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CONTRACTIVE EXTENSION PROBLEMS FOR MATRIX VALUED ALMOST PERIODIC FUNCTIONS OF SEVERAL VARIABLES

LEIBA RODMAN, ILYA M. SPITKOVSKY, and HUGO J. WOERDEMAN

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Abstract. Problems of Nehari type are studied for matrix valued k-variable almost periodic Wiener functions: Find contractive k-variable almost periodic Wiener functions having prespecified Fourier coefficients with indices in a given halfspace of $\mathbb{R}^k$. We characterize the existence of a solution, give a construction of the solution set, and exhibit a particular solution that has a certain maximizing property. These results are used to obtain various distance formulas and multivariable almost periodic extensions of Sarason's theorem. In the periodic case, a generalization of Sarason's theorem is proved using a variation of the commutant lifting theorem. The main results are further applied to a model-matching problem for multivariable linear filters.

Keywords: *Almost periodic matrix functions, contractive extensions, Besikovitch space, Hankel operators, Sarason's Theorem, band method, commutant lifting, model matching.*

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1. INTRODUCTION

In this paper we study problems of Nehari type for matrices whose entries belong to various subalgebras of the class (AP^k) of almost periodic complex function of k real variables. As is well known, (AP^k) may be characterized as the closed subalgebra of $L^\infty(\mathbb{R}^k)$ generated by the functions $e_\lambda(t) = \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle}$. Here $\mathbb{R}$ is the set of reals, $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, $t = (t_1, \dots, t_k) \in \mathbb{R}^k$, and

$$\langle\lambda, t\rangle = \sum_{j=1}^{k} \lambda_j t_j$$

is the standard inner product of λ and t. For $f \in (\mathrm{AP}^k)$ the *Fourier series* is defined by the formal sum

$$\sum_{\lambda} f_\lambda \mathrm{e}^{\mathrm{i}\langle \lambda, t\rangle}, \tag{1.1}$$

where

$$f_\lambda = \lim_{T\to\infty} \frac{1}{(2T)^k} \int\limits_{[-T,T]^k} \mathrm{e}^{-\mathrm{i}\langle \lambda, t\rangle} f(t)\,\mathrm{d}t, \quad \lambda \in \mathbb{R}^k, \tag{1.2}$$

and the sum in (1.1) is taken over the *spectrum* $\sigma(f) = \{\lambda \in \mathbb{R}^k : f_\lambda \neq 0\}$ of f. The *mean* $M\{f\}$ of $f \in (\mathrm{AP}^k)$ is defined by $M\{f\} = f_0 = \lim\limits_{T\to\infty} \frac{1}{(2T)^k} \int\limits_{[-T,T]^k} f(t)\,\mathrm{d}t$. The spectrum of every $f \in (\mathrm{AP}^k)$ is at most a countable set. The *Wiener algebra* (APW^k) is defined as the set of all $f \in (\mathrm{AP}^k)$ such that the Fourier series of f converges absolutely. The Wiener algebra is a Banach algebra with respect to the *Wiener norm* $\|f\|_{\mathrm{W}} = \sum\limits_{\lambda\in\mathbb{R}^k} |f_\lambda|$. For a non-empty subset Δ of $\mathbb{R}^k$ we denote by $(\mathrm{AP}^k)_\Delta$ and $(\mathrm{APW}^k)_\Delta$ the subspace of (AP^k) and (APW^k), respectively, of functions whose spectrum is contained in Δ. For the general theory of almost periodic functions of one and several variables we refer the reader to the books [6], [21], [22] and to Chapter 1 in [26]. If (X) is a set (typically a Banach space or an algebra), we denote by $(X)^{m\times n}$ the set of $m \times n$ matrices with entries in (X). The norm on $(\mathrm{AP}^k)^{m\times n}$ is given by $\|f\|_\infty = \sup\limits_{t\in\mathbb{R}^k} \|f(t)\|$, where $\|A\|$ denotes the largest singular value of the matrix A. When $\|f\|_\infty < 1$ we say that f is *strictly contractive.*

Analogously to [19], we introduce the notion of a halfspace. A subset S of $\mathbb{R}^k$ is called a *halfspace* of $\mathbb{R}^k$ if S satisfies the following properties:

(i) $\mathbb{R}^k = S \cup (-S)$;
(ii) $S \cap (-S) = \{0\}$;
(iii) if $x, y \in S$ then $x + y \in S$;
(iv) if $x \in S$ and α is a nonnegative real number, then $\alpha x \in S$.

A standard example of a halfspace is given by

$$\begin{aligned} E_k = \{&(x_1, \ldots, x_k)^{\mathrm{T}} \in \mathbb{R}^k \setminus \{0\} : \\ &x_1 = x_2 = \cdots = x_{j-1} = 0,\, x_j \neq 0 \Rightarrow x_j > 0\} \cup \{0\}. \end{aligned} \tag{1.3}$$

(The vectors in $\mathbb{R}^k$ are understood as column vectors; the superscript $^{\mathrm{T}}$ denotes the transpose.) It is known that any halfspace S of $\mathbb{R}^k$ may be retrieved as the image of E_k under an invertible linear map $A : \mathbb{R}^k \to \mathbb{R}^k$, i.e.,

$$S = AE_k \stackrel{\mathrm{def}}{=} \{Ax : x \in E_k\}; \tag{1.4}$$

see Section 2 of [29], for example. A halfspace of Λ, where Λ is a subgroup of $\mathbb{R}^k$, is the intersection of a halfspace in $\mathbb{R}^k$ with Λ.

We are now ready to state the contractive extension problem that we consider. Fix a halfspace $S \subset \mathbb{R}^k$ and additive subgroups $\Lambda \subseteq \Lambda'$ of $\mathbb{R}^k$, and let $f \in (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda}$ be given. We say that h is a *strictly contractive extension* in $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$ of f, if h belongs to $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$, and

(1) h is strictly contractive;
(2) $h_\lambda = f_\lambda$, $\lambda \in (-S) \cap \Lambda'$.

The problem we are considering asks when strictly contractive extensions of a function exist, and in that case how to construct one and how to describe the set of all solutions. This problem has been studied in [28] for the case $k = 1$, where background on the problem may be found. We shall provide a solution to this problem using the abstract band method (see [15], [16], [17], also Chapter XXXIV of [14] for the development of the method), which is an algebraic scheme dealing with positive and contractive extension problems. One of the main hurdles in applying the band method concerns the question whether a solution to a linear equation has an inverse that lies in a certain subspace. In the classical Carathéodory-Toeplitz problem this hurdle is overcome by applying Szegö's theorem on the location of zeros of orthogonal polynomials (see Theorem 6.1 in [9] for more details). After succesfully overcoming this obstacle using different techniques in our multivariable almost periodic setting, we obtain characterization of existence in terms of strict contractiveness of an associated Hankel operator on a Besikovitch Hilbert space. In addition, a construction of a specific solution characterized by a maximizing property and a linear fractional description for the set of all solutions are obtained. We shall also consider the case of non-strict contractive extensions, which leads us in a natural way to a solution in a Besikovitch space that extends (AP^k). These main results will be presented in Sections 2–5. We shall also consider various consequences of these results. In Section 6 we shall extract distance formulas from the main results and use them to prove multivariable and almost periodic generalizations of Sarason's theorem. In addition, a commutant lifting approach to the periodic ($\Lambda = \mathbb{Z}^k$) case will be explored in this section. In Section 7 a joint norm bound result is given, and in Section 8 we apply the results to a model-matching problem for filters indexed by a subgroup of $\mathbb{R}^k$, which includes 2D systems as in [12].

Standard notation will be used throughout: $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{T}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{D}$ stand for the sets of integers, of non-negative integers, of unimodular complex numbers, of real numbers, of complex numbers, and for the open unit disc, respectively.

2. THE MAIN RESULTS

Before we can state the main results we need to introduce the Besikovitch Hilbert space. Define an inner product on (AP^k) by the formula

$$\langle f, g\rangle = M\{fg^*\}, \quad f, g \in (\mathrm{AP}^k). \tag{2.1}$$

The completion of (AP^k) with respect to this inner product is called the *Besikovitch space* and is denoted by (B^k). Thus (B^k) is a Hilbert space. For a nonempty set $\Lambda \subseteq \mathbb{R}^k$, define the projection

$$\Pi_\Lambda\left(\sum_{\lambda\in\sigma(f)} f_\lambda \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle}\right) = \sum_{\lambda\in\sigma(f)\cap\Lambda} f_\lambda \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle},$$

where $f \in (\mathrm{APW}^k)$. The projection Π_Λ extends by continuity to the orthogonal projection (also denoted Π_Λ) on (B^k). We denote by $(B^k)_\Lambda$ the range of Π_Λ, or, equivalently, the completion of $(\mathrm{AP}^k)_\Lambda$ with respect to the inner product (2.1). The vector valued Besikovitch space $(B^k)^{n\times 1}$ consists of $n \times 1$ columns with components in (B^k), with the standard Hilbert space structure. Similarly, $(B^k)^{n\times 1}_\Lambda$ is the Hilbert space of $n \times 1$ columns with components in $(B^k)_\Lambda$. The matrix Besikovitch space $(B^k)^{m\times p}_\Delta$ will be considered as a Hilbert space with the inner product

$$\langle g, h\rangle = \sum_{i=1,\,j=1}^{i=m,\,j=p} \langle g_{ij}, h_{ij}\rangle = \lim_{T\to\infty} \frac{1}{(2T)^k} \int\limits_{[-T,T]^k} \mathrm{trace}\,(g(t)(h(t))^*)\,\mathrm{d}t, \tag{2.2}$$

where g_{ij} and h_{ij} are the (i, j) entries of g and h, respectively, and the inner products in the middle of (2.2) are taken in $(B^k)_\Delta$.

Fix a halfspace $S \subset \mathbb{R}^k$ and an additive subgroup Λ of $\mathbb{R}^k$. Let $f \in (\mathrm{APW}^k)^{m\times n}_\Lambda$ be given. For any additive group $\Lambda' \supseteq \Lambda$ and integer $p \geqslant 1$, define the following generalized Hankel operator

$$\mathbf{H}(f)_{\Lambda'} : (B^k)^{n\times p}_{S\cap\Lambda'} \to (B^k)^{m\times p}_{(-S)\cap\Lambda'}$$

by

$$\mathbf{H}(f)_{\Lambda'} g = \Pi_{-S}(fg), \quad g \in (B^k)^{n\times p}_{S\cap\Lambda'}. \tag{2.3}$$

We suppress the dependence of $\mathbf{H}(f)_{\Lambda'}$ on S in our notation, although different choices of S yield totally different Hankels. It is not hard to see that the norm of $\mathbf{H}(f)_{\Lambda'}$ is independent of the choice of the positive integer p. It is convenient, however, to allow $\mathbf{H}(f)_{\Lambda'}$ to act on matrix valued functions with different number of columns.

THEOREM 2.1. *Let $f \in (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda}$ be given, and let Λ' be a supergroup of Λ. The following statements are equivalent:*

(i) *f has a strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$;*
(ii) *f has a strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_{\Lambda}$;*
(iii) *the generalized Hankel operator $\mathbf{H}(f)_{\Lambda'}$ is a strict contraction;*
(iv) *the generalized Hankel operator $\mathbf{H}(f)_{\Lambda}$ is a strict contraction.*

When one (and thus all) of (i)–(iv) *is satisfied, then put*

$$\begin{aligned}
\widehat{\alpha}(t) &= \left[I - \mathbf{H}(f)_\Lambda(\mathbf{H}(f)_\Lambda)^*\right]^{-1}(I_m),\\
\widehat{\beta}(t) &= \mathbf{H}(f)_\Lambda\left[I - (\mathbf{H}(f)_\Lambda)^*\mathbf{H}(f)_\Lambda\right]^{-1}(I_n),\\
\widehat{\gamma}(t) &= (\mathbf{H}(f)_\Lambda)^*\left[I - (\mathbf{H}(f)_\Lambda(\mathbf{H}(f)_\Lambda)^*\right]^{-1}(I_m),\\
\widehat{\delta}(t) &= \left[I - (\mathbf{H}(f)_\Lambda)^*\mathbf{H}(f)_\Lambda)\right]^{-1}(I_n),
\end{aligned}$$

where I_r stands for the constant matrix function on $\mathbb{R}^k$ with value I_r for all $t \in \mathbb{R}^k$. Further, let

$$\alpha(t) = \widehat{\alpha}(t)M\{\widehat{\alpha}\}^{-\frac{1}{2}}, \quad \beta(t) = \widehat{\beta}(t)M\{\widehat{\delta}\}^{-\frac{1}{2}}; \tag{2.4}$$

$$\gamma(t) = \widehat{\gamma}(t)M\{\widehat{\alpha}\}^{-\frac{1}{2}}, \quad \delta(t) = \widehat{\delta}(t)M\{\widehat{\delta}\}^{-\frac{1}{2}}. \tag{2.5}$$

Then the function

$$g_0(t) = \beta(t)\delta(t)^{-1} = [\alpha(t)^*]^{-1}\gamma(t)^*, \quad t \in \mathbb{R}^k, \tag{2.6}$$

is a strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_{\Lambda}$ $(\subset (\mathrm{APW}^k)^{m\times n}_{\Lambda'})$ *of f.*

It will be shown in the course of the proof of Theorem 2.1 that $\widehat{\alpha}$ is invertible in $(\mathrm{APW}^k)^{m\times m}_{\Lambda}$ and $M\{\widehat{\alpha}\}$ is positive definite. Similarly, it will be shown that $\widehat{\delta}$ is invertible in $(\mathrm{APW}^k)^{n\times n}_{\Lambda}$ and $M\{\widehat{\delta}\} > 0$. Thus, the formulas (2.4)–(2.6) make sense. In fact, we shall show that $\widehat{\alpha}^{-1} \in (\mathrm{APW}^k)^{m\times m}_{(-S)\cap\Lambda}$ and $\widehat{\delta}^{-1} \in (\mathrm{APW}^k)^{n\times n}_{S\cap\Lambda}$. It should be noted that

$$g_0(t) = \widehat{\beta}(t)\widehat{\delta}(t)^{-1} = [\widehat{\alpha}(t)^*]^{-1}\widehat{\gamma}(t)^*, \quad t \in \mathbb{R}^k, \tag{2.7}$$

so that the introduction of the functions α, β, γ and δ was not strictly necessary in the above theorem. However, the normalization performed in (2.4) and (2.5) is important in the following theorem.

Next, a parameterization of the set of all strictly contractive extensions in $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$ of a function $f \in (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda}$ will be given. Introduce the parameter set

$$(\mathrm{CAPW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'} = \Big\{g \in (\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'} : \sup_{t\in\mathbb{R}^k}\|g(t)\| < 1\Big\}.$$

Theorem 2.2. *Let $f \in (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda}$ be given, and let Λ' be a supergroup of Λ. Suppose that $\|\mathbf{H}(f)_\Lambda\| < 1$. Define $\alpha(t), \beta(t), \gamma(t)$, and $\delta(t)$, as in Theorem 2.1. Then each strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$ of f is of the form*

$$T(g) = (\alpha g + \beta)(\gamma g + \delta)^{-1}, \tag{2.8}$$

where $g \in (\mathrm{CAPW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'}$. Moreover, this correspondence between the set $(\mathrm{CAPW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'}$ and the set of strictly contractive extensions in $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$ of f is one-to-one.

Aside from the crucial role that g_0 plays in finding the linear fractional description, the extension g_0 has also some other outstanding properties. Before we describe them we have to introduce some additional notions.

