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DUALITY FOR ACTIONS OF WEAK KAC ALGEBRAS AND CROSSED PRODUCT INCLUSIONS OF II_1 FACTORS

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Abstract. Weak Kac algebras generalize both finite dimensional Kac algebras and groupoid algebras. They naturally arise as symmetries of depth 2 inclusions of II_1 factors ([16]). We show that indecomposable weak Kac algebras are free over their counital subalgebras and prove a duality theorem for their actions. Using this result, for any biconnected weak Kac algebra we construct a minimal action on the hyperfinite II_1 factor. The corresponding crossed product inclusion of II_1 factors has depth 2 and an integer index. Its first relative commutant is, in general, non-trivial, so we derive some arithmetic properties of weak Kac algebras from considering reduced subfactors.

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1. INTRODUCTION

It is well understood now that Kac algebras (Hopf C^*-algebras) are closely related with the subfactors theory: it was announced by Ocneanu and was proved in [22], [12], [5] and [4] that irreducible depth 2 inclusions of type II factors come from crossed products with Kac algebras. This result was recently extended to the case of general (i.e., not necessarily irreducible) finite index depth 2 subfactors in [16]. It was shown then that if $N \subset M \subset M_1 \subset M_2 \subset \cdots$ is the Jones tower constructed from such a subfactor $N \subset M$, then $K = M' \cap M_2$ has a natural structure of a finite-dimensional weak Kac algebra or weak Hopf C^*-algebra and there is a minimal action of K on M_1 such that M is the fixed point subalgebra of M_1 and M_2 is isomorphic to the crossed product of M_1 and K. This result establishes an injective correspondence between finite index depth 2 subfactors of a given II_1 factor and weak Hopf C^*-algebras.

It is natural to ask if this correspondence is one-to-one in the case of the hyperfinite II_1 factor. Note that in [24] Yamanouchi constructed an outer action of any finite dimensional Kac algebra K on the hyperfinite II_1 factor R, where the outerness means that $R' \cap R \rtimes K = \mathbb{C}$, i.e., that the first relative commutant of the crossed product inclusion $R \subset R \rtimes K$ is minimal. His construction used the Takesaki duality for actions of Kac algebras ([6]).

In this work we extend this result to weak Kac algebras, i.e., we show that any weak Kac algebra has a minimal action on R. Finite dimensional weak Kac algebras generalize both finite groupoid algebras and usual Kac algebras. Note that a weak Kac algebra is a special case of a weak Hopf C^*-algebra introduced in [3] and [14], which is characterized by the property $S^2 = \mathrm{id}$. It was shown in [17] that the category of weak Kac algebras is equivalent to the categories of generalized Kac algebras of T. Yamanouchi ([25]) and Kac bimodules (an algebraic version of Hopf bimodules ([8])). Compared with these objects, the advantage of the language of weak Kac algebras is that their definition is transparently self-dual, so it is easy to work with both weak Kac algebra and its dual simultaneously.

The paper is organized as follows.

In Section 2 we collect the necessary definitions and facts about weak Kac algebras, their actions and crossed products, and their counital subalgebras; we also give a brief description of the basic construction for $*$-algebras.

In Section 3 we introduce a λ-Markov condition for weak Kac algebras. A weak Kac algebra K satisfies the λ-Markov condition if the normalized Haar trace on K is the λ-Markov trace for the inclusion $K_{\mathrm{s}} \subset K$, where K_{s} is the source counital subalgebra of K. This condition is automatically satisfied if K is indecomposable, i.e., not isomorphic to the direct sum of two weak Kac algebras. Theorem 3.5 shows that being λ-Markov is equivalent to the freeness of K over its counital subalgebras; in particular, λ^{-1} must be a positive integer. As a corollary, we obtain that indecomposable weak Kac algebras of prime dimension are group algebras of cyclic groups, which extends the well-known result of Kac ([11]).

Also in this section we introduce and study basic properties of connected and biconnected weak Kac algebras, i.e., those for which the inclusion $K_{\mathrm{s}} \subset K$ is connected (respectively inclusions $K_{\mathrm{s}} \subset K$ and $K_{\mathrm{s}}^* \subset K^*$ are connected). The latest class of indecomposable weak Kac algebras is the most important for the applications to subfactors in Section 5, so we describe a way of constructing biconnected weak Kac algebras from usual Kac algebra actions on C^*-algebras (this procedure generalizes a construction of a groupoid from a group acting on a space).

The central result of Section 4 is a duality theorem for actions of weak Kac algebras. This theorem is an analogue of the well known duality results for actions of locally compact groups ([13]), Kac algebras ([6]), and Hopf algebras ([2]). It states that if K satisfies the λ-Markov condition and acts on a C^*-algebra (von Neumann algebra) A, then the dual crossed product algebra $(A \rtimes K) \rtimes K^*$ is isomorphic to $A \otimes M_{\lambda^{-1}}(\mathbb{C})$. Let us note that a similar result for depth 2 inclusions of von Neumann algebras was proved in [8].

In Section 5 for any biconnected weak Kac algebra K we construct a minimal action on the hyperfinite II_1 factor R (where the minimality means that the relative commutant $R' \cap R \rtimes K$ is minimal). The resulting crossed product inclusion $R \subset R \rtimes K$ of II_1 factors has depth 2 and an integer index λ^{-1}. We compute the standard invariant of this inclusion, and show, in particular, that the first relative commutant is isomorphic to the counital subalgebra of K: $R' \cap R \rtimes K \cong K_{\mathrm{s}}$.

Finally, in Section 6 we construct irreducible subfactors reducing the inclusion $R \subset R \rtimes K$ by the minimal projection in $K_{\mathrm{s}} = R' \cap R \rtimes K$. In this way we can associate an irreducible finite depth subfactor of R with every irreducible representation of K or $K_{\rm s}$. This allows us to derive certain arithmetic properties of biconnected weak Kac algebras.

2. PRELIMINARIES

2.1. WEAK KAC ALGEBRAS ([3] and [17]). Throughout this paper all weak Kac algebras are supposed to be finite-dimensional.

The notion of a weak Kac algebra ([17]) is a special case of a more general concept of weak C^*-Hopf algebra introduced in [3]; see [17] for a discussion on equivalence of weak Kac algebras with other algebraic versions of a quantum groupoid (generalized Kac algebras of T. Yamanouchi ([25]) and Kac bimodules).

A *weak Kac algebra* K is a finite dimensional C^*-algebra equipped with the following linear maps:

(i) *comultiplication* $\Delta : K \to K \otimes K$;
(ii) *counit* $\varepsilon : K \to \mathbb{C}$;
(iii) *antipode* $S : K \to K$;

where Δ is a (not necessarily unital) homomorphism of C^*-algebras, ε is a positive (not necessarily multiplicative) functional, S is a $*$-preserving anti-multiplicative and anti-comultiplicative involution (i.e., $S^2 = \mathrm{id}$) such that the following identities hold (we denote $\varepsilon_{\rm s}(x) = (\mathrm{id} \otimes \varepsilon)((1 \otimes x)\Delta(1))$ and $\varepsilon_{\rm t}(x) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(x \otimes 1))$):

(1) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$, i.e., K is a coalgebra;
(2) $\varepsilon_{\rm s}(x)y = (\mathrm{id} \otimes \varepsilon)((1 \otimes x)\Delta(y))$;
(3) $(\varepsilon_{\rm s} \otimes \mathrm{id})\Delta(x) = (1 \otimes x)\Delta(1)$;
(4) $m(S \otimes \mathrm{id})\Delta(x) = \varepsilon_{\rm s}(x)$ with $x, y \in K$;

where m denotes multiplication.

The following identities are equivalent to the above axioms (2)–(4) respectively:

(2′) $x\varepsilon_{\rm t}(y) = (\varepsilon \otimes \mathrm{id})(\Delta(x)(y \otimes 1))$;
(3′) $(\mathrm{id} \otimes \varepsilon_{\rm t})\Delta(x) = \Delta(1)(x \otimes 1)$;
(4′) $m(\mathrm{id} \otimes S)\Delta(x) = \varepsilon_{\rm t}(x)$ with $x, y \in K$.

The dual vector space K^* has a natural structure of a weak Kac algebra given by dualizing the structure operations of K:

$$
\begin{array}{ll}
\langle \varphi\psi, x\rangle = \langle \varphi \otimes \psi, \Delta(x)\rangle & \text{(multiplication)},\\
\langle \Delta(\varphi), x \otimes y\rangle = \langle \varphi, xy\rangle & \text{(comultiplication)},\\
\langle S(\varphi), x\rangle = \langle \varphi, S(x)\rangle & \text{(antipode)},\\
\langle \varphi^*, x\rangle = \overline{\langle \varphi, S(x^*)\rangle} & \text{($*$-operation)},
\end{array}
$$

for all $\varphi, \psi \in K^*$, $x, y \in K$. The unit is given by ε and counit by $\varphi \mapsto \langle \varphi, 1\rangle$.

Below we collect the most important results of the theory of weak Kac algebras. The proofs can be found in [17].

The maps $\varepsilon_{\rm s}$ and $\varepsilon_{\rm t}$ are called *source* and *target counital maps* respectively and we have $\varepsilon_{\rm s}^2 = \varepsilon_{\rm s}$ and $\varepsilon_{\rm t}^2 = \varepsilon_{\rm t}$. Their images are unital C^*-subalgebras, called *counital subalgebras* of K:

$$K_{\rm s} = \{x \in K \mid \varepsilon_{\rm s}(x) = x\} = \{x \in K \mid \Delta(x) = 1_{(1)} \otimes x1_{(2)}\},$$
$$K_{\rm t} = \{x \in K \mid \varepsilon_{\rm t}(x) = x\} = \{x \in K \mid \Delta(x) = 1_{(1)}x \otimes 1_{(2)}\}.$$

The counital subalgebras commute: $[K_{\rm s}, K_{\rm t}] = 0$; also we have $S \circ \varepsilon_{\rm s} = \varepsilon_{\rm t} \circ S$ and $S(K_{\rm s}) = K_{\rm t}$.

