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DECOMPOSABILITY AND STRUCTURE OF NONNEGATIVE BANDS IN INFINITE DIMENSIONS

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ABSTRACT. A semigroup in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is a collection of operators which is closed under multiplication. A band will denote a semigroup of idempotents. The question whether a band of infinite-rank operators on an infinite-dimensional Hilbert space is reducible is still unsolved. Here, a negative answer to this problem is given as far as decomposability of a band is concerned. Furthermore, conditions leading to decomposability of such bands are discussed. Also, the structure of a maximal nonnegative band of constant rank r is given is under special condition.

KEYWORDS: *Semigroups, decomposability, standard subspaces and bands.*

MSC (2000): 47A15.

INTRODUCTION

In this paper $\mathcal{X}$ will be a separable, locally compact Hausdorff space and μ a Borel measure on $\mathcal{X}$. $\mathcal{L}^2(\mathcal{X})$ will denote the Hilbert space of (equivalence classes of) complex-valued measurable functions on $\mathcal{X}$ which are square-integrable relative to μ and $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, the space of all bounded linear operators on $\mathcal{L}^2(\mathcal{X})$. We also assume for simplicity that $\mu(\mathcal{X}) < \infty$. This is not a great restriction and almost all our considerations will be valid for the case of a σ-finite measure with obvious modifications.

A function $f \in \mathcal{L}^2(\mathcal{X})$ is said to be nonnegative (respectively positive), written $f \geqslant 0$ (respectively $f > 0$) if $\mu\{x \in \mathcal{X} : f(x) < 0\} = 0$ (respectively $\mu\{x \in \mathcal{X} : f(x) \leqslant 0\} = 0$).

A subspace of $\mathcal{L}^2(\mathcal{X})$ is a norm-closed linear manifold in $\mathcal{L}^2(\mathcal{X})$. A *standard subspace* of $\mathcal{L}^2(\mathcal{X})$ is a subspace of the form

$$\mathcal{L}^2(U) = \{f \in \mathcal{L}^2(\mathcal{X}) : f = 0 \text{ a.e. on } U^{\mathrm{c}}\}$$

for some Borel subset U of $\mathcal{X}$. This space is nontrivial if $\mu(U)\cdot\mu(U^{c})>0$. An operator on $\mathcal{L}^2(\mathcal{X})$ is said to be *decomposable* if there exists a nontrivial standard subspace of $\mathcal{L}^2(\mathcal{X})$ invariant under A. This definition is extended in the obvious manner to define decomposability of a semigroup in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$.

A *band* in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is a semigroup of idempotents i.e. operators E on $\mathcal{L}^2(\mathcal{X})$ such that $E=E^2$.

Suppose $\mathcal{X}_1$ and $\mathcal{X}_2$ are Borel subsets of $\mathcal{X}$. An operator A from $\mathcal{L}^2(\mathcal{X}_1)$ to $\mathcal{L}^2(\mathcal{X}_2)$ is called *nonnegative* if $Af\geqslant 0$ whenever $f\geqslant 0$ in $\mathcal{L}^2(\mathcal{X}_1)$. Similarly, A is called *positive* if $Af>0$ whenever $0\neq f\geqslant 0$ in $\mathcal{L}^2(\mathcal{X}_1)$. For any function f, we define the support of f as $\operatorname{supp} f=\{x\in\mathcal{X}: f(x)\neq 0\}$. If f is a member of $\mathcal{L}^2(\mathcal{X})$, then $\operatorname{supp} f$ is defined up to a set of measure zero. When no confusion is likely to arise, we simply write $\operatorname{supp} f$ for any $f\in\mathcal{L}^2(\mathcal{X})$ to mean $\operatorname{supp} f_0$, where f_0 is a function representing f.

Triangularizability (simultaneous) of a semigroup means the existence of a chain $\mathcal{C}$ of closed subspaces of $\mathcal{H}$ such that:

(a) $\mathcal{C}$ is maximal (as a chain of closed subspaces of $\mathcal{H}$); and
(b) every member of $\mathcal{C}$ is invariant for $\mathcal{S}$.