Recall from [29] definitions of canonical factorizations with respect to a half-space S. Let $G \in (\mathrm{AP}^k)^{n\times n}$. A representation

$$G(t) = G_+(t)G_-(t), \quad t \in \mathbb{R}^k, \tag{2.9}$$

where

$$G_+^{\pm 1} \in (\mathrm{AP}^k)_S^{n\times n}, \quad G_-^{\pm 1} \in (\mathrm{AP}^k)_{-S}^{n\times n} \tag{2.10}$$

is called a (left) AP_S *canonical factorization* of G. We say that (2.9) is an APW_S canonical factorization of G if $G_\pm$ satisfy stronger than (2.10) conditions $G_+^{\pm 1} \in (\mathrm{APW}^k)_S^{n\times n}$, $G_-^{\pm 1} \in (\mathrm{APW}^k)_{-S}^{n\times n}$. If Λ is an additive subgroup of $\mathbb{R}^k$, then a representation (2.9) is called a canonical $(\mathrm{AP}_S)_\Lambda$ factorization if $G_+^{\pm 1} \in (\mathrm{AP}^k)_{S\cap\Lambda}^{n\times n}$, $G_-^{\pm 1} \in (\mathrm{AP}^k)_{(-S)\cap\Lambda}^{n\times n}$, and a canonical $(\mathrm{APW}_S)_\Lambda$ factorization if $G_+^{\pm 1} \in (\mathrm{APW}^k)_{S\cap\Lambda}^{n\times n}$, $G_-^{\pm 1} \in (\mathrm{APW}^k)_{(-S)\cap\Lambda}^{n\times n}$. Of course, G must belong to $(\mathrm{APW}^k)^{n\times n}$ (respectively, $(\mathrm{AP}^k)_\Lambda^{n\times n}$, $(\mathrm{APW}^k)_\Lambda^{n\times n}$) in order to potentially admit a canonical APW_S (respectively, $(\mathrm{AP}_S)_\Lambda$, $(\mathrm{APW}_S)_\Lambda$) factorization.

A $n \times n$ matrix valued function G defined on $\mathbb{R}^k$ is called *strictly positive* if there is an $\varepsilon > 0$ such that $G(t) \geqslant \varepsilon I$ for all $t \in \mathbb{R}^k$. By Corollary 5.2 in [29] we know that any strictly positive $G \in (\mathrm{APW}^k)_{S\cap\Lambda}^{n\times n}$ has a left APW_S canonical factorization. For a strictly positive $G \in (\mathrm{APW}^k)_{S\cap\Lambda}^{n\times n}$ we define

$$D(G) = M\{G_+\}M\{G_-\},$$

where $G = G_+G_-$ is an APW_S canonical factorization. It is straightforward to check that $D(G)$ is well-defined (i.e., does not depend on the choice of the APW_S canonical factorization) and is in addition positive definite.

We now characterize the strictly contractive extension g_0 given by (2.6) via a maximizing property. For hermitian matrices we shall use the Loewner partial ordering $\geqslant$, i.e., $A \geqslant B$ if and only if $A - B$ is positive semidefinite.

THEOREM 2.3. *Let $f \in (\mathrm{APW}^k)_{(-S)\cap\Lambda}^{m\times n}$ be given, and let Λ' be a supergroup of Λ. Suppose that $\|\mathbf{H}(f)_\Lambda\| < 1$. Let g_0 be given by (2.6). Then g_0 is the unique strictly contractive extension in $(\mathrm{APW}^k)_{\Lambda'}^{m\times n}$ of f such that*

$$g_0(t)\left(I_n - g_0(t)^*g_0(t)\right)^{-1} \in (\mathrm{APW}^k)_{(-S)\cap\Lambda'}^{m\times n},$$

or, equivalently,

$$\left(I_m - g_0(t)g_0(t)^*\right)^{-1} g_0(t) \in (\mathrm{APW}^k)_{(-S)\cap\Lambda'}^{m\times n}.$$

Moreover, if g is a strictly contractive extension in $(\mathrm{APW}^k)_{\Lambda'}^{m\times n}$ of f, then

$$D\left(I_n - g_0(t)^*g_0(t)\right) \geqslant D\left(I_n - g(t)^*g(t)\right) \tag{2.11}$$

with equality if and only if $g(t) = g_0(t)$, $t \in \mathbb{R}^k$. In particular, if g is a strictly contractive extension in $(\mathrm{APW}^k)_{\Lambda'}^{m\times n}$ of f, then

$$M\{\log\det(I_n - g_0(t)^*g_0(t))\} \geqslant M\{\log\det(I_n - g(t)^*g(t))\} \tag{2.12}$$

with equality if and only if $g(t) = g_0(t)$, $t \in \mathbb{R}$.

The proofs of Theorems 2.1–2.3 will be given in the next section.

3. PROOFS OF THE MAIN RESULTS

We start this section with some basic results on almost periodic functions.

PROPOSITION 3.1. *Let Δ be a subset of $\mathbb{R}^k$ such that $\lambda, \mu \in \Delta \Rightarrow \lambda+\mu \in \Delta$ and $\Delta \cap (-\Delta) = \{0\}$. Then $(\mathrm{AP}^k)_\Delta$ and $(\mathrm{APW}^k)_\Delta$ are closed subalgebras of (AP^k) and of (APW^k), respectively, and the mean is an additive and multiplicative continuous functional on $(\mathrm{AP}^k)_\Delta$ and on $(\mathrm{APW}^k)_\Delta$:*

$$M\{f+g\} = M\{f\} + M\{g\}, \quad M\{f \cdot g\} = M\{f\} \cdot M\{g\},$$

where $f, g \in (\mathrm{AP}^k)_\Delta$ or $f, g \in (\mathrm{APW}^k)_\Delta$.

Proof. The algebraic closedness properties follow from the observations that $\sigma(f+g) \subseteq \sigma(f) \cup \sigma(g) \subseteq \Delta$ and $\sigma(fg) \subseteq \sigma(f) + \sigma(g) \subseteq \Delta + \Delta \subseteq \Delta$. The statement $M\{f+g\} = M\{f\} + M\{g\}$ follows directly from the definition of the mean, while $M\{f \cdot g\} = M\{f\} \cdot M\{g\}$ follows from the observation that $\delta_1 + \delta_2 = 0$ and $\delta_1, \delta_2 \in \Delta$ imply $\delta_1 = \delta_2 = 0$. ∎

Proposition 3.1 will be applied to the case when Δ is the intersection of a halfspace and of an additive subgroup.

PROPOSITION 3.2. *Let $\Delta \subseteq \mathbb{R}^k$ be closed under addition, and assume that Δ contains zero and is a subset of a halfspace S in $\mathbb{R}^k$. Let be given $F \in (\mathrm{AP}^k)_\Delta^{n\times n}$ (respectively $F \in (\mathrm{APW}^k)_\Delta^{n\times n}$) such that $\inf_{t\in\mathbb{R}^k} |\det F(t)| > 0$ and $M\{F\} = I$. Assume, in addition, that*

$$F^{-1} \in (\mathrm{AP}^k)_\Delta^{n\times n}. \tag{3.1}$$

Then there is a continuous branch f of $\log(\det F)$. Every such branch f belongs to $(\mathrm{AP}^k)_\Delta$ (respectively $(\mathrm{APW}^k)_\Delta$), and one of them has the property that $M\{f\} = 0$.

Proof. We will prove the proposition for the case when $F \in (\mathrm{APW}^k)_\Delta^{n\times n}$ (the proof for the case when $F \in (\mathrm{AP}^k)_\Delta^{n\times n}$ is analogous).

Let $\Delta' = \Delta - \Delta$ be the additive subgroup generated by Δ. First, we observe that $(\det(F))^{-1} \in (\mathrm{APW}^k)_{\Delta'}$. This follows from the condition $\inf_{t\in\mathbb{R}^k} |\det F(t)| > 0$ and the inverse closedness of $(\mathrm{APW}^k)_{\Delta'}$ in $(\mathrm{AP}^k)_{\Delta'}$; see, e.g., Proposition 2.2 in [29]. Therefore, $F^{-1} \in (\mathrm{APW}^k)_{\Delta'}^{n\times n}$. In view of $\Delta' \cap \Delta = \Delta$, the condition (3.1) now guarantees that $F^{-1} \in (\mathrm{APW}^k)_\Delta^{n\times n}$. Since $(\mathrm{APW}^k)_\Delta$ is a subalgebra of (APW^k), we now have $(\det(F))^{-1} = \det(F^{-1}) \in (\mathrm{APW}^k)_\Delta$. Moreover, by the additive and multiplicative properties of the mean (Proposition 3.1) $M\{\det F\} = \det M\{F\} = 1$. Therefore, it suffices to consider the case of a scalar function F.

Denote by Θ the set of all scalar functions F for which the statement of Proposition 3.2 holds. In other words, $\Theta = \{F = \mathrm{e}^g : g \in (\mathrm{APW}^k)_\Delta,\ M\{g\} = 0\}$. As in the proof of Proposition 2.9 in [28], we verify that Θ is open and closed in the set

$$X = \left\{F \in (\mathrm{APW}^k)_\Delta : F^{-1} \in (\mathrm{APW}^k)_\Delta;\ M\{F\} = 1\right\}.$$

We now prove that the set X is connected. Using small (in the norm of (APW^k)) perturbations, we may restrict ourselves to proving that every (AP^k)-polynomial, i.e., almost periodic function with finite spectrum, in X is connected within X to the constant 1. Let S be a halfspace that contains Δ. Without loss of generality, we may assume that $S = E_k$, given by (1.3) (otherwise, consider the topologically equivalent set $\widetilde{X} = \{F \circ (A^{\mathrm{T}})^{-1} : F \in X\}$, where $S = AE_k$). If $F \in X$ is an (AP^k)-polynomial, then

$$F_\alpha(t) = F\left(t_1 + \mathrm{i}\frac{\alpha}{1-\alpha}, t_2, \ldots, t_k\right), \quad \alpha \in [0,1]$$

is a continuous path (in the APW^k norm) connecting $F_0 = F$ with $F_1 = M_1\{F\}$. Here $M_i\{f\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{\mathrm{T}} f(t_1, \ldots, t_k)\,\mathrm{d}t_i$ corresponds to taking the mean with respect to the ith variable. In fact, $F_\alpha(t) \in X$ for every $\alpha \in [0,1]$. To verify this, write

$$F(t) = \sum_{\lambda\in\Delta} f_\lambda \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle}; \quad F(t)^{-1} = \sum_{\lambda\in\Delta} g_\lambda \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle}.$$

Then, denoting $\ell = (1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^k$, we have

$$F_\alpha(t) = \sum_{\lambda\in\Delta} f_\lambda \mathrm{e}^{-\frac{\alpha}{1-\alpha}\langle\lambda,\ell\rangle} \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle}.$$

Since $\langle\lambda,\ell\rangle \geqslant 0$ for every $\lambda \in \Delta$, the function

$$G_\alpha(t) := \sum_{\lambda\in\Delta} g_\lambda \mathrm{e}^{-\frac{\alpha}{1-\alpha}\langle\lambda,\ell\rangle} \mathrm{e}^{\mathrm{i}\langle\lambda,t\rangle}$$

belongs to (APW^k) for every $\alpha \in [0,1]$, and one easily verifies that $G_\alpha = F_\alpha^{-1}$. Thus, $F_\alpha \in X$. Repeating this argument, we connect $M_1\{F\}$ with $M_2\{M_1\{F\}\}$, $M_2\{M_1\{F\}\}$ with $M_3\{M_2\{M_1\{F\}\}\}$, etc., all within X. Ultimately, we have connected F with

$$M_k\{\cdots\{M_2\{M_1\{F\}\}\cdots\} = M\{F\} = 1,$$

proving the connectedness of X.

Being an open, closed, and non-empty subset of X, Θ coincides with X. ∎

Note that the hypothesis (3.1) is essential in Proposition 3.2. If $\Delta = S \cap \Lambda$, where S is a halfspace and Λ is an additive subgroup of $\mathbb{R}^k$, then (3.1) follows from a formally weaker requirement that $F^{-1} \in (\mathrm{AP}^k)_S^{n\times n}$ (given that $F \in (\mathrm{AP}^k)_\Delta^{n\times n}$).

Proof of Theorem 2.1. The directions (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) (use $\mathbf{H}(f)_\Lambda = \Pi_{(-S)\cap\Lambda}\mathbf{H}(f)_{\Lambda'}\Pi_{S\cap\Lambda}$) are trivial. For (i) $\Rightarrow$ (iii) observe that if $h \in (\mathrm{APW}^k)_\Lambda^{m\times n}$ is a strictly contractive extension of f, we have that the multiplication operator $M_h : (B^k)_{\Lambda'}^{n\times p} \to (B^k)_{\Lambda'}^{m\times p}$ defined by $M_h(g) = hg$ is a strict contraction. But then $\mathbf{H}(f)_{\Lambda'} = \Pi_{-S} M_h \Pi_S$ is also a strict contraction.

It remains to show (iv) $\Rightarrow$ (ii). For this we shall use the Abstract Band Method (see Chapter XXXIV of [14]). Assume that $\|\mathbf{H}(f)_\Lambda\| < 1$. Let

$$\mathcal{M} = (\mathrm{APW}^k)_\Lambda^{(m+n)\times(m+n)} = \mathcal{M}_1 \dotplus \mathcal{M}_2^0 \dotplus \mathcal{M}_\mathrm{d} \dotplus \mathcal{M}_3^0 \dotplus \mathcal{M}_4, \tag{3.2}$$

where

$$\mathcal{M}_1 = \begin{bmatrix} 0 & (\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda} \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_2^0 = \begin{bmatrix} (\mathrm{APW}^k)^{m\times m}_{(S\setminus\{0\})\cap\Lambda} & (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda} \\ 0 & (\mathrm{APW}^k)^{n\times n}_{(S\setminus\{0\})\cap\Lambda} \end{bmatrix},$$

$$\mathcal{M}_{\mathrm{d}} = \begin{bmatrix} (\mathrm{APW}^k)^{m\times m}_{\{0\}} & 0 \\ 0 & (\mathrm{APW}^k)^{n\times n}_{\{0\}} \end{bmatrix}, \quad \mathcal{M}_3^0 = (\mathcal{M}_2^0)^*, \quad \mathcal{M}_4 = (\mathcal{M}_1)^*.$$

Then $\mathcal{M}$ is an algebra with band structure (3.2) as defined in Chapter XXXIV, Section 1 in [14]. By the existence of canonical factorizations for positive elements of $\mathcal{M}$ one sees that the definition of positivity used here ($f(t) \geqslant \varepsilon I > 0$ for all t) and the definition of positivity used in Chapter XXXIV, Section 1 in [14] coincide. We let

$$\mathcal{M}_2 = \mathcal{M}_2^0 \dot{+} \mathcal{M}_{\mathrm{d}}, \quad \mathcal{M}_3 = \mathcal{M}_3^0 \dot{+} \mathcal{M}_{\mathrm{d}}, \quad \mathcal{M}_{\mathrm{c}} = \mathcal{M}_2^0 \dot{+} \mathcal{M}_{\mathrm{d}} \dot{+} \mathcal{M}_3^0.$$

Let P_2 be the projection on $\mathcal{M}_2$ along $\mathcal{M}_3^0 \dot{+} \mathcal{M}_1 \dot{+} \mathcal{M}_4$. Projections P_3, P_2^0, P_4 etc. are defined analogously. Let

$$k_\varepsilon = \begin{bmatrix} I & \varepsilon f \\ \varepsilon f^* & I \end{bmatrix} \in \mathcal{M}_{\mathrm{c}}, \quad \varepsilon \in [0,1].$$

We solve for $x_\varepsilon \in \mathcal{M}_2$ the equation

$$P_2(k_\varepsilon x_\varepsilon) = I. \tag{3.3}$$

We seek for x_ε in the form $x_\varepsilon = \begin{bmatrix} I & -\beta_\varepsilon \\ 0 & \delta_\varepsilon \end{bmatrix}$. Then

$$\begin{aligned} k_\varepsilon x_\varepsilon &= \begin{bmatrix} I & -\beta_\varepsilon + \varepsilon f \delta_\varepsilon \\ \varepsilon f^* & -\varepsilon f^* \beta_\varepsilon + \delta_\varepsilon \end{bmatrix} \\ &\in \begin{bmatrix} I + (\mathrm{APW}^k)^{m\times m}_{((-S)\cap\Lambda)\setminus\{0\}} & (\mathrm{APW}^k)^{m\times n}_{(S\cap\Lambda)\setminus\{0\}} \\ (\mathrm{APW}^k)^{n\times m}_{\Lambda} & I + (\mathrm{APW}^k)^{n\times n}_{((-S)\cap\Lambda)\setminus\{0\}} \end{bmatrix}. \end{aligned} \tag{3.4}$$

Apply $\Pi_{(-S)\cap\Lambda}$ to the $(1,2)$ block position in (3.4) and $\Pi_{S\cap\Lambda}$ to the $(2,2)$ block position in (3.4). This yields:

$$-\beta_\varepsilon + \varepsilon \mathbf{H}(f)_\Lambda(\delta_\varepsilon) = 0, \quad -\varepsilon(\mathbf{H}(f)_\Lambda)^*(\beta_\varepsilon) + \delta_\varepsilon = I.$$

Take

$$\delta_\varepsilon = (I - \varepsilon^2 (\mathbf{H}(f)_\Lambda)^* \mathbf{H}(f)_\Lambda)^{-1}(I), \quad \beta_\varepsilon = \varepsilon \mathbf{H}(f)_\Lambda(\delta_\varepsilon).$$

Then the equation (3.3) is satisfied. We shall see later that indeed $x_\varepsilon \in \mathcal{M}_2$.