Like usual finite-dimensional Kac algebras (= Hopf C^*-algebras), weak Kac algebras have integrals in the following sense.

There exists a unique projection $p_\varepsilon \in K$, called a *Haar projection*, such that for all $x \in K$:

$$p_\varepsilon x = p_\varepsilon \varepsilon_{\rm s}(x), \quad x p_\varepsilon = \varepsilon_{\rm t}(x) p_\varepsilon, \quad \varepsilon_{\rm s}(p_\varepsilon) = \varepsilon_{\rm t}(p_\varepsilon) = 1.$$

There exists a unique faithful trace τ on K, called a *normalized Haar trace*, such that

$$(\tau \otimes \mathrm{id})\Delta = (\tau \otimes \varepsilon_{\rm s})\Delta, \quad (\mathrm{id} \otimes \tau)\Delta = (\varepsilon_{\rm t} \otimes \tau)\Delta, \quad \tau \circ \varepsilon_{\rm s} = \tau \circ \varepsilon_{\rm t} = \varepsilon.$$

The normalized Haar projection and trace are unimodular, i.e. $S(p_\varepsilon) = p_\varepsilon$ and $\tau \circ S = \tau$. By duality, τ is the normalized Haar projection for the dual weak Kac algebra K^*.

The maps

$$\begin{array}{ll} E_{\rm s} : K \to K_{\rm s} & E_{\rm s}(x) = (\tau \otimes \mathrm{id})\Delta(x), \\ E_{\rm t} : K \to K_{\rm t} & E_{\rm t}(x) = (\mathrm{id} \otimes \tau)\Delta(x) \end{array}$$

define τ-preserving conditional expectations (see 2.3 for the definition) from K to counital subalgebras.

To fix the notation in what follows, let

$$K \cong \bigoplus_{i=1}^{N} M_{d_i}(\mathbb{C}), \quad K_{\rm s} \cong K_{\rm t} \cong \bigoplus_{\alpha=1}^{L} M_{m_\alpha}(\mathbb{C}),$$

and let $\{e_{kl}^{(i)}\}$ ($i = 1, \ldots, N$; $k, l = 1, \ldots, d_i$) be a system of matrix units in K, $\{f_{rs}^{(\alpha)}\}$ in $K_{\rm s}$, and $\{g_{rs}^{(\alpha)}\}$ in $K_{\rm t}$ ($\alpha = 1, \ldots, L$; $r, s = 1, \ldots, m_\alpha$). By [17] we have :

$$\Delta(p_\varepsilon) = \sum_i \frac{1}{d_i} \sum_{k,l} e_{kl}^{(i)} \otimes S(e_{lk}^{(i)}),$$
$$\Delta(1) = \sum_\alpha \frac{1}{m_\alpha} \sum_{r,s} f_{rs}^{(\alpha)} \otimes S(f_{sr}^{(\alpha)}) = \sum_\alpha \frac{1}{m_\alpha} \sum_{r,s} S(g_{sr}^{(\alpha)}) \otimes g_{rs}^{(\alpha)}.$$

In particular, p_ε is cocommutative, i.e., $\Delta(p_\varepsilon) = \varsigma\Delta(p_\varepsilon)$, where ς is the flip on the tensor product $K \otimes K$.

Also we denote $\Lambda = (\Lambda_{ij})$ the $(L \times N)$ inclusion matrix ([9]) of $K_{\rm s}$ (or $K_{\rm t}$) into K.

2.2. ACTIONS, DUAL ACTIONS, AND CROSSED PRODUCTS ([15]). By a $*$-algebra we understand an associative algebra over $\mathbb{C}$ equipped with a conjugate linear anti-automorphism of order 2 (involution), $x \mapsto x^*$.

The notion of an action of a weak C^*-Hopf algebra on a $*$-algebra was defined in [15]. We slightly modify that definition, since we consider only those actions for which the map $x \mapsto (x \triangleright 1)$ (respectively $x \mapsto (1 \triangleleft x)$) is injective on a counital subalgebra. We need definitions of left and right actions.

A *left* (respectively *right*) *action* of a weak Kac algebra K on a unital $*$-algebra A is a linear map

$$K \otimes A \ni h \otimes a \mapsto (h \triangleright a) \in A, \quad \text{respectively } A \otimes K \ni a \otimes h \mapsto (a \triangleleft h) \in A,$$

defining a structure of a left (respectively right) K-module on A such that:

(i) $h \triangleright ab = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$ (respectively $ab \triangleleft h = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)})$);
(ii) $(h \triangleright a)^* = Sh^* \triangleright a^*$ (respectively $(a \triangleleft h)^* = a^* \triangleleft Sh^*$);
(iii) $h \triangleright 1 = \varepsilon_{\mathrm{t}}(h) \triangleright 1$, and $h \triangleright 1 = 0$ iff $\varepsilon_{\mathrm{t}}(h) = 0$ (respectively $1 \triangleleft h = 1 \triangleleft \varepsilon_{\mathrm{s}}(h)$, and $1 \triangleleft h = 0$ iff $\varepsilon_{\mathrm{s}}(h) = 0$).

If A is a C^*-algebra or a von Neumann algebra then we also require that the map $a \mapsto (h \triangleright a)$ (respectively $a \mapsto (a \triangleleft h)$) to be norm continuous or weakly continuous for all $h \in K$.

Note that the map $z \mapsto (z \triangleright 1)$ (respectively $z \mapsto (1 \triangleleft z)$) defines an injective $*$-homomorphism from K_{t} (respectively K_{s}) to A. Thus, A must contain a $*$-subalgebra isomorphic to a counital subalgebra of K.

A *trivial* left (respectively right) action of K on K_{t} (respectively K_{s}) is given by

$$h \triangleright a = \varepsilon_{\mathrm{t}}(ha) \text{ (respectively } a \triangleleft h = \varepsilon_{\mathrm{s}}(ah)), \quad h \in K, a \in K_{\mathrm{t}} \text{ (respectively } K_{\mathrm{s}}).$$

A *dual* left (respectively right) action of K^* on K is given by

$$\varphi \triangleright h = h_{(1)} \langle \varphi, h_{(2)} \rangle, \quad \text{respectively } h \triangleleft \varphi = \langle \varphi, h_{(1)} \rangle h_{(2)}, \quad \varphi \in K^*,\ h \in K.$$

Given a left (respectively right) action of K on a $*$-algebra A, there is a left (respectively right) *crossed product* $*$-algebra $A \rtimes K$ (respectively $K \ltimes A$) constructed as follows. As a $\mathbb{C}$-vector space it is $A \otimes_{K_{\mathrm{t}}} K$ (respectively $K \otimes_{K_{\mathrm{s}}} A$), where K is a left (respectively right) K_{t}-module (respectively K_{s}-module) via multiplication and A is a right K_{t}-module (respectively left K_{s}-module) via multiplication by image of K_{t} (respectively K_{s}) under $z \mapsto (z \triangleright 1)$ (respectively $z \mapsto (1 \triangleleft z)$); that is, we identify

$$a(z \triangleright 1) \otimes h \equiv a \otimes zh, \quad \text{respectively } hz \otimes a \equiv h \otimes (1 \triangleleft z)a,$$

for all $a \in A$, $h \in K$, $z \in K_{\mathrm{t}}$ (respectively K_{s}). Let $[a \otimes h]$ (respectively $[h \otimes a]$) denote the class of $a \otimes h$ (respectively $h \otimes a$). A $*$-algebra structure is defined by

$$[a \otimes h][b \otimes k] = [a(h_{(1)} \triangleright b) \otimes h_{(2)}k], \quad [a \otimes h]^* = [(h^*_{(1)} \triangleright a^*) \otimes h^*_{(2)}],$$

respectively

$$[h \otimes a][k \otimes b] = [hk_{(1)} \otimes (a \triangleleft k_{(2)})b], \quad [h \otimes a]^* = [h^*_{(1)} \otimes (a^* \triangleleft h^*_{(2)})],$$

for all $a, b \in A$, $h, k \in K$. The maps $i_A : a \mapsto [a \otimes 1_K]$ (respectively $a \mapsto [1_K \otimes a]$) and $i_K : h \mapsto [1_A \otimes h]$ (respectively $h \mapsto [h \otimes 1_A]$) are inclusions of $*$-algebras such that $A \rtimes K = i_A(A)i_K(K)$ (respectively $K \ltimes A = i_K(K)i_A(A)$). Moreover, if A is a C^*-algebra (von Neumann algebra), then the crossed product is naturally $*$-isomorphic to a norm closed (weakly closed) $*$-algebra of operators on some Hilbert space, i.e., it becomes a C^*-algebra (von Neumann algebra).

For the crossed products constructed from the trivial actions of K on counital subalgebras we have

$$K_{\rm t} \rtimes K \cong K \quad \text{and} \quad K \ltimes K_{\rm s} \cong K.$$

A left (respectively right) *dual action* of K^* on the crossed product $A \rtimes K$ (respectively $K \ltimes A$) is defined as

$$\varphi \triangleright [a \otimes h] = [a \otimes (\varphi \triangleright h)], \quad \text{respectively } [h \otimes a] \triangleleft \varphi = [(h \triangleleft \varphi) \otimes a],$$

for all $a \in A$, $h \in K$, $\varphi \in K^*$. The action of K^* on K defined above is dual to the trivial action of K on the counital subalgebra.

