_{i=1}^k$ is a basis of $\mathcal{S}x_1$. Suppose $\mathcal{S}=\operatorname{span}\{A_1,\dots,A_n\}$. There exists a unique $k\times n$ complex matrix (a_{ij}) so that $A_jx_1=\sum\limits_{i=1}^{k}a_{ij}A_ix_1$, $j=1,\dots,n$, and if $j\leqslant k$, $a_{jj}=1$ and $a_{ij}=0$, $i\neq j$. Suppose $T^{(m)}\in\operatorname{ref}(\mathcal{S}^{(m)})$; in the following we prove that $T\in\mathcal{S}$. For any $x_2,\dots,x_m\in H$, there exist scalars $t_1,\dots,t_n$ such that

$$\begin{pmatrix}Tx_1\\ \vdots\\ Tx_m\end{pmatrix}=t_1\begin{pmatrix}A_1x_1\\ \vdots\\ A_1x_m\end{pmatrix}+\dots+t_n\begin{pmatrix}A_nx_1\\ \vdots\\ A_nx_m\end{pmatrix}. \tag{2.6}$$

Since $Tx_1\in\operatorname{span}\{A_1x_1,\dots,A_nx_1\}$, there exist $\mu_1,\dots,\mu_k$ such that

$$Tx_1=\sum_{i=1}^{k}\mu_iA_ix_1. \tag{2.7}$$

By (2.6) and (2.7), we have

$$Tx_g=\sum_{i=1}^{k}\mu_iA_ix_g+\sum_{j=1}^{n}t_j\Big(A_j-\sum_{i=1}^{k}a_{ij}A_i\Big)x_g,\quad g=2,\dots,m. \tag{2.8}$$

Let

$$T_1=T-\sum_{i=1}^{k}\mu_iA_i\quad\text{and}\quad B_j=A_j-\sum_{i=1}^{k}a_{ij}A_i. \tag{2.9}$$

Note $B_j=0$ for $j=1,\dots,k$. By (2.8) and (2.9), we have

$$\begin{pmatrix}T_1x_2\\ \vdots\\ T_1x_m\end{pmatrix}=t_{k+1}\begin{pmatrix}B_{k+1}x_2\\ \vdots\\ B_{k+1}x_m\end{pmatrix}+\dots+t_n\begin{pmatrix}B_nx_2\\ \vdots\\ B_nx_m\end{pmatrix}.$$

By the induction hypothesis, we have that $\operatorname{span}\{B_{k+1},\dots,B_n\}$ is $[\sqrt{2(n-k)}]$-reflexive. Since $k\geqslant\{\sqrt{2n}-1/2\}$, we have $[\sqrt{2n}]-1=m-1\geqslant[\sqrt{2(n-k)}]$. It follows that $T_1\in\operatorname{span}\{B_{k+1},\dots,B_n\}$. Therefore $T\in\mathcal{S}$. ∎

PROPOSITION 2.12. *If* $\dim \operatorname{clin}(\mathcal{S}H) = k$, *then* $\mathcal{S}$ *is* k*-reflexive.*

Proof. Since $\dim \mathcal{S} = n$, $\mathcal{S} \subseteq F(H)$, and $\dim \operatorname{clin}(\mathcal{S}H) = k$, there exists an orthogonal projection P satisfying $\dim PH = m < \infty$ and $P\mathcal{S}P = \mathcal{S}$. So we may assume that $\mathcal{S}$ is a subspace of $M_m(\mathbb{C})$. Let $\{e_1, \dots, e_k\}$ be an orthonormal basis of $\mathcal{S}\mathbb{C}^m$. Extend this to an orthonormal basis $\{e_1, \dots, e_k, e_{k+1}, \dots, e_m\}$ of $\mathbb{C}^m$. Clearly $\mathcal{S}$ is a subspace of $\mathcal{R} = \{(r_{ij}) \in M_m(\mathbb{C}) : r_{ij} = 0, \text{ for any } i > k\}$. It is easy to prove that $\mathcal{R}^*$ is reflexive. Since $\mathcal{R}^{*(k)}$ has a separating vector, it follows that $\mathcal{R}^{*(k)}$ is elementary, by Proposition 3.2 from [1]. By Proposition 2.10 from [1], it follows that $\mathcal{S}^{*(k)}$ is reflexive. Hence $\mathcal{S}^{(k)}$ is reflexive. ∎