Next, we show that δ_ε is an $(\mathrm{APW}^k)^{n\times n}_{S\cap\Lambda}$-valued function of ε that depends analytically on the parameter $\varepsilon \in \overline{\mathbb{D}}$, where $\overline{\mathbb{D}} = \{z : |z| \leqslant 1\}$ is the closed unit disk.

As in Section 3 of [2] we introduce the following notation. Denote by $\mathcal{F}_{\mathrm{B}}$ the Fourier-Bohr transform, that is, the isometric isomorphism of $(B^k)^{n\times 1}_\Lambda$ onto the space $l_2^{n\times 1}(\Lambda)$ defined by the formula $(\mathcal{F}_{\mathrm{B}}f)(\lambda) = f_\lambda, \quad \lambda \in \Lambda$. Let $\mathfrak{B}\left((B^k)_\Lambda^{n\times 1}\right)$ be the C^*-algebra of all linear and bounded operators acting on the Hilbert space $(B^k)_\Lambda^{n\times 1}$, and let $\mathcal{R}(\Lambda)$ be its C^*-subalgebra generated by all operators of the form $\Psi(a) = \mathcal{F}_{\mathrm{B}}^{-1} M_a \mathcal{F}_{\mathrm{B}}$, $a \in l_\infty^{n\times n}(\Lambda)$. Here M_a is the multiplication operator with symbol a. In addition, introduce the Banach algebra $\mathcal{B}_{\mathrm{W}}(\Lambda)$ consisting of operators

$$T = \sum_{\lambda\in\Lambda} A_\lambda M_{e_\lambda I}, \tag{3.5}$$

where $A_\lambda \in \mathcal{R}(\Lambda)$ and

$$\|T\|_{\mathcal{B}_{\mathrm{W}}(\Lambda)} \stackrel{\text{def}}{=} \sum_{\lambda\in\Lambda} \|A_\lambda\| < \infty.$$

The operator $\mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda = \Pi_{S\cap\Lambda} f^* \Pi_{(-S)\cap\Lambda} f \Pi_{S\cap\Lambda}$ belongs to $\mathcal{B}_{\mathrm{W}}(\Lambda)$, because $\Pi_{(\pm S)\cap\Lambda} \in \mathcal{R}(\Lambda)$ and $f \in (\mathrm{APW}^k)_\Lambda^{m\times n}$ (so that $f^* \in (\mathrm{APW}^k)_\Lambda^{n\times m}$). Hence, $I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda \in \mathcal{B}_{\mathrm{W}}(\Lambda)$. The condition $\|\mathbf{H}(f)_\Lambda\| < 1$ implies that the operator $I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda$ is invertible for all $\varepsilon \in [0,1]$. Lemma 3.2 in [2] then implies:

$$(I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda)^{-1} = \sum_{\lambda\in\Lambda} A_\lambda M_{e_\lambda I}, \tag{3.6}$$

where $A_\lambda \in \mathcal{R}(\Lambda)$ and $\sum_{\lambda\in\Lambda} \|A_\lambda\| < \infty$.

Let $\varphi_{j,\mu} = (I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda)^{-1}(e_\mu E_j)$, where E_j is the jth column of the identity matrix I ($j = 1, \dots, n$; $\mu \in \Lambda$). Then, of course, $\varphi_{j,\mu} \in (B^k)_\Lambda^{n\times 1}$. However, more can be said: due to (3.6),

$$\varphi_{j,\mu} = \sum_{\lambda\in\Lambda} A_\lambda M_{e_{\lambda+\mu} I} E_j.$$

By the definition of $\mathcal{R}(\lambda)$, each A_λ has the form $A_\lambda = \mathcal{F}_{\mathrm{B}}^{-1} M_{a_\lambda} \mathcal{F}_{\mathrm{B}}$, where $a_\lambda \in l_\infty^{n\times n}(\Lambda)$. Hence, $A_\lambda M_{e_{(\lambda+\mu)}} E_j = \mathcal{F}_{\mathrm{B}}^{-1} M_{a_\lambda} \mathcal{F}_{\mathrm{B}} M_{e_{\lambda+\mu}} E_j = a_\lambda(\lambda+\mu) E_j e_\lambda$, which of course lies in $(\mathrm{APW}^k)_\Lambda^{n\times 1}$. Since $\|a_\lambda(\lambda+\mu)\| \leqslant \|A_\lambda\|$, $\varphi_{j,\mu}$ is represented as an absolutely convergent series with terms in $(\mathrm{APW}^k)_\Lambda^{n\times 1}$, and therefore itself belongs to $(\mathrm{APW}^k)_\Lambda^{n\times 1}$.

An arbitrary vector $g \in (\mathrm{APW}^k)_{S\cap\Lambda}^{n\times 1}$ can be represented as an absolutely convergent series $\sum_{j=1}^n \sum_{\mu_{js}\in\Lambda} c_{js} e_{\mu_{js}} E_j$. Hence, the (unique) solution ξ of the equation $(I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda)\xi = g$ in $(B^k)_{S\cap\Lambda}^{n\times 1}$ equals $\sum_{j=1}^n \sum_{\mu_{js}\in\Lambda} c_{js} \varphi_{j,\mu_{js}}$, and therefore automatically lies in $(\mathrm{APW}^k)_\Lambda^{n\times 1} \cap (B^k)_{S\cap\Lambda}^{n\times 1} = (\mathrm{APW}^k)_{S\cap\Lambda}^{n\times 1}$.

In other words, for all $\varepsilon \in \overline{\mathbb{D}}$, the operators $I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda$ are invertible on $(\mathrm{APW}^k)_{S\cap\Lambda}^{n\times 1}$. Then $(I - \varepsilon^2 \mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda)^{-1}$ is analytic in a neighborhood of $\overline{\mathbb{D}}$ as an operator valued function on $(\mathrm{APW}^k)_{S\cap\Lambda}^{n\times 1}$. In particular,

$(I-\varepsilon^2\mathbf{H}(f)^*_\Lambda\mathbf{H}(f)_\Lambda)^{-1}g$ is analytic as a function from $\overline{\mathbb{D}}$ into $(\mathrm{APW}^k)^{n\times 1}_{S\cap\Lambda}$ for any fixed $g\in(\mathrm{APW}^k)^{n\times 1}_{S\cap\Lambda}$. Since

$$\delta_\varepsilon=\left[(I-\varepsilon^2\mathbf{H}(f)^*_\Lambda\mathbf{H}(f)_\Lambda)^{-1}E_1,\dots,(I-\varepsilon^2\mathbf{H}(f)^*_\Lambda\mathbf{H}(f)_\Lambda)^{-1}E_n\right],$$

it is a matrix function in $(\mathrm{APW}^k)^{n\times n}_{S\cap\Lambda}$, depending analytically on $\varepsilon\in\overline{\mathbb{D}}$.

It follows that $\beta_\varepsilon=\varepsilon\mathbf{H}(f)_\Lambda(\delta_\varepsilon)$ is a matrix function in $(\mathrm{APW}^k)^{n\times n}_{(-S)\cap\Lambda}$, depending analytically on $\varepsilon\in\overline{\mathbb{D}}$. Note that we have established that $x_\varepsilon\in\mathcal{M}_2$.

Similarly, the solution

$$y_\varepsilon=\begin{bmatrix}\alpha_\varepsilon & 0\\ -\gamma_\varepsilon & I\end{bmatrix}\in\mathcal{M}_3$$

of the equation $P_3(y_\varepsilon k_\varepsilon)=I$ exists for all $\varepsilon\in\overline{\mathbb{D}}$, where $\alpha_\varepsilon\in(\mathrm{APW}^k)^{n\times n}_{(-S)\cap\Lambda}$ and $\gamma_\varepsilon\in(\mathrm{APW}^k)^{n\times n}_{S\cap\Lambda}$ depend analytically on the parameter ε.

Observe that $(\delta_\varepsilon)_0$ is a compression of $(I-\varepsilon^2(\mathbf{H}(f)_\Lambda)^*\mathbf{H}(f)_\Lambda)^{-1}$ onto $\mathbb{C}^{n\times n}$ (considered as a subspace of constant matrix functions in $(B^k)^{n\times n}_{S\cap\Lambda})$). For all $\varepsilon\in[0,1]$, the operator $(I-\varepsilon^2(\mathbf{H}(f)_\Lambda)^*\mathbf{H}(f)_\Lambda)^{-1}$ is positive definite. Hence, $(\delta_\varepsilon)_0$ $(\in\mathbb{C}^{n\times n})$ also is positive definite for these values of ε. Similarly, $(\alpha_\varepsilon)_0$ is positive definite for $\varepsilon\in[0,1]$.

If $\varepsilon=0$, then $I-\varepsilon^2(\mathbf{H}(f)_\Lambda)^*\mathbf{H}(f)_\Lambda$ turns into the identity operator and therefore $\delta_0=\alpha_0=I$. From the continuity of δ_ε, α_ε as functions of ε it follows that δ_ε, α_ε are invertible in $(\mathrm{APW}^k)^{n\times n}_{S\cap\Lambda}$ and $(\mathrm{APW}^k)^{n\times n}_{(-S)\cap\Lambda}$, respectively, for $\varepsilon\in[0,\sigma)$ and a sufficiently small $\sigma>0$.

For such ε we get by Theorems 1.1, 1.2 and 1.3 in Chapter XXXIV of [14] that

$$\begin{aligned}&\begin{bmatrix}I & -\beta_\varepsilon\\ 0 & \delta_\varepsilon\end{bmatrix}^{*-1}\begin{bmatrix}I & 0\\ 0 & (\delta_\varepsilon)_0\end{bmatrix}\begin{bmatrix}I & -\beta_\varepsilon\\ 0 & \delta_\varepsilon\end{bmatrix}^{-1}\\ &\qquad=\begin{bmatrix}\alpha_\varepsilon & 0\\ -\gamma_\varepsilon & I\end{bmatrix}^{*-1}\begin{bmatrix}(\alpha_\varepsilon)_0 & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\alpha_\varepsilon & 0\\ -\gamma_\varepsilon & I\end{bmatrix}^{-1}\end{aligned}\tag{3.7}$$

is the band extension of k_ε. Taking the inverses of both sides of (3.7) and computing the $(1,1)$ and $(2,2)$ block entries we get

$$\begin{aligned}\alpha_\varepsilon\left((\alpha_\varepsilon)_0\right)^{-1}(\alpha_\varepsilon)^*&=I+\beta_\varepsilon\left((\delta_\varepsilon)_0\right)^{-1}(\beta_\varepsilon)^*,\\ \delta_\varepsilon\left((\delta_\varepsilon)_0\right)^{-1}(\delta_\varepsilon)^*&=I+\gamma_\varepsilon\left((\alpha_\varepsilon)_0\right)^{-1}(\gamma_\varepsilon)^*,\quad\varepsilon\in[0,\sigma).\end{aligned}\tag{3.8}$$

Since ε in the equalities (3.8) is real, we can rewrite them as

$$\begin{aligned}\alpha_\varepsilon\left((\alpha_\varepsilon)_0\right)^{-1}(\alpha_{\overline{\varepsilon}})^*&=I+\beta_\varepsilon\left((\delta_\varepsilon)_0\right)^{-1}(\beta_{\overline{\varepsilon}})^*,\\ \delta_\varepsilon\left((\delta_\varepsilon)_0\right)^{-1}(\delta_{\overline{\varepsilon}})^*&=I+\gamma_\varepsilon\left((\alpha_\varepsilon)_0\right)^{-1}(\gamma_{\overline{\varepsilon}})^*.\end{aligned}\tag{3.9}$$

Both sides of the equalities (3.9) are analytic matrix valued functions of ε on $\{\varepsilon\in\overline{\mathbb{D}}:(\delta_\varepsilon)_0,(\alpha_\varepsilon)_0\text{ are invertible}\}$. Since the latter set contains $[0,1]$, all the expressions in (3.9) are analytic on some neighborhood U of $[0,1]$ in $\mathbb{C}$. Due to (3.8), the equalities (3.9) actually hold on $[0,\sigma)$, that is, a non-isolated subset of U. Hence, the equalities (3.9) hold on U; in particular, they hold for all $\varepsilon\in[0,1]$.

In other words, the equalities (3.8) can be extended from $[0,\sigma)$ onto $[0,1]$. The right hand sides of these equalities are positive definite, and therefore invertible in $(\mathrm{APW}^k)_\Lambda^{n\times n}$. This implies the invertibility of α_ε and δ_ε in $(\mathrm{APW}^k)_\Lambda^{n\times n}$ for all $\varepsilon\in[0,1]$. The reasoning above in this paragraph followed the proof of Theorem II.1.1 of [15].

We now use Theorem 7.1 of [28]. By this theorem, the invertibility of α_ε and δ_ε in $(\mathrm{APW}^k)_\Lambda^{n\times n}$ for all $\varepsilon\in[0,1]$, when combined with the invertibility of α_ε in $(\mathrm{APW}^k)_{(-S)\cap\Lambda}^{n\times n}$ and δ_ε in $(\mathrm{APW}^k)_{S\cap\Lambda}^{n\times n}$ for $\varepsilon\in[0,\sigma)$, imply that actually $(\alpha_\varepsilon)^{-1}\in(\mathrm{APW}^k)_{(-S)\cap\Lambda}^{n\times n}$ and $(\delta_\varepsilon)^{-1}\in(\mathrm{APW}^k)_{S\cap\Lambda}^{n\times n}$ for *all* $\varepsilon\in[0,1]$.

Conclusion: We have found a solution $x_\varepsilon\in\mathcal{M}_2$ of (3.3) and a solution $y_\varepsilon\in\mathcal{M}_3$ of $P_3(y_\varepsilon k_\varepsilon)=I$ such that $x_\varepsilon^{-1}\in\mathcal{M}_+$ and $y_\varepsilon^{-1}\in\mathcal{M}_-$, for every $\varepsilon\in[0,1]$. Now we may apply Theorems 1.1, 1.2 and 1.3 in [14] again to show that for all $\varepsilon\in[0,1]$ we have that (3.7) is the positive extension of k_ε.

Note that $\widehat{\alpha}=\alpha_1$, $\widehat{\beta}=\beta_1$, $\widehat{\gamma}=\gamma_1$ and $\widehat{\delta}=\delta_1$. This finishes the proof. ∎

Proof of Theorem 2.3. Note that from the proof of Theorem 2.1 we get that

$$\begin{bmatrix} I & g_0 \\ g_0^* & I\end{bmatrix}^{-1} = \begin{bmatrix} * & g_0(I-g_0^*g_0)^{-1} \\ * & * \end{bmatrix}^{-1} \in \mathcal{M}_2^0 \dotplus \mathcal{M}_{\mathrm{d}} \dotplus \mathcal{M}_3^0.$$

Thus $g_0(I-g_0^*g_0)^{-1}\in(\mathrm{APW}^k)_{(-S)\cap\Lambda}^{m\times n}$. Moreover, by Theorem 1.3 in Chapter XXXIV of [14], we get that for any strictly contractive extension g of f with

$$\begin{bmatrix} I & g \\ g^* & I\end{bmatrix}^{-1} \in \mathcal{M}_2^0 \dotplus \mathcal{M}_{\mathrm{d}} \dotplus \mathcal{M}_3^0,$$

we have $g=g_0$. But then the first statement follows.

For the second statement, let $\mathcal{R}=(\mathrm{AP}^k)_\Lambda^{(n+m)\times(n+m)}$. Then $\mathcal{M}$ is "an algebra with band structure (3.2) in the unital C^*-algebra $\mathcal{R}$", as defined in Section XXXIV.1 in [14]. It is straightforward to check that the decomposition (3.2) satisfies Axioms (C1) and (C2) in Chapter XXXIV, Section 4 of [14]. Applying now Theorem 4.2 in Chapter XXXIV in [14] we obtain the result stating that inequality (2.11) holds, and equality occurs if and only if $g_0=g$.

