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ENDOMORPHISMS OF CONTINUOUS CUNTZ ALGEBRAS

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Abstract. We establish a one to one correspondence between endomorphisms of Arveson's continuous analogues $C^*(E)$ of the Cuntz algebras and certain cocycles. For the analogues of quasi-free automorphism groups there are no positive gauge invariant KMS-weights, whereas for the gauge action there exists a non lower semi-continuous ground weight on $C^*(E)$. Crossed products by quasi-free actions are often simple.

Keywords: *Continuous Cuntz algebras (spectral algebras), endomorphisms, quasi-free automorphisms.*

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1. INTRODUCTION

In [3] a class of C^*-algebras, called spectral algebras is defined and studied which may be viewed as a continuous analogue of the Cuntz algebras $\mathcal{O}_n$. Many techniques established for the Cuntz algebras have parallels in the continuous situation. For instance, recall [5] that there is a 1-1-correspondence between unitaries and unital endomorphisms of $\mathcal{O}_n$ defined as follows: Let $s_1, \dots, s_n$ be a fixed sequence of generators of $\mathcal{O}_n$. Then by simplicity for any $u \in \mathcal{O}_n$ unitary the map $s_i \mapsto us_i$, $i = 1, \dots, n$ extends to a unital endomorphism ρ_u of $\mathcal{O}_n$. Conversely, any unital endomorphism defines a unitary $u_\rho = \sum\limits_{i=1}^{n} \rho(s_i)s_i^*$ such that $\rho_{u_\rho} = \rho$ and $u_{\rho_u} = u$.

In this paper we first try to generalize this correspondence to Arveson's spectral algebras. It turns out that in case of $C^*(E)$ we have to replace unitaries by certain cocycles of a semigroup and unital endomorphisms by endomorphisms whoses images contain approximate units.

We then consider an analogue of the quasi-free automorphisms in [6], using a definition in [1] and show that there are no KMS-states and no reasonable KMS-weights for 1-parameter groups of quasi-free actions. In the discrete case KMS-states are closely related to diagonal subalgebras which play an important role in the groupoid picture of $\mathcal{O}_n$ ([5] and [11]). Their absence in the continuous setting at least indicates that here diagonals and a groupoid description must be of a different nature.

For the analogue of the gauge action, we find a non lower semi-continuous ground weight and show that there are no ground states. (We do not know whether this is the first such example. Problems of this type have been posed by Sakai in [12].)

In the last section we consider crossed products by quasi-free actions. In the discrete case they provide examples of simple projectionless C^*-algebras with traces coming from KMS-states. Although this has no continuous analogue, we show that in the continuous case crossed products are also often simple. In particular, any separable locally compact abelian group can act on $C^*(E_\infty)$ so that the crossed product is simple.

Let us recall briefly the definition and basic properties of continuous analogues of Cuntz algebras. For a complete discussion we refer to [3] and [15]. First we need the following definition (Definition 1.4 of [1]) Suppose we have a measurable field of separable Hilbert spaces over $(0,\infty)$, i.e. a standard Borel space E together with a Borel measurable map $p : E \to (0,\infty)$ such that $E(t) := p^{-1}(t)$ are separable Hilbert spaces and $E \cong E(t_0) \times (0,\infty)$ as Borel fibrations. This means that there exists a trivialization, i.e. a sequence of measurable sections $t \mapsto e_n(t) \in E(t)$ such that $(e_n(t)) \subseteq E(t)$ is an orthonormal basis for all $t > 0$. Assume further that there is a measurable product $E \times E \to E$ s.t $p(ef) = p(e)+p(f)$ and the map $E(s) \otimes E(t) \ni e \otimes f \mapsto ef \in E(s+t)$ extends to an isomorphism of Hilbert spaces for $s,t > 0$. Then E is called a *product system.* Such a structure is a continuous analogue of the tensor powers of a single Hilbert space, i.e. the monoid $\bigcup_{n\in\mathbb{N}} H^{\otimes^n}$. A section $t \mapsto u(t) \in E(t)$ such that $\|u(t)\| = 1$, $u(s+t) = u(s)u(t)$, $s,t > 0$ is called a *(normalized) unit.*

A representation of a product system is a measurable map $\phi : E \to \mathcal{B}(H)$, fiberwise linear and multiplicative such that $\phi(e)^*\phi(e) = \|e\|^2 1 \ \forall e \in E$. It follows that any measurable section $f \in L^1(E)$ defines a bounded operator $\xi \mapsto \phi(f)\xi := \int_0^\infty \mathrm{d}t\, \phi(f(t))\xi$. We may define the *spectral algebra or continuous Cuntz algebra* $C^*(E)$ associated to E as the norm closure of the $*$-algebra generated by $\Big\{ \bigoplus_\phi \phi(f)\phi(g)^* : f,g \in L^1(E) \Big\}$, where the direct sum runs over a representative set of representations of E. By results of [3] there is a 1-1-correspondence between representations of E and $C^*(E)$.

For technical reasons, we assume that all product systems in this paper contain nontrivial units. We also suppose that $\dim(E(t)) > 1$, hence infinite for one and therefore each $t > 0$.

Then by Section 8 of [3], $C^*(E)$ is simple and by [15] and [16] KK-contractible, and it contains infinite projections. Therefore it does not admit any (lower semi-continuous) traces. By Section 4.2.19 of [15], $C^*(E)$ is also the C^*-algebra generated by $\{\phi(f) : f \in L^1(E)\}$ and E is contained in the multiplier algebra of $C^*(E)$.

The most canonical representation of a product system is the regular representation on $L^2(E) = \int^{\oplus}_{(0,\infty)} E(t)\mathrm{d}t$ given by

$$(\lambda(e)\xi)(t) = \begin{cases} e\xi(t-s) & \text{if } t > s, \\ 0 & \text{otherwise,} \end{cases}$$

where $e \in E(s)$. By simplicity we may identify $C^*(E)$ with $\lambda(C^*(E))$ and know that $\lambda(E) \subseteq \mathcal{M}(C^*(E))$ and $\lambda(L^1(E)) \subseteq C^*(E)$. Let λ_t be the corresponding e_0-semigroup, i.e. $\lambda_t(A) = \sum_n \lambda(e_n(t))A\lambda(e_n(t))^*$ with $(e_n(t))$ a fixed trivialization of E and the sum taken weakly or strongly. A λ-cocycle is a strongly measurable (equivalently ultraweakly continuous) family $(U_t)_{t\in\mathbb{R}_+} \subseteq \mathcal{B}(L^2(E))$ such that $U_t^*U_t = \lambda_t(\mathbb{1}) = P(t)$ which is the projection onto $\int^{\oplus}_{(t,\infty)} E(s)\,\mathrm{d}s$ and $U_s\lambda_s(U_t) = U_{s+t}$ for all $s, t > 0$.

2. COCYCLES AND ENDOMORPHISMS

Definition Let A be a C^*-algebra and ρ be a $*$-endomorphism of A. We call ρ *unital* if $\rho(A)$ contains an approximate unit for A. $\mathrm{End}_1(A)$ denotes the set of unital endomorphisms of A.

Remark (i) The definition is consistent with the usual meaning for A unital.

(ii) Any $\rho \in \mathrm{End}_1(A)$ extends uniquely to the multiplier algebra $\mathcal{M}(A)$: Because $\rho(A)$ contains an approximate unit for A, we have $\rho(A)A\rho(A) = A$ (even without taking the closure by Cohen's factorization theorem ([7], (32.26)). Thus for any $m \in \mathcal{M}(A)$, $\overline{\rho}(m)\rho(a_1)b\rho(a_2) := \rho(ma_1)b\rho(a_2)$ and $\rho(a_1)b\rho(a_2)\overline{\rho}(m) := \rho(a_1)b\rho(a_2m)$ defines a multiplier $\overline{\rho}(m)$ on A, and it is clear that $\overline{\rho}$ is homomorphic. Of course $\overline{\rho}(1) = 1$.

Suppose we have a semigroup $t \mapsto s(t)$, $t > 0$ of isometries in $\mathcal{B}(H)$, where $A \subseteq \mathcal{B}(H)$ is a separable C^*-algebra acting nondegenerately on the separable Hilbert space H.