For a semigroup $\mathcal{S}$ on an infinite dimensional Hilbert space, let $\operatorname{Lat}'\mathcal{S}$ denote the lattice of all standard subspaces which are invariant under every member of $\mathcal{S}$. By Zorn's lemma, it can be shown that $\operatorname{Lat}'\mathcal{S}$ has a maximal chain. This chain may be nontrivial invariant or trivial according as $\mathcal{S}$ has a nontrivial subspace or not. Each chain in $\operatorname{Lat}'\mathcal{S}$ gives rise to a block triangularization for $\mathcal{S}$ and since the members in the chain are standard subspaces, we shall call it a *standard block triangularization*.

Reducibility of semigroups of operators (viz. existence of nontrivial invariant subspaces) has been the subject of study in recent times ([6] and [7]). In this paper, we aim to study decomposability of nonnegative semigroups (especially bands) of operators since existence of standard invariant subspaces yield a nice structure for some special type of bands. Under certain conditions, semigroups of nonnegative quasinilpotent operators have been proved to be not only decomposable but simultaneously triangularizable with a maximal subspace consisting of standard subspaces ([1]). Even in finite dimensions, where we know that a band is triangularizable, the structure of bands is still not at all well understood. Some attempts have been made to study the structure of bands, e.g. in [2] and [3].

The initial results given in this paper are infinite dimensional analogues of some of the results which led to the decomposability of the nonnegative semigroups and in particular nonnegative bands of $n\times n$ matrices with entries in $\mathbb{R}$ (or $\mathbb{C}$), mentioned in the paper written by the author ([5]). The possibility that operators in a band can have infinite rank gives a new perspective to the study of their decomposability. It is proved that a nonnegative band with each member having rank greater than one and containing at least one finite-rank operator is decomposable. An example of a nonnegative band in $\mathcal{B}(l^2)$ with constant infinite rank is given which is not decomposable. It is also shown that under the additional hyphothesis of finiteness, an infinite-rank nonnegative band is decomposable.

1. PRELIMINARY RESULTS

The following propositions which shall be used in the sequel are stated without proof.

PROPOSITION 1.1. *For any two nonnegative functions* f, g *in* $\mathcal{L}^2(\mathcal{X})$,

$$\langle f, g\rangle = 0 \textit{ if and only if } \mu\{\operatorname{supp} f \cap \operatorname{supp} g\} = 0.$$

PROPOSITION 1.2. *For any* $f \in \mathcal{L}^2(\mathcal{X})$,

$$f \geqslant 0 \Leftrightarrow \langle f, g\rangle \geqslant 0, \textit{ for all } g \geqslant 0 \textit{ in } \mathcal{L}^2(\mathcal{X}).$$

PROPOSITION 1.3. *For any* A *in* $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, $A \geqslant 0 \Leftrightarrow A^* \geqslant 0$.

PROPOSITION 1.4. *Let* S *be a nonnegative operator on* $\mathcal{L}^2(\mathcal{X})$ *and* U, V *be any Borel subsets of* $\mathcal{X}$. *Then* $\langle S\chi_U, \chi_V\rangle = 0$ *if and only if* $\langle Sf, g\rangle = 0$ *for all* $f \in \mathcal{L}^2(U)$ *and for all* $g \in \mathcal{L}^2(V)$.

PROPOSITION 1.5. *A nonnegative operator* S *on* $\mathcal{L}^2(\mathcal{X})$ *is decomposable if and only if there exists a Borel subset* U *of* $\mathcal{X}$ *with* $\mu(U) \cdot \mu(U^{\mathrm{c}}) > 0$ *such that*

$$\langle S\chi_U, \chi_{U^{\mathrm{c}}}\rangle = 0.$$

REMARK. The propositions given above for a single nonnegative operator hold true for semigroups of nonnegative operators on $\mathcal{L}^2(U)$.

2. DECOMPOSABILITY OF NONNEGATIVE SEMIGROUPS

It was seen in the finite-dimensional case that the existence of common zero entry in a semigroup of nonnegative matrices led to its decomposability ([5]). In this section, we aim to establish an analog of this fact for nonnegative semigroups in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ which shall result into their decomposability. For this, we need a couple of simple propositions and a lemma.