For the last statement use that $M\{\log\det(f)\}=\log\det D(f)$ (Proposition 3.1), and the fact that $\log\det$ is strictly concave on the cone of positive definite matrices (see, e.g., Section 16.F in [23]). ∎

Proof of Theorem 2.2. We take the same setup as in the proof of Theorem 2.3. Note that Axiom A in Chapter XXXIV of [14] is satisfied: in other words if $F\in\mathcal{M}_+$ is such that $\sup_{t\in\mathbb{R}^k}\|F(t)\|<1$, then $(I-F)^{-1}\in\mathcal{M}_+$ (use that $(I-F)^{-1}=I+F+F^2+\cdots$ converges in $(\mathrm{AP}^k)_{S\cap\Lambda}^{(n+m)\times(n+m)}$, and that $(\mathrm{APW}^k)_\Lambda^{(n+m)\times(n+m)}$ is inverse closed in $(\mathrm{AP}^k)_\Lambda^{(n+m)\times(n+m)}$). By Theorem 2.1 of Chapter XXXIV in [14], we get that all positive extensions of

$$\begin{bmatrix} I & f \\ f^* & I\end{bmatrix} \tag{3.10}$$

are given (in a one-to-one correspondence) by

$$
\begin{aligned}
\widehat{T}\left(\begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix}\right) &:= \left(\begin{bmatrix} I & -\beta \\ 0 & \delta \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ -\gamma & I \end{bmatrix}\begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix}\right)^{*-1} \\
&\quad \cdot \begin{bmatrix} I & 0 \\ 0 & I - g^*g \end{bmatrix}\left(\begin{bmatrix} I & -\beta \\ 0 & \delta \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ -\gamma & I \end{bmatrix}\begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix}\right)^{-1} \\
&= \begin{bmatrix} I & T(g) \\ T(g)^* & I \end{bmatrix},
\end{aligned}
\tag{3.11}
$$

where $g \in (\mathrm{CAPW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$. It should be noted that the $(2,2)$ entry of the product indeed has to equal I since this product is a positive extension of (3.10). This proves the result. ∎

4. A POINT EXCLUDING VARIATION

In this section we give a solution to the following variation of the contractive extension problem.

Fix a halfspace $S \subset \mathbb{R}^k$ and an additive subgroup Λ of $\mathbb{R}^k$. Given is an $f \in (\mathrm{APW}^k)^{m\times n}_{(-S\setminus\{0\})\cap\Lambda}$. When does there exist a *strictly contractive extension* $h \in (\mathrm{APW}^k)^{m\times n}_{\Lambda'}$ of f, where $\Lambda' \supseteq \Lambda$ is an additive subgroup or $\mathbb{R}^k$, i.e., an h such that

(1) h is strictly contractive;
(2) $h_\lambda = f_\lambda, \lambda \in (-S \setminus \{0\}) \cap \Lambda'$.

If such h exists, how do we construct one/all?

This problem has been studied in [28] for the case $k = 1$.

To give an answer to this question we need to introduce the following generalized Hankel operators:

$$\widetilde{\mathbf{H}}(f)_{\Lambda'} : (B^k)^{n\times p}_{S\cap\Lambda'} \to (B^k)^{m\times p}_{(-S\setminus\{0\})\cap\Lambda'}, \qquad g \mapsto \Pi_{(-S\setminus\{0\})}(fg), \tag{4.1}$$

$$\widetilde{\widetilde{\mathbf{H}}}(f)_{\Lambda'} : (B^k)^{n\times p}_{(S\setminus\{0\})\cap\Lambda'} \to (B^k)^{m\times p}_{-S\cap\Lambda'}, \qquad g \mapsto \Pi_{-S}(fg). \tag{4.2}$$

THEOREM 4.1. *Let $f \in (\mathrm{APW}^k)^{m\times n}_{(-S\setminus\{0\})\cap\Lambda}$ and let Λ' be a supergroup of Λ. The following statements are equivalent:*

(i) *f has a strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_{\Lambda'}$;*
(ii) *f has a strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_{\Lambda}$;*
(iii) *the generalized Hankel operator $\widetilde{\mathbf{H}}(f)_{\Lambda'}$ is a strict contraction;*
(iv) *the generalized Hankel operator $\widetilde{\widetilde{\mathbf{H}}}(f)_{\Lambda'}$ is a strict contraction;*
(v) *the generalized Hankel operator $\widetilde{\mathbf{H}}(f)_{\Lambda}$ is a strict contraction;*
(iv) *the generalized Hankel operator $\widetilde{\widetilde{\mathbf{H}}}(f)_{\Lambda}$ is a strict contraction.*

When one (and thus all) of (i)–(vi) *is satisfied, then put*

$$\begin{aligned}
\widehat{\alpha}(t) &= \big[I - \widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda(\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda)^*\big]^{-1}(I_m),\\
\widehat{\beta}(t) &= \widetilde{\mathbf{H}}(f)_\Lambda\big[I - (\widetilde{\mathbf{H}}(f)_\Lambda)^*\widetilde{\mathbf{H}}(f)_\Lambda\big]^{-1}(I_n),\\
\widehat{\gamma}(t) &= (\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda)^*\big[I - (\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda(\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda)^*\big]^{-1}(I_m),\\
\widehat{\delta}(t) &= \big[I - (\widetilde{\mathbf{H}}(f)_\Lambda)^*\widetilde{\mathbf{H}}(f)_\Lambda)\big]^{-1}(I_n),
\end{aligned}$$

where I_r stands for the constant matrix function on $\mathbb{R}^k$ with value I_r for all $t \in \mathbb{R}^k$. Further, let

$$\alpha(t) = \widehat{\alpha}(t)M\{\widehat{\alpha}\}^{-\frac{1}{2}}, \quad \beta(t) = \widehat{\beta}(t)M\{\widehat{\delta}\}^{-\frac{1}{2}};$$
$$\gamma(t) = \widehat{\gamma}(t)M\{\widehat{\alpha}\}^{-\frac{1}{2}}, \quad \delta(t) = \widehat{\delta}(t)M\{\widehat{\delta}\}^{-\frac{1}{2}}.$$

Then the function

$$g_0(t) = \beta(t)\delta(t)^{-1} = [\alpha(t)^*]^{-1}\gamma(t)^*, \quad t \in \mathbb{R}^k$$

is a strictly contractive extension in $(\mathrm{APW}^k)^{m\times n}_\Lambda$ of f.

Proof. Let

$$\mathcal{M} = (\mathrm{APW}^k)^{(m+n)\times(m+n)}_\Lambda = \mathcal{M}_1 \dotplus \mathcal{M}_2^0 \dotplus \mathcal{M}_\mathrm{d} \dotplus \mathcal{M}_3^0 \dotplus \mathcal{M}_4, \tag{4.3}$$

where

$$\mathcal{M}_1 = \begin{bmatrix} 0 & (\mathrm{APW}^k)^{m\times n}_{S\cap\Lambda} \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_2^0 = \begin{bmatrix} (\mathrm{APW}^k)^{m\times m}_{(S\setminus\{0\})\cap\Lambda} & (\mathrm{APW}^k)^{m\times n}_{(-S\setminus\{0\})\cap\Lambda} \\ 0 & (\mathrm{APW}^k)^{n\times n}_{(S\setminus\{0\})\cap\Lambda} \end{bmatrix},$$

$$\mathcal{M}_\mathrm{d} = \begin{bmatrix} (\mathrm{APW}^k)^{m\times m}_{\{0\}} & 0 \\ 0 & (\mathrm{APW}^k)^{n\times n}_{\{0\}} \end{bmatrix}, \quad \mathcal{M}_3^0 = (\mathcal{M}_2^0)^*, \quad \mathcal{M}_4 = (\mathcal{M}_1)^*.$$

Then $\mathcal{M}$ is an algebra with band structure (4.3) as defined in Chapter XXXIV, Section 1 in [14]. One may now proceed as in the proof of Theorem 2.1 with straightforward modifications. These include using in the formulas for δ and β the operator $\widetilde{\mathbf{H}}(f)_\Lambda$ instead of $\mathbf{H}(f)_\Lambda$, and in the formulas for α and γ the operator $\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda$ instead of $\mathbf{H}(f)_\Lambda$. The arguments remain exactly the same. ∎

To obtain valid analogues of Theorems 2.2 and 2.3 for the point excluding version of the strictly contractive extension problem considered in this section the following changes need to be made.

(1) Replace $(\mathrm{APW}^k)_{(-S)\cap\Lambda}$ by $(\mathrm{APW}^k)_{(-S\setminus\{0\})\cap\Lambda}$.
(2) Replace the reference to Theorem 2.1 by a reference to Theorem 4.1.
(3) Replace $\mathbf{H}(f)_\Lambda$ by $\widetilde{\mathbf{H}}(f)_\Lambda$.
(4) Replace in Theorem 2.2 $(\mathrm{CAPW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ by $(\mathrm{CAPW}^k)^{m\times n}_{S\cap\Lambda}$.

In the proofs one uses decomposition (4.3) of the proof of Theorem 4.1. The remaining arguments are exactly analogous as in the proofs of Theorems 2.2 and 2.3.

5. NON-STRICTLY CONTRACTIVE EXTENSIONS

The existence part of the main results in Section 2 can be extended also to the case of contractive extensions that are not necessarily strict. However, such extensions may belong to a space larger than $(\mathrm{APW}^k)_\Lambda^{m\times n}$.

To set up the appropriate framework, we recall several basic facts concerning Bohr compactifications and Besikovitch spaces (see, e.g., Chapter 1 in [26]). The discrete abelian group $\mathbb{R}^k$ has a *Bohr compactification* $\mathbb{R}^k_{\mathrm{B}}$, i.e., $\mathbb{R}^k_{\mathrm{B}}$ is a compact abelian group which contains an isomorphic image of $\mathbb{R}^k$ as a dense subgroup. The space (AP^k) is characterized as the set of those bounded, continuous (in the standard topology) functions on $\mathbb{R}^k$ which extend continuously to $\mathbb{R}^k_{\mathrm{B}}$ (see Theorem 1.2 of Chapter 1 in [26]). As is the case for any compact, abelian group, $\mathbb{R}^k_{\mathrm{B}}$ has a unique invariant measure μ satisfying the normalization condition $\mu(\mathbb{R}^k_{\mathrm{B}}) = 1$, i.e., μ is a unique normalized Borel measure on $\mathbb{R}^k_{\mathrm{B}}$ such that $\mu(s+E) = \mu(E)$ for each $s \in \mathbb{R}^k_{\mathrm{B}}$ and each μ-measurable set E. Moreover, it turns out that the mean $M\{f\}$ of an $f \in (\mathrm{AP}^k)$ can alternatively be computed as

$$M\{f\} = \int\limits_{\mathbb{R}^k_{\mathrm{B}}} \widetilde{f}(x)\,\mathrm{d}\mu(x)$$

where $\widetilde{f}$ is the unique continuous extension of f to $\mathbb{R}^k_{\mathrm{B}}$. The Besikovitch space (B^k) then can be viewed alternatively as the L^2-space $(B^k) = L^2(\mathbb{R}^k_{\mathrm{B}}, \mu)$. For $\Delta \subseteq \mathbb{R}^k$, the set $(B^k)_\Delta$ is defined as the closed subspace of (B^k) consisting of those f in $L^2(\mathbb{R}^k_{\mathrm{B}}, \mu)$ for which $\int\limits_{\mathbb{R}^k_{\mathrm{B}}} f(t)\mathrm{e}^{-\mathrm{i}\langle\lambda,t\rangle}\,\mathrm{d}\mu(t) = 0$ for all $\lambda \in \mathbb{R}^k \setminus \Delta$. If Δ is a subgroup, then $(B^k)_\Delta$ can be alternatively viewed as $L^2(\Delta_{\mathrm{B}}, \mu_\Delta)$, where Δ_{B} is the Bohr compactification of the discrete Abelian group Δ, with the corresponding normalized invariant measure μ_Δ. E.g., when $\Delta = \mathbb{Z}^k$ then $\Delta_{\mathrm{B}} = \mathbb{T}^k$ and $(B^k)_{\mathbb{Z}^k}$ may be identified with $L^2(\mathbb{T}^k)$, the Lebesgue space of square integrable functions on the k-torus.

We shall also need the L^∞ and L^1 versions of the Besikovitch space:

$$(B^k_\infty) = \{f : \mathbb{R}_{\mathrm{B}} \to \mathbb{C} : f \text{ is } \mu\text{-measurable}, \|f\|_{(B^k_\infty)} = \operatorname*{ess\,sup}_{t\in\mathbb{R}_{\mathrm{B}}} |f(t)| < \infty\},$$

and

$$(B^k_1) = L^1(\mathbb{R}^k_{\mathrm{B}}, \mu).$$

For Δ a (non-empty) subset of $\mathbb{R}^k$, the closed subspace $(B^k_\infty)_\Delta$ of (B^k_∞) consists of those functions $f \in (B^k_\infty)$ for which $M\{f\mathrm{e}^{-\mathrm{i}\langle\lambda,\cdot\rangle}\} = 0$ for all $\lambda \in \mathbb{R}^k \setminus \Delta$. Analogously the closed subspace $(B^k_1)_\Delta$ of (B^k_1) is defined. Note that $(B^k_\infty)_{\mathbb{Z}^k}$ may be identified with $L^\infty(\mathbb{T}^k)$, the Lebesgue space of essentially bounded functions on the k-torus.

THEOREM 5.1. *Let $f \in (\mathrm{AP}^k)^{m\times n}_{(-S)\cap\Lambda}$ be given, and let Λ' be a supergroup of Λ. The following statements are equivalent:*

(i) *f has a contractive extension in $(B^k_\infty)^{m\times n}_{\Lambda'}$, that is, there exists $h \in (B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'}$ such that $\|f+h\|_{(B^k_\infty)^{m\times n}_{\Lambda'}} \leqslant 1$;*

(ii) *f has a contractive extension in $(B^k_\infty)^{m\times n}_\Lambda$;*

(iii) *the generalized Hankel operator $\mathbf{H}(f)_{\Lambda'}$ given by* (2.3) *is a contraction, i.e., $\|\mathbf{H}(f)_{\Lambda'}\| \leqslant 1$;*

(iv) *the generalized Hankel operator $\mathbf{H}(f)_\Lambda$ is a contraction.*

Proof. We start with the easier parts of the proof. The implication (ii) $\Rightarrow$ (i) is trivial. To prove (i) $\Rightarrow$ (iii), let $h \in (B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'}$ be such that $f+h$ is contractive (in the $(B^k_\infty)^{m\times n}_{\Lambda'}$ norm). From the definition of the Hankel operators it follows that $\mathbf{H}(f)_{\Lambda'} = \mathbf{H}(f+h)_{\Lambda'}$. Now

$$\|\mathbf{H}(f)_{\Lambda'}\| = \|\mathbf{H}(f+h)_{\Lambda'}\| \leqslant \|f+h\|_{(B^k_\infty)^{m\times n}_{\Lambda'}} \leqslant 1,$$

where the first inequality holds because $\mathbf{H}(f+h)_{\Lambda'}$ is a compression of the operator of multiplication by $f+h$ whose norm is equal to $\|f+h\|_{(B^k_\infty)^{m\times n}_{\Lambda'}}$. This proves (iii). Analogously, (ii) $\Rightarrow$ (iv) is proved.