2.3. THE BASIC CONSTRUCTION FOR $*$-ALGEBRAS ([23]). Let B be a unital $*$-algebra, A be its $*$-subalgebra containing the unit of B. A *conditional expectation* $E : B \to A$ is a faithful (i.e, such that $E(Bb) = 0$ implies $b = 0$, for $b \in B$) linear $*$-preserving map satisfying

$$E(ab) = aE(b), \quad E(ba) = E(b)a, \quad \text{and} \quad E(a) = a,$$

for all $a \in A$, $b \in B$. A finite family $\{u_1, \dots, u_n\} \subset B$ is called a *quasi-basis* for E if

$$b = \sum_i u_i E(u_i^* b) \quad \text{for all } b \in B.$$

It is called a *basis* if "the coefficients" $E(u_i^* b)$ are unique, i.e. if $\sum_i u_i a_i = 0$, $a_i \in A \Leftrightarrow a_i = 0$ $(\forall i)$. A conditional expectation E is of *finite-index type* if there exists a quasi-basis for E. In this case the *index* of E is defined as

$$\operatorname{Index} E = \sum_i u_i u_i^* \in B.$$

$\operatorname{Index} E$ belongs to the center of B and does not depend on the choice of quasi-basis.

The *basic construction* for E is a $*$-algebra $B \otimes_A B$ with the multiplication and involution given by

$$(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 E(b_2 b_3) \otimes b_4, \quad (b_1 \otimes b_2)^* = b_2^* \otimes b_1^*,$$

for all b_1, b_2, b_3, b_4 in B. Note that the unit of this algebra is $\sum_i u_i \otimes u_i^*$, where $\{u_i\}$ is the quasi-basis for E.

In what follows we consider only conditional expectations of finite-index type for which a basis (not just a quasi-basis) exists.

In this case $B \otimes_A B$ is canonically isomorphic to $\operatorname{End}^{\rm r}_A(B)$, the algebra of endomorphisms of B viewed as a right A-module, with the isomorphism $\varphi : B \otimes_A B \to \operatorname{End}^{\rm r}_A(B)$ given by

$$\varphi(b_1 \otimes b_2)(b) = b_1 E(b_2 b), \quad b, b_1, b_2 \in B.$$

B is canonically identified with the subalgebra of left multiplication operators ($x \mapsto bx$ for $x \in B$) in $\operatorname{End}^{\rm r}_A(B)$. Clearly, $\operatorname{End}^{\rm r}_A(B) \cong M_n(A)$, the $*$-algebra of $(n \times n)$ matrices over A, since B is free of rank n over A.

Note that $e_A = E \in \operatorname{End}^{\rm r}_A(B)$ is a projection such that:

(i) $e_A b e_A = E(b) e_A$ for all $b \in B$,

(ii) the map $A \ni a \mapsto ae_A \in \mathrm{End}^{\mathrm{r}}_A(B)$ is injective,

and $\mathrm{End}^{\mathrm{r}}_A(B)$ is generated by B and e_A.

Conversely, if C is a $*$-algebra containing B as a unital $*$-subalgebra and generated by B and some projection e_A satisfying properties (i) and (ii) above, then C is canonically $*$-isomorphic to $B \otimes_A B$ with the isomorphism given by $b_1 e_A b_2 \mapsto b_1 \otimes b_2$.

Due to this fact, we will denote the basic construction for E by $\langle B, e_A \rangle$.

When A and B are C^* -algebras (von Neumann algebras) then $\langle B, e_A \rangle$ naturally becomes a C^*-algebra (von Neumann algebra).

3. λ-Markov condition and connected weak Kac algebras

We show that indecomposable weak Kac algebras have the Markov property (i.e., $E_{\mathrm{s}}(p_\varepsilon)$ is a scalar). We prove that this property is equivalent to existence of a basis for E_{s}, i.e., to freeness of K over the counital subalgebra K_{s}, and can be expressed in terms of the inclusion matrix of $K_{\mathrm{s}} \subset K$.

We also introduce notions of connected and biconnected weak Kac algebras which are important for applications to the theory of subfactors.

Definition 3.1. A weak Kac algebra K is *decomposable* if it is isomorphic to the direct sum of two weak Kac algebras, $K \cong K_1 \oplus K_2$; otherwise K is *indecomposable.*

These properties can be expressed in terms of the algebra $K_{\mathrm{s}} \cap K_{\mathrm{t}} \cap Z(K)$, the *hypercenter* of K ([15]) (here $Z(K)$ is the center of K).

Proposition 3.2. *K is indecomposable iff the hypercenter of K is trivial, i.e., $K_{\mathrm{s}} \cap K_{\mathrm{t}} \cap Z(K) = \mathbb{C}$.*

Proof. If q is a projection in $K_{\mathrm{s}} \cap K_{\mathrm{t}} \cap Z(K)$, then $S(q) = q$, $\Delta(q) = (q \otimes q)\Delta(1)$, and $\Delta(qh) = (q \otimes q)\Delta(h)$ for all $h \in K$. Therefore, qK and $(1-q)K$ are weak Kac algebras and $K = qK \oplus (1-q)K$. Conversely, if K is decomposable and K_1 is its direct summand, then $\varepsilon_{\mathrm{s}}(1_{K_1}) = \varepsilon_{\mathrm{t}}(1_{K_1}) = 1_{K_1}$ and the unit of K_1 belongs to the hypercenter of K. ∎

It turns out that indecomposable weak Kac algebras satisfy an important λ-Markov condition.

Definition 3.3. A weak Kac algebra K satisfies a λ-Markov condition if

$$E_{\mathrm{s}}(p_\varepsilon) = E_{\mathrm{t}}(p_\varepsilon) = \lambda$$

for some positive λ (note that since p_ε is cocommutative, we always have $E_{\mathrm{s}}(p_\varepsilon) = E_{\mathrm{t}}(p_\varepsilon)$).

Proposition 3.4. *If K is indecomposable, then it satisfies the λ-Markov condition for some λ.*

Proof. Note that

$$E_{\mathrm{s}}(p_\varepsilon) = E_{\mathrm{t}}(p_\varepsilon) = \sum_i \frac{1}{d_i} \sum_k e^{(i)}_{kk} \tau(e^{(i)}_{kk}) = \sum_i \frac{\tau_i}{d_i} \sum_k e^{(i)}_{kk} \in Z(K),$$

where $\tau_i = \tau(e^{(i)}_{kk})$ does not depend on k. Therefore, if $E_{\mathrm{s}}(p_\varepsilon) \neq \lambda$, then the hypercenter is non-trivial and K is decomposable by Proposition 3.2. ∎

The following theorem describes the λ-Markov condition in several equivalent ways.

THEOREM 3.5. *The following conditions are equivalent:*

(i) *K satisfies the λ-Markov condition;*

(ii) *$\tau = \lambda\,\mathrm{Tr}$ where Tr is the trace of the left regular representation of K on itself;*

(iii) *$(\Lambda\Lambda^{\mathrm{t}})\vec{m} = \lambda\vec{m}$, where $\vec{m} = (m_1, \ldots, m_L)$ is the dimension vector of a counital subalgebra, and Λ is the $L \times N$ inclusion matrix of $K_{\mathrm{s}} \subset K$;*

(iv) *$n = \lambda^{-1}$ is an integer and there is a basis $\{x_\nu\}_{\nu=1,\ldots,n}$ for E_{s}, i.e., a basis of K over K_{s} such that $x = \sum_\nu x_\nu E_{\mathrm{s}}(x_\nu^* x)$ for all $x \in K$;*

(v) *$n = \lambda^{-1}$ is an integer and there is a basis $\{y_\nu\}_{\nu=1,\ldots,n}$ for E_{t}, i.e., a basis of K over K_{t} such that $y = \sum_\nu y_\nu E_{\mathrm{t}}(y_\nu^* y)$ for all $y \in K$.*

Proof. (i) $\Leftrightarrow$ (ii) As we saw in the proof of Proposition 3.4, $E_{\mathrm{s}}(p_\varepsilon) = \lambda$ iff there exists λ such that $\tau(e^{(i)}_{kk}) = \lambda d_i$, i.e., $\tau = \lambda\,\mathrm{Tr}$.

(ii) $\Leftrightarrow$(iii) It suffices to prove that (iii) holds true if and only if Tr is normalized by conditions $(\mathrm{Tr}\otimes\mathrm{id})\Delta(1) = (\mathrm{id}\otimes\mathrm{Tr})\Delta(1) = \lambda^{-1}$ (it was shown in [17] that $(\mathrm{Tr}\otimes\mathrm{id})\Delta = (\mathrm{Tr}\otimes\varepsilon_{\mathrm{s}})\Delta$ and $(\mathrm{id}\otimes\mathrm{Tr})\Delta = (\varepsilon_{\mathrm{t}}\otimes\mathrm{Tr})\Delta$). Since $\mathrm{Tr}\circ S = \mathrm{Tr}$, we have

$$(\mathrm{Tr}\otimes\mathrm{id})\Delta(1) = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \sum_r \mathrm{Tr}(g^{(\alpha)}_{rr}) g^{(\alpha)}_{rr} = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \Big(\sum_i \Lambda_{\alpha i} d_i\Big) \sum_r g^{(\alpha)}_{rr},$$

$$(\mathrm{id}\otimes\mathrm{Tr})\Delta(1) = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \sum_r f^{(\alpha)}_{rr}\,\mathrm{Tr}(f^{(\alpha)}_{rr}) = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \Big(\sum_i \Lambda_{\alpha i} d_i\Big) \sum_r f^{(\alpha)}_{rr}.$$

This shows that (ii) is equivalent to the following condition:

$$\sum_i \Lambda_{\alpha i} d_i = \lambda^{-1} m_\alpha, \quad \alpha = 1, \ldots, L.$$

But $d_i = \sum_{\beta=1}^{L} \Lambda_{\beta i} m_\beta$ since the inclusion $K_{\mathrm{s}} \subset K$ is unital. Hence, we can rewrite the last condition as

$$\sum_{i=1}^{N} \sum_{\beta=1}^{L} \Lambda_{\alpha i} \Lambda_{\beta i} m_\beta = \lambda^{-1} m_\alpha, \quad \alpha = 1, \ldots, L,$$

which means precisely that $(\Lambda\Lambda^{\mathrm{t}})\vec{m} = \lambda^{-1}\vec{m}$.