PROPOSITION 2.1. *Let* $B : \mathcal{L}^2(\mathcal{X}) \to \mathcal{L}^2(\mathcal{Y})$ *be a nonnegative operator such that* $Bf_0 = 0$ *for some* $f_0 > 0$ *in* $\mathcal{L}^2(\mathcal{X})$. *Then* $B = 0$.

COROLLARY 2.2. *Let* B *be a nonnegative operator in* $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. *If there exists a vector* h *in the kernel of* B *which is nonzero and nonnegativ, then* B *is decomposable.*

PROPOSITION 2.3. *Let* $\mathcal{S}$ *be a semigroup of nonnegative operators on* $\mathcal{L}^2(\mathcal{X})$. *Then* $\mathcal{S}$ *is decomposable if and only if* $\mathcal{S}^*$ *is decomposable.*

LEMMA 2.4. *Let* $\mathcal{A}$ *be a collection of nonnegative vectors in* $\mathcal{L}^2(\mathcal{X})$. *Then there exists a minimal Borel subset* $\mathcal{G}$ *in* $\mathcal{X}$ *(defined up to a null set) such that all the vectors in* $\mathcal{A}$ *vanish on* $\mathcal{G}^{\mathrm{c}}$.

Proof. Since $\mathcal{L}^2(\mathcal{X})$ is a separable metric space, so is $\mathcal{A}$. Let $\mathcal{M}$ be a countable dense subset of $\mathcal{A}$. Suppose $\mathcal{M} = \{f_1, f_2, \ldots\}$ where $f_1, f_2, \ldots$ are chosen representatives of the equivalence classes of functions in $\mathcal{M}$. Consider $\mathcal{G} = \bigcup_i \operatorname{supp} f_i$.

Let $f \in \mathcal{A}$, then $\overline{\mathcal{M}} = \mathcal{A}$ implies that there exists a subsequence $\{f_{n_k}\}$ in $\mathcal{M}$ such that $f_{n_k} \to f$ pointwise a.e. (cf. [9], p. 68, Theorem 3.12).

By considering $\mathcal{G}_0 = \bigcup_k \operatorname{supp} f_{n_k} \subseteq \mathcal{G}$, we can show that $\mathcal{A} \subseteq \mathcal{L}^2(\mathcal{G})$. Also $\mathcal{G}$ has no subset of positive measure on which all the vectors in $\mathcal{A}$ vanish. Thus $\mathcal{G}$ is the minimal subset of $\mathcal{X}$, up to a null set, on whose complement all the vectors in $\mathcal{A}$ vanish. ∎

We now prove the main lemma.

LEMMA 2.5. *Let $\mathcal{S}$ be a semigroup of nonnegative operators on $\mathcal{L}^2(\mathcal{X})$ with the property that $\langle A\chi_E, \chi_F\rangle = 0$ for all $A \in \mathcal{S}$, where E, F are Borel subsets of $\mathcal{X}$ with $\mu(E)\cdot\mu(F) > 0$. Then $\mathcal{S}$ is decomposable.*

Proof. We distinguish two cases:
(i) $\mu(E \cap F) = 0$;
(ii) $\mu(E \cap F) > 0$.

We prove case (i) and show that the second case can be reduced to the first. In case (i), we can assume with no loss of generality that $E \cap F = \varphi$. Thus, we can write

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G),$$

where E, F, G can be assumed mutually disjoint with $\mu(G) > 0$. Then, with respect to some choice of bases for $\mathcal{L}^2(E), \mathcal{L}^2(F)$ and $\mathcal{L}^2(G)$, every $A \in \mathcal{S}$ has the matrix representation $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$, where $A_{21} = 0$, by hypothesis and Proposition 1.4.