For the rest of the proof, we follow the approach used in the proof of Theorem 4.9 of [2]. An appropriate weak-$*$ topology on $(B^k_\infty)^{m\times n}_{\Lambda'}$ is introduced with respect to which the unit ball of $(B^k_\infty)^{m\times n}_{\Lambda'}$ is compact. In general, it is known that if X is a Banach space with the dual space X^*, and Y is a closed subspace of X, then

$$Y^\perp = \{x^* \in X^* : \langle y, x^*\rangle = 0 \text{ for all } y \in Y\}$$

(the *annihilator* of Y) is a closed subspace of X^*, and is itself the Banach space dual of the quotient Banach space X/Y:

$$Y^\perp = (X/Y)^* \tag{5.1}$$

(see e.g. p. 133 of [13], or Section 5.2 of [5]). We apply these facts with $X = (B^k_1)^{n\times m}$, $X^* = (B^k_\infty)^{m\times n}$, the pairing

$$\langle y, x^*\rangle = \int\limits_{\mathbb{R}^k_{\mathrm{B}}} \operatorname{trace}\{y(t)x^*(t)\}\,\mathrm{d}\mu(t), \tag{5.2}$$

and $Y = (B^k_1)^{n\times m}_{\Lambda'}$. Then $Y^\perp$, under the pairing given by (5.2), can be easily seen to be $Y^\perp = (B^k_\infty)^{m\times n}_{\Lambda'}$, and hence, by the general principle (5.1) we have

$$(B^k_\infty)^{m\times n}_{\Lambda'} = \left((B^k_1)^{n\times m}/(B^k_1)^{n\times m}_{\Lambda'}\right)^*.$$

As is the case for any dual Banach space, the unit ball of $(B^k_\infty)^{m\times n}_{\Lambda'}$ is compact in the weak-$*$ topology.

We now prove (iii) $\Rightarrow$ (i). Let $\{f^{(q)}\}_{q=1}^\infty$ be a sequence of functions $f^{(q)} \in (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda'}$ such that $f^{(q)} \to f$ as $q \to \infty$ in the $(\mathrm{AP}^k)^{m\times n}$ norm. (We mention in passing that for $g \in (\mathrm{AP}^k)^{m\times n}$ the $(\mathrm{AP}^k)^{m\times n}$ and $(B^k_\infty)^{m\times n}$ norms of g coincide.) Then

$$\|\mathbf{H}(f^{(q)})_{\Lambda'}\| \to \|\mathbf{H}(f)_{\Lambda'}\| \leqslant 1,$$

and by scaling $f^{(q)}$, if necessary, we can assume that $\|\mathbf{H}(f^{(q)})_{\Lambda'}\| < 1$ for all q. By Theorem 2.1, there exist $h^{(q)} \in (\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda'}$ such that $\|f^{(q)} +$

$h^{(q)}\|_{(B_\infty^k)^{m\times n}} < 1$. By the compactness (see the preceding paragraph), some subnet $\{F_\alpha = f_\alpha + h_\alpha\}$ of the sequence $\{F_q = f^{(q)} + h^{(q)}\}_{q=1}^\infty$ converges in the weak-$*$ topology to an element F of $(B_\infty^k)_{\Lambda'}^{m\times n}$. Since f_α converges (in the $(B_\infty^k)_{\Lambda'}^{m\times n}$ norm, and hence also in the weak-$*$ topology) to f, we have that h_α converges to an element $h \in (B_\infty^k)_{\Lambda'}^{m\times n}$. Note that $(B_\infty^k)_{(S\setminus\{0\})\cap\Lambda'}^{m\times n}$ is closed in the weak-$*$ topology (this follows from the condition $M\{y\mathrm{e}^{-\mathrm{i}\langle\lambda,\cdot\rangle}\} = 0$ for every $\lambda \notin (S\setminus\{0\})\cap\Lambda'$, which describes the inclusion $y \in (B_\infty^k)_{(S\setminus\{0\})\cap\Lambda'}$, and taking into account that $\mathrm{e}^{-\mathrm{i}\langle\lambda,\cdot\rangle} \in L^1(\mathbb{R}_{\mathrm{B}}^k, \mu)$). Consequently, $h \in (B_\infty^k)_{(S\setminus\{0\})\cap\Lambda'}^{m\times n}$. Finally, we use the (easily proved) semicontinuity of norm in weak-$*$ topology: If a net $\{x_\alpha\}$ in a dual Banach space X^* weakly-$*$ converges to $x \in X^*$, then $\liminf_\alpha \|x_\alpha\| \geqslant \|x\|$. Applying this fact with $x_\alpha = f_\alpha + h_\alpha$ and $X^* = (B_\infty^k)^{m\times n}$, we conclude that $\|f + h\|_{(B_\infty^k)^{m\times n}} \leqslant 1$. This proves (i).

The implication (iv) $\Rightarrow$ (ii) is proved similarly. ∎

Analogously the existence part of the point excluding variation (Theorem 4.1) can be extended to non-strictly contractive extensions. We state only the result, omitting a proof.

THEOREM 5.2. *Let $f \in (\mathrm{AP}^k)_{(-S\setminus\{0\})\cap\Lambda}^{m\times n}$ be given, and let Λ' be a supergroup of Λ. The following statements are equivalent:*

(i) *f has a contractive extension in $(B_\infty^k)_{\Lambda'}^{m\times n}$;*
(ii) *f has a contractive extension in $(B_\infty^k)_{\Lambda}^{m\times n}$;*
(iii) *the generalized Hankel operator $\widetilde{\mathbf{H}}(f)_{\Lambda'}$ given by (4.1) is a contraction;*
(iv) *the generalized Hankel operator $\widetilde{\mathbf{H}}(f)_{\Lambda}$ is a contraction;*
(v) *the generalized Hankel operator $\widetilde{\widetilde{\mathbf{H}}}(f)_{\Lambda'}$ given by (4.2) is a contraction;*
(vi) *the generalized Hankel operator $\widetilde{\widetilde{\mathbf{H}}}(f)_{\Lambda}$ is a contraction.*

6. DISTANCE FORMULAS, SARASON'S THEOREM AND COMMUTANT LIFTING: THE MULTIVARIABLE CASE

We interpret some of the results in the previous section as distance formulas. For $f \in (B_\infty^k)_{\Lambda}^{m\times n}$ we define $\mathbf{H}(f)$, $\widetilde{\mathbf{H}}(f)$ and $\widetilde{\widetilde{\mathbf{H}}}(f)$ by the same formulas as in the beginning of Sections 2 and 4. If X is a Banach space with a closed subspace Ω, we denote by $\mathrm{dist}(f, \Omega)$ the distance from $f \in X$ to Ω. Note that in general Banach space context the distance is not always attained, i.e., there need not exist $\omega \in \Omega$ such that $\mathrm{dist}(f, \Omega) = \|f - \omega\|$. However, if X is a dual Banach space and Ω is closed in the weak-$*$ topology, then $\mathrm{dist}(f, \Omega)$ is always attained. This fact can be easily verified using compactness of the unit ball in X in the weak-$*$ topology.

THEOREM 6.1. *Let $f \in (\mathrm{AP}^k)_{\Lambda}^{m\times n}$. Then*

$$\mathrm{dist}(f, (\mathrm{AP}^k)_{(S\setminus\{0\})\cap\Lambda}^{m\times n}) = \mathrm{dist}(f, (B_\infty^k)_{(S\setminus\{0\})\cap\Lambda}^{m\times n}) = \|\mathbf{H}(f)\|, \tag{6.1}$$

and

$$\mathrm{dist}(f, (\mathrm{AP}^k)_{S\cap\Lambda}^{m\times n}) = \mathrm{dist}(f, (B_\infty^k)_{S\cap\Lambda}^{m\times n}) = \|\widetilde{\mathbf{H}}(f)\| = \|\widetilde{\widetilde{\mathbf{H}}}(f)\|. \tag{6.2}$$

The distances $\operatorname{dist}(f,(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})$ *and* $\operatorname{dist}(f,(B^k_\infty)^{m\times n}_{S\cap\Lambda})$ *are attained.*

Proof. We shall prove (6.1). The equalities (6.2) follow in an analogous manner. Let $f\in(\mathrm{APW}^k)^{m\times n}_\Lambda$ and $\varepsilon>0$. Theorem 2.1 shows that there exists a $g\in(\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ so that $\|f-g\|_\infty<\|\mathbf{H}(f)\|+\varepsilon$. This shows that

$$\operatorname{dist}(f,(\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})\leqslant\|\mathbf{H}(f)\|.$$

Next, suppose that $h\in(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$. Then

$$\|f-h\|_{B^k_\infty}=\|M_{f-h}\|\geqslant\|\mathbf{H}(f-h)\|=\|\mathbf{H}(f)\|,$$

where $M_{f-h}:(B^k)^n_\Lambda\to(B^k)^m_\Lambda$ is the multiplication operator with the symbol $f-h$. But then it follows that $\operatorname{dist}(f,(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})\geqslant\|\mathbf{H}(f)\|$. Thus

$$\begin{aligned}\|\mathbf{H}(f)\|&\leqslant\operatorname{dist}(f,(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})\leqslant\operatorname{dist}(f,(\mathrm{AP}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})\\&\leqslant\operatorname{dist}(f,(\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})\leqslant\|\mathbf{H}(f)\|,\end{aligned}$$

yielding (6.1) when $f\in(\mathrm{APW}^k)^{m\times n}_\Lambda$. Since elements of $(\mathrm{AP}^k)^{m\times n}_\Lambda$ may be approximated in the $\|\cdot\|_\infty$ norm by elements in $(\mathrm{APW}^k)^{m\times n}_\Lambda$, the result also follows for $f\in(\mathrm{AP}^k)^{m\times n}_\Lambda$.

Finally, observe that $(B^k_\infty)^{m\times n}$ is a dual Banach space, and its subspaces $(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ and $(B^k_\infty)^{m\times n}_{S\cap\Lambda}$ are closed in the weak-$*$ topology (see the proof of Theorem 5.1). In view of the observation made before Theorem 6.1, the distances to $(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ and to $(B^k_\infty)^{m\times n}_{S\cap\Lambda}$ are attained. ∎

Using the above distance results we obtain the following generalization of the well-known Sarason's theorem ([31]).

THEOREM 6.2. (i) $(\mathrm{AP}^k)^{m\times n}_\Lambda+(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ *is closed in* $(B^k_\infty)^{m\times n}_\Lambda$.
(ii) $(\mathrm{AP}^k)^{m\times n}_\Lambda+(B^k_\infty)^{m\times n}_{S\cap\Lambda}$ *is closed in* $(B^k_\infty)^{m\times n}_\Lambda$.

Proof. We use a familiar type of argument (see [32]). Consider the map

$$i:(\mathrm{AP}^k)^{m\times n}_\Lambda/(\mathrm{AP}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}\to(B^k_\infty)^{m\times n}_\Lambda/(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda},$$

defined by

$$i(f+(\mathrm{AP}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda})=f+(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}.$$

It follows from Theorem 6.1 that this is an isometry, and thus its range is closed. Letting

$$\pi:(B^k_\infty)^{m\times n}_\Lambda\to(B^k_\infty)^{m\times n}_\Lambda/(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$$

denote the canonical projection, we get that

$$\pi^{-1}(i((\mathrm{AP}^k)^{m\times n}_\Lambda/(\mathrm{AP}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}))$$

is closed (using the continuity of π). Since this preimage equals $(\mathrm{AP}^k)^{m\times n}_\Lambda+(B^k_\infty)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ the first part of the theorem is proven.

The second part follows analogously. Alternatively, use the fact that the spaces in (i) and (ii) differ by a finite dimensional subspace, and hence they are closed only simultaneously. ∎

When we specify the above theorem for the case $\Lambda = \mathbb{Z}^k$, we get the following. We let $(L_\infty(\mathbb{T}^k))^{m\times n}_\Delta$ denote the Lebesgue space of essentially bounded $m \times n$ matrix valued functions on the k-torus that have Fourier spectrum in Δ. The space $(C(\mathbb{T}^k))^{m\times n}$ consists of continuous $m \times n$ matrix valued functions on the k-torus.

COROLLARY 6.3. *The space $(L_\infty(\mathbb{T}^k))^{m\times n}_S + (C(\mathbb{T}^k))^{m\times n}$ is closed in $(L_\infty(\mathbb{T}^k))^{m\times n}$, where S is a halfspace in $\mathbb{Z}^k$.*

The original Sarason's theorem ([31]) asserts that $H_\infty(\mathbb{T}) + C(\mathbb{T})$ is closed in $L_\infty(\mathbb{T})$. Since then, it has been extended in many ways. We mention here [27] and [18], where versions of almost periodic Sarason's theorem in one variable (different from Theorem 6.2) are given. The paper [30] contains a general result on closedness of the sum of two closed subspaces in a Banach space; it has been used to reprove the original Sarason's theorem and obtain several of its generalizations.

It is instructive to explore the Commutant Lifting approach in our context. When $\Lambda = \mathbb{Z}^k$ and $S = E_k$ one may use an adjusted commutant lifting approach as follows. We start off with a commutant lifting theorem where the lifting is required to commute with additional unitary operators. This result will be used to establish a distance result in a multivariable setting. We shall use the terminology and notation of [11]. For a Hilbert space K with a subspace H (all subspaces are assumed closed), we denote the orthogonal projection onto H by P_H. An operator $B : K \to K'$ is called a *lifting* of $A : H \to H'$ if $H \subseteq K$, $H' \subseteq K'$, and $AP_H = P_{H'}B$; in other words, with respect to the orthogonal decompositions $H \oplus (K \ominus H)$, $H' \oplus (K' \ominus H')$, the operator B has the form $B = \begin{bmatrix} A & 0 \\ * & * \end{bmatrix}$. Recall that an invariant subspace H_1 for an operator A is called *reducing*, if H_1 is invariant for A^* as well.

THEOREM 6.4. *Let $A : H \to H'$, $T : H \to H$ and $T' : H' \to H'$ be contractive Hilbert space operators that satisfy the intertwining relation $AT = T'A$. Let $U : K \to K$ and $U' : K' \to K'$ be minimal isometric dilations of T and T', respectively (i.e., U and U' are isometries, $T^n = P_H U^n|H$, $K = \bigvee_0^\infty U^n[H]$, and ${T'}^n = P_{H'}{U'}^n|H'$, $K' = \bigvee_0^\infty {U'}^n[H']$ for all $n = 0, 1, \ldots$). Further, let $W_i : K \to K$ and $W_i' : K' \to K'$, $i \in J$, be unitary operators that satisfy $W_iU = UW_i$, $W_i'U' = U'W_i'$, $i \in J$, and that have H and H' as a reducing invariant subspace, respectively. Moreover, assume that $V_i'A = AV_i$, $V_iT = TV_i$ and $V_i'T' = T'V_i'$ for all $i \in J$, where V_i and V_i' are the restrictions $V_i = P_HW_i|H$, $V_i' = P_{H'}W_i'|H'$. Then there exists a contractive lifting $B : K \to K'$ of A that satisfies $U'B = BU$ and $W_i'B = BW_i$ for all $i \in J$.*

Proof. We shall follow the proof of the classical commutant lifting theorem as it is presented in Section VII.3 in [11], and show that the lifting may be chosen so that the additional intertwining relations $W_i'B = BW_i$, $i \in J$, are satisfied.

$$
\begin{tikzcd}
\boxed{K} \arrow[d, "{U, W_i}"'] \arrow[rrrrr, bend left, "B"] & \supseteq & \boxed{H} \arrow[r, "A"] \arrow[d, "{T, V_i}"] & \boxed{H'} \arrow[d, "{T', V_i'}"'] & \subseteq & \boxed{K'} \arrow[d, "{U', W_i'}"] \\
\boxed{K} \arrow[rrrrr, bend right, "B"'] & \supseteq & \boxed{H} \arrow[r, "A"'] & \boxed{H'} & \subseteq & \boxed{K'}
\end{tikzcd}
$$

Figure 1. Spaces and operators in Theorem 6.4.

First observe that since minimal isometric dilations are unique up to an isomorphism, we may assume that U and U' are the standard minimal isometric dilations of T and T' as defined in (3.4) of Section VI.3 in [11]. Thus, for example,

$$
U = \begin{bmatrix} T & 0 & 0 & 0 & 0 & \cdots \\ D_T & 0 & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} : H \oplus \ell^2(\overline{\operatorname{Im} D_T}) \to H \oplus \ell^2(\overline{\operatorname{Im} D_T}),
$$

where $D_T = (I - T^*T)^{1/2}$ and $\ell^2(M)$ is the Hilbert space consisting of sequences $(m_i)_{i=0}^\infty$, $m_i \in M$, that are square summable in norm. Analogous formula holds for U'. Consequently, $K = H \oplus \ell^2(\overline{\operatorname{Im} D_T})$ and $K' = H' \oplus \ell^2(\overline{\operatorname{Im} D_{T'}})$. We shall first show that $H \oplus (\overline{\operatorname{Im} D_T})^p$, $p = 0, 1, \ldots$, is a reducing invariant subspace for W_i. Here $H \oplus (\overline{\operatorname{Im} D_T})^p$ stands for the orthogonal sum of H and p copies of $\overline{\operatorname{Im} D_T}$, considered as a subspace of K.