Lemma *If t ranges over $(0,\infty)$, the following conditions are equivalent:*
(i) *$t \mapsto s(t)\xi$ is $\mathrm{d}t$-measurable for all $\xi \in H$;*
(ii) *$t \mapsto s(t)x$, $xs(t)$ are $\mathrm{d}t$-measurable for all $x \in A$;*
(iii) *$t \mapsto s(t)x$, $xs(t)$ are continuous for all $x \in A$;*
(iv) *$t \mapsto s(t)\xi$ is continuous for all $\xi \in H$.*

Proof. (i) ⇔ (iv) and (ii) ⇔ (iii) follow from 10.2.3 of [8], (iii) ⇒ (iv) and (ii) ⇒ (i) are evident because A acts nondegenerately. For (i) ⇒ (ii) we can follow the proof of Pettis' theorem. We include the argument for the readers convenience: Suppose $t \mapsto s(t)\xi$ is measurable for all $\xi \in H$. Let $(\xi_n) \subseteq H$ be norm dense in the unit sphere $\{\xi \in H : \|\xi\| = 1\}$. Then $\|T\| = \sup_{n,m} |\langle \xi_n, T\xi_m\rangle|$ for each $T \in \mathcal{B}(H)$. Thus for any $\varepsilon > 0$ and $t_0 \in (0,\infty)$ we have for $x \in A$

$$\Delta_{\varepsilon,t_0} := \{t : \|(s(t) - s(t_0))x\| < \varepsilon\} = \bigcap_{n,m} \{t : |\langle \xi_n, (s(t) - s(t_0))x\xi_m\rangle| < \varepsilon\}.$$

In particular, Δ_{ε,t_0} is measurable. Let $(t_l) \subseteq (0,\infty)$ be a sequence such that $\{x_l = s(t_l)x : l \in \mathbb{N}\} \subseteq \{s(t)x : t \in (0,\infty)\}$ is dense in the latter. Then $\bigcup_l \Delta_{\varepsilon,t_l} = (0,\infty)$ and we define f_ε by induction as follows: $f_\varepsilon|\Delta_{\varepsilon,t_0} = x_0$, $f_\varepsilon(t) = x_l$ if $t \in \Delta_{\varepsilon,t_l} \setminus (\Delta_{\varepsilon,t_{l-1}} \cup \cdots \cup \Delta_{\varepsilon,t_0})$. We have $\|f_\varepsilon - s(\cdot)x\|_\infty < \varepsilon$ and f_ε is a simple function. Similarly one shows that $t \mapsto xs(t)$ is measurable. ∎

LEMMA *Let $s(t)$ be a semigroup of isometries as in one of the conditions in Lemma 2.3. Consider the strong integral $s_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}t\, s(t)$ in $\mathcal{B}(H)$ and suppose $s_\varepsilon \in A$ for each $\varepsilon > 0$. Then $s_\varepsilon x \to x$, $xs_\varepsilon \to x$ and $s(t)x \to x$, $xs(t) \to x$ for any $x \in A$ in norm, whenever $\varepsilon \to 0$, $t \to 0$. Moreover, $s(t) \in \mathcal{M}(A)$ for each $t > 0$.*

Proof. For any $x \in A$ and $t, \varepsilon, \delta > 0$ we have $s_\varepsilon s(t)x = s(t)s_\varepsilon x \stackrel{\varepsilon\to 0}{\longrightarrow} s(t)x$. Thus $x^* s_\varepsilon y = (s(t)x)^* s_\varepsilon (s(t)y) \stackrel{\varepsilon\to 0}{\longrightarrow} (s(t)x)^*(s(t)y) = x^*y$. In the same way one obtains $x^* s_\varepsilon^* y, x^* s_\varepsilon^* s_\varepsilon y \stackrel{\varepsilon\to 0}{\longrightarrow} x^*y$ which implies $(s_\varepsilon x - x)^*(s_\varepsilon x - x) = x^* s_\varepsilon^* s_\varepsilon x - x^* s_\varepsilon x - x^* s_\varepsilon^* x + x^*x \to 0$. So s_ε is a left, and because $\|s_\varepsilon^* s_\delta - s_\delta\| \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ also a right approximate unit for A. We have $s(t)s_\varepsilon x = s_\varepsilon s(t)x \stackrel{t\to 0}{\longrightarrow} s_\varepsilon x$. The same holds whenever x is on the other side. Now it is easy to see that the non-selfadjoint algebra generated by the s_ε is dense in $S := \{\int \mathrm{d}t\, f(t)s(t) : f \in L^1(\mathbb{R}_+)\}$. Let $D := SAS$. Then $s(t)D$, $Ds(t) \subseteq D$ and $s(t)d$, $ds(t) \to d$ $\forall d \in D$ provided $t \to 0$. D is dense in A because s_ε is an approximate unit. Hence $s(t) \in \mathcal{M}(A)$ and $s(t) \stackrel{t\to 0}{\longrightarrow} 1$ strictly. ∎

PROPOSITION *There is a 1-1-correspondence between:*
(i) *unital endomorphisms $\rho \in \mathrm{End}_1(C^*(E))$,*
(ii) *λ-cocycles $(U_t) \subseteq \mathcal{B}(L^2(E))$ such that $\int_0^\infty U_t \lambda(f(t))\,\mathrm{d}t \in C^*(E)$ for each $f \in L^1(E)$.*

Proof. (i)⇒(ii): Let $\rho \in \mathrm{End}_1(C^*(E))$. $\lambda \circ \rho$ defines a representation of $C^*(E)$ which is nondegenerate because ρ is unital. The claim would follow from a nonunital version of 3.18 of [1]. We offer the following proof: Since $\lambda(E) \subseteq \mathcal{M}(C^*(E))$, the extension $\overline{\rho}$ of ρ is defined on $\lambda(E)$. Thus $\overline{\rho}(\lambda(e_n(t)))$ is an isometry in $L^2(E)$ and we can form $U_t = \sum_{n=1}^\infty \overline{\rho}(\lambda(e_n(t)))\lambda(e_n(t))^*$ in the strong operator topology. It is clear that $t \mapsto U_t\xi$ is measurable provided $\xi \in L^2(E)$. For any $s, t > 0$ and ξ as above we have:

$$U_t\lambda_t(U_s)\xi = \sum_n \overline{\rho}(\lambda(e_n(t)))\lambda(e_n(t))^* \sum_k \lambda(e_k(t)) \sum_l \overline{\rho}(\lambda(e_l(s)))\lambda(e_l(s))^*\lambda(e_k(t))^*\xi,$$

where all the sums are taken in the strong sense. We get the convergent sum

$$\sum_n \sum_l \overline{\rho}(\lambda(e_n(t)e_l(s)))\lambda(e_n(t)e_l(s))^*\xi.$$

Because $(e_n(t)e_l(s))$ is an orthonormal basis of $E(t+s)$, this equals U_{t+s}. $\overline{\rho}(1) = 1$ implies that $U_t^*U_t = \sum \lambda(e_n(t))\overline{\rho}(1)\lambda(e_n(t))^* = \lambda_t(1) = P(t)$. Furthermore,

$$\rho(\lambda(f)) = \int_0^\infty \mathrm{d}t\, U_t\lambda(f(t))$$

for f of the form $t \mapsto \alpha(t)e_i(t)$, $\alpha \in L^1(\mathbb{R}_+)$, hence for all $f \in L^1(E)$.

(ii)⇒(i) For any cocycle U_t as in (ii), the map $e(t) \mapsto U_t\lambda(e(t))$, $e(t) \in E(t)$ defines a representation of E on $L^2(E)$ and thus gives a faithful representation of $C^*(E)$. The assumption implies that its image lies in $\lambda(C^*(E))$ and we get a $*$-endomorphism ρ of $C^*(E)$. By Lemma 2.3, $s(t) := U_t\lambda(u(t))$ is a strictly continuous semigroup of isometries such that all integrals $\int \mathrm{d}t\, \alpha(t)s(t)$, where $\alpha \in L^1(\mathbb{R}_+)$ are in $C^*(E)$. By Lemma 2.4, $s_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}t\, s(t)$ is an approximate unit in $C^*(E)$ which lies in $\rho(C^*(E))$ and thus ρ is unital. ∎

We denote the above cocycle by ${}_\rho U$ and write ρ_U for the endomorphism given by U. The above proof shows that ${}_\rho U_t \in C^*(E)^{**}$ for all $t > 0$. Let ρ^{**} be the bitransposed endomorphism. We write $\rho({}_\sigma U_t)$ for $\rho^{**}({}_\sigma U_t)$.