THEOREM 2.13. *If* $\dim \mathcal{S} = n$, $k(\mathcal{S}) = k$, *then* $\mathcal{S}$ *is* k*-reflexive.*

Proof. If $\mathcal{S}$ has property A, by Lemma 2.11 we have that $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive. Since $k \geqslant \{\sqrt{2n} - 1/2\} \geqslant [\sqrt{2n}]$, it follows that $\mathcal{S}$ is k-reflexive.

Step 1. Suppose $\mathcal{S}$ does not have property A. Thus there exists a subspace $\mathcal{S}_1$ of $\mathcal{S}$ such that $k(\mathcal{S}_1) < \{\sqrt{2n} - 1/2\}$. By Lemma 2.10, $\mathcal{S}_1$ has property B. Hence $\mathcal{S}$ has property B.

Step 2. Let M be a maximal subspace of H such that $k(\mathcal{S}_M) = \dim M$. Let P be the orthogonal projection of H onto M.

If $\mathcal{S}_M \nsubseteq \mathcal{S}$, we prove next that $P^\perp \mathcal{S}$ has property A. If property A fails, then Step 1 implies that $P^\perp \mathcal{S}$ has property B. Thus there exists a subspace N of H such that

$$k((P^\perp \mathcal{S})_N) = \dim N. \tag{2.10}$$

By (2.10), we have $N \subseteq P^\perp H$. Let $\widetilde{M} = M \oplus N$. By Proposition 2.3,

$$\begin{aligned} k(\mathcal{S}_{\widetilde{M}}) &\geqslant k((\mathcal{S}_{\widetilde{M}})_M) + k(P^\perp (\mathcal{S}_{\widetilde{M}})^{\mathrm{c}}_M) = k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_{\widetilde{\mathcal{M}}}) \\ &= k(P^\perp \mathcal{S}_{\widetilde{M}}) + \dim M = k((P^\perp \mathcal{S})_{\widetilde{M}}) + \dim M \\ &= k((P^\perp \mathcal{S})_N) + \dim M = \dim N + \dim M = \dim \widetilde{M}. \end{aligned}$$

So $k(\mathcal{S}_{\widetilde{M}}) = \dim \widetilde{M}$, contradicting the maximality of M.

Suppose $\dim M = m$ and $\dim(P^\perp \mathcal{S}) = l$. Let $r = [\sqrt{2l}]$. We show $\mathcal{S}$ is $(m + r)$-reflexive by induction on l.

If $l = 0$, then $\operatorname{clin}(\mathcal{S}H) = M$. By Proposition 2.12, it follows $\mathcal{S}$ is m-reflexive.

Suppose the statement is true for all $\dim(P^\perp \mathcal{S}) \leqslant l - 1$, $l \geqslant 1$. Suppose $\dim P^\perp \mathcal{S} = l$. Since $\mathcal{S} = \mathcal{S}_M + \mathcal{S}^{\mathrm{c}}_M$, we have $P^\perp \mathcal{S} = P^\perp \mathcal{S}^{\mathrm{c}}_M$. If $\{A_1, \dots, A_s\}$ is a basis of $\mathcal{S}^{\mathrm{c}}_M$, we can easily prove that $\{P^\perp A_i\}_{i=1}^s$ is linearly independent, so $s = l$. If $k(P^\perp \mathcal{S}) = J$, then there exists an $x_1 \in H$ and $\{A_1, \dots, A_J\} \subseteq \mathcal{S}^{\mathrm{c}}_M$ so that $\{P^\perp A_1 x_1, \dots, P^\perp A_J x_1\}$ is linearly independent. Let $\{A_{l+1}, \dots, A_n\}$ be a basis of $\mathcal{S}_M$; it follows that $\{A_1, \dots, A_n\}$ is a basis of $\mathcal{S}$. Since $P^\perp A_j x_1 \in \operatorname{span}\{P^\perp A_1 x_1, \dots, P^\perp A_J x_1\}$, $J + 1 \leqslant j \leqslant n$, we have