Indeed, for $p = 0$, H is a reducing subspace for W_i by assumption. Thus

$$
W_i = \begin{bmatrix} V_i & 0 \\ 0 & (W_{k\ell}^{(i)})_{k,\ell=1}^\infty \end{bmatrix} : H \oplus \ell^2(\overline{\operatorname{Im} D_T}) \to H \oplus \ell^2(\overline{\operatorname{Im} D_T}).
$$

Writing out the equation $UW_i = W_iU$, one sees directly that $W_{k\ell}^{(i)} = 0$ for $k \neq \ell$, $D_T V_i = W_{11}^{(i)}$, and, for a fixed i,

$$
W_{kk}^{(i)} = W_{\ell\ell}^{(i)}, \quad k, \ell = 1, 2, \ldots. \tag{6.3}
$$

Moreover, it also follows immediately from the equation $UW_i = W_iU$ that the restriction $W_i^{(p)} := W_i|H \oplus (\overline{\operatorname{Im} D_T})^p$ commutes with

$$T^{(p)} := \begin{bmatrix} T & 0 & 0 & \cdots & 0 & 0 \\ D_T & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} : H \oplus (\overline{\operatorname{Im} D_T})^p \to H \oplus (\overline{\operatorname{Im} D_T})^p.$$

Similarly, the restriction

$$W_i'^{(p)} := W_i'|H' \oplus (\overline{\operatorname{Im} D_{T'}})^p$$

satisfies $T'^{(p)}W_i'^{(p)} = W_i'^{(p)}T'^{(p)}$, $i \in J$, where

$$T'^{(p)} = \begin{bmatrix} T' & 0 & 0 & \cdots & 0 & 0 \\ D_{T'} & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix} : H' \oplus (\overline{\operatorname{Im} D_{T'}})^p \to H \oplus (\overline{\operatorname{Im} D_{T'}})^p.$$

Next we will show by induction on p that we may choose contractive p-step intertwining liftings A_p of A that satisfy in addition $W_i'^{(p)}A_p = A_pW_i^{(p)}$, $i \in J$. Recall from Section VII.3 in [11] that A_p is a contractive p-step intertwining lifting of A if $T'^{(p)}A_p = A_pT^{(p)}$ and A_p is a contractive lifting of A. For $p = 0$ we have $A_0 = A$. Suppose that $A_p : H \oplus (\overline{\operatorname{Im} D_T})^p \to H' \oplus (\overline{\operatorname{Im} D_{T'}})^p$ is a contractive p-step intertwining lifting of A that satisfies in addition $W_i'^{(p)}A_p = A_pW_i^{(p)}$, $i \in J$. We follow the construction of a contractive $p+1$ step intertwining lifting of A as is explained in Section V.1 in [11].

Let

$$\begin{aligned} F_p &= \overline{\{D_{A_p}T^{(p)}h \oplus D_{T^{(p)}}h : h \in H \oplus (\overline{\operatorname{Im} D_T})^p\}} \\ &\subseteq \left(H \oplus (\overline{\operatorname{Im} D_T})^p\right) \oplus \left(H \oplus (\overline{\operatorname{Im} D_T})^p\right), \end{aligned}$$

and define $\Gamma_p : F_p \to \overline{\operatorname{Im} D_{T'}}$ via

$$\Gamma_p(D_{A_p}T^{(p)}h \oplus D_{T^{(p)}}h) = \begin{bmatrix} 0 \ 0 \ \cdots 0 \ I_{\overline{\operatorname{Im} D_{T'}}} \end{bmatrix} D_{T'^{(p)}}A_ph, \quad h \in H \oplus (\overline{\operatorname{Im} D_T})^p.$$

A standard argument (see, e.g., Section V.1 of [11]) shows that Γ_p is a well-defined contraction.

Next, we introduce the operators $Q_p : \overline{\operatorname{Im} D_T} \to H \oplus (\overline{\operatorname{Im} D_T})^p$ by

$$Q_py = 0_H \oplus 0_{\overline{\operatorname{Im} D_T}} \oplus \cdots \oplus 0_{\overline{\operatorname{Im} D_T}} \oplus y, \quad y \in \overline{\operatorname{Im} D_T}$$

if $p > 0$, and $Q_0y = y$ for $p = 0$. Introduce also $E_p : H \oplus (\overline{\operatorname{Im} D_T})^{p+1} \to H \oplus (\overline{\operatorname{Im} D_T})^p$ defined by $E_p(x \oplus y) = x$, where $x \in H \oplus (\overline{\operatorname{Im} D_T})^p$, $y \in \overline{\operatorname{Im} D_T}$. Finally, put

$$A_{p+1} := \begin{bmatrix} A_pE_p \\ \Gamma_pP_{F_p}(D_{A_p} \oplus Q_p) \end{bmatrix} : H \oplus (\overline{\operatorname{Im} D_T})^{p+1} \to H' \oplus (\overline{\operatorname{Im} D_{T'}})^{p+1},$$

where P_{F_p} is the orthogonal projection on F_p. Since A_p is a lifting of A, so is A_{p+1}.

We have to prove that A_{p+1} is a contraction, that $T'^{(p+1)}A_{p+1} = A_{p+1}T^{(p+1)}$, and that

$$W_i'^{(p+1)}A_{p+1} = A_{p+1}W_i^{(p+1)}, \quad i \in J. \tag{6.4}$$

Consider the contraction property of A_{p+1} first. If $p = 0$, a standard result (see, e.g., Theorem V.1.2 in [11]) shows that A_1 is contractive. We give details for the case $p > 0$. For $(x, y) \in \big(H \oplus (\overline{\operatorname{Im} D_T})^p\big) \oplus \overline{\operatorname{Im} D_T}$ we have that

$$\begin{aligned}
&\langle A_{p+1}(x, y), A_{p+1}(x, y)\rangle \\
&\quad = \langle A_p x, A_p x\rangle + \langle \Gamma_p P_{F_p}(D_{A_p} \oplus Q_p)(x, y), \Gamma_p P_{F_p}(D_{A_p} \oplus Q_p)(x, y)\rangle \\
&\quad = \langle A_p x, A_p x\rangle + \langle \Gamma_p P_{F_p}(D_{A_p}x \oplus y_0), \Gamma_p P_{F_p}(D_{A_p}x \oplus y_0)\rangle,
\end{aligned} \tag{6.5}$$

where we denote $y_0 = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus y$. Fix a positive $\varepsilon \leqslant \|D_{A_p}x \oplus y_0\|$ (if $\|D_{A_p}x \oplus y_0\| = 0$, everything is trivial, and we leave this case aside), and let $h \in H \oplus (\overline{\operatorname{Im} D_T})^p$ be such that $\|P_{F_p}(D_{A_p}x \oplus y_0) - (D_{A_p}T^{(p)}h \oplus D_{T^{(p)}}h)\| < \varepsilon$. Then (6.5) can be rewritten as (we use the contractiveness of Γ_p in the first inequality)

$$\begin{aligned}
&\langle A_{p+1}(x, y), A_{p+1}(x, y)\rangle \\
&\quad \leqslant \langle A_p x, A_p x\rangle + \langle \Gamma_p(D_{A_p}h \oplus D_{T^{(p)}}h), \Gamma_p(D_{A_p}h \oplus D_{T^{(p)}}h)\rangle \\
&\qquad + 2\varepsilon\|D_{A_p}x \oplus y_0\| + \varepsilon^2 \\
&\quad = \langle A_p x, A_p x\rangle + \langle [0 \ \cdots \ 0 \ I]D_{T'^{(p)}}A_p h, [0 \ \cdots \ 0 \ I]D_{T'^{(p)}}A_p h\rangle \\
&\qquad + 2\varepsilon\|D_{A_p}x \oplus y_0\| + \varepsilon^2.
\end{aligned} \tag{6.6}$$

On the other hand, since P_{F_p} is an orthogonal projection, we have

$$\begin{aligned}
\|D_{A_p}T^{(p)}h \oplus D_{T^{(p)}}h\|^2 &\leqslant \varepsilon^2 + 2\varepsilon\|D_{A_p}x \oplus y_0\| + \|P_{F_p}(D_{A_p}x \oplus y_0)\|^2 \\
&\leqslant \varepsilon^2 + 2\varepsilon\|D_{A_p}x \oplus y_0\| + \|D_{A_p}x \oplus y_0)\|^2,
\end{aligned}$$

or, letting $\delta = \varepsilon^2 + 2\varepsilon\|D_{A_p}x \oplus y_0\|$,

$$\langle D_{A_p}T^{(p)}h, D_{A_p}T^{(p)}h\rangle + \langle D_{T^{(p)}}h, D_{T^{(p)}}h\rangle \leqslant \delta + \langle D_{A_p}x, D_{A_p}x\rangle + \langle y_0, y_0\rangle.$$

This inequality can be rewritten in the form (using $A_pT^{(p)} = T'^{(p)}A_p$)

$$\langle h, h\rangle - \langle T'^{(p)*}T'^{(p)}A_p h, A_p h\rangle \leqslant \delta + \langle D_{A_p}x, D_{A_p}x\rangle + \langle y_0, y_0\rangle.$$

Combining with (6.6), we obtain:

$$\begin{aligned}
\langle A_{p+1}(x, y), A_{p+1}(x, y)\rangle &\leqslant \delta + \langle A_p x, A_p x\rangle + \langle D_{T'^{(p)}}A_p h, D_{T'^{(p)}}A_p h\rangle \\
&= \delta + \langle A_p x, A_p x\rangle + \langle A_p h, A_p h\rangle - \langle T'^{(p)*}T'^{(p)}A_p h, A_p h\rangle \\
&\leqslant \delta + \langle A_p x, A_p x\rangle + \langle h, h\rangle - \langle T'^{(p)*}T'^{(p)}A_p h, A_p h\rangle \\
&\leqslant \langle A_p x, A_p x\rangle + 2\delta + \langle D_{A_p}x, D_{A_p}x\rangle + \langle y_0, y_0\rangle \\
&= 2\delta + \langle x, x\rangle + \langle y_0, y_0\rangle.
\end{aligned}$$

Since ε can be chosen arbitrarily close to zero, it follows that

$$\langle A_{p+1}(x, y), A_{p+1}(x, y)\rangle \leqslant \langle (x, y), (x, y)\rangle,$$

i.e., A_{p+1} is a contraction.

Next, we verify

$$T'^{(p+1)}A_{p+1} = A_{p+1}T^{(p+1)}, \tag{6.7}$$

assuming $p > 0$ (if $p = 0$, it is standard: Theorem V.1.2 in [11]). Partition (in a self-explanatory notation)

$$T^{(p+1)} = \begin{bmatrix} & T^{(p)} & & & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}, \quad T'^{(p+1)} = \begin{bmatrix} & T'^{(p)} & & & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}.$$

Writing out the equality (6.7) using this partition, we note that the definition of E_p implies $T'^{(p)}A_pE_p = A_pE_p \begin{bmatrix} & T^{(p)} & & & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}$, so it remains to verify

$$[\,0 \quad \cdots \quad 0 \quad I\,]\, A_pE_p = \Gamma_p P_{F_p}(D_{A_p} \oplus Q_p) \begin{bmatrix} & T^{(p)} & & & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix}. \tag{6.8}$$

Since $E_p = [I\ 0]$, (6.8) boils down to

$$[\,0 \quad \cdots \quad 0 \quad I\,]\, A_p = \Gamma_p P_{F_p}\big(D_{A_p}T^{(p)} \oplus Q_p\,[\,0 \quad \cdots \quad 0 \quad I\,]\big). \tag{6.9}$$

But for every $x \in H \oplus (\overline{\operatorname{Im} D_T})^p$ we have $Q_p\,[\,0 \quad \cdots \quad 0 \quad I\,]\,x = D_{T^{(p)}}x$, and therefore (by definitions of F_p and of Γ_p)

$$\Gamma_p P_{F_p}\big(D_{A_p}T^{(p)}x \oplus Q_p\,[\,0 \quad \cdots \quad 0 \quad I\,]\,x\big) = D_{T'^{(p)}}A_px.$$

It remains to observe that $[\,0 \quad \cdots \quad 0 \quad I\,]\,D_{T'^{(p)}} = [\,0 \quad \cdots \quad 0 \quad I\,]$, and (6.7) follows.

Finally, we verify the equalities (6.4). Details will be given for the case $p = 0$ (if $p > 0$, the verification is analogous); i.e., we verify

$$W_i'^{(1)}A_1 = A_1W_i^{(1)}, \quad i \in J. \tag{6.10}$$

To this end, it suffices to check that

$$\Gamma_0 P_{F_0}(D_A \oplus Q_0)W_i^{(1)} = W'^{(i)}_{1,1}\Gamma_0 P_{F_0}(D_A \oplus Q_0). \tag{6.11}$$

First observe that, since $A^*AV_i = A^*V_i'A = V_iA^*A$, we have that

$$(D_A \oplus Q_0)W_i^{(1)} = (I_H \oplus Q_0)W_i^{(1)}(D_A \oplus I_{\overline{\operatorname{Im} D_T}}). \tag{6.12}$$

Next we show that

$$P_{F_0}(I_H \oplus Q_0)W_i^{(1)} = W_i^{(1)}P_{F_0}(I_H \oplus Q_0). \tag{6.13}$$

For this purpose, suppose that h_0, h_1 and $h \in H$ are such that $P_{F_0}(D_Ah_0 \oplus Q_0h_1) = D_ATh \oplus D_Th$. (In the general case, one has to approximate $P_{F_0}(D_Ah_0 \oplus Q_0h_1)$ by $D_ATh \oplus D_Th$, as in the above proof of contractive property of A_{p+1} when $p > 0$; we omit the details.) Then we claim that

$$P_{F_0}(V_iD_Ah_0 \oplus Q_0W_{11}^{(i)}Q_0h_1) = V_iD_ATh \oplus Q_0W_{11}^{(i)}D_Th.$$

Indeed, for any $g \in H$, we have

$$\begin{aligned}&\langle V_iD_Ah_0 \oplus Q_0W_{11}^{(i)}Q_0h_1 - V_iD_ATh \oplus Q_0W_{11}^{(i)}D_Th, D_ATg \oplus D_Tg\rangle \\ &\qquad = \langle D_Ah_0 \oplus Q_0h_1 - D_ATh \oplus D_Th, D_ATV_i^*g \oplus D_TV_i^*g\rangle = 0,\end{aligned}$$

where we used that $W_{11}^{(i)} D_T = D_T V_i$. This shows that (6.13) holds. Lastly, we get that

$$\Gamma_0 W_i^{(1)} = W'^{(i)}_{1,1} \Gamma_0. \tag{6.14}$$

Indeed, if we let $q = D_A T h \oplus D_T h$, then

$$\begin{aligned}\Gamma_0 W_i^{(1)} q &= \Gamma_0 W_i^{(1)} (D_A T h \oplus D_T h) = \Gamma_0 (D_A T V_i h \oplus D_T V_i h) \\ &= D_{T'} A(V_i h) = D_{T'^{(1)}} V_i' A h = W'^{(i)}_{1,1} D_{T'} A h = W'^{(i)}_{1,1} \Gamma_0 q.\end{aligned}$$

Combining now (6.12), (6.13) and (6.14) gives (6.11). This proves (6.10).