Remark (i) ${}_{\rho\sigma}U_t = \rho({}_\sigma U_t)\,{}_\rho U_t$ and $\rho_U\rho_V = \rho_{\rho_U(V)U}$.

(ii) If $\rho = \mathrm{Ad}(W)$ for some unitary $W \in \mathcal{M}(C^*(E))$, then $U_t = W\lambda_t(W^*)$ is the corresponding cocycle.

(iii) Let us mention without proof that there exists a partial extension of the above correspondence to certain completely positive maps. Let $\varphi : L^1(E) \to A$ be any bounded homomorphism into a C^*-algebra A. Then φ extends uniquely to a completely positive map $\phi : C^*(E) \to A$. Conversely, any completely positive map ψ such that $\psi|L^1(E)$ is homomorphic has ψ as its unique extension. If $A = C^*(E)$, then $a_t = \sum \overline{\phi}(\lambda(e_n(t)))\lambda(e_n(t))^*$ is a family of operators such that $a_t\lambda_t(a_s) = a_{t+s}$ (here $\overline{\phi}$ denotes the extension of ϕ to say $C^*(E)^{**}$). Conversely, any such "subcocycle" with $\int \mathrm{d}t\, a_t\lambda(f(t)) \in C^*(E)$, $\forall f \in L^1(E)$ defines a completely positive map of $C^*(E)$ into itself. This may be viewed as a continuous analogue of the correspondence between contractions in and certain completely positive maps on $\mathcal{O}_n$ studied in [4]. On the other hand, there should exist completely positive maps on $C^*(E)$ which are not multiplicative on $L^1(E)$.

3. WEIGHTS AND QUASI-FREE AUTOMORPHISMS

An *automorphism* of a product system E is a measurable fiberwise unitary map α preserving the product of E. α defines an automorphism of $C^*(E)$ also denoted by α and such automorphisms are called *quasi-free.* For instance one always has the quasi-free gauge automorphism $\gamma_s(e(t)) := \mathrm{e}^{\mathrm{i}st}e(t)$, $e(t) \in E(t)$. Consider the exponential product systems E_n with $E_n(t) = \mathcal{F}^s(L^2((0,t),\mathbb{C}^n))$. They are exactly those generated by their units i.e. $E_n(t) = [u_1(t_1)\cdots u_k(t_k) | k \in \mathbb{N},\ u_i \text{ units},\ t_1 + \cdots + t_k = t]$. The index n is a complete cocycle conjugacy invariant in this case. In Section 8 of [1] Arveson showed that for E_n, $n = 1,\ldots,\infty$, all quasi-free automorphisms are given by

$$\alpha(\exp(f)) = \mathrm{e}^{\mathrm{i}st - t\|\xi\|^2/2 - \int_0^t \langle \xi, uf(x)\rangle\,\mathrm{d}x} \exp(uf + \chi_{(0,t)} \otimes \xi),$$

where $f \in L^2((0,t),\mathbb{C}^n)$, u is a unitary on $\mathbb{C}^n$ and $\xi \in \mathbb{C}^n$. The group formed by them is therefore $U(n)\rtimes\mathbb{C}^n \times \mathbb{R}$. Let α_t be a 1-parameter group of quasi-free automorphisms of $C^*(E)$. Then $\alpha_t|E(s) : E(s) \to E(s)$ is a unitary group on the Hilbert space $E(s)$ i.e. $\alpha_t|E(s) = \mathrm{e}^{\mathrm{i}tH_s}$. We have $H_s \otimes H_r \cong H_{s+r}$. By Theorem 3.4 of [2] the maps $\mathrm{e}^{-\beta H_s}$ are not compact for any $\beta \in \mathbb{R}$. Thus we have:

REMARK Fix $\beta \in \mathbb{R}$ and $s \in \mathbb{R}_+$. For any $\varepsilon_1 > 0$ and $c > 0$ we can find $N \in \mathbb{N}$ and $v_k(s) \in E(s)$ unit vectors, analytic for the action α_t such that $\sum_{k=0}^{N} \|\mathrm{e}^{-\beta H_s} v_k(s) - \mathrm{e}^{-\beta\lambda_k} v_k(s)\| < \varepsilon_1$ and $\sum_{k=0}^{N} \mathrm{e}^{-\beta\lambda_k} > c$.

It is easy to see that there are no KMS-states:

PROPOSITION *Let α_t be a 1-parameter quasi-free automorphism group. Then there is no nonzero β-KMS state φ_β on $C^*(E)$ for any value of β.*

Proof. Suppose the contrary and let φ_β be such a state. It extends to $\mathcal{M}(C^*(E))$. Let $c > 0$, $\varepsilon_1 > 0$ and take N and $v_k(s)$ as in Remark 3.1. Note that $v_k(s) \in \mathcal{M}(C^*(E))$ by 4.2.20 of [15]. The β-KMS condition implies:

$$\begin{aligned} 1 = \varphi_\beta(1) &\geqslant \sum_{k=0}^{N} \varphi_\beta(v_k(s)v_k(s)^*) = \sum_{k=0}^{N} \varphi_\beta(v_k(s)^*\alpha_{\mathrm{i}\beta}(v_k(s))) \\ &\geqslant \sum_{k=0}^{N} \mathrm{e}^{-\lambda_k\beta}\varphi_\beta(v_k(s)^*v_k(s)) - \varepsilon_1 = \sum_{k=0}^{N} \mathrm{e}^{-\lambda_k\beta} - \varepsilon_1 \end{aligned}$$

and $\sum_{k=0}^{N} \mathrm{e}^{-\lambda_k\beta} \geqslant c$. For c big enough we thus obtain a contradiction. ∎

Recall that for any weight φ on a C^*-algebra which we always assume to be positive, we have the left ideal $N_\varphi = \{x \in A : \varphi(x^*x) < \infty\}$ and the hereditary subalgebra $\mathrm{Dom}(\varphi) := N_\varphi^* N_\varphi$ on which φ is finite. φ is called *lower semi-continuous* (l.s.c. for short) if $\{x \in A_+ : \varphi(x) \leqslant d\}$ is norm closed in A_+ for each $d > 0$. In this case we have $\varphi(x) = \sup\{\omega(x) : \omega \in A_+^*, \omega \leqslant \varphi\}$ for any positive element. As β-KMS condition we require that $\varphi(xy) = \varphi(y\alpha_{\mathrm{i}\beta}(x))$ for all analytic elements $x, y \in \mathrm{Dom}(\varphi)$. Note that KMS-weights are always invariant under the respective group.

In order to show the absence of KMS-weights for quasi-free actions on $C^*(E)$, we need certain domain conditions. Let us call a section $t \mapsto e(t) \in E(t)$ continuous if $t \mapsto \lambda(e(t))$ is a strongly continuous family of operators (equivalently in any other nonzero representation). Using Proposition 2.5 of [1] we can find a trivialization (e_n) of E such that $t \mapsto e_n(t)$ is continuous for each $n \in \mathbb{N}$. We denote the dense subspace of $L^1(E)$ consisting of the continuous sections of compact support by $C_c(E)$. Then $\mathrm{span}(C_c(E)C_c(E)^*)$ is a norm dense $*$-subalgebra.

Consider the unbounded weight $T : C^*(E) \to \mathcal{B}(L^2(E))$ defined as the weak integral over the gauge group. More precisely: Let $\mathrm{Dom}(T) := \Big\{x \in C^*(E) : \exists y \in \mathcal{B}(L^2(E))\ \forall \xi, \eta \in L^2(E) : \langle \xi, y\eta\rangle = \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} \mathrm{d}t\, \langle \xi, \lambda(\gamma_t(x))\eta\rangle \Big\}$ and define $T(x) = y$ for $x \in \mathrm{Dom}(T)$.