Let $A \in \mathcal{S}$ be arbitrary and $B \in \mathcal{S}$ be fixed, where

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}.$$

Then $BA \in \mathcal{S}$ implies that $(BA)_{21} = 0$, and thus

$$B_{23}A_{31} = 0 \quad \text{for all } A \in \mathcal{S}. \tag{2.1}$$

Consider the set

$$\mathcal{A} = \{A_{31}(f) : A \in \mathcal{S},\ f \in \mathcal{L}^2(E),\ f \geqslant 0\}.$$

If $A_{31}(f) = 0$ for all A and for all $f \geqslant 0$ in $\mathcal{L}^2(E)$, then $A_{31} = 0$ for all A, and so $\mathcal{L}^2(E)$ is a standard invariant subspace for $\mathcal{S}$. Therefore, we can assume that there exists at least one $A \in \mathcal{S}$ and some $f \in \mathcal{L}^2(E)$, $f \geqslant 0$, such that $A_{31}(f) \neq 0$.

Consider the closed linear span $\widehat{\mathcal{A}}$ of $\mathcal{A}$. Since it is a proper subspace of $\mathcal{L}^2(G)$, by Lemma 2.4 we can find a minimal subset G_0 of G, up to a null set, on whose complement all the vectors in $\widehat{\mathcal{A}}$ and hence in $\mathcal{A}$ vanish, or equivalently

$$\langle A\chi_E, \chi_{G\setminus G_0}\rangle = 0 \quad \text{for all } A \in \mathcal{S}.$$

Thus, with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G_0) \oplus \mathcal{L}^2(G \setminus G_0)$, the matrix representation of any $A \in \mathcal{S}$ is given by

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & A_{43} & A_{44} \end{pmatrix}.$$

Consider the new matrix of B with respect to the decomposition above. Using the facts that $BA \in \mathcal{S}$ for all $A \in \mathcal{S}$ and that $(BA)_{21} = 0$, we get

$$B_{23}(\widehat{\mathcal{A}}) = 0. \tag{2.2}$$

The minimality of G_0 implies that there exists $f \in \widehat{\mathcal{A}}$ such that $\operatorname{supp} f = G_0$; in other words, $f > 0$ on G_0.

From (2.2), we get $B_{23}(f) = 0$ where $f > 0$ in $\mathcal{L}^2(G_0)$.

By Proposition 2.1, $B_{23} = 0$. This is true for all $B \in \mathcal{S}$. Further, using the fact that $(BA)_{41} = 0$ for all $A \in \mathcal{S}$, we get $B_{43} = 0$ for all $B \in \mathcal{S}$. This shows that $\mathcal{L}^2(E) \oplus \mathcal{L}^2(G_0) \in \operatorname{Lat}' \mathcal{S}$ and hence $\mathcal{S}$ is decomposable.

(ii) Next, consider the case when $\mu(E \cap F) > 0$. This is subdivided into two cases according as $\mu(E\Delta F)$ is zero or positive, where $E\Delta F = (E \setminus F) \dot\cup (F \setminus E)$.

(a) If $\mu(E\Delta F) = 0$, then $E = F$ with no loss of generality and we can write $\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(E^{\mathrm{c}})$. Since $\langle A\chi_E, \chi_E\rangle = 0$ for all $A \in \mathcal{S}$, every $A \in \mathcal{S}$ has a representation $A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with respect to the decomposition above. For a fixed $B \in \mathcal{S}$, $(BA)_{11} = 0 \Rightarrow B_{12}A_{21} = 0$, where $A_{21} : \mathcal{L}^2(E) \to \mathcal{L}^2(E^{\mathrm{c}})$.

Again by Lemma 2.4, applied to the set $\mathcal{A}_1 = \{A_{21}(f) : A \in \mathcal{S}, f \in \mathcal{L}^2(E), f \geqslant 0\}$, we can find a minimal subset $\mathcal{N}$ of E^{c} having positive measure such that $\langle A\chi_E, \chi_{E^{\mathrm{c}}\setminus\mathcal{N}}\rangle = 0$ for all $A \in \mathcal{S}$ where $\mathcal{N}$ is the union of the supports of all vectors in a countable dense subset of the closed linear span $\widehat{\mathcal{A}}_1$ of $\mathcal{A}_1$. Then, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(\mathcal{N}) \oplus \mathcal{L}^2(E^{\mathrm{c}} \setminus \mathcal{N}),$$

any $A \in \mathcal{S}$ has the matrix representation

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}.$$

For a fixed $B \in \mathcal{S}$, $(BA)_{11} = 0 \Rightarrow B_{12}A_{21} = 0$ and $(BA)_{31} = 0 \Rightarrow B_{32}A_{21} = 0$, where $A_{21} : \mathcal{L}^2(E) \to \mathcal{L}^2(\mathcal{N})$.