$$P^\perp A_j x_1 = \sum_{i=1}^{J} a_{ij} P^\perp A_i x_1,\ J+1 \leqslant j \leqslant l \text{ and } P^\perp A_j x_1 = 0,\ l+1 \leqslant j \leqslant n. \tag{2.11}$$

If $T \in B(H)$, then $T^{(m+r)} \in \operatorname{ref}(\mathcal{S}^{(m+r)})$. For any $x_2, \ldots, x_{m+r} \in H$, there exist $t_1, \ldots, t_n$ so that

$$\begin{pmatrix} Tx_1 \\ \vdots \\ Tx_{m+r} \end{pmatrix} = t_1 \begin{pmatrix} A_1x_1 \\ \vdots \\ A_1x_{m+r} \end{pmatrix} + \cdots + t_n \begin{pmatrix} A_nx_1 \\ \vdots \\ A_nx_{m+r} \end{pmatrix}. \tag{2.12}$$

Since $Tx_1 \in \operatorname{span}\{A_1x_1, \ldots, A_nx_1\}$, it follows that $P^\perp Tx_1 \in \operatorname{span}\{P^\perp A_1x_1, \ldots, P^\perp A_Jx_1\}$. Hence there exist $v_1, \ldots, v_J$ so that

$$P^\perp Tx_1 = \sum_{i=1}^{J} v_i P^\perp A_i x_1. \tag{2.13}$$

By (2.11) to (2.13), we have

$$Tx_g = \sum_{i=1}^{J} \Big(v_i - \sum_{j=J+1}^{l} t_j a_{ij}\Big) A_i x_g + \sum_{i=J+1}^{n} t_i A_i x_g, \quad g = 2, \ldots, m+r. \tag{2.14}$$

Let

$$\begin{gathered} C = T - \sum_{i=1}^{J} v_i A_i, \quad B_j = A_j - \sum_{i=1}^{J} a_{ij} A_i, \\ J+1 \leqslant j \leqslant l, \quad B_j = A_j, \quad l+1 \leqslant j \leqslant n. \end{gathered} \tag{2.15}$$

By (2.14) and (2.15), we have

$$\begin{pmatrix} Cx_2 \\ \vdots \\ Cx_{m+r} \end{pmatrix} = t_{J+1} \begin{pmatrix} B_{J+1}x_2 \\ \vdots \\ B_{J+1}x_{m+r} \end{pmatrix} + \cdots + t_n \begin{pmatrix} B_nx_2 \\ \vdots \\ B_nx_{m+r} \end{pmatrix}.$$

Let $\widetilde{\mathcal{S}} = \operatorname{span}\{B_{J+1}, \ldots, B_n\}$. Then $\dim P^\perp \widetilde{\mathcal{S}} \leqslant l - J$ and $k(\widetilde{\mathcal{S}}_M) = k(\mathcal{S}_M) = \dim M$. Since $P^\perp \mathcal{S}$ has property A, we have that $J \geqslant \{\sqrt{2l} - 1/2\}$. So $m+r-1 \geqslant m + [\sqrt{2(l-J)}] \geqslant m + [\sqrt{2 \dim P^\perp \widetilde{\mathcal{S}}}]$. By the induction hypothesis, we have $C \in \operatorname{span}\{B_{J+1}, \ldots, B_n\}$. Hence $T \in \operatorname{span}\{A_1, \ldots, A_n\} = \mathcal{S}$. By Proposition 2.4, $k = k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^{\mathrm{c}}) = m + k(P^\perp \mathcal{S})$. Since $P^\perp \mathcal{S}$ has property A, $k(P^\perp \mathcal{S}) \geqslant \{\sqrt{2l} - 1/2\}$, it follows $k \geqslant m + \{\sqrt{2l} - 1/2\} \geqslant m + [\sqrt{2l}]$. Hence $\mathcal{S}$ is k-reflexive.