LEMMA $\mathrm{span}(C_c(E)C_c(E)^*) \subseteq \mathrm{Dom}(T)$ *and for* $f, g \in C_c(E)$ *we have*

$$T(\lambda(f)\lambda(g)^*) = \int_0^\infty \mathrm{d}t\, \lambda(f(t))\lambda(g(t))^*.$$

Proof. For $f, g \in C_c(E)$ and $\xi, \eta \in L^2(E)$, we have

$$\frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} \mathrm{d}r \langle \xi, \gamma_r(\lambda(f)\lambda(g)^*)\eta\rangle = \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} \mathrm{d}r \int \mathrm{d}t \int \mathrm{d}s\, \mathrm{e}^{\mathrm{i}r(s-t)} \langle \xi, \lambda(f(s))\lambda(g(t))^*\eta\rangle$$
$$= \int_0^\infty \mathrm{d}t\, \langle \xi, \lambda(f(t))\lambda(g(t))^*\eta\rangle,$$

because the function $(s,t) \mapsto \langle \xi, \lambda(f(s))\lambda(g(t))^*\eta\rangle$ is continuous with compact support. ∎

Let $\mathcal{A}_E \subseteq \mathcal{M}(C^*(E))$ be the C^*-algebra generated by the set

$$\Big\{ \int_0^\infty \mathrm{d}t\, \lambda(f(t))\lambda(g(t))^* : f, g \in L^2(E) \Big\}.$$

From now on we may restrict T to $D := \{x \in \mathrm{Dom}(T) : T(x) \in \mathcal{A}_E\}$ and T remains densely defined. We call a weight φ on $C^*(E)$ gauge invariant if it admits a factorization $\varphi = \overline{\varphi} \circ T$, where $\overline{\varphi}$ is a weight on $\mathcal{A}_E$.

THEOREM *Let* φ *be a lower semi-continuous weight on* $C^*(E)$ *such that* $C_c(E)C_c(E)^* \subseteq N_\varphi^* N_\varphi = \mathrm{Dom}(\varphi)$.

(i) φ *is gauge invariant iff* $\varphi \circ \gamma_t = \varphi$ *for all* $t \in \mathbb{R}$ *and in this case there exists a positive Borel measure* μ *on* $\mathbb{R}_+$ *and a measurable* μ*-a.e. bounded family* $t \mapsto A(t) \in \mathcal{B}(E(t))$ *such that* $\varphi(\lambda(f)\lambda(g)^*)) = \int_0^\infty \mathrm{d}\mu(t)\ \mathrm{tr}\big[A(t)(f(t) \otimes \overline{g(t)})\big]$.

*Suppose further that φ is a β-*KMS*-weight for the quasi-free action α and some finite β.*

(ii) *If φ is gauge invariant, then $\varphi = 0$.*

(iii) *If there exists a unit u and $\lambda \neq 0$ such that $\alpha_t(u)(s) = \mathrm{e}^{\mathrm{i}\lambda st}u(s)$ for all $s > 0$, then $\varphi = 0$.*

Proof. (i) Let φ be any weight on $C^*(E)$ such that $C_{\mathrm{c}}(E) \subseteq N_\varphi^*$. We define the positive sesquilinear form

$$b : C_{\mathrm{c}}(E) \times C_{\mathrm{c}}(E) \to \mathbb{C}, \quad b(g, f) := \varphi(\lambda(f)\lambda(g)^*).$$

The associated Hilbert space is a subspace of the GNS-space $\mathcal{H}$ for φ. The form b determines φ provided $\mathrm{span}(C_{\mathrm{c}}(E)C_{\mathrm{c}}(E)^*)$ acts non-degenerately, in particular if φ is l.s.c. which we assume. Any continuous real valued function $\alpha \in C_{\mathrm{c}}(\mathbb{R}_+)$ with compact support acts on $C_{\mathrm{c}}(E)$ by $(\alpha f)(t) := \alpha(t)f(t)$. Now suppose φ is gauge invariant. We can write $\varphi = \overline{\varphi} \circ T$ with some weight $\overline{\varphi}$ on $\mathcal{A}_E$ such that $\left\{ \int_0^\infty \mathrm{d}t\, \lambda(f(t))\lambda(g(t))^* \,:\, f, g \in C_{\mathrm{c}}(E) \right\} \subseteq \mathrm{Dom}(\overline{\varphi})$. Clearly, $\varphi \circ \gamma_t = \varphi\ \forall t \in \mathbb{R}$, and the gauge invariance implies $b(\alpha g, f) = b(g, \alpha f)$ for any α as above. If on the other hand, we assume $\varphi \circ \gamma_t = \varphi\ \forall t \in \mathbb{R}$, then the group $(U_t f)(s) := \mathrm{e}^{\mathrm{i}st}f(s)$ on $C_{\mathrm{c}}(E)$ extends to a strongly continuous unitary group on $\mathcal{H}_\varphi$ implementing γ. It follows that $b(\alpha g, f) = b(g, \alpha f)$ for Fourier transforms of L^1-functions, and in fact all real valued $\alpha \in C_{\mathrm{c}}(\mathbb{R}_+)$. It remains to show that this condition implies the existence of μ and $t \mapsto A(t)$.

Taking a trivialization (e_n) of E which consists of continuous sections, the functionals $C_{\mathrm{c}}(\mathbb{R}_+) \ni \alpha \mapsto b(\alpha e_i, e_i)$ are positive and hence define positive Radon measures on $\mathbb{R}_+$. By the polarization identity, we obtain complex Borel measures μ_{ij} on $\mathbb{R}_+$ such that $\mu_{ij}(\alpha) = b(\alpha e_i, e_j)$. Let μ be any regular Borel measure whose class dominates $\mathrm{d}t$ and $|\mu_{ij}|$ for all $i, j \in \mathbb{N}$. Then each μ_{ij} has a density ρ_{ij} with respect to μ and for μ-almost all $t \in \mathbb{R}_+$, the form $\sum \overline{a}_i b_j \rho_{ij}(t)$ is finite whenever $(a_i), (b_j) \in \ell^2(\mathbb{N})$ (or $\mathbb{C}$ if E is trivial). Hence the matrix $(\rho_{ij}(t))$ defines a measurable family of positive operators $A(t)$ on $E(t)$ which are μ-a.e. bounded and such that $\varphi(\lambda(f)\lambda(g)^*) = \int_0^\infty \mathrm{d}\mu(t)\ \mathrm{tr}[A(t)(f(t) \otimes \overline{g(t)})]$. Note that $t \mapsto \|A(t)\|$ is not necessarily in $L^\infty(\mu)$.

(ii) Let φ be l.s.c., gauge invariant and β-KMS, β finite such that $C_{\mathrm{c}}(E) \subseteq N_\varphi^*$. We can find μ and $t \mapsto A(t)$ as in (i). We have for $K, L \in \mathrm{span}(C_{\mathrm{c}}(E)C_{\mathrm{c}}(E)^*)$:

$$\begin{aligned}
\varphi(KL) = &\int_0^\infty \mathrm{d}\mu(t)\ \mathrm{tr}\left[A(t)(KL)(t,t)\right] = \int_0^\infty \mathrm{d}\mu(t) \int_0^\infty \mathrm{d}s \int_0^{\min(s,t)} \mathrm{d}\lambda \\
&\qquad \mathrm{tr}\Big[A(t)\big[K(t,s)(L(s-\lambda, t-\lambda) \otimes 1_\lambda) + (K(t-\lambda, s-\lambda) \otimes 1_\lambda)L(s,t)\big]\Big] \\
= &\int_0^\infty \mathrm{d}\mu(t) \int_0^\infty \mathrm{d}s \int_0^{\min(s,t)} \mathrm{d}\lambda \\
&\qquad \mathrm{tr}\Big[A(t)\big[\mathrm{e}^{\beta H_t}\mathrm{e}^{-\beta H_t}K(t,s)\mathrm{e}^{\beta H_s}\mathrm{e}^{-\beta H_s}(L(s-\lambda, t-\lambda) \otimes 1_\lambda)
\end{aligned}$$

$$+ \mathrm{e}^{\beta H_t}\mathrm{e}^{-\beta H_t}(K(t-\lambda, s-\lambda)\otimes 1_\lambda)\mathrm{e}^{\beta H_s}\mathrm{e}^{-\beta H_s}L(s,t)\big]\Big]$$

$$(*) \qquad = \int_0^\infty \mathrm{d}\mu(t) \int_0^\infty \mathrm{d}s \int_0^{\min(s,t)} \mathrm{d}\lambda$$

$$\operatorname{tr}\Big[\mathrm{e}^{-\beta H_s}\big[(L(s-\lambda, t-\lambda)\otimes 1_\lambda)A(t)\mathrm{e}^{\beta H_t}(\alpha_{\mathrm{i}\beta}K)(t,s)$$

$$+ L(s,t)A(t)\mathrm{e}^{\beta H_t}((\alpha_{\mathrm{i}\beta}K)(t-\lambda, s-\lambda)\otimes 1_\lambda)\big]\Big] = \varphi(L\alpha_{\mathrm{i}\beta}K),$$

where the last equality follows from the β-KMS-condition. On the other hand,

$$(**) \qquad \varphi(L\alpha_{\mathrm{i}\beta}K) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}\mu(s) \int_0^{\min(s,t)} \mathrm{d}\lambda \operatorname{tr}\Big[A(s)\big[L(s,t)((\alpha_{\mathrm{i}\beta}K)(t-\lambda, s-\lambda)\otimes 1_\lambda)$$

$$+ (L(s-\lambda, t-\lambda)\otimes 1_\lambda)(\alpha_{\mathrm{i}\beta}K)(t,s)\big]\Big].$$