Now $B_{12}(\widehat{\mathcal{A}}_1) = 0 = B_{32}(\widehat{\mathcal{A}}_1)$. Following the argument in case (i), by the minimality of $\mathcal{N}$ (or otherwise), we show the existence of a vector g in $\widehat{\mathcal{A}}_1$ such that $g > 0$ on $\mathcal{N}$. Therefore, $B_{12}(g) = 0 = B_{32}(g)$.

By Proposition 2.1, $B_{12} = 0 = B_{32}$. This is true for all $B \in \mathcal{S}$. Thus $\mathcal{L}^2(E) \oplus \mathcal{L}^2(\mathcal{N}) \in \operatorname{Lat} \mathcal{S}$ and hence $\mathcal{S}$ is decomposable.

(b) Next, suppose that $\mu(E\Delta F) > 0$, in which case either $E \setminus F$ or $F \setminus E$ must have positive measure. By considering $\mathcal{S}^*$, if necessary, we can assume with no loss of generality that $\mu(F \setminus E) > 0$. Then, we can write

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F \setminus E) \oplus \mathcal{L}^2(\mathcal{X} \setminus (E \cup F)),$$

where we have $\langle A\chi_E, \chi_{F\setminus E}\rangle = 0$ for all $A \in \mathcal{S}$. With respect to this decomposition, any $A \in \mathcal{S}$ has a matrix representation

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

This reduces to case (i) and hence $\mathcal{S}$ is decomposable. ∎

PROPOSITION 2.6. *If $\mathcal{S}$ is an indecomposable semigroup of nonnegative operators in $\mathcal{L}^2(\mathcal{X})$, then so is every nonzero ideal of $\mathcal{S}$.*

Proof. If $\mathcal{J}$ be a nonzero decomposable ideal of $\mathcal{S}$, then there exists a Borel subset U of $\mathcal{X}$ with $\mu(U)\cdot\mu(U^{c}) > 0$ such that $\mathcal{L}^2(U)$ is invariant under every member of $\mathcal{J}$. This is equivalent to saying that $\langle J\chi_U, \chi_{U^c}\rangle = 0$ for all $J \in \mathcal{J}$. Thus, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^{c}), \tag{2.3}$$

every member J of $\mathcal{J}$ assumes the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Pick a nonzero J of this form and let $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ be an arbitrary element of $\mathcal{S}$ with respect to the decomposition (2.3). Then

$$SJ = \begin{pmatrix} S_{11}A & S_{11}B + S_{12}C \\ S_{21}A & S_{21}B + S_{22}C \end{pmatrix}.$$

Since $\mathcal{J}$ is an ideal, $SJ \in \mathcal{J}$ and therefore, we must have

$$S_{21}A = 0. \tag{2.4}$$

If A is nonzero, then A being a nonnegative operator on $\mathcal{L}^2(U)$, there exists a nonzero, nonnegative function in its range f_0, say. There must exist some $\varepsilon > 0$ for which the set $E = \{x \in U : f_0(x) \geqslant \varepsilon\}$ has positive measure. Then χ_E is a nonzero characteristic function in $\mathcal{L}^2(U)$ and is such that $f_0(x) \geqslant \varepsilon\chi_E(x)$ for all $x \in U$ i.e., $\chi_E \leqslant \alpha f_0$, $\alpha = \frac{1}{\varepsilon} > 0$. From equation (2.4), $S_{21}\chi_E \leqslant \alpha S_{21} f_0 = 0$ i.e., $S_{21}\chi_E = 0$. Thus $\langle S_{21}\chi_E, \chi_F\rangle = 0$ for any Borel subset F in U^{c} of positive measure. Therefore, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G),$$

where $G = (U \setminus E) \dot\cup (U^{c} \setminus F)$, any $S \in \mathcal{S}$ has the following representation

$$S = \begin{pmatrix} S'_{11} & S'_{12} & S'_{13} \\ 0 & S'_{22} & S'_{23} \\ S'_{31} & S'_{32} & S'_{33} \end{pmatrix}.$$

Thus $\langle S\chi_E, \chi_F\rangle = 0$ for all $S \in \mathcal{S}$, which implies by Lemma 2.5 that $\mathcal{S}$ is decomposable which is a contradiction.