(iii) $\Rightarrow$ (iv) It is clear that λ^{-1} is a positive rational number since all entries of $(\Lambda\Lambda^{\mathrm{t}})$ and $\vec{m}$ are positive integers. On the other hand, λ^{-1} is an algebraic integer, since it is an eigenvalue of the integer matrix $(\Lambda\Lambda^{\mathrm{t}})$, therefore, λ^{-1} is an integer.

For all $\alpha = 1, \ldots, L$ and $r = 1, \ldots, m_\alpha$ define $K_{\alpha r} = K f^{(\alpha)}_{rr}$. Then

$$\dim(K_{\alpha r}) = \mathrm{Tr}(f^{(\alpha)}_{rr}) = \sum_i \Lambda_{\alpha i} d_i = n m_\alpha.$$

For all $y, z \in K_{\alpha r}$ we have:

$$E_{\rm s}(y^*z) = f_{rr}^{(\alpha)} E_{\rm s}(y^*z) f_{rr}^{(\alpha)} = (y,z) f_{rr}^{(\alpha)},$$

where (y,z) is a scalar since $f_{rr}^{(\alpha)}$ is minimal in $K_{\rm s}$. Clearly, $(\,\cdot\,,\,\cdot\,)$ defines an inner product in $K_{\alpha r}$, which is non-degenerate since $E_{\rm s}$ is faithful. Let us choose an orthonormal basis $\{x_\mu^{\alpha r}\}, (\mu = 1,\dots,nm_\alpha)$ in $K_{\alpha r}$, $\alpha = 1,\dots L$, $r = 1,\dots,m_\alpha$ in such a way that

$$x_\mu^{\alpha t} = x_\mu^{\alpha r} f_{rt}^{(\alpha)} \qquad \text{for all } t, r = 1,\dots,m_\alpha,\ \mu = 1,\dots,nm_\alpha.$$

Then we have the following relation

$$E_{\rm s}((x_\mu^{\alpha r})^* x_{\mu'}^{\alpha' r'}) = \delta_{\alpha\alpha'}\delta_{\mu\mu'} f_{rr'}^{(\alpha)} \qquad \text{for all } \alpha, \alpha', \mu, \mu', r, r'.$$

We claim that

$$x_\nu = \sum_\alpha \sum_{r,s} \frac{1}{\sqrt{m_\alpha}} \exp\Big(\frac{2sr\pi}{m_\alpha}\mathrm{i}\Big) x_{\nu+(s-1)n}^{\alpha r}, \qquad \nu = 1,\dots,n,$$

is a basis of K over $K_{\rm s}$. Indeed:

$$\sum_\nu x_\nu E_{\rm s}(x_\nu^* x_\mu^{\beta t}) = \sum_\nu \sum_{\alpha,r,s} \frac{1}{m_\alpha} x_{\nu+(s-1)n}^{\alpha r} E_{\rm s}((x_{\nu+(s-1)n}^{\alpha r})^* x_\mu^{\beta t}) = \frac{1}{m_\beta}\sum_r x_\mu^{\beta r} f_{rt}^{(\beta)} = x_\mu^{\beta t}$$

for all $\beta = 1,\dots,K$, $t = 1,\dots,m_\beta$, $\mu = 1,\dots,nm_\beta$. Next,

$$\begin{aligned} E_{\rm s}(x_\nu^* x_\kappa) &= \sum_{\alpha,r,r',s,s'} \frac{1}{m_\alpha} \exp\Big(\frac{2(sr - s'r')\pi}{m_\alpha}\mathrm{i}\Big) E_{\rm s}((x_{\nu+(s-1)n}^{\alpha r})^* x_{\kappa+(s'-1)n}^{\alpha r'}) \\ &= \delta_{\nu\kappa} \sum_{\alpha,r,r',s} \frac{1}{m_\alpha} \exp\Big(\frac{2s(r-r')\pi}{m_\alpha}\mathrm{i}\Big) f_{rr'}^{(\alpha)} = \delta_{\nu\kappa} \sum_{\alpha,r} f_{rr}^{(\alpha)} = \delta_{\nu\kappa}. \end{aligned}$$

Since '$E_{\rm s}$ -orthogonality' implies linear independence over $K_{\rm s}$, we conclude that $\{x_\nu\}$ is a basis for $E_{\rm s}$.

(iv) $\Rightarrow$ (iii) If there is a basis for $E_{\rm s} : K \to K_{\rm s}$ then the basic construction $\langle K, e_{K_{\rm s}}\rangle$ is isomorphic to $M_n(K_{\rm s})$. This means that the inclusion matrix B of the inclusion $K_{\rm s} \subset \langle K, e_{K_{\rm s}}\rangle$ satisfies $B\vec{m} = \lambda^{-1}\vec{m}$. But $B = \Lambda\Lambda^{\rm t}$ ([9]).

(iv) $\Leftrightarrow$ (v) We will prove (iv) $\Rightarrow$ (v), the converse implication is completely analogous. If $x = \sum_\nu x_\nu E_{\rm s}(x_\nu^* x)$ then $Sx^* = \sum_\nu Sx_\nu^* E_{\rm t}(Sx_\nu Sx^*)$, since $E_{\rm t} = S \circ E_{\rm s} \circ S$, and we can take $y_\nu = Sx_\nu^*$, $\nu = 1,\dots,\lambda^{-1}$ as a basis for $E_{\rm t}$. ∎

Corollary 3.6. *If the equivalent conditions of Theorem* 3.5 *are satisfied then τ is a λ^{-1}-Markov trace for the inclusion $K_{\rm t} \subset K$ ($K_{\rm s} \subset K$).*

Proof. We need to show that $\Lambda^{\rm t}\Lambda\vec{t} = \lambda^{-1}\vec{t}$, where $\vec{t}$ is the "trace-vector" corresponding to τ (3.2.3 (ii) of [10]). Since $\tau = \lambda\,\mathrm{Tr}$, we have $\vec{t} = \lambda\vec{d}$, where $\vec{d} = (d_1,\dots,d_N)$ is the "dimension-vector" of K. Using Theorem 3.5 (iii) we compute

$$\Lambda^{\rm t}\Lambda\vec{t} = \lambda\Lambda^{\rm t}\Lambda\vec{d} = \lambda\Lambda^{\rm t}\Lambda\Lambda^{\rm t}\vec{m} = \Lambda^{\rm t}\vec{m} = \vec{d} = \lambda^{-1}\vec{t}. \quad ∎$$

Remark 3.7. (i) Proposition 3.2 says that K is indecomposable iff the matrix Λ is indecomposable in the sense of [9]. In this case, Theorem 3.5 (iii) implies that $\vec{m}$ is the Perron-Frobenius eigenvector of the matrix $(\Lambda\Lambda^{\rm t})$. It is well-known that in this case the corresponding eigenvalue λ^{-1} is equal to the spectral radius of $(\Lambda\Lambda^{\rm t})$, so

$$\lambda^{-1} = \|\Lambda\Lambda^{\rm t}\| = \|\Lambda\|^2.$$

(ii) Theorem 3.5 (iv) and (v) show that an indecomposable weak Kac algebra K is free over its counital subalgebras $K_{\rm s}$ and $K_{\rm t}$. In particular, $\dim K_{\rm s}$ divides $\dim K$ and

$$\lambda^{-1} = \frac{\dim K}{\dim K_{\rm s}}.$$

(iii) Conditional expectations $E_{\rm s}$ and $E_{\rm t}$ are of index-finite type and their index is an integer scalar: $\mathrm{Index}\, E_{\rm s} = \mathrm{Index}\, E_{\rm t} = \lambda^{-1}$.

COROLLARY 3.8. *If K is indecomposable and $\dim K = p$, where p is a prime, then $K \cong \mathbb{C}\mathbb{Z}_p$, a group algebra of a simple abelian group.*

Proof. Remark 3.7 (ii) implies that counital subalgebras of K must be 1-dimensional, so K is an ordinary Kac algebra. But in this case the result is well-known ([11]). ∎

The λ-Markov condition is invariant under duality.

PROPOSITION 3.9. *K satisfies the λ-Markov condition iff K^* satisfies the λ-Markov condition (with the same λ).*

Proof. Since K satisfies the λ-Markov condition iff every its indecomposable component does, it sufficed to prove this statement in the case when K is indecomposable. But this is trivial by Proposition 3.4 and Remark 3.7 (ii), since $\dim K_{\rm s} = \dim K_{\rm s}^*$. ∎

Connected weak Kac algebras (i.e., those with connected Bratteli diagram of the inclusion $K_{\rm s} \subset K$) form a subclass of indecomposable weak Kac algebras important for the applications to subfactors in Section 5.

DEFINITION 3.10. A weak Kac algebra K is *connected* if the inclusion $K_{\rm s} \subset K$ is connected, i.e., $K_{\rm s} \cap Z(K) = \mathbb{C}$ (or, equivalently, $K_{\rm t} \cap Z(K) = \mathbb{C}$), where $Z(\,\cdot\,)$ denotes the center of an algebra. K is *biconnected* if both K and K^* are connected.