It follows that if $f, g \in C_{\mathrm{c}}(E)$, now also $\varphi(\lambda(\alpha_1 f)\lambda(\alpha_2 g)^*)$ is defined for bounded Borel functions α_1 and α_2. In particular, the families $\chi_{(a,b]}K\chi_{(c,d]}$ and $\chi_{(a,b]}L\chi_{(c,d]}$ are in $\mathrm{Dom}(\varphi)$. The Radon-Nikodym theorem implies that for any regular Borel measure ν and $f \in L^1(\mathbb{R}_+, \nu)$, we have $f(t) = \lim_{\varepsilon\to 0} \frac{1}{\nu(t-\varepsilon, t]} \int_{t-\varepsilon}^{t} \mathrm{d}\nu f$, ν-a.e. Now for $a > \varepsilon > 0$ we replace L by $\chi_{(a-\varepsilon,a]}L$ and K by $K\chi_{(a-\varepsilon,a]}$. Using a little calculation (compare Proposition 3.5 (i)), the first expression $(*)$ multiplied by ε^{-2} converges to $\int_0^\infty \mathrm{d}\mu(t)\ \operatorname{tr}[\mathrm{e}^{-\beta H_a}L(a,t)A(t)\mathrm{e}^{\beta H_t}\alpha_{\mathrm{i}\beta}K(t,a)]$ for almost all $a > 0$ and the second $(**)$ multiplied by $(\varepsilon\mu(a-\varepsilon, a])^{-1}$ converges to $\int_0^\infty \mathrm{d}t\ \operatorname{tr}[A(a)L(a,t)\alpha_{\mathrm{i}\beta}K(t,a)]$ for μ-almost all $a > 0$ and $\varepsilon \to 0$. Thus we may replace μ by the Lebesgue measure and conclude then $A(t) = \mathrm{e}^{-\beta H_t}$ for almost all $t > 0$.

On the other hand, $\mathrm{e}^{-\beta H_t}$ are noncompact operators and we can show that they do not define positive weights: Let $K = \lambda(f)\lambda(f)^*$, $f \in C_{\mathrm{c}}(E)$ which is positive. Define for $\delta > 0$ and $n \in \mathbb{N}$ the section $fe_n(t) = f(t-\delta)e_n(\delta)$ if $t > \delta$ and 0 otherwise.

Then $K_N := \sum_{n=0}^{N} \lambda(fe_n)\lambda(fe_n)^* = \lambda(f)\Big(\sum_{n=0}^{N}\lambda(e_n(\delta))\lambda(e_n(\delta))^*\Big)\lambda(f)^* \leqslant K$ and

$$\varphi(K) \geqslant \varphi(K_N) = \sum_{n=0}^{N} \int_\delta^\infty \mathrm{d}t\ \operatorname{tr}\big[\mathrm{e}^{-\beta H_t}((fe_n)(t)\otimes(\overline{fe_n})(t))\big]$$

$$= \sum_{n=0}^{N} \int_0^\infty \mathrm{d}t\ \operatorname{tr}\big[\mathrm{e}^{-\beta H_t}(f(t)\otimes\overline{f(t)})\big] \operatorname{tr}\big[\mathrm{e}^{-\beta H_\delta}(e_n(\delta)\otimes\overline{e_n(\delta)})\big]$$

$$\geqslant \varphi(K)\sum_{n=0}^{N} \operatorname{tr}\big[\mathrm{e}^{-\beta H_\delta}(e_n(\delta)\otimes\overline{e_n(\delta)})\big].$$

Since the last sum diverges for $N \to \infty$, we conclude $\varphi(K) = 0$. Using the polarization identity, it follows that $\varphi(C_c(E)C_c(E)^*) = 0$ which finishes the proof.

(iii) Suppose φ is as in the assumption with GNS-construction $(\pi, \Lambda, \mathcal{H})$. Then $\Lambda : N_\varphi \to \mathcal{H}$ is a closed linear map ([13], 2.1.11). Let u be a unit such that $\alpha_t(u)(s) = e^{i\lambda st}u(s)$, where $\lambda \neq 0$. Then α leaves the C^*-algebra generated by $\{u(\alpha) : \alpha \in C_c(\mathbb{R}_+)\}$ invariant. This algebra is the Wiener-Hopf algebra $\mathcal{W}$ and also the C^*-algebra $C^*(E_0)$ of the trivial product system. But $\alpha_t = \gamma_{\lambda t}$ on this subalgebra, hence $\varphi|\mathcal{W}$ is gauge invariant. Using (ii), $\varphi(u(\alpha)u(\beta)^*) = 0$ whenever $\alpha, \beta \in C_c(\mathbb{R}_+)$. Let $(\alpha_n) \subseteq C_c(\mathbb{R}_+)$ be a sequence of positive functions such that $\operatorname{supp}(\alpha_n) \subseteq \left(0, \frac{1}{n}\right)$ and $\int_0^{n^{-1}} dt\, \alpha_n = 1$. Then $u_n := u(\alpha_n)u(\alpha_n)^*$ is an approximate unit in $C^*(E)$ and we have $\langle \Lambda(xu_n), \Lambda(y)\rangle = \varphi(u_n x^* y) = 0$, whenever $x, y \in N_\varphi$ by the Cauchy-Schwarz inequality. Thus $\Lambda(xu_n) = 0$ for all n and therefore $\Lambda(x) = 0$ because Λ is closed. But this means $\varphi = 0$. ∎

We now define a counterpart of the ground state of $\mathcal{O}_\infty$ or the Toeplitz-Cuntz algebras $\mathcal{T}_n$ on $C^*(E)$. Note that according to Voiculescu ([14]), $\mathcal{T}_n$ is the reduced free product of n Toeplitz algebras with respect to the ground state on $\mathcal{T}_1$. One would expect to obtain $C^*(E_n)$ as a kind of reduced free product of $n+1$ Wiener-Hopf algebras with respect to the weight we are going to consider. Although our weights have somewhat unusual properties, i.e. they are not lower semi-continuous, this property is necessary if we want to obtain irreducible GNS-representations. We keep the assumption that E contains a unit u and put:

$$\xi_\varepsilon = \begin{cases} \frac{1}{\varepsilon}u(t) & \text{if } t < \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\xi_\varepsilon\|^2 = \frac{1}{\varepsilon}$. The same section considered as an element in $L^1(E)$ has norm 1 and is an approximate unit denoted by u_ε. We have the family of vector functionals $\varphi_\varepsilon(x) = \langle \xi_\varepsilon, \lambda(x)\xi_\varepsilon\rangle$. If the limit for $\varepsilon \to 0$ exists, we denote it by $\varphi_u(x)$.