Let now

$$B = \text{strong} \lim_{p \to \infty} A_p P_{H \oplus (\overline{\operatorname{Im} D_T})^p}.$$

Then, by Section VII.3 in [11], we have that B is a contractive lifting of A that satisfies $BU = U'B$. By (6.4) it now follows that in addition we have that $BW_i = W_i' B$, $i \in J$. ∎

Analogously to (1.3), we define $F_k \subseteq \mathbb{Z}^k$ by

$$\begin{aligned} F_k = \{ &(x_1, \ldots, x_k)^{\mathrm{T}} \in \mathbb{Z}^k \setminus \{0\} : \\ &x_1 = x_2 = \cdots = x_{j-1} = 0,\, x_j \neq 0 \Rightarrow x_j > 0\} \cup \{0\}. \end{aligned}$$

THEOREM 6.5. *Let $f \in (L_\infty(\mathbb{T}^k))^{m \times n}$. Then*

$$\operatorname{dist}(f, (L_\infty(\mathbb{T}^k))^{m\times n}_{F_k}) = \|\Gamma_f\|,$$

where $\Gamma_f : (L_2(\mathbb{T}^k))^n_{F_k} \to (L_2(\mathbb{T}^k))^m_{\mathbb{Z}^k \setminus F_k}$ is defined by

$$(\Gamma_f g)(z) = \Pi_{\mathbb{Z}^k \setminus F_k} (f(z) g(z)), \quad g \in (L_2(\mathbb{T}^k))^n_{F_k}.$$

In fact, the distance is attained.

While this paper was in preparation, we have learned about the paper [8], in which the result of Theorem 6.5 is proved (using theory of group characters) for compact abelian groups.

Proof. In case $\Gamma_f = 0$, we have that $f \in (L_\infty(\mathbb{T}^k))^{m\times n}_{F_k}$ and the theorem follows trivially. Therefore, we assume that $\Gamma_f \neq 0$. By downward induction we prove that there exist operators

$$H_j : (L_2(\mathbb{T}^k))^n_{F_j \times \mathbb{Z}^{k-j}} \to (L_2(\mathbb{T}^k))^m_{(\mathbb{Z}^j \setminus F_j) \times \mathbb{Z}^{k-j}}, \quad j = k, k-1, \ldots, 0,$$

so that

$$H_j M_{z_p} = M_{z_p} H_j, \quad p \geqslant j+1; \tag{6.15}$$

$$\Pi_{F_j \times \mathbb{Z}^{k-j}} M_{z_j} H_j = H_j M_{z_j}; \tag{6.16}$$

$$\Pi_{(\mathbb{Z}^{j+1} \setminus F_{j+1}) \times \mathbb{Z}^{k-j-1}} H_j \Pi_{F_{j+1} \times \mathbb{Z}^{k-j-1}} = H_{j+1}; \tag{6.17}$$

$$\|H_j\| = \|H_{j+1}\|, \quad 0 \leqslant j \leqslant k-1, \quad H_k = \Gamma_f. \tag{6.18}$$

By convention $\mathbb{Z}^0 \setminus F_0 = \mathbb{Z}^0 = \{0\}$, and M_{z_j} is the multiplication operator with the jth variable. Clearly $H_k = \Gamma_f$ satisfies the requirements. Suppose now that $H_k, H_{k-1}, \dots, H_j$ have been constructed. We apply Theorem 6.4 with

$$A = \frac{1}{\|H_j\|} H_j, \quad T = M_{z_j}, \quad T' = \Pi_{F_j \times \mathbb{Z}^{k-j}} M_{z_j}, \quad W_p = M_{z_p}, \quad W'_p = M_{z_p},$$

where $p = j+1, \dots, k$. Note that $U = T$ and $U' = M_{z_j}$. This yields an operator $B : (L_2(\mathbb{T}^k))^n_{F_j \times \mathbb{Z}^{k-j}} \to (L_2(\mathbb{T}^k))^m_{(\mathbb{Z}^{j-1} \setminus F_{j-1}) \times \mathbb{Z}^{k-j+1}}$ satisfying

$$BM_{z_p} = M_{z_p} B, \quad p \geqslant j, \quad \Pi_{(\mathbb{Z}^j \setminus F_j) \times \mathbb{Z}^{k-j}} B = \frac{1}{\|H_j\|} H_j, \quad \|B\| = 1.$$

Observe that M_{z_j} on $(L_2(\mathbb{T}^k))^n_{F_{j-1} \times \mathbb{Z}^{k-j+1}}$ is the minimal unitary extension of M_{z_j} on $(L_2(\mathbb{T}^k))^n_{F_j \times \mathbb{Z}^{k-j}}$ (for the definition see Section VI.2 in [11]). Let

$$\widetilde{B} : (L_2(\mathbb{T}^k))^n_{F_{j-1} \times \mathbb{Z}^{k-j+1}} \to (L_2(\mathbb{T}^k))^m_{(\mathbb{Z}^{j-1} \setminus F_{j-1}) \times \mathbb{Z}^{k-j+1}}$$

be defined by

$$\widetilde{B} = \text{strong} \lim_{p \to \infty} M^{*p}_{z_j} B M^p_{z_j} \Pi_{F_j \times \mathbb{Z}^{k-j} - pe_j}, \tag{6.19}$$

where e_j is the jth unit vector in $\mathbb{Z}^k$. Formula (6.19) and Corollary VI.2.4 in [11] show that

$$\widetilde{B} M_{z_p} = M_{z_p} \widetilde{B}, \quad p \geqslant j, \quad \|\widetilde{B}\| = 1, \quad \widetilde{B} | (L_2(\mathbb{T}^k))^n_{F_j \times \mathbb{Z}^{k-j}} = B.$$

Put $H_{j-1} = \|H_j\| \widetilde{B}$. We need to show that

$$\Pi_{F_{j-1} \times \mathbb{Z}^{k-j+1}} M_{z_{j-1}} H_{j-1} = H_{j-1} M_{z_{j-1}}, \tag{6.20}$$

since the other required properties are easily checked. To prove the equality in (6.20) let $(p_1, \dots, p_k) \in F_{j-1} \times \mathbb{Z}^{k-j+1}$ and $(q_1, \dots, q_k) \in (\mathbb{Z}^{j-1} \setminus F_{j-1}) \times \mathbb{Z}^{k-j+1}$. Then $(p_1, \dots, p_{j-1}+1, \dots, p_k) \in F_k \subset F_j \times \mathbb{Z}^{k-j}$, so

$$\begin{aligned}
&\langle H_{j-1} M_{z_{j-1}} (z_1^{p_1} \cdots z_k^{p_k}), z_1^{q_1} \cdots z_k^{q_k} \rangle \\
&\quad = \langle \|H_j\| B((z_1^{p_1} \cdots z_{j-1}^{p_{j-1}+1} \cdots z_k^{p_k}), z_1^{q_1} \cdots z_k^{q_k} \rangle \\
&\quad = \langle z_j^{-q_j-1} \|H_j\| B((z_1^{p_1} \cdots z_{j-1}^{p_{j-1}+1} \cdots z_k^{p_k}), z_1^{q_1} \cdots z_j^{-1} \cdots z_k^{q_k} \rangle \\
&\quad = \langle \|H_j\| B(z_1^{p_1} \cdots z_{j-1}^{p_{j-1}+1} z_j^{p_j-q_j-1} \cdots z_k^{p_k}), z_1^{q_1} \cdots z_j^{-1} \cdots z_k^{q_k} \rangle \\
&\quad = \langle H_j (z_1^{p_1} \cdots z_{j-1}^{p_{j-1}+1} z_j^{p_j-q_j-1} \cdots z_k^{p_k}), z_1^{q_1} \cdots z_j^{-1} \cdots z_k^{q_k} \rangle \\
&\quad = \widehat{f}(q_1 - p_1, \dots, q_{j-1} - p_{j-1} - 1, \dots, q_k - p_k),
\end{aligned}$$

where we used that $(q_1, \dots, q_{j-1}, -1, q_{j+1}, \dots, q_k) \in \mathbb{Z}^k \setminus F_k \subseteq (\mathbb{Z}^j \setminus F_j) \times \mathbb{Z}^{k-j}$. On the other hand,

$$\begin{aligned}
&\langle M_{z_{j-1}} H_{j-1} (z_1^{p_1} \cdots z_k^{p_k}), z_1^{q_1} \cdots z_k^{q_k} \rangle \\
&\quad = \langle H_{j-1} (z_1^{p_1} \cdots z_j^{1} \cdots z_k^{p_k}, z_1^{q_1} \cdots z_{j-1}^{q_{j-1}-1} z_j^{q_j-p_j-1} \cdots z_k^{q_k} \rangle \\
&\quad = \langle H_j (z_1^{p_1} \cdots z_j^{1} \cdots z_k^{p_k}, z_1^{q_1} \cdots z_{j-1}^{q_{j-1}-1} z_j^{q_j-p_j-1} \cdots z_k^{q_k} \rangle \\
&\quad = \widehat{f}(q_1 - p_1, \dots, q_{j-1} - p_{j-1} - 1, \dots, q_k - p_k),
\end{aligned}$$

where we used that $(p_1, \dots, p_{j-1}, 1, p_{j+1}, \dots, p_k) \in F_k$ and $(q_1, \dots, q_{j-1} - 1, q_j - p_j - 1, \dots, q_k) \in \mathbb{Z}^k \setminus F_k$. Thus (6.20) has been established, and therefore the existence of the operators $H_k, \dots, H_0$ has been proven.

Since $H_0 : (L_2(\mathbb{T}^k))^n \to (L_2(\mathbb{T}^k))^m$ commutes with all multiplication operators $M_{z_1}, \dots, M_{z_k}$, there is (see Theorem IX.1.1 in [11] for the case when $k = 1$; the proof for general k is analogous) a function $h \in (L_\infty(\mathbb{T}^k))^{m\times n}$ so that H_0 is the multiplication operator with symbol h. In fact, in the scalar case $m = n = 1$, h is given by $h = H_0 1$, where 1 stands for the constant function 1. In particular, $\|h\|_\infty = \|H_0\|$. Moreover, since $\Pi_{\mathbb{Z}^k\setminus F_k} H_0 \Pi_{F_k} = \Gamma_f$ we get that $\Gamma_h = \Gamma_f$, and thus $f - h \in (L_\infty(\mathbb{T}^k))^{m\times n}_{F_k}$. This shows that

$$\operatorname{dist}(f, (L_\infty(\mathbb{T}^k))^{m\times n}_{F_k} \leqslant \|h\|_\infty = \|H_0\| = \|\Gamma_f\|.$$

The other inequality,

$$\operatorname{dist}(f, (L_\infty(\mathbb{T}^k))^{m\times n}_{F_k} \geqslant \|\Gamma_f\|,$$

follows directly from the observation that for $g \in (L_\infty(\mathbb{T}^k))^{m\times n}_{F_k}$ we have that

$$\|f - g\|_\infty \geqslant \|\Gamma_{f-g}\| = \|\Gamma_f\|. \quad \blacksquare$$

By a useful trick (see Section 5.3 in [4], or p. 478 in [3] for the case $k = 2$) one may replace F_k in Theorem 6.5 by $\mathbb{Z}^k \cap A[E_k]$ (and $\mathbb{Z}^k \setminus F_k$ by $\mathbb{Z}^k \setminus A[E_k]$) where A is an invertible $k \times k$ matrix with rational entries. It is an interesting open problem whether the general halfspace case (and the general B^k_∞ case) may be covered by some type of commutant lifting approach as well.

Note that in [10] other distance formulas are considered, e.g., the distance $\operatorname{dist}(f, (H_\infty(\mathbb{T}^k))^{m\times n})$ to the space $(H_\infty(\mathbb{T}^k))^{m\times n}$ of $m\times n$ matrix valued bounded analytic functions on $\mathbb{D}^k = \{z : |z| < 1\}^k$. This distance does *not* equal the norm of the related Hankel operator. It is important in this regard to note that in our case both $(L_\infty(\mathbb{T}^k))_{F_k}$ and $(L_\infty(\mathbb{T}^k))_{\mathbb{Z}^k\setminus F_k}$ are closed under multiplication. The space $(L_\infty(\mathbb{T}^k))_{\mathbb{Z}^k\setminus\mathbb{Z}^k_+}$, which is the one that appears naturally when one considers $\operatorname{dist}(f, (H_\infty(\mathbb{T}^k))^{m\times n})$, is *not* closed under multiplication (when $k > 1$). The case of $\operatorname{dist}(f, (L_\infty(\mathbb{T}^k))_{\mathbb{Z}_+\times\mathbb{Z}^{k-1}})$ may be reduced to an operator valued one-variable case, and the results in [25] then yield the formula

$$\operatorname{dist}(f, (L_\infty(\mathbb{T}^k))_{\mathbb{Z}_+\times\mathbb{Z}^{k-1}}) = \|\Pi_{(\mathbb{Z}\setminus\mathbb{Z}_+)\times\mathbb{Z}^{k-1}} M_f \Pi_{\mathbb{Z}_+\times\mathbb{Z}^{k-1}}\|.$$

See also [7] and [10].

7. A JOINT NORM BOUND RESULT

For $f \in (B^k)^{m\times n}$ let $\|f\|_{B^k} = [\mathrm{trace}(M\{f^*f\})]^{1/2}$ be the Besikovitch norm.

THEOREM 7.1. *Let* $f \in (\mathrm{APW}^k)^{m\times n}_{(-S)\cap\Lambda}$ *be given, where* Λ *is a subgroup of* $\mathbb{R}^k$. *Denote* $\widehat{d}_2 = \|f\|_{B^k}$, $\widehat{d}_\infty = \|\mathbf{H}(f)_\Lambda\|$. *For each* $\varepsilon > 1$ *there exists an* $\widetilde{f} \in (\mathrm{APW}^k)^{m\times n}_{(S\setminus\{0\})\cap\Lambda}$ *such that*

$$\|f+\widetilde{f}\|_\infty \leqslant \varepsilon\widehat{d}_\infty \quad \textit{and} \quad \|f+\widetilde{f}\|_{B^k} \leqslant \frac{\varepsilon\widehat{d}_2}{\sqrt{\varepsilon^2-1}}.$$

Theorem 7.1 is a generalization of a result proved in [20], where the case $k=1$, $\Lambda=\mathbb{Z}$ was considered. Sharper bounds on $\|\cdot\|_\infty$ and $\|\cdot\|_{B^k}$ involving the entropy have been obtained in [1] (again, for the case $k=1$, $\Lambda=\mathbb{Z}$).

Proof. The proof is analogous to the proof of Theorem 11.1 in [28]. Without loss of generality we may assume that $\varepsilon\widehat{d}_\infty = 1$ (excluding the trivial case $f\equiv 0$), since we may divide f by $\varepsilon\widehat{d}_\infty$. But then $\|\mathbf{H}(f)_\Lambda\| = \frac{1}{\varepsilon} < 1$. Let now $\widetilde{f}(t) = \beta(t)\delta(t)^{-1} - f(t)$, where β and δ are as in Theorem 2.1.