PROPOSITION 3.11. (cf. [15])] *The following conditions are equivalent:*
(i) *K is connected;*
(ii) *$K_{\rm s}^* \cap K_{\rm t}^* = \mathbb{C}$;*
(iii) *p_ε is a minimal projection in K (i.e the counital representation of K (Section 2.2 of [17]) is irreducible).*

Proof. (i) ⇒ (ii) Suppose that there is $\beta \in K_{\rm s}^* \cap K_{\rm t}^*$, $\beta \notin \mathbb{C}$. Since the counital subalgebras commute, β must belong to $Z(K_{\rm s}^*)$, the center of $K_{\rm s}^*$. Consider the element $b \in K \cong K^{**}$ defined as $\langle b, \varphi\rangle = \langle 1, \beta\varphi\rangle$ for all $\varphi \in K^*$. We can compute:

$$\langle b, \varphi_{(1)}\rangle\varphi_{(2)} = \langle 1, \beta\varphi_{(1)}\rangle\varphi_{(2)} = \langle 1, \beta_{(1)}\varphi_{(1)}\rangle\beta_{(2)}\varphi_{(2)} = \beta\varphi,$$
$$\varphi_{(1)}\langle b, \varphi_{(2)}\rangle = \varphi_{(1)}\langle 1, \beta\varphi_{(2)}\rangle = \varepsilon_{(1)}\varphi\langle 1, \beta\varepsilon_{(2)}\rangle = \beta\varphi,$$

therefore $b \in Z(K)$. Also, for all $\varphi \in K^*$ we have

$$\langle \varepsilon_{\mathrm{s}}(b), \varphi \rangle = \langle b, \varepsilon_{\mathrm{s}}(\varphi) \rangle = \langle 1, \beta \varepsilon_{\mathrm{s}}(\varphi) \rangle = \langle 1, \beta \varepsilon_{(1)} \rangle \langle 1, \varepsilon_{(2)} \varphi \rangle = \langle 1, \beta \varphi \rangle = \langle b, \varphi \rangle,$$

therefore $\varepsilon_{\mathrm{s}}(b) = b$ and $b \in K_{\mathrm{s}}$. Thus $Z(K) \cap K_{\mathrm{s}} \neq \mathbb{C}1$, so K is not connected.

(ii) $\Rightarrow$(i) If K is not connected, then there exists $b \in Z(K) \cap K_{\mathrm{t}}$, $b \notin \mathbb{C}$. Define $\beta \in K^*$ by $\beta : x \mapsto \varepsilon(bx)$. We have, for all $x \in K$:

$$\begin{aligned} \langle \beta, \varepsilon_{\mathrm{s}}(x) \rangle &= \varepsilon(b1_{(1)}) \varepsilon(x1_{(2)}) = \varepsilon(x \varepsilon_{\mathrm{t}}(b)) = \varepsilon(xb), \\ \langle \beta, \varepsilon_{\mathrm{t}}(x) \rangle &= \varepsilon(b1_{(2)}) \varepsilon(1_{(1)}x) = \varepsilon(bx) = \varepsilon(xb), \end{aligned}$$

from where $\varepsilon_{\mathrm{s}}(\beta) = \beta = \varepsilon_{\mathrm{t}}(\beta)$ and $K^*_{\mathrm{s}} \cap K^*_{\mathrm{t}} \neq \mathbb{C}\varepsilon$.

(i) $\Rightarrow$ (iii) If there is a proper subprojection q of p_ε then from the formula for $\Delta(p_\varepsilon)$ we get $\varepsilon_{\mathrm{s}}(q) \neq 1$ and $\varepsilon_{\mathrm{s}}(q) \in Z(K)$, so K is not connected.

(iii) $\Rightarrow$ (i) Let P_ε be the central support of p_ε. It was shown in [17] that the quotient map $K \mapsto P_\varepsilon K$ (which is a homomorphism of weak Kac algebras) is one-to-one on the counital subalgebras. Therefore $K_{\mathrm{s}} \cap Z(K)$ is contained in $Z(P_\varepsilon K)$, and $K_{\mathrm{s}} \cap Z(K) = \mathbb{C}$ when p_ε is minimal. ∎

The following construction generalizes transformation groupoids arising from group actions on spaces ([20]). We associate a weak Kac algebra with any finite dimensional C^*-algebra carrying an action of a usual Kac algebra. Our method uses two-sided crossed products introduced in [14].

Namely, let H be a usual finite-dimensional Kac algebra (i.e., finite-dimensional Hopf C^*-algebra) acting on the right on a finite-dimensional C^*-algebra A via $a \otimes h \mapsto (a \triangleleft h)$, where $a \in A$, $h \in H$. Then H also acts on the left on A^{op}, the C^*-algebra opposite to A, via $(h \triangleright a) = a \triangleleft S(h)$, where $a \in A^{\mathrm{op}}$, $h \in H$.

DEFINITION 3.12. A *two-sided crossed product C^*-algebra* $A^{\mathrm{op}} \rtimes H \ltimes A$ is defined as vector space $A^{\mathrm{op}} \otimes H \otimes A$ with multiplication and involution given by

$$\begin{aligned} (b \otimes h \otimes a)(b' \otimes h' \otimes a') &= (h_{(1)} \triangleright b')b \otimes h_{(2)} h'_{(1)} \otimes (a \triangleleft h'_{(2)})a' \\ (b \otimes h \otimes a)^* &= (h^*_{(1)} \triangleright b^*) \otimes h^*_{(2)} \otimes (a^* \triangleleft h^*_{(3)}), \end{aligned}$$

for all $a, a' \in A$, $b, b' \in A^{\mathrm{op}}$, $h, h' \in H$.

Let $\{f^\alpha_{rs}\}$ be a system of matrix units in $A = \oplus_\alpha M_{m_\alpha}(\mathbb{C})$. Then the element $e \in A \otimes A^{\mathrm{op}}$ and the functional $\omega \in A^*$ defined by

$$e = \sum_{\alpha, r, s} \frac{1}{m_\alpha} f^\alpha_{rs} \otimes f^\alpha_{sr}, \quad \omega(f^\alpha_{rs}) = \delta_{rs} m_\alpha$$

do not depend on the choice of matrix units. Moreover, one can directly check that

$$\omega(a(h \triangleright b)) = \omega(b(a \triangleleft h)), \quad e^{(1)} \otimes (h \triangleright e^{(2)}) = (e^{(1)} \triangleleft h) \otimes e^{(2)},$$

where $a \in A$, $b \in A^{\mathrm{op}}$, $h \in H$, and $e = e^{(1)} \otimes e^{(2)}$ (with the summation sign suppressed).

PROPOSITION 3.13. (cf. [14]) *There is a structure of weak Kac algebra on* $K = A^{\mathrm{op}} \rtimes H \ltimes A$ *defined by*

$$\begin{aligned}
&\Delta(b \otimes h \otimes a) = (b \otimes h_{(1)} \otimes e^{(1)}) \otimes ((h_{(2)} \triangleright e^{(2)}) \otimes h_{(3)} \otimes a),\\
&\varepsilon(b \otimes h \otimes a) = \omega(a(h \triangleright b)),\\
&S(b \otimes h \otimes a) = a \otimes S(h) \otimes b,
\end{aligned}$$

where $a \in A$*,* $b \in A^{\mathrm{op}}$*,* $h \in H$*, and the canonical anti-isomorphism* $b \mapsto b$ *between* A^{op} *and* A *is implicitly used.*

Proof. The verification of all the axioms is straightforward and is left to the reader. ∎

The source and target counital subalgebras of K are

$$K_{\mathrm{s}} = \{1 \otimes 1 \otimes a \mid a \in A\}, \quad K_{\mathrm{t}} = \{b \otimes 1 \otimes 1 \mid b \in A^{\mathrm{op}}\}.$$

Clearly, $K_{\mathrm{s}} \cap K_{\mathrm{t}} = \mathbb{C}$, so K^* is connected by Proposition 3.11. It is easy to see that K is biconnected iff the fixed points algebra

$$A^H = \{a \in A \mid a \triangleleft h = \varepsilon(h)a, \forall h \in H\}$$

is trivial.

In the special case when $H = \mathbb{C}$ acts trivially on A, K^* is isomorphic to the full matrix algebra $M_d(\mathbb{C})$, $d = \dim A$. Such weak Kac algebras were classified in [17].

EXAMPLE 3.14. Let A be a right coideal C^*-subalgebra of H^* and the action of H on A be induced by the dual action of H on H^*:

$$a \triangleleft h = \langle h, a_{(1)} \rangle a_{(2)}.$$

Then $K = A^{\mathrm{op}} \rtimes H \ltimes A$ is a biconnected weak Kac algebra and

$$\lambda^{-1} = (\dim H)(\dim A).$$

In Section 6 we derive some arithmetic properties of biconnected weak Kac algebras from the existence of a minimal action of any such algebra on the hyperfinite II_1 factor.

4. DUALITY FOR ACTIONS

In this section K is a weak Kac algebra satisfying the λ-Markov condition (e.g., indecomposable) and acting on a C^*-algebra A. Left actions are assumed everywhere; the right counterparts of the results below can be obtained similarly and are left to the reader.