Thus assume that $A = 0$ for all $J \in \mathcal{J}$. Then

$$JS = \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} BS_{21} & BS_{22} \\ CS_{21} & CS_{22} \end{pmatrix} \in \mathcal{J}.$$

We have $BS_{21} = 0 = CS_{21}$. Since $\mathcal{S}$ is indecomposable, we can pick an element S in $\mathcal{S}$ for which $S_{21} \neq 0$. Then S_{21}^* is also a nonzero, nonnegative operator from $\mathcal{L}^2(U^c)$ into $\mathcal{L}^2(U)$ and we consider $(BS_{21})^* = S_{21}^* B^*$. As argued above for A, B^* being nonzero would imply decomposability of $\mathcal{S}^*$ and consequently of $\mathcal{S}$. Thus B^* and consequently B is zero. By a similar reasoning $C = 0$, in other words, $J = 0$, a contradiction. Hence every nonzero ideal of $\mathcal{S}$ must be indecomposable. ∎

PROPOSITION 2.7. *Let $\mathcal{S}$ be a collection of nonnegative operators from $\mathcal{L}^2(\mathcal{X})$ into $\mathcal{L}^2(\mathcal{Y})$. Let A and B be nonzero, nonnegative operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{Y}))$ and $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ respectively, satisfying $A\mathcal{S}B = \{0\}$. Then there exist Borel subsets $E \subseteq \mathcal{X}$ and $F \subseteq \mathcal{Y}$ with positive measures such that $\langle S\chi_E, \chi_F\rangle = 0$, for all $S \in \mathcal{S}$.*

Proof. The hypothesis $A\mathcal{S}B = \{0\}$ gives that

$$\langle ASBf, g\rangle = 0 \Rightarrow \langle SBf, A^*g\rangle = 0 \quad \text{for all } f \in \mathcal{L}^2(\mathcal{X}) \text{ and for all } g \in \mathcal{L}^2(\mathcal{Y}). \tag{2.5}$$

Now, since B is nonnegative and nonzero, its range must contain a nonzero, nonnegative element, say f_0. Similarly, there exists a nonzero, nonnegative function g_0 in the range of A^*.

Further, f_0 being nonnegative and nonzero, we can find Borel subsets $E \in \mathcal{X}$ and F in $\mathcal{Y}$ of positive measure such that $\chi_E \leqslant \alpha f_0$ and $\chi_F \leqslant \beta g_0$ for some positive scalars α and β.

For any $S \in \mathcal{S}$, since S is a nonnegative operator, we have $S\chi_E \leqslant \alpha S f_0$. By the property of monotonicity for integrals,

$$\langle S\chi_E, \chi_F\rangle \leqslant \langle \alpha S f_0, \beta g_0\rangle = \alpha\beta\langle S f_0, g_0\rangle = 0 \quad \text{for all } S \in \mathcal{S} \text{ (from (2.5))}$$

which proves the proposition. ∎

COROLLARY 2.8. *A nonnegative semigroup of operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is decomposable if and only if there exist nonzero, nonnegative operators A and B on $\mathcal{L}^2(\mathcal{X})$, not necessarily in $\mathcal{S}$, such that $A\mathcal{S}B = \{0\}$.*

Proof. By the preceding proposition, the condition $A\mathcal{S}B = \{0\}$ implies $\langle S\chi_E, \chi_F\rangle = 0$ for all $S \in \mathcal{S}$ and for Borel subsets E, F of $\mathcal{X}$ with $\mu(E)\cdot\mu(F) > 0$. This gives decomposability of $\mathcal{S}$ by Lemma 2.5.