PROPOSITION (i) *If $f \in L^1(E)$ is a section, then*

$$\varphi_u(\lambda(f)) = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon dt\, \langle u(t), f(t)\rangle =: \frac{1}{2}\omega_u(f).$$

(ii) *For $f, g \in L^1(E)$ and one of them bounded near* 0 *we have $\omega_u(f * g) = 0 = \varphi_u(\lambda(f)\lambda(g))$.*

(iii) *Let f, g be sections (not necessarily L^1) such that $\lambda(f)\lambda(g)^* \in C^*(E)$ and $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon dt\int_0^\varepsilon ds\, \langle u(t), f(t)\rangle\langle g(s), u(s)\rangle$ exists, then it is equal to $2\varphi_u(\lambda(f)\lambda(g)^*)$. For $f, g \in L^1(E)$, $\varphi_u(\lambda(f)\lambda(g)^*) = 0$.*

(iv) *$C^*(E)\lambda(L^1\cap L^2)+\lambda(L^1\cap L^2)$ is a dense left ideal and $\lambda(L^1\cap L^2)^*C^*(E)^\sim\cdot\lambda(L^1\cap L^2)$ is a dense hereditary subalgebra on which $\varphi(\lambda(f)^*(\lambda(x)+\alpha 1)\lambda(g)) := \langle f, \lambda(x)g\rangle + \alpha\langle f, g\rangle$ defines an extension of φ_u to a non lower semi-continuous weight having the regular representation as its non degenerate* GNS*-representation. In particular, φ_u does not depend on the choice of the unit.*

(v) *For the* GNS*-map $\Lambda_\varphi : N_\varphi \to \mathcal{H}_\varphi$ we have $\langle \Lambda_\varphi(\lambda(f)), \Lambda_\varphi(\lambda(g))\rangle = \varphi(\lambda(f)^*\lambda(g)) = \langle f, g\rangle$, $\forall f, g \in L^1 \cap L^2$. Let an operator on $L^2(E)$ be defined* *by* $(e^{izN}g)(t) = e^{izt}g(t)$. *Then* $\varphi(\lambda(f)^*\gamma_z\lambda(g)) = \langle f, e^{izN}g\rangle$ *which extends to the upper half plane.* γ_z *is the analytic extension of the gauge group.*

Proof. (i) For $f \in L^1(E)$, $\langle \xi_\varepsilon, \lambda(f)\xi_\varepsilon\rangle = \frac{1}{\varepsilon^2}\int_0^\varepsilon dt \int_0^t dx\, \langle u(t), f(x)u(t-x)\rangle = \frac{1}{\varepsilon^2}\int_0^\varepsilon dt \int_0^t dx\, \langle u(x), f(x)\rangle$. If $\omega_u(f)$ exists, then $\int_0^\varepsilon dt\, t\Big[\frac{1}{t}\int_0^t dx\, \langle u(x), f(x)\rangle - \omega_u(f)\Big] = o(\varepsilon^2)$ shows the claim.

(ii) For $f, g \in L^1(E)$, $\langle \xi_\varepsilon, \lambda(f*g)\xi_\varepsilon\rangle = \frac{1}{\varepsilon^2}\int_0^\varepsilon dt \int_0^t dx\, \langle u(t), (f*g)(x)u(t-x)\rangle = \frac{1}{\varepsilon^2}\int_0^\varepsilon dt \int_0^t dx \int_0^x ds\, \langle u(s), f(s)\rangle\langle u(x-s), g(x-s)\rangle$. The conclusion follows from the fact that for a continuous $\psi \in C_c[0,\infty)$ we have $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_0^\varepsilon dt \int_0^t dx\, \psi(x) = \frac{1}{2}\psi(0)$.

(iii) $\langle \lambda(f)^*\xi_\varepsilon, \lambda(g)^*\xi_\varepsilon\rangle = \frac{1}{\varepsilon^2}\int_0^\varepsilon dx \int_0^{\varepsilon-x} dt\, \langle u(t), f(t)\rangle \int_0^{\varepsilon-x} ds\, \langle g(s), u(s)\rangle = \frac{1}{\varepsilon^2}\int_0^\varepsilon dx \int_0^x dt \int_0^x ds\, \langle u(t), f(t)\rangle\langle g(s), u(s)\rangle$ and the first claim follows as in (i). The second claim follows because $x \mapsto \int_0^x dt \int_0^x ds\, \langle u(t), f(t)\rangle\langle g(s), u(s)\rangle$ is continuous and converges to 0 for $x \to 0$.

(iv) We first check that φ given by the formula $\varphi(\lambda(f)^*\lambda(x)\lambda(g)) = \langle f, \lambda(x)g\rangle$ is well-defined. We only have to show that the map $\lambda(x)\lambda(f) \mapsto \lambda(x)f \in L^2(E)$, $x \in C^*(E)$, $f \in L^1 \cap L^2$ is well-defined i.e. $\sum_{i=1}^k \lambda(x_i)\lambda(f_i) = 0$ implies $\sum_{i=1}^k \lambda(x_i)f_i = 0$. But $\sum_{i=1}^k \lambda(x_i)\lambda(f_i) = 0$ implies $\sum_{i=1}^k \lambda(x_i)\lambda(f_i)\xi_\varepsilon = \sum_{i=1}^k \lambda(x_i)(f_i * u_\varepsilon) = 0$ for all $\varepsilon > 0$ and $f_i * u_\varepsilon \to f_i$ in $L^2(E)$ shows that $\sum_{i=1}^k \lambda(x_i)f_i$ must be 0. φ extends φ_ε because $\langle f, \lambda(x)g\rangle = \lim_{\varepsilon\to 0}\langle f * u_\varepsilon, \lambda(x)g * u_\varepsilon\rangle = \lim_{\varepsilon\to 0}\langle \xi_\varepsilon, \lambda(f)^*\lambda(x)\lambda(g)\xi_\varepsilon\rangle = \varphi_u(\lambda(f)^*\lambda(x)\lambda(g))$ for all $x \in C^*(E)$ and $f, g \in L^1 \cap L^2$. The same calculations are valid if we replace $\lambda(x)$ by 1.

The corresponding GNS-representation is obviously the regular representation which is known to be irreducible ([3], Theorem 5.2).

Suppose finally that φ is l.s.c. Then for any positive $x \in C^*(E)$, we have $\varphi(x) = \lim\{\omega(x) : \omega \in C^*(E)^*_+ \text{ such that } \omega \leqslant \varphi\}$ and for any $0 \leqslant \omega \leqslant \varphi$ there exists a positive contraction $T_\omega \in \pi_\varphi(C^*(E))'$ such that $\omega(x^*y) = \langle \Lambda_\varphi(x), T_\omega\Lambda_\varphi(y)\rangle$. But the commutant is trivial and thus φ bounded which is a contradiction.

(v) This follows from (iv). ∎

In case of $E = E_n$ there is a nice dense subalgebra of $\mathrm{Dom}(\varphi)$ which is the polynomial $*$-algebra generated by the units in E_n. If u is a unit in a product system E and f a continuous function on $[0,\infty)$ of compact support, denote the section $t \mapsto f(t)u(t)$ by $u(f)$. Let $\mathcal{S}_1$ be the algebra and $\mathcal{S}$ be the $*$-algebra generated by them. One can see that $\mathcal{S}$ is dense in $C^*(E)$ iff $E = E_n$ for some n. Note that for any two units u and v the function $t \mapsto \langle u(t), v(t)\rangle$ is multiplicative and hence of the form $t \mapsto \mathrm{e}^{-c(u,v)t}$. c is the conditionally positive definite covariance function of [1].

REMARK Each element in $\mathcal{S}$ is a linear combination of words of the form $u_1(f_1)\cdots u_k(f_k)v_l(g_l)^*\cdots v_1(g_1)^*$. For each $f \in \mathcal{S}_1$ the strong limit $\operatorname*{s-lim}_{t\to 0}\lambda(f(t))$ exists and lies in $\mathbb{C}1$.

Proof. Let $f_1, g_1 \in C_c[0,\infty)$ and u, v units. Then we have omitting λ:

$$\begin{aligned} &v(g_1)^* u(f_1)\\ &= \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}s\, \overline{g}_1(t) f_1(s) \mathrm{e}^{-c(v,u)s} v(t-s)^* + \int_0^\infty \mathrm{d}s \int_0^s \mathrm{d}t\, \overline{g}_1(t) f_1(s) \mathrm{e}^{-c(v,u)t} u(s-t)\\ &= v(f_2)^* + u(g_2), \end{aligned}$$

where $f_2, g_2 \in C_c[0,\infty)$. The first claim follows from this. For the second, observe first that $\lim_{t\to 0}\lambda(f(t))$ exists and is in $\mathbb{C}1$ if $f = u(f_1)$ for some $f_1 \in C_c[0,\infty)$. Next suppose that for $f \in \mathcal{S}_1$, $\lim_{t\to 0}\lambda(f(t))\xi = \alpha\xi$ for any $\xi \in L^2(E)$. Then $\lambda((u(f_1) * f)(t))\xi = \int_0^t \mathrm{d}x\, f_1(x)u(x)\cdot\lambda(f(t-x))\xi \to 0$, whenever $\xi \in L^2(E)$ and $f_1 \in C_c[0,\infty)$. ∎

The weight φ is now given on $\mathcal{S}$ as follows:

$$\varphi(\lambda(f))1 = \overline{\varphi(\lambda(f)^*)}1 = \operatorname*{s-lim}_{t\to 0}\lambda(f(t)), \quad \varphi(\lambda(f)\lambda(g)^*) = 0.$$

Note that we could replace here λ by any other representation.