Lemma 4.1. *For all $a \in A$ we have*

$$(n \triangleright a) = a(n \triangleright 1), \quad n \in K_{\rm s}, \qquad \text{and} \qquad (n \triangleright a) = (n \triangleright 1)a, \quad n \in K_{\rm t}.$$

Proof. For all $n \in K_{\rm s}$ we compute

$$n \triangleright a = (n_{(1)} \triangleright a)(n_{(2)} \triangleright 1) = (1_{(1)} \triangleright a)(1_{(2)} n \triangleright 1) = a(n \triangleright 1),$$

and similarly the second statement. ∎

Proposition 4.2. *The map $E_A : A \rtimes K \to A$ defined as*

$$E_A([a \otimes h]) = a(E_{\rm t}(h) \triangleright 1), \quad a \in A,\ h \in K,$$

is a faithful conditional expectation. If $\{y_\nu\}_{\nu=1,\dots,n}$ is a basis for $E_{\rm t}$ as in Theorem 3.5 (v), *then $\{[1 \otimes y_\nu]\}_{\nu=1,\dots,n}$ is a basis for E_A.*

Proof. For all $z \in K_{\rm t}$ we compute

$$\begin{aligned} E_A([a \otimes zh]) &= a(E_{\rm t}(zh) \triangleright 1) = a(zE_{\rm t}(h) \triangleright 1) \\ &= a(z \triangleright 1)(E_{\rm t}(h) \triangleright 1) = E_A([a(z \triangleright 1) \otimes h]), \end{aligned}$$

therefore E_A is well-defined on $A \rtimes K$. Clearly, $E_A|A = {\rm id}_A$. Let us check other properties (using Lemma 4.1):

$$\begin{aligned} E_A([a \otimes 1][b \otimes h][c \otimes 1]) &= E_A([ab(h_{(1)} \triangleright c) \otimes h_{(2)}]) = ab(h_{(1)} \triangleright c)(E_{\rm t}(h_{(2)}) \triangleright 1) \\ &= ab(E_{\rm t}(h) \triangleright c) = ab(E_{\rm t}(h) \triangleright 1)c = aE_A([b \otimes h])c, \end{aligned}$$

for all $a, b, c \in A$ and $h \in K$, so E_A is a conditional expectation. We have $h = \sum_\nu y_\nu E_{\rm t}(y_\nu^* h) = \sum_\nu E_{\rm t}(hy_\nu)y_\nu^*$ for all $h \in K$ by Theorem 3.5 (v), so

$$\begin{aligned} [a \otimes h] &= \sum_\nu [a \otimes E_{\rm t}(hy_\nu)y_\nu^*] = \sum_\nu [a(E_{\rm t}(hy_\nu) \triangleright 1) \otimes 1][1 \otimes y_\nu^*] \\ &= \sum_\nu [E_A([a \otimes h][1 \otimes y_\nu^*]) \otimes 1][1 \otimes y_\nu^*], \end{aligned}$$

applying the involution we get

$$[a \otimes h] = \sum_\nu [1 \otimes y_\nu][E_A([1 \otimes y_\nu^*][a \otimes h]) \otimes 1], \quad a \in A, h \in K.$$

Therefore, every $x \in A \rtimes K$ can be written as $x = \sum_\nu [1 \otimes y_\nu][a_\nu \otimes 1]$ for some a_ν, $\nu = 1, \dots, n$. Since $E_A([1 \otimes y_\nu^* y_\kappa]) = \delta_{\nu\kappa}$, we have

$$E_A(x^*x) = \sum_{\nu,\kappa} E_A([a_\nu^* \otimes 1][1 \otimes y_\nu^*][1 \otimes y_\kappa][a_\kappa \otimes 1]) = \sum_\nu a_\nu^* a_\nu,$$

and $x = 0$ iff $E_A(x^*x) = 0$ iff $a_\nu = 0$ $(\forall \nu)$. This proves that E_A is faithful and $\{[1 \otimes y_\nu]\}_{\nu=1,\dots,n}$ is a basis for E_A. ∎

REMARK 4.3. $\operatorname{Index} E_A = \operatorname{Index} E_{\mathrm{t}} = \lambda^{-1}$.

In what follows we consider C^*-algebras $A, K, K^*, A \rtimes K$, and $K \rtimes K^*$ as subalgebras of $(A \rtimes K) \rtimes K^*$ in an obvious way with inclusion maps denoted by i_A, i_K etc.

LEMMA 4.4. *Let* $e_A = i_{K^*}(\tau) \in (A \rtimes K) \rtimes K^*$. *Then:*
(i) $e_A i_{A \rtimes K}(x) e_A = i_A(E_A(x)) e_A$ *for all* $x \in A \rtimes K$*;*
(ii) *the map* $A \ni a \mapsto i_A(a) e_A \in (A \rtimes K) \rtimes K^*$ *is injective.*
Moreover, $E_{A \rtimes K}(e_A) = \lambda$.

Proof. For all $a \in A$, $h \in K$ we compute

$$e_A i_{A \rtimes K}([a \otimes h]) e_A = [\tau_{(1)} \triangleright [a \otimes h] \otimes \tau_{(2)} \tau] = [\tau_{(1)} \triangleright [a \otimes h] \otimes \varepsilon_{\mathrm{t}}(\tau_{(2)})][1_{A \rtimes K} \otimes \tau]$$
$$= [\tau \triangleright [a \otimes h] \otimes \varepsilon][1_{A \rtimes K} \otimes \tau] = i_A(E_A([a \otimes h])) e_A,$$

which proves (i). Next, we compute

$$E_{A \rtimes K}(i_A(a) e_A) = E_{A \rtimes K}([a \otimes 1] \otimes \tau) = [a \otimes 1](\lambda \varepsilon \triangleright [1 \otimes 1]) = \lambda i_A(a),$$

thus proving that the map $a \mapsto i_A(a) e_A$ is injective. Taking $a = 1$ in the last formula, we obtain $E_{A \rtimes K}(e_A) = \lambda$. ∎

PROPOSITION 4.5. $(A \rtimes K) \rtimes K^* = (A \rtimes K) e_A (A \rtimes K)$.

Proof. Observe that for all $a \in A$, $g, h \in K$

$$i_{A \rtimes K}([a \otimes h]) e_A i_K(g) = i_A(a)(i_K(h) e_A i_K(g)).$$

Since $(A \rtimes K) \rtimes K^* = \operatorname{span}\{i_A(a) i_{K \rtimes K^*}(x) \mid a \in A, x \in K \rtimes K^*\}$, it suffices to show that $K \rtimes K^* = K e_K K$ (here $e_K = [1_K \otimes \tau] \in K \rtimes K^*$).

For this purpose, we need to show that every element of $K \rtimes K^*$ can be written as a linear combination of elements $i_K(h) e_K i_K(g)$, $h, g \in K$.

Let $\{\varphi^\gamma_{ij}\}$ be a system of matrix units in K^*. Since τ is the normalized Haar projection in K^*, we have

$$\Delta(\tau) = \sum_\gamma \frac{1}{c_\gamma} \sum_{i,j} \varphi^\gamma_{ij} \otimes S(\varphi^\gamma_{ji}),$$

for some integers c_γ. Let $\{v^\gamma_{ij}\}$ be the system of comatrix units in K, dual to $\{\varphi^\gamma_{ij}\}$: $\Delta(v^\gamma_{ij}) = \sum_k v^\gamma_{ik} \otimes v^\gamma_{kj}$, $\varepsilon(v^\gamma_{ij}) = \delta_{ij}$.

Fix $x \in K$ and let $h_k = x S(v^\gamma_{pk})$, $g_k = c_\gamma v^\gamma_{kl}$ for some γ, p, l $(k = 1, \dots, m_\gamma)$. Then

$$\sum_k i_K(h_k) e_K i_K(g_k) = \sum_{k,i,j,m} [x S(v^\gamma_{pk}) v^\gamma_{km} \otimes \langle \varphi^\gamma_{ij}, v^\gamma_{ml} \rangle S(\varphi^\gamma_{ji})]$$
$$= \sum_{k,m} [x S(v^\gamma_{pk}) v^\gamma_{km} \otimes S(\varphi^\gamma_{lm})] = \sum_m [x \varepsilon_{\mathrm{s}}(v^\gamma_{pm}) \otimes S(\varphi^\gamma_{lm})]$$
$$= \Big[x \otimes \sum_m \langle \varepsilon_{(1)}, v^\gamma_{pm} \rangle \varepsilon_{(2)} S(\varphi^\gamma_{lm}) \Big].$$

Since $x \in K$ is arbitrary, it remains to show that the elements of the form $\psi^\gamma_{lp} = \sum_m \langle \varepsilon_{(1)}, v^\gamma_{pm} \rangle \varepsilon_{(2)} S(\varphi^\gamma_{lm})$ form a linear basis for K^*. We have

$$\begin{aligned}\langle S(\psi^\gamma_{lp}), v^\beta_{pq}\rangle &= \sum_m \langle \varphi^\gamma_{lm}\varepsilon_{(1)}, v^\beta_{pq}\rangle\langle \varepsilon_{(2)}, S(v^\gamma_{pm})\rangle = \sum_{m,j}\langle \varphi^\gamma_{lm}, v^\beta_{pj}\rangle\langle \varepsilon, v^\beta_{jq} S(v^\gamma_{pm})\rangle \\ &= \delta_{\gamma\beta}\delta_{lp}\sum_m \langle \varepsilon, v^\gamma_{mq} S(v^\gamma_{pm})\rangle = \delta_{\gamma\beta}\delta_{lp}\varepsilon(S(v^\gamma_{pq})) = \delta_{\gamma\beta}\delta_{lp}\delta_{pq},\end{aligned}$$

therefore, $\psi^\gamma_{lp} = S(\varphi^\gamma_{lp})$. ∎

Corollary 4.6. *$(A \rtimes K) \rtimes K^* \cong \langle A \rtimes K,\ e_A\rangle$, i.e., $(A \rtimes K) \rtimes K^*$ is the basic construction for the conditional expectation E_A.*

Proof. Propositions 4.2 and Proposition 4.5 show that $(A \rtimes K) \rtimes K^*$ is generated by $A \rtimes K$ and projection e_A in the way characterizing the basic construction (see Subsection 2.3). ∎

The following result is an analogue of the Takesaki duality theorem for actions of Kac algebras ([6]) and Hopf algebras ([2]).