If $\mathcal{S}_M = \mathcal{S}$, then $\mathcal{S}$ is k-reflexive by Proposition 2.12. ∎

THEOREM 2.14. *If* $\dim \mathcal{S} = n$, *then* $\mathcal{S}$ *is* $[\sqrt{2n}]$*-reflexive.*

Proof. If $n = 1, 2, 3$, Theorem 3 from [13] implies the result. Suppose the result holds for $\dim \mathcal{S} \leqslant n-1$, $n \geqslant 4$. Let $\dim \mathcal{S} = n$ and suppose $k(\mathcal{S}) = k$. If $k \leqslant [\sqrt{2n}]$, by Theorem 2.13 it follows that $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive.

If $k > [\sqrt{2n}]$ then $k \geqslant \{\sqrt{2n} - 1/2\}$. If $k = n$, then $\mathcal{S}$ is 2-reflexive. Hence $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive. If $[\sqrt{2n}] < k \leqslant n-1$, using the same argument as in Lemma 2.11, we have $\dim \operatorname{span}\{B_{k+1}, \ldots, B_n\} \leqslant n-k$. By the induction hypothesis, it follows that $\operatorname{span}\{B_{k+1}, \ldots, B_n\}$ is $[\sqrt{2(n-k)}]$-reflexive. Since $k \geqslant \{\sqrt{2n} - 1/2\}$, it follows that $[\sqrt{2n}] - 1 \geqslant [\sqrt{2(n-k)}]$. Thus $\operatorname{span}\{B_{k+1}, \ldots, B_n\}$ is $([\sqrt{2n}] - 1)$-reflexive, so $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive. ∎

EXAMPLE 2.15. Let $\mathcal{S}_k$ be the set of all $k\times k$ upper triangular matrices with zero trace. We may show $\dim \mathcal{S}_k = \frac{k(k+1)}{2} - 1$ and $\mathcal{S}_k$ is not $(k-1)$-reflexive. For any positive integer l, one can easily show that there exists a positive integer k such that

$$\frac{k(k+1)}{2} - 1 \leqslant l < \frac{(k+1)(k+2)}{2} - 1. \tag{2.16}$$

For any positive integer l, choose k such that (2.16) holds and let

$$m = l - \left(\frac{k(k+1)}{2} - 1\right).$$

Let $\mathcal{S} = \mathcal{S}_k \oplus \mathcal{A}_m$, where $\mathcal{A}_m = \{\mathrm{diag}(a_1, \ldots, a_m) : a_i \in \mathbb{C}\}$. It is easy to prove that $\mathcal{S}$ is not $([\sqrt{2l}] - 1)$-reflexive.

REMARKS 2.16. (i) By Theorem 2.14 and Example 2.15, it follows that $[\sqrt{2n}]$ is the smallest integer such that all n-dimensional subspaces of $B(H)$ are $[\sqrt{2n}]$-reflexive. Thus we answer a question of Magajna ([15]).

(ii) By the proof of Theorem 2.14, we have that if $k(\mathcal{S}) \geqslant n-1$, then $\mathcal{S}$ is 2-reflexive and that if $k(\mathcal{S}) \geqslant n-4$, then $\mathcal{S}$ is 3-reflexive. This improves Theorem 3.6 from [2].

In the following, so we give an application of Theorem 2.14.

THEOREM 2.17. *If* $\Phi(\cdot) = \sum\limits_{i=1}^{n} a_i(\cdot)b_i$, $\{a_i\}, \{b_i\}$ *are subsets of a* C^**-algebra* $\mathcal{A}$, *then* Φ *is completely positive if and only if* Φ *is* $\max\{[\sqrt{2(n-1)}], 1\}$*-positive.*

The proof is similar to the proof of Theorem 6 from [13]; we leave it to the reader.

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