It should be possible to define our weight if E has no units using Proposition 3.3 (iv).

PROPOSITION *There is no ground state for the gauge action.*

Proof. Let ω be a ground state on $C^*(E)$. Then $\omega(\gamma_t(x)) = \omega(x)\ \forall x \in C^*(E)$ ([12], 4.2.2). We may consider ω as a state on the Wiener-Hopf algebra $\mathcal{W}$ generated by a unit u and suppose that in the GNS-representation of $\mathcal{W}$ with respect to ω there is an invariant subspace of the form $L^2(\mathbb{R}_+)$ on which $u(t)$ acts like a shift by t. Otherwise ω vanishes on the ideal of compact operators and defines a translation invariant state on $\mathcal{W}/\mathcal{K} = C_0(\mathbb{R})$ which does not exist. We have $\omega(x) = \langle\Omega, \pi_\omega(x)\Omega\rangle$, $x \in \mathcal{W}$ and because Ω is cyclic, the component of Ω in the subspace $L^2(\mathbb{R}_+)$ is not zero and may be identified with a function $f \in L^2(\mathbb{R}_+)$. But then $\langle f, \gamma_s u(t) f\rangle = \int_t^\infty \mathrm{d}r\, \mathrm{e}^{\mathrm{i}st}\overline{f(r)}f(r-t)$ is independent of s for any fixed positive t. Thus $\int_t^\infty \mathrm{d}r\, \overline{f(r)}f(r-t) = 0$ for $t \neq 0$ hence $f = 0$ which is a contradiction. ∎

Sakai remarked that if we have a unital C^*-algebra on a Hilbert space, any 1-parameter automorphism group which is implemented by a positive generator admits a ground state ([12], 4.2.13). As we can see here the assertion is false in the nonunital case.

4. SIMPLICITY OF CROSSED PRODUCTS

In this section we consider a strongly continuous automorphism group $w : G \to \mathrm{Aut}(E)$ (i.e. $w|E(t)$ is a strongly continuous unitary group for each $t > 0$), where G is a separable locally compact abelian group. The corresponding quasi-free automorphism group is denoted by $\alpha : G \to \mathrm{Aut}(C^*(E))$. We assume that there exists a unit u_0 in E such that $w_g(u_0) = u_0$ for all $g \in G$. For instance, this assumption is fulfilled for the $U(n)$-part of the quasi-free group on E_n. Let $S_t := \mathrm{sp}(w|E(t)) \subseteq \widehat{G}$. Then $\overline{S_t S_r} = S_{t+r}$ for $r, t > 0$. We denote $S := \bigcap_{t>0} \overline{\bigcup_{r<t} S_r}$ and $T := \overline{\bigcup_{t>0} S_t} \supseteq S$. Then we have:

Theorem $G \times_\alpha C^*(E)$ *is simple if* $S = \widehat{G}$ *and is not simple if* $T \neq \widehat{G}$.

The proof is an adaptation of Arveson's arguments which lead to the simplicity of $C^*(E)$ for any E containing a unit. It will be outlined in the rest of this section.

Notice first that the regular representation is covariant for α and denoted by (λ, U_λ). The following shows in particular the second claim of Theorem 4.1.

Proposition $\lambda \times U_\lambda : G \times_\alpha C^*(E) \to \mathcal{B}(L^2(E))$ *is faithful if* $S = \widehat{G}$ *and is not faithful if* $T \neq \widehat{G}$.

Proof. We have $\mathrm{sp}(U_\lambda) = T$. If $T \neq \widehat{G}$, then $\ker(\lambda \times U_\lambda)$ contains the nontrivial subalgebra generated by $\{\varphi \otimes x : x \in C^*(E),\ \mathrm{supp}(\widehat{\varphi}) \cap T = \emptyset\}$ and is therefore not faithful.

Suppose $S = \widehat{G}$. For each $\gamma \in \widehat{G}$ and any decreasing sequence (Ω_k) of neighborhoods of γ such that $\bigcap_k \Omega_k = \{\gamma\}$, choose a sequence of unitvectors $\xi_k \in E(t_k)$, $t_k \searrow 0$ such that $\mathrm{sp}_w(\xi_k) \subseteq \Omega_k$.

Recall the weak integral representation of elements in $\lambda(C^*(E))$: For $f, g \in L^1(E) \cap L^2(E)$ and $\xi, \eta \in L^2(E)$ we have ([3], 6.4):

$$\langle \xi, \lambda(f)\lambda(g)^*\eta \rangle = \int_0^\infty \mathrm{d}t\, \langle \xi, \lambda_t^{\mathrm{op}}(f \otimes \overline{g})\eta \rangle,$$

where λ_t^{op} is the e_o-semigroup coming from the right anti-representation r of E on $L^2(E)$ defined by $r(e)\xi(t) = \xi(t-s)e$, if $t > s$ and 0 otherwise, where $e \in E(s)$. This formula implies

$$[\lambda(f)\lambda(g)^*, r(\xi_k)] = \int_0^{t_k} \mathrm{d}t\, \lambda_t^{\mathrm{op}}(f \otimes \overline{g}) r(\xi_k),$$

which is a sequence of compact operators converging to 0 in norm. We also have for $\varphi \in L^1(G)$ and $\psi \in L^2(E)$

$$\int_G \mathrm{d}g\, [\varphi(g) U_\lambda(g) r(\xi_k) - \varphi(g)\langle \gamma, g \rangle r(\xi_k) U_\lambda(g)]\psi \overset{k\to\infty}{\longrightarrow} 0.$$

Thus $[(\lambda\times U_\lambda)(y)r(\xi_k)r(\xi_k)^*-r(\xi_k)(\lambda\times U_\lambda)(\widehat{\alpha}_\gamma(y))r(\xi_k)^*]\psi \overset{k\to\infty}{\longrightarrow} 0$ if $y=\sum_{i=1}^{k}\varphi_i\otimes \lambda(f_i)\lambda(g_i)^*$. Using the fact that the commutators above converge to 0 it follows that $\ker(\lambda\times U_\lambda)$ is an $\widehat{\alpha}$-invariant ideal which is not the whole algebra. It must be 0 because $C^*(E)$ is simple. ∎

THEOREM (Compare [3], 6.1) *Let* $A=(\lambda\times U_\lambda)(G\times_\alpha C^*(E))$. *Then:*
(i) $A\cap\mathcal{K}=\{0\}$*;*
(ii) $[A,r(E)]\subseteq\mathcal{K}$*;*
(iii) $\|P(t)yP(t)\|=\|y\|$ *for all* $t\geqslant 0$ *and* $y\in A$.

Proof. (i) We have $U_\lambda(g)r(e)=r(w_g e)U_\lambda(g)$ for any $e\in E$ and thus $\mathrm{Ad}(U_\lambda(g))\circ\lambda_t^{\mathrm{op}}=\lambda_t^{\mathrm{op}}\circ\mathrm{Ad}(U_\lambda(g))$. From the mentioned integral representation we obtain for $f,h\in L^1(E)$

$$\begin{aligned}(\lambda\times U_\lambda)(\varphi\otimes\lambda(f)\lambda(h)^*)&=\int_0^\infty \mathrm{d}t\int_G \mathrm{d}g\,\varphi(g)\lambda_t^{\mathrm{op}}(f\otimes\overline{h})U_\lambda(g)\\&=\int_0^\infty \mathrm{d}t\int_G \mathrm{d}g\,\varphi(g)U_\lambda(g)\lambda_t^{\mathrm{op}}(U_\lambda(g)^*(f\otimes\overline{h})U_\lambda(g))\\&=\int_0^\infty \mathrm{d}t\int_G \mathrm{d}g\,\varphi(g)U_\lambda(g)\lambda_t^{\mathrm{op}}((w_{g^{-1}}f)\otimes(\overline{w_{g^{-1}}h}))\end{aligned}$$

in the same weak sense as before. Because $\lambda_\varepsilon^{\mathrm{op}}(A)$ has infinite multiplicity for each $\varepsilon>0$ and nonzero A, the assertion follows.