Theorem 4.7. (Duality for actions) *Let K be a weak Kac algebra satisfying the λ-Markov condition, acting on a C^*-algebra A. Then*

$$(A \rtimes K) \rtimes K^* \cong A \otimes M_n(\mathbb{C}), \quad \textit{where } n = \lambda^{-1}.$$

Proof. By Proposition 4.2 there is a basis for E_A, therefore $\langle A \rtimes K, e_A\rangle \cong A \otimes M_n(\mathbb{C})$, and the result follows from Corollary 4.6. ∎

Lemma 4.8. *Let K be a weak Kac algebra acting on the right on a $*$-algebra A. Then $K_t \subset A' \cap K \ltimes A$.*

Proof. If $z \in K_t$, then

$$i_A(a) i_K(z) = [z_{(1)} \otimes (a \triangleleft z_{(2)})] = [z 1_{(1)} \otimes (a \triangleleft 1_{(2)})] = [z \otimes a] = i_K(z) i_A(a),$$

thus $K_t \subset A' \cap K \ltimes A$. ∎

Definition 4.9. A right action of K on A is *minimal* if $K_t = A' \cap K \ltimes A$.

5. CONSTRUCTION OF A MINIMAL ACTION OF A BICONNECTED WEAK KAC ALGEBRA ON THE HYPERFINITE II_1 FACTOR

In this section we assume that K is a biconnected weak Kac algebra, in particular that it satisfies the λ-Markov condition for some $\lambda = n^{-1}$.

LEMMA 5.1. *Let K act on a finite-dimensional C^*-algebra A. Suppose that* tr *is a trace on $A \rtimes K$, and E_A from Proposition* 4.2 *is the* tr*-preserving conditional expectation. Then* $\mathrm{tr}_1 = \mathrm{tr} \circ E_{A \rtimes K}$ *is a trace on $\langle A \rtimes K, e_A \rangle$, extending* tr *and satisfying* $\mathrm{tr}_1(e_A) = \lambda$. *In other words, if* tr *is a trace on $A \rtimes K$ such that E_A preserves it, then* tr *is a λ-Markov trace for the inclusion $A \subset A \rtimes K$, and* tr_1 *is its λ-Markov extension to $\langle A \rtimes K,\ e_A \rangle$.*

Proof. Clearly, tr_1 is a positive functional on $\langle A \rtimes K, e_A \rangle$ extending tr. Let us show that tr_1 is a trace. By Lemma 4.4, $E_{A \rtimes K}(e_A) = \lambda$, therefore

$$\begin{aligned}\mathrm{tr}_1((x_1 e_A y_1)(x_2 e_A y_2)) &= \mathrm{tr}_1((x_1 E_A(y_1 x_2) e_A y_2) = \lambda\, \mathrm{tr}(E_A(x_1 y_2) E_A(y_1 x_2)) \\ &= \mathrm{tr}_1((x_2 e_A y_2)(x_1 e_A y_1)),\end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in A \rtimes K$. Since $\langle A \rtimes K,\ e_A \rangle$ is spanned by elements of the form $x e_A y, (x, y \in A \rtimes K)$ the result follows from Subsection 3.2.5 of [10]. ∎

REMARK 5.2. In conditions of Lemma 5.1, e_A is the Jones projection for the inclusion $A \subset A \rtimes K$ with respect to the Markov trace tr and $E_{A \rtimes K} : \langle A \rtimes K, e_A \rangle \to A \rtimes K$ is the tr-preserving conditional expectation.

Note that the map $\varphi \mapsto (\varphi \triangleright 1)$ gives an isomorphism between K_{t}^* and K_{s} in the crossed product algebra $K \rtimes K^*$.

PROPOSITION 5.3. *Let K be a connected weak Kac algebra and let* tr *be the unique Markov trace for the inclusion $i_K(K) \subset K \rtimes K^*$. Then*

$$\begin{array}{ccc} i_K(K) & \subset & K \rtimes K^* \\ \cup & & \cup \\ i_K(K_{\mathrm{s}}) \equiv i_{K^*}({K_{\mathrm{t}}}^*) & \subset & i_{K^*}(K^*) \end{array}$$

is a symmetric commuting square with respect to tr.

Proof. By Corollary 3.6, τ is a Markov trace for the inclusion $K_{\mathrm{t}} \subset K$, and E_{t} is the τ-preserving conditional expectation.

Since $K \rtimes K^* = (K_{\mathrm{t}} \rtimes K) \rtimes K^*$, it follows from Lemma 5.1 that tr extends τ and $E_K : K \rtimes K^* \to K$ is the tr-preserving conditional expectation. We have

$$E_K(i_{K^*}(\varphi)) = E_K([1 \otimes \varphi]) = i_K(\varphi \triangleright 1) \in i_K(K_{\mathrm{s}}),$$

for all $\varphi \in K^*$. This proves that the square is commuting. It is symmetric since $K \rtimes K^* = i_K(K) i_{K^*}(K^*)$. ∎

Corollary 4.6 implies that the sequence

$$K_{\mathrm{t}} \subset K \subset K \rtimes K^* \subset K \rtimes K^* \rtimes K \subset \cdots \subset M$$

is the Jones tower for the inclusion $K_{\mathrm{t}} \subset K$. When K is connected, all the inclusions in this sequence are connected and the union of these C^*-algebras admits a unique tracial state. Consequently, its von Neumann algebra completion M with respect to this trace is a copy of the hyperfinite II_1 factor. Using the standard procedure of iterating the basic construction we can construct a von Neumann subalgebra $N \subset M$ from the above symmetric commuting square.

PROPOSITION 5.4. *The lattice of C^*-algebras obtained by iterating the basic construction (in the horizontal direction) for the symmetric commuting square from Proposition* 5.3 *is given by two sequences of alternating crossed products with K and K^*:*

$$\begin{array}{ccccccccc} K & \subset & K \rtimes K^* & \subset & K \rtimes K^* \rtimes K & \subset & \cdots & \subset & M \\ \cup & & \cup & & \cup & & & & \cup \\ K_{\mathrm{s}} \equiv K_{\mathrm{t}}^* & \subset & K^* & \subset & K^* \rtimes K & \subset & \cdots & \subset & N, \end{array}$$

where we identify all C^*-subalgebras with their images in M.

Proof. Identities $K^* \rtimes K = \langle K, e_K \rangle$, $K^* \rtimes K \rtimes K^* = \langle K^* \rtimes K, e_{K \rtimes K^*} \rangle$ etc. follow immediately from Proposition 4.5. ∎

PROPOSITION 5.5. *There is a $*$-isomorphism between finite dimensional C^*-algebras*

$$A^r = \underbrace{K \rtimes K^* \rtimes \cdots \rtimes K \rtimes K^*}_{2r\ factors} \quad \text{and} \quad B^r = \underbrace{K \ltimes K^* \ltimes \cdots \ltimes K \ltimes K^*}_{2r\ factors}$$

given by the "identity" map

$$[h^1 \otimes \varphi^1 \otimes \cdots \otimes h^r \otimes \varphi^r] \mapsto [h^1 \otimes \varphi^1 \otimes \cdots \otimes h^r \otimes \varphi^r],$$

where $h^i \in K$, $\varphi^i \in K^*$.

Proof. By the definition of crossed product, the above algebras are isomorphic to

$$K \underset{K_{\mathrm{t}} = K_{\mathrm{s}}^*}{\otimes} K^* \underset{K_{\mathrm{t}}^* = K_{\mathrm{s}}}{\otimes} \cdots \underset{K_{\mathrm{t}} = K_{\mathrm{s}}^*}{\otimes} K^*$$

as vector spaces. By Theorem 4.7, we know that these algebras are isomorphic to $M_{n^r}(\mathbb{C}) \otimes K_{\mathrm{s}}$, where $n = \lambda^{-1}$. To see that the "identity" map defines a $*$-algebra isomorphism, it suffices to note that

$$\begin{aligned} &[h^1 \otimes \varphi^1 \otimes \cdots \otimes h^r \otimes \varphi^r] \cdot_{A^r} [g^1 \otimes \psi^1 \otimes \cdots \otimes g^r \otimes \psi^r] \\ &\quad = [h^1(\varphi^1_{(1)} \triangleright g^1) \otimes \varphi^1_{(2)}(h^2_{(1)} \triangleright \psi^1) \otimes \cdots \otimes h^r_{(2)}(\varphi^r_{(1)} \triangleright g^r) \otimes \varphi^r_{(2)}\psi^r] \\ &\quad = [h^1 g^1_{(1)} \otimes \langle \varphi^1_{(1)}, g^1_{(2)} \rangle \varphi^1_{(2)} \psi^1_{(1)} \otimes \cdots \otimes \langle \psi^{r-1}_{(2)}, h^r_{(1)} \rangle h^r_{(2)} g^r_{(1)} \otimes \langle \varphi^r_{(1)}, g^r_{(2)} \rangle \varphi^r_{(2)} \psi^r] \\ &\quad = [h^1 g^1_{(1)} \otimes (\varphi^1 \triangleleft g^1_{(2)}) \psi^1_{(1)} \otimes \cdots \otimes (h^r \triangleleft \psi^{r-1}_{(2)}) g^r_{(1)} \otimes (\varphi^r \triangleleft g^r_{(2)}) \psi^r] \\ &\quad = [h^1 \otimes \varphi^1 \otimes \cdots \otimes h^r \otimes \varphi^r] \cdot_{B^r} [g^1 \otimes \psi^1 \otimes \cdots \otimes g^r \otimes \psi^r], \end{aligned}$$

for all $h^i, g^i \in K$, $\varphi^i, \psi^i \in K^*$, $i = 1, \ldots, r$, i.e. multiplications in A^r and B^r are the same. ∎

COROLLARY 5.6. *The lattice of algebras from Proposition* 5.4 *is isomorphic to*

$$\begin{array}{ccccccccc} K & \subset & K \ltimes K^* & \subset & K \ltimes K^* \ltimes K & \subset & \cdots & \subset & M \\ \cup & & \cup & & \cup & & & & \cup \\ K_{\mathrm{s}} & \subset & K^* & \subset & K^* \ltimes K & \subset & \cdots & \subset & N. \end{array}$$

Proof. Clearly, the isomorphisms constructed in Proposition 5.5 are compatible with all inclusions of the lattice from Proposition 5.4. ∎

Our next goal is to show that there is a right action of K on N such that $M \cong K \ltimes N$.