Conversely, suppose $\mathcal{S}$ is decomposable. Then, with respect to some decomposition of $\mathcal{L}^2(\mathcal{X})$, every $S \in \mathcal{S}$ has the matrix representation

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}.$$

If with respect to the same decomposition, we define two nonzero, nonnegative operators

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

then it is easily verified that $ASB = 0$ for all $S \in \mathcal{S}$ i.e., $A\mathcal{S}B = \{0\}$. ∎

3. WHEN IS A NONNEGATIVE BAND DECOMPOSABLE?

This section is devoted to studying the decomposability of nonnegative bands in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. We shall first establish the decomposability of a single nonnegative idempotent which is already a proven result (cf. Zhong, [10]). We are including the proof here for the sake of completeness and also because it has a slightly different approach from Zhong's and works for a more general class of nonnegative idempotents.

LEMMA 3.1. *Let A be a nonnegative idempotent on $\mathcal{L}^2(\mathcal{X})$ and let f be a nonnegative element in the range of A. Fix a nonnegative representative of f (and still denote it by f). If $U = \operatorname{supp} f = \{x : f(x) > 0\}$, then $\mathcal{L}^2(U) \in \operatorname{Lat} A$.*

Proof. It suffices to prove that $\langle A\chi_U, \chi_{U^c}\rangle = 0$. By hypothesis,

$$\begin{aligned}&\langle Af, \chi_{U^c}\rangle = 0 \quad (\text{as } Af = f \text{ and } \operatorname{supp} f = U)\\ &\Rightarrow \langle f, A^*\chi_{U^c}\rangle = 0 \Rightarrow \int_U (A^*\chi_{U^c})(x) f(x)\mu(\mathrm{d}x) = 0\\ &\Rightarrow (A^*\chi_{U^c})(x) f(x) = 0 \quad \text{a.e. on } U \text{ as } A^* \geqslant 0.\end{aligned}$$

But $f(x) > 0$ a.e. on U. Therefore, $(A^*\chi_{U^c})(x) = 0 \Rightarrow \langle A^*\chi_{U^c}, \chi_U\rangle = 0 \Rightarrow \langle A\chi_U, \chi_{U^c}\rangle = 0$ for almost all $x \in U$. ∎

LEMMA 3.2. *Let A be as in the preceding lemma. If an element f in the range of A is real, then there exists a nonnegative element h in $\mathcal{L}^2(\mathcal{X})$ such that $Ah = 0$ and $f^+ + h, f^- + h$ are in the range of A.*

Proof. For the proof see [10]. ∎

LEMMA 3.3. *If an element f in $\mathcal{L}^2(\mathcal{X})$ belongs to the range of a nonnegative idempotent A, then the real part* $\operatorname{Re} f$ *and the imaginary part* $\operatorname{Im} f$ *of f are also in the range of A.*

Proof. This is so because $\operatorname{Re} f + \mathrm{i}\operatorname{Im} f = f = Af = A(\operatorname{Re} f) + \mathrm{i}A(\operatorname{Im} f)$ and A being nonnegative, it sends real valued functions to real valued functions. ∎

DEFINITION 3.4. By $\ker \mathcal{A}$, for any collection $\mathcal{A}$ of operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, we mean $\{f \in \mathcal{L}^2(\mathcal{X}) : Sf = 0 \text{ for all } S \in \mathcal{A}\}$.

THEOREM 3.5. *Let A be a nonnegative idempotent in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ of rank at least two. Then A is decomposable.*

Proof. If $Ah = 0$ for some nonzero, nonnegative h, then A is zero on $\mathcal{L}^2(\operatorname{supp} h)$ and is thus decomposable (by Corollary 2.2). Therefore assume that $\ker A$ contains no nonzero, nonnegative element. By Lemma 3.3, if an element is in the range of A, then so are its real and imaginary parts. Thus, we can obtain a basis of the range of A consisting of real elements. Further, with our assumption together with Lemma 3.2, we can obtain a basis of the range of A containing nonnegative elements.

Since $\operatorname{rank} A \geqslant 2$, A has at least two nonnegative, nonzero linearly independent elements in its range, say f and g. If either of them is zero on a set of positive measure, we are done by Lemma 3.1. Therefore, assume that both f and g are positive.