Using the easily derived inequality

$$\mathrm{trace}\, M^*M \leqslant -\log\det(I - M^*M),$$

which holds for every $M \in \mathbb{C}^{m\times n}$ with $\|M\|<1$, we have:

$$\begin{aligned}
\|f+\widetilde{f}\|^2_{B^k} &= \lim_{T\to\infty}\frac{1}{(2T)^k}\int\limits_{[-T,T]^k}\mathrm{trace}\big[((f+\widetilde{f})(t))^*(f+\widetilde{f})(t)\big]\,\mathrm{d}t\\
&\leqslant -\lim_{T\to\infty}\frac{1}{(2T)^k}\int\limits_{[-T,T]^k}\log\det\big(I-((f+\widetilde{f})(t))^*(f+\widetilde{f})(t)\big)\,\mathrm{d}t\\
&= -\lim_{T\to\infty}\frac{1}{(2T)^k}\int\limits_{[-T,T]^k}\log\det(I-(\beta(t)\delta(t)^{-1})^*\beta(t)\delta(t)^{-1})\,\mathrm{d}t.
\end{aligned}$$

By (3.11) with $g=0$ it follows that

$$I-(\beta(t)\delta(t)^{-1})^*\beta(t)\delta(t)^{-1} = \delta(t)^{-1*}(\delta(t)^*\delta(t)-\beta(t)^*\beta(t))\delta(t)^{-1} = \delta(t)^{-1*}\delta(t)^{-1}.$$

Since $\big(\delta(t)M\{\delta\}^{-1}\big)^{\pm 1} \in I + (\mathrm{APW}^k)^{n\times n}_{S\setminus\{0\}}$, we get from Proposition 3.2 that

$$M\left\{\log\big(\delta(t)M\{\delta\}^{-1}\big)\right\} = 0.$$

So then we obtain

$$\|f+\widetilde{f}\|^2_{B^k} \leqslant M\left\{\log\det\left(\delta(t)\delta(t)^*\right)\right\} = \log\det(M\{\widehat{\delta}\}) = \mathrm{trace}\log M\{\widehat{\delta}\}, \tag{7.1}$$

where $\widehat{\delta}$ is as in Theorem 2.1. Note that

$$\begin{aligned}
M\{\widehat{\delta}\} &= M\left\{\left(I-(\mathbf{H}(f))^*_\Lambda\mathbf{H}(f)_\Lambda\right)^{-1}(I_n)\right\}\\
&= M\left\{\big(I+(\mathbf{H}(f)_\Lambda)^*(I-\mathbf{H}(f)_\Lambda(\mathbf{H}(f)_\Lambda)^*)^{-1}\mathbf{H}_\Lambda(f)\big)(I_n)\right\}.
\end{aligned}$$

From the inequality $\log(1+r) \leqslant r$ valid for $r \geqslant 0$ we get that

$$\log M\{\widehat{\delta}\} \leqslant M\left\{\left(\mathbf{H}(f)_\Lambda^*(I - \mathbf{H}(f)_\Lambda \mathbf{H}(f)_\Lambda^*)^{-1}\mathbf{H}(f)_\Lambda\right)(I_n)\right\}.$$

Since $\|\mathbf{H}(f)\| \leqslant \widehat{d}_\infty = \frac{1}{\varepsilon}$, from here and from (7.1) it follows that:

$$\|f + \widetilde{f}\|_{B^k}^2 \leqslant \frac{1}{1 - \widehat{d}_\infty^2} \operatorname{trace} M\left\{\left(\mathbf{H}(f)_\Lambda^* \mathbf{H}(f)_\Lambda\right)(I_n)\right\} = \frac{\varepsilon^2}{\varepsilon^2 - 1}\|f\|_{B^k}^2 = \frac{\varepsilon^2 \widehat{d}_2^2}{\varepsilon^2 - 1}. \quad \blacksquare$$

By using Theorem 4.1 instead of Theorem 2.1 one can prove a point excluding variation of Theorem 7.1, where

$$f \in (\mathrm{APW}^k)^{m\times n}_{(-S\setminus\{0\})\cap\Lambda}, \quad \widetilde{f} \in (\mathrm{APW}^k)^{m\times n}_{S\cap\Lambda}, \quad \text{and} \quad \widetilde{d}_\infty = \|\widetilde{\mathbf{H}}(f)_\Lambda\|.$$

We omit a statement of this variation.

8. THE MODEL MATCHING PROBLEM FOR A CLASS OF MULTIVARIABLE LINEAR FILTERS

In this section we consider filters acting on square summable sequences indexed by an additive group in $\mathbb{R}^k$. The case of the group $\mathbb{Z}$ in $\mathbb{R}$ is the familiar case, treated extensively in the literature (see, e.g., [24]).

Let Λ be an additive subgroup of $\mathbb{R}^k$. For $\Delta \subseteq \Lambda$ we let $\ell_2^n(\Delta)$ denote the Hilbert space of sequences $(v_\lambda)_{\lambda\in\Delta}$ where at most countably many $v_\lambda \in \mathbb{C}^n$ are nonzero and which are square summable in norm, i.e., $\sum_{\lambda\in\Delta} \|v_\lambda\|^2 < \infty$. By $\ell_1^{n\times n}(\Delta)$ we denote the Banach space of sequences $(f_\lambda)_{\lambda\in\Delta}$ where at most countably many $f_\lambda \in \mathbb{C}^{n\times n}$ are nonzero and which are summable in norm, i.e., $\sum_{\lambda\in\Delta} \|f_\lambda\| < \infty$.

Fix a halfspace S of $\mathbb{R}^k$. With S we associate an ordering $\leqslant_S$ on Λ by $q \leqslant_S p$ if and only if $p - q \in S$. We shall use the interval notation with the usual conventions. So, for instance, $S \cap \Lambda = [0, \infty)$. With an element $f \in \ell_1^{n\times n}([0,\infty))$, we associate a filter $\Sigma_f : \ell_2^n([0,\infty)) \to \ell_2^n([0,\infty))$, defined by

$$\Sigma_f((u_\lambda)_{\lambda\in[0,\infty)}) = (y_\lambda)_{\lambda\in[0,\infty)}, \quad y_\lambda = \sum_{\alpha\in[0,\lambda]} f_\alpha u_{\lambda-\alpha}.$$

We shall depict the filter as

$$\xrightarrow{(u_\lambda)_\lambda} \boxed{\quad \Sigma_f \quad} \xrightarrow{(y_\lambda)_\lambda},$$

and call $(u_\lambda)_\lambda$ the *input* and $(y_\lambda)_\lambda$ the *output* of the filter. The concatenation of two filters

$$\xrightarrow{u} \boxed{\quad \Sigma_f \quad} \xrightarrow{w} \boxed{\quad \Sigma_h \quad} \xrightarrow{v}$$

results in the *product filter* $\Sigma_h \Sigma_f$. The difference filter $\Sigma_f - \Sigma_h$ may be depicted as in Figure 2.

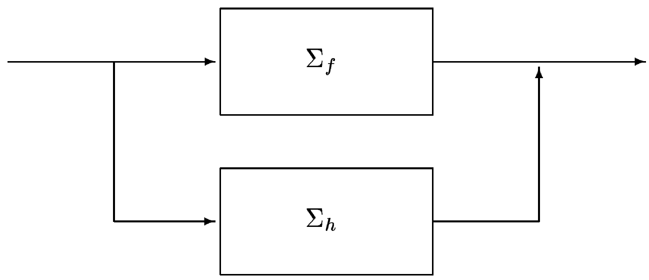

Figure 2.

With an element $f = (f_\lambda)_{\lambda\in[0,\infty)} \in \ell_1^{n\times n}([0,\infty))$ we may associate a member of $(\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$, which with a slight abuse of notation we shall also denote by f, and which is defined via $f(t) = \sum_{\lambda\in[0,\infty)} f_\lambda e^{i\langle\lambda,t\rangle}$, $t \in \mathbb{R}^k$. Note that $\Sigma_h\Sigma_f = \Sigma_{hf}$ and $\Sigma_h - \Sigma_f = \Sigma_{h-f}$. For a filter Σ_f we define its norm by

$$\|\Sigma_f\| = \sup_{u\neq 0} \frac{\|\Sigma_f(u)\|}{\|u\|}.$$

It is not hard to see that $\|\Sigma_f\| = \|f\|_\infty := \sup_{t\in\mathbb{R}^k} \|f(t)\|$.

In this section we consider the model matching problem for linear filters, i.e., given filters $\Sigma_{f_1}, \Sigma_{f_2}, \Sigma_{f_3}$, find a filter Σ_h so that the filter $\Sigma_{f_1} - \Sigma_{f_2}\Sigma_h\Sigma_{f_3}$ depicted in Figure 3 has minimal possible norm.

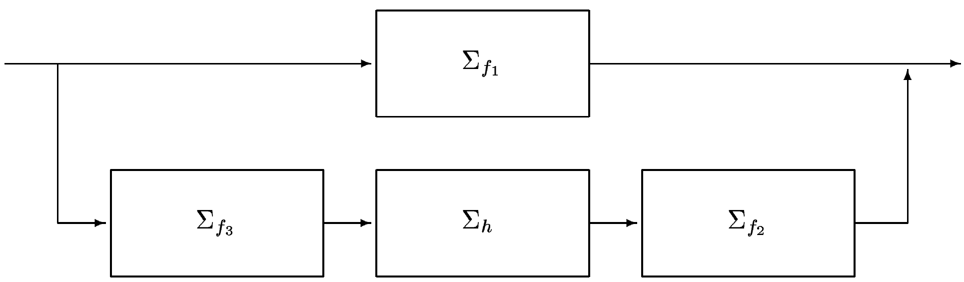

Figure 3.

Equivalently, given f_1, f_2 and f_3 in $(\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$, find $h \in (\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$ so that $\|f_1 - f_2hf_3\|_\infty$ is as small as possible. In the case that

$$\inf_{t\in\mathbb{R}^k} |\det f_2(t)| > 0, \quad \inf_{t\in\mathbb{R}^k} |\det f_3(t)| > 0, \tag{8.1}$$

we shall provide a solution to the *suboptimal problem*: Let

$$\nu > \inf_h \|f_1 - f_2 h f_3\|_\infty,$$

construct one/all $h \in (\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$ such that

$$\|f_1 - f_2 h f_3\|_\infty < \nu. \tag{8.2}$$

PROPOSITION 8.1. *Let f_1, f_2 and f_3 be given elements of $(\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$ so that* (8.1) *is satisfied. Let $f_{2,+}(t)^* f_{2,+}(t)$ and $f_{3,+}(t) f_{3,+}(t)^*$ be a right and left canonical factorization of $f_2(t)^* f_2(t)$ and $f_3(t) f_3(t)^*$, respectively, i.e.,*

$$f_{2,+}(t)^* f_{2,+}(t) = f_2(t)^* f_2(t), \quad f_{3,+}(t) f_{3,+}(t)^* = f_3(t) f_3(t)^*,$$

where $f_{2,+}^{\pm 1}, f_{3,+}^{\pm 1} \in (\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$. Let $f_{2,i} = f_2 f_{2,+}^{-1}$ and $f_{3,i} = f_{3,+}^{-1} f_3$. Then

$$\inf_h \|f_1 - f_2 h f_3\|_\infty = \|\widetilde{\mathbf{H}}(f_{2,i}^* f_1 f_{3,i}^*)_\Lambda\|.$$

Here $\widetilde{\mathbf{H}}$ is defined by (4.1).

Proof. Since $f_{2,i} f_{2,i}^* = f_2 f_{2,+}^{-1} f_{2,+}^{*-1} f_2^* = f_2 (f_{2,+}^* f_{2,+})^{-1} f_2^* = I$, it follows that $f_{2,i}(t)$ is unitary for all t. Similarly, $f_{3,i}(t)$ is unitary for all t. Therefore,

$$\|f_1 - f_2 h f_3\|_\infty = \|f_1 - f_{2,i} f_{2,+} h f_{3,+} f_{3,i}\|_\infty = \|f_{2,i}^* f_1 f_{3,i}^* - f_{2,+} h f_{3,+}\|_\infty.$$

Apply Theorem 4.1 to obtain that for $\varepsilon > 0$ there exists a $g_\varepsilon \in (\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$ so that

$$\|f_{2,i}^* f_1 f_{3,i}^* - g_\varepsilon\| < \|\widetilde{\mathbf{H}}(f_{2,i}^* f_1 f_{3,i}^*)\| + \varepsilon.$$

Put $h_\varepsilon = f_{2,+}^{-1} g_\varepsilon f_{3,+}^{-1} \in (\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$. Then $\|f_1 - f_2 h_\varepsilon f_3\| < \|\widetilde{\mathbf{H}}(f_{2,i}^* f_1 f_{3,i}^*)\| + \varepsilon$. This proves the inequality $\leqslant$. The opposite inequality is trivial. ∎

Recall the definition of the parameter set

$$(\mathrm{CAPW}^k)^{n\times n}_{S\cap\Lambda} = \Big\{ g \in (\mathrm{APW}^k)^{n\times n}_{S\cap\Lambda} : \sup_{t\in\mathbb{R}^k} \|g(t)\| < 1 \Big\}.$$

THEOREM 8.2. *Let f_1, f_2 and f_3 be given elements of $(\mathrm{APW}^k)^{n\times n}_{\Lambda\cap S}$ so that* (8.1) *is satisfied. Introduce $f_{2,+}, f_{3,+}, f_{2,i}$ and $f_{3,i}$ as in Proposition* 8.1. *Let $\nu > \|\widetilde{\mathbf{H}}(f_{2,i}^* f_1 f_{3,i}^*)_\Lambda\|$. Let $f = \Pi_{\Lambda\cap(-S\setminus\{0\})}(f_{2,i}^* f_1 f_{3,i}^*)$, and put*

$$\begin{aligned}
\widehat{\alpha}(t) &= \big[\nu^2 I - \widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda (\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda)^*\big]^{-1}(I_m),\\
\widehat{\beta}(t) &= \widetilde{\mathbf{H}}(f)_\Lambda \big[\nu^2 I - (\widetilde{\mathbf{H}}(f)_\Lambda)^* \widetilde{\mathbf{H}}(f)_\Lambda\big]^{-1}(I_n),\\
\widehat{\gamma}(t) &= (\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda)^* \big[\nu^2 I - (\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda (\widetilde{\widetilde{\mathbf{H}}}(f)_\Lambda)^*\big]^{-1}(I_m),\\
\widehat{\delta}(t) &= \big[\nu^2 I - (\widetilde{\mathbf{H}}(f)_\Lambda)^* \widetilde{\mathbf{H}}(f)_\Lambda)\big]^{-1}(I_n),
\end{aligned}$$

where I_r stands for the constant matrix function on $\mathbb{R}^k$ with value I_r for all $t \in \mathbb{R}^k$. Further, let

$$\begin{aligned}
\alpha(t) &= \widehat{\alpha}(t) M\{\widehat{\alpha}\}^{-\frac12}, \quad \beta(t) = \widehat{\beta}(t) M\{\widehat{\delta}\}^{-\frac12},\\
\gamma(t) &= \widehat{\gamma}(t) M\{\widehat{\alpha}\}^{-\frac12}, \quad \delta(t) = \widehat{\delta}(t) M\{\widehat{\delta}\}^{-\frac12}.
\end{aligned}$$

Then each solution h to the suboptimal model matching problem (8.2) *is of the form*

$$h = f_2^{-1} f_1 f_3^{-1} - \nu f_{2,+}^{-1}(\alpha g + \beta)(\gamma g + \delta)^{-1} f_{3,+}^{-1}, \tag{8.3}$$

where $g \in \nu(\mathrm{CAPW}^k)^{n\times n}_{S\cap\Lambda}$. Moreover, this correspondence between the set $\nu(\mathrm{CAPW}^k)^{n\times n}_{S\cap\Lambda}$ and the set of solutions h is one-to-one.

Proof. As the proof of Proposition 8.1 shows, we have that h is a solution to the model matching problem (8.2) if and only if

$$\frac{1}{\nu}(-f_{2,+} h f_{3,+} + f_{2,i}^* f_1 f_{3,i}^*) \tag{8.4}$$

is a strictly contractive extension of $\frac{1}{\nu}\Pi_{\Lambda\cap(-S\setminus\{0\})}(f_{2,i}^* f_1 f_{3,i}^*)$. By the point excluding variation (see Section 4) of Theorem 2.2 these are parametrized by

$$(\alpha\nu\widetilde{g} + \beta)(\gamma\nu\widetilde{g} + \delta)^{-1}, \tag{8.5}$$

where $\widetilde{g} \in (\mathrm{CAPW}^k)^{n\times n}_{\Lambda\cap S}$. Equating (8.4) and (8.5) and solving for h yields (8.3) with $g = \nu\widetilde{g}$. ∎

A version of Theorem 8.2 in which S (respectively $-S\setminus\{0\}$) is replaced by $S\setminus\{0\}$ (respectively $-S$) can be stated and proved analogously.

In the paper [12] systems of the form

$$\begin{aligned} x(h+1,k+1) &= A_1 x(h,k+1) + A_2 x(h+1,k) + B_1 u(h,k+1) + B_2 u(h+1,k),\\ y(h,k) &= Cx(h,k), \end{aligned}$$

appear, where the initial conditions on the state are given on the diagonal $\{(i,-i) : i \in \mathbb{Z}\}$. When the joint spectral radius of A_1 and A_2 is smaller than one, this system generates (with a minor modification of the location of the initial conditions) the type of filter that we are considering in this section (take $\Lambda = \mathbb{Z}^2$ and $S = \{(m,n) : m+n > 0 \text{ or } (m+n=0 \text{ and } m \geqslant 0)\}$).

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LEIBA RODMAN
Department of Mathematics
P.O. Box 8795
The College of William and Mary
Williamsburg, VA 23187–8795
USA
E-mail: lxrodm@math.wm.edu

ILYA M. SPITKOVSKY
Department of Mathematics
P.O. Box 8795
The College of William and Mary
Williamsburg, VA 23187–8795
USA
E-mail: ilya@math.wm.edu

HUGO J. WOERDEMAN
Department of Mathematics
P.O. Box 8795
The College of William and Mary
Williamsburg, VA 23187–8795
USA
E-mail: hugo@math.wm.edu

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