PROPOSITION 5.7. *Let* $i_K : h \mapsto [h \otimes \varepsilon \otimes 1 \otimes \cdots]$ *be the inclusion of* K *in* M, $E_N : M \to N$ *be the trace preserving conditional expectation. Then the map*

$$x \triangleleft h = \lambda^{-1} E_N(i_K(p_\varepsilon) x i_K(h)), \quad x \in N,\, h \in K$$

defines a right action of K *on* N *such that* $M = K \ltimes N$ (cf. Section 5 of [22]).

Proof. There is a right action of K on the $*$-subalgebra given by the union of the generating sequence of C^*-algebras of N:

$$[\varphi \otimes g \otimes \cdots] \triangleleft h = [(\varphi \triangleleft h) \otimes g \otimes \cdots], \quad h, g \in K,\, \varphi \in K^*.$$

We have

$$\begin{aligned}[\varphi \otimes g \otimes \cdots] \triangleleft h &= \lambda^{-1}[(\varepsilon \triangleleft E_{\mathrm{s}}(p_\varepsilon))(\varphi \triangleleft h) \otimes g \otimes \cdots] \\ &= \lambda^{-1} E_N([p_\varepsilon \otimes (\varphi \triangleleft h) \otimes g \otimes \cdots]) = \lambda^{-1} E_N(i_K(p_\varepsilon)[\varphi \otimes g \otimes \cdots] i_K(h)),\end{aligned}$$

therefore the map $x \triangleleft h = \lambda^{-1} E_N(i_k(p_\varepsilon) x i_k(h))$ extends the above action to a weakly continuous action of K on N. Clearly, $K \ltimes N = i_k(K)N = M$. ∎

COROLLARY 5.8. $[M : N] = \lambda^{-1}$.

Proof. Follows from Remark 4.3 and Proposition 5.1.9 in [10]. ∎

Let us compute the higher relative commutants of the inclusion $N \subset M$.

LEMMA 5.9. *Let* K *act on the left on a* C^**-algebra* A*; then*

$$i_{K^*}(K^*)' \cap i_{A \rtimes K}(A \rtimes K) \cap (A \rtimes K) \rtimes K^* = i_A(A).$$

Proof. Let $C = i_{K^*}(K^*)' \cap i_{A \rtimes K}(A \rtimes K) \cap (A \rtimes K) \rtimes K^*$ and $x \in C$. Recall that $e_A = i_{K^*}(\tau)$. Then $x e_A = e_A x e_A = E_A(x) e_A$ and since the map $A \rtimes K \ni x \mapsto i_{A \rtimes K}(x) e_A$ is injective (Lemma 4.4), it follows that $x \in i_A(A)$ and $C \subset i_A(A)$.

Conversely, for all $a \in A, \varphi \in K^*$ we have

$$\begin{aligned} i_{K^*}(\varphi) i_A(a) &= [1_{A \rtimes K} \otimes \varphi][[a \otimes 1] \otimes \varepsilon] = [(\varphi_{(1)} \triangleright [a \otimes 1]) \otimes \varphi_{(2)}] \\ &= [[a \otimes 1](\varphi_{(1)} \triangleright [1 \otimes 1]) \otimes \varphi_{(2)}] = [[a \otimes 1] \otimes \varphi] = i_A(a) i_{K^*}(\varphi),\end{aligned}$$

therefore $i_A(A) = C$. ∎

PROPOSITION 5.10. *Let* $N \subset M = M_0 \subset M_1 \subset M_2 \cdots$ *be the Jones tower constructed from the inclusion* $N \subset M$*. Then*

$$N' \cap M_n \cong \underbrace{\cdots \ltimes K \ltimes K^*}_{n \ factors} \ltimes K_{\mathrm{t}}, \quad n \geqslant 0$$

$$M' \cap M_n \cong \underbrace{\cdots \ltimes K^* \ltimes K}_{(n-1)\ factors} \ltimes K_{\mathrm{t}}^*, \quad n \geqslant 1.$$

In particular, the action of K *is minimal.*

Proof. Iterating the basic construction for the commuting square from Proposition 5.3 in the vertical direction and using Proposition 5.5, we get the lattice

$$\begin{array}{ccc} \cdots & & \cdots \\ \cup & & \cup \\ K^* \rtimes K & \subset & K^* \rtimes K \rtimes K^* \\ \cup & & \cup \\ K & \subset & K \rtimes K^* \\ \cup & & \cup \\ K_{\mathrm{t}}^* \equiv K_{\mathrm{s}} & \subset & K^*. \end{array}$$

The Ocneanu compactness argument ([10]) and Lemma 5.9 imply that

$$N'\cap M = K_{\mathrm t},\quad N'\cap M_1 = K^*,\quad N'\cap M_2 = K\rtimes K^* \quad \dots$$

Similarly, one computes the relative commutants for M. ∎

Corollary 5.11. ([15]) *The inclusion $N\subset M$ is of depth* 2.

Proof. We have seen in Section 4 that $K\rtimes K^*\cong K_{\mathrm t}\otimes M_n(\mathbb{C})$, where $n=\lambda^{-1}$. Therefore, $\dim Z(N'\cap M)=\dim Z(N'\cap M_2)$, and so $N\subset M$ is of depth 2. ∎

Corollary 5.12. *The λ-lattice of higher relative commutants* ([19]) *of the inclusion $N\subset M$ is given by*

$$\begin{array}{ccccccccccc}
\mathbb{C} & \subset & K_{\mathrm t}^*\equiv K_{\mathrm s} & \subset & K & \subset & K^*\ltimes K & \subset & K\ltimes K^*\ltimes K & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \cup & & \\
 & & \mathbb{C} & \subset & K_{\mathrm t}\equiv K_{\mathrm s}^* & \subset & K^* & \subset & K\ltimes K^* & \subset & \cdots .
\end{array}$$

Remark 5.13. In a similar way one can construct a left minimal action of a biconnected weak Kac algebra on the hyperfinite II_1 factor.

6. Examples of subfactors and arithmetic properties of biconnected weak Kac algebras

Let K be a biconnected weak Kac algebra. Recall the notation

$$K\cong\bigoplus_{i=1}^{N} M_{d_i}(\mathbb{C}),\quad K_{\mathrm s}\cong K_{\mathrm t}\cong\bigoplus_{\alpha=1}^{L} M_{m_\alpha}(\mathbb{C}),$$

from Subsection 2.1. Let us also denote $d=\dim K_{\mathrm s}$. We have $\dim K=d\lambda^{-1}$.

Reducing the inclusion $N\subset M=K\ltimes N$ constructed in Section 5 by a minimal projection $q\in N'\cap M=K_{\mathrm t}$, we get an irreducible inclusion $qN\subset qMq$ of hyperfinite II_1 factors with index $[qMq:qN]=\tau(q)^2\lambda^{-1}$, where τ is the normalized trace on M ($qN\subset qMq$ is of finite depth ([1]), and therefore extremal, see [18]). But $\tau(q)=\frac{m_\alpha}{d}$, when $q\in M_{m_\alpha}(\mathbb{C})$, therefore

$$[qMq:qN]=\frac{m_\alpha^2\lambda^{-1}}{d^2}.$$

Note that since $qN\subset qMq$ has a finite depth, its index is an algebraic integer. But by Theorem 3.5, λ^{-1} is an integer, so $[qMq:qN]$ is rational. Therefore, $[qMq:qN]$ is an integer. Thus, we proved

Proposition 6.1. *d^2 divides $m_\alpha^2\lambda^{-1}$ for all α.*

Corollary 6.2. *If $\lambda^{-1}=p$ is a prime, then $K\cong\mathbb{C}\mathbb{Z}_p$.*

Proof. By the previous proposition we must have $d=1$, so $\dim K=d\lambda^{-1}=p$ and the result follows from Corollary 3.9. ∎

Next, reducing the inclusion $M \subset M_2$ by a minimal projection q from the relative commutant $M' \cap M_2 = K$ we get an irreducible inclusion $qM \subset qM_2q$. Clearly, this inclusion depends only on the equivalence class of q, so inclusions of the above type are in one-to-one correspondence with irreducible representations of K. The index is

$$[qM_2q : qM] = \tau(q)^2[M_2 : M] = \tau(q)^2\lambda^{-2} = \left(\frac{d_i}{d}\right)^2,$$

whenever $q \in M_{d_i}(\mathbb{C})$. Again, the index must be an integer, so we get the following arithmetic property of biconnected weak Kac algebras.

COROLLARY 6.3. *The dimension of a counital subalgebra of K divides the degree of any irreducible representation of K, i.e., d divides d_i for all i. In particular, d^2 divides* $\dim K$, *and d divides* $\lambda^{-1} = [M : N]$.

Finally, let us remark that considering the biconnected weak Kac algebra $K = H \rtimes H^* \ltimes H$ constructed from a Kac algebra H as in Example 3.14, we can associate an irreducible subfactor with any irreducible representation of H (since we have $K_{\mathrm{t}} = H$ in this case).

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