(ii) Let u_0 be the unit in E such that $w_g u_0(t)=u_0(t)$. Then $\lambda(u_0(t))$ and $r(u_0(t))$ both commute with $U_\lambda(g)$. For $x=(\lambda\times U_\lambda)(\varphi\otimes\lambda(f)\lambda(h)^*)$ we obtain:

$$[x,r(u_0(t))]=\int_0^t \mathrm{d}s\int_G \mathrm{d}g\,\varphi(g)U_\lambda(g)\lambda_s^{\mathrm{op}}((w_{g^{-1}}f)\otimes(\overline{w_{g^{-1}}h}))r(u_0(t))$$

which is compact because for any fixed $g\in G$, the ds-integral is a compact operator, and it is easy to see that a strong Bochner integral over compact operators is compact.

(iii) For $x_i=\lambda(f_i)\lambda(h_i)^*$ we have

$$\begin{aligned}\Big\|P(t)(\lambda\times U_\lambda)\Big(\sum_{i=1}^k\varphi_i\otimes x_i\Big)P(t)\Big\|&\geqslant\Big\|r(u_0(t))^*(\lambda\times U_\lambda)\Big(\sum_{i=1}^k\varphi_i\otimes x_i\Big)r(u_0(t))\Big\|\\&=\Big\|\sum_{i=1}^k r(u_0(t))^*x_i r(u_0(t))\int_G \mathrm{d}g\,\varphi_i(g)U_\lambda(g)\Big\|\\&=\Big\|\sum_{i=1}^k(x_i+k_i)\int_G \mathrm{d}g\,\varphi_i(g)U_\lambda(g)\Big\|\end{aligned}$$

for some compact operators k_i using (ii). So this norm equals $\Big\|(\lambda\times U_\lambda)\Big(\sum_{i=1}^k \varphi_i\otimes x_i\Big)\Big\|$ using (i). ∎

Let $\overline{\gamma}$ be the extension of the gauge group to $G\times_\alpha C^*(E)$. The proof of 7.1 from [3] can easily be generalized to show the following:

Theorem *If (π,U) is a nonzero covariant representation of $(G,C^*(E))$ on a separable Hilbert space H, then for each $y\in G\times_\alpha C^*(E)$ we have*

$$\sup_{t\in\mathbb{R}}\|(\pi\times U)(\overline{\gamma}_t(y))\|\geqslant\|(\lambda\times U_\lambda)(y)\|.$$

Proof. We follow Section 7 of [3] closely and only indicate the modifications. We start with a covariant representation (π,U) of $(G,C^*(E))$. Instead of just representations $\overline{\pi}$ and π_+ in Arveson's proof, we consider the covariant representations $(\overline{\pi},\overline{U})=\Big(\int^\oplus\pi\circ\gamma_t\,\mathrm{d}t,\mathrm{id}\otimes U(g)\Big)$ on $L^2(\mathbb{R},H)$ and define (π_+,U_+) on $L^2(\mathbb{R}_+,H)$ as follows. Let π_+ be the unique representation of $C^*(E)$ given by $\phi_+: E\to\mathcal{B}(L^2(\mathbb{R}_+,H))$, where $(\phi_+(e)\xi)(x):=\phi(e)\xi(x-t)$ if $x>t$ and 0 otherwise. Let $U_+(g):=\mathrm{id}\otimes U(g)$. Then $W: L^2(E)\otimes H\to L^2(\mathbb{R}_+,H)$ defined by $W(f\otimes\xi)(x)=\phi(f(x))\xi$ is a unitary equivalence between the covariant representation $(\lambda\otimes\mathbb{1},U_\lambda\otimes U)$ and a covariant subrepresentation of (π_+,U_+). For each $a\geqslant 0$ we have $W(P(a)\otimes\mathbb{1})=\chi_a W$, where χ_a is the multiplication with $\chi_{[a,\infty)}$ in $L^2(\mathbb{R}_+,H)$. χ_a commutes with $U_+(g)$ and $P(a)\otimes\mathbb{1}$ with $U_\lambda(g)\otimes U(g)$. We finally obtain similarly to the end of Section 7 in [3] for each $y\in G\times_\alpha C^*(E)$:

$$\sup_{t\in\mathbb{R}}\|(\pi\times U)(\overline{\gamma}_t(y))\|\geqslant\|(\lambda\otimes\mathbb{1})\times(U_\lambda\otimes U)(y)\|.$$

Because G is amenable, we can find a sequence of unit vectors $(\xi_l)\subseteq H$ such that $\|U(g)\xi_l-\xi_l\|\to 0$ uniformly on compact subsets of G. If ξ and η run through the unit vectors in $L^2(E)$, then we have $\Big\|(\lambda\otimes\mathbb{1})\times(U_\lambda\otimes U)\Big(\sum_{i=1}^k\varphi_i\otimes x_i\Big)\Big\|\geqslant \sup_{\xi,\eta}\lim_{l\to\infty}\Big\langle\xi\otimes\xi_l,(\lambda\otimes\mathbb{1})\times(U_\lambda\otimes U)\Big(\sum_{i=1}^k\varphi_i\otimes x_i\Big)\eta\otimes\xi_l\Big\rangle=\sup_{\xi,\eta}\Big\langle\xi,(\lambda\times U_\lambda)\Big(\sum_{i=1}^k\varphi_i\otimes x_i\Big)\eta\Big\rangle=\Big\|(\lambda\times U_\lambda)\Big(\sum_{i=1}^k x_i\otimes\varphi_i\Big)\Big\|$, provided the $\varphi_i\in L^1(G)$ are of compact support. ∎

Corollary *If $S=\widehat{G}$, then $G\times_\alpha C^*(E)$ is $\overline{\gamma}$-simple.*

Proof of 4.1. We can now use Section 8 of [3] to conclude the simplicity of $G\times_\alpha C^*(E)$ whenever $S=\widehat{G}$. To this end let (π,U) be any nonzero covariant representation of $(G,C^*(E))$ on a separable Hilbert space H. By [3], 8.2 there exists a sequence of unit vectors $(\xi_k)\subseteq H$ such that $\langle\xi_k,\pi(x)\xi_k\rangle\to\omega_{u_0}(x)$ for each $x\in C^*(E)$ where ω_{u_0} is the state such that $\omega_{u_0}(\lambda(f)\lambda(g)^*)=\int\mathrm{d}t\,\langle u_0(t),f(t)\rangle\int\mathrm{d}s\,\langle g(s),u_0(s)\rangle$. But ω_{u_0} is G-invariant and thus defines also a covariant representation (π_{u_0},U_{u_0}). Moreover, we have $\|(\pi_{u_0}\times U_{u_0})(y)\|\leqslant$

$\|(\pi \times U)(y)\|$ for all $y \in G \times_\alpha C^*(E)$ using the approximation by vector states. The same holds for the units u_0^t given by $u_0^t(s) = \mathrm{e}^{ist} u_0(s)$ and thus

$$\|(\pi_{u_0} \times U_{u_0})(\overline{\gamma}_t(y))\| = \|(\pi_{u_0^t} \times U_{u_0^t})(y)\| = \|(\pi_{u_0} \times U_{u_0})(y)\|,$$

independently of $t \in \mathbb{R}$. Therefore $\ker(\pi \times U)$ is contained in a $\overline{\gamma}$-invariant ideal which must be 0 by 4.5. Thus $\pi \times U$ is faithful and $G \times_\alpha C^*(E)$ simple, provided $S = \widehat{G}$ because $\lambda \times U_\lambda$ is faithful in this case. ∎

REMARK For a quasi-free action of $\mathbb{R}$, we can only conclude that there is no l.s.c. trace on $\mathbb{R} \times_\alpha C^*(E)$ which is scaled by $\widehat{\alpha}$ because such a trace would correspond to a l.s.c. KMS-weight on $C^*(E)$ ([10]). There could exist other traces. Under the assumption $S = \mathbb{R}$ however, we obtain simple and KK-contractible C^*-algebras. Note that $\mathbb{R} \times_\gamma C^*(E)$ is stably projectionless and prime without traces ([15]).

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