Consider the following nonempty subsets of reals $S_1 = \{r : f - rg > 0\}$ and $S_2 = \{r : f - rg < 0\}$.

Let $r_0 = \sup S_1$ and $s_0 = \inf S_2$. The linear independence f and g ensures that $r_0 < s_0$. We can pick a number p such that $r_0 < p < s_0$. Therefore, $f - pg \ngtr 0$ and $f - pg \nless 0$. Hence $f - pg$ is a mixed vector and it is clearly in the range of A. Existence of such a vector in the range of A gives decomposability of A, for if u is such that $Au = u, u = u^+ - u^-, u^+, u^-$ nonzero, then by Lemma 3.2, we can find $h \geqslant 0$ in $\mathcal{L}^2(\mathcal{X})$, $Ah = 0$ such that $u^+ + h$ and $u^- + h$ are in the range of A. But by our assumption, $h = 0$. Therefore, u^+, u^- are in the range of A. By Lemma 3.1, $\mathcal{L}^2(\operatorname{supp} u^+)$ is a nontrivial standard invariant subspace for A. Hence A is decomposable. ∎

Having established the decomposability of a single nonnegative idempotent with rank at least two, we now prove that it has a very special standard block triangularization. This will require a couple of lemmas and some definitions.

LEMMA 3.6. *An indecomposable, nonnegative rank-one operator on $\mathcal{L}^2(\mathcal{X})$ is positive.*

Proof. Let A be an indecomposable, nonnegative rank-one operator on $\mathcal{L}^2(\mathcal{X})$. Then $A = u \otimes v$, where u, v are nonzero, nonnegative vectors in $\mathcal{L}^2(\mathcal{X})$, so that $Af = \langle f, v\rangle u$ for all $f \in \mathcal{L}^2(\mathcal{X})$.

Suppose A is not positive. Then there exists a nonzero, nonnegative vector f in $\mathcal{L}^2(\mathcal{X})$ for which Af is not positive. In other words, the set $E = \{x \in \mathcal{X} : (Af)(x) = 0\}$ has positive measure. Also, if $Af = 0$, then $A \equiv 0$ on $\mathcal{L}^2(\operatorname{supp} f)$ and is thus decomposable (Corollary 2.2) which is not possible. Therefore $Af \neq 0$. Now $E = \{x \in \mathcal{X} : u(x) = 0\}$ (because $Af \neq 0$). Since $\mu(E) > 0$, χ_E is a nonzero, nonnegative vector and is such that $A^*\chi_E = 0$. This implies that $A^* \equiv 0$ on $\mathcal{L}^2(E)$ (by Proposition 2.1). Thus A^* and consequently A is decomposable which is a contradiction. Hence A must be positive. ∎

DEFINITION 3.7. A nonnegative semigroup $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ will be called a *full semigroup* if neither $\ker \mathcal{S}$ nor $\ker \mathcal{S}^*$ has a nonzero, nonnegative vector. A single nonnegative operator is called *full* if the semigroup generated by it is full.

DEFINITION 3.8. A chain of subspaces of $\mathcal{L}^2(\mathcal{X})$ is called *maximal* if it is not properly contained in any other chain of subspaces of $\mathcal{L}^2(\mathcal{X})$.

If $\mathcal{C}$ is any chain of subspaces and $\mathcal{M} \in \mathcal{C}$, then we define $\mathcal{M}_-$ to be the closed linear span of all those members of $\mathcal{C}$ which are properly contained in $\mathcal{M}$. It is not difficult to see ([8]) that a subspace chain is maximal if and only if

(i) $\mathcal{C}$ is closed under arbitrary spans and intersections,
(ii) for each $\mathcal{M}$ in $\mathcal{C}$, $\mathcal{M} \ominus \mathcal{M}_-$ is at most one-dimensional.

A maximal chain $\mathcal{C}$ is said to be *continuous* if $\mathcal{M} = \mathcal{M}_-$ for each $\mathcal{M}$ in $\mathcal{C}$, in other words, $\mathcal{C}$ has no gaps in it.

DEFINITION 3.9. A collection of operators $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is said to have a *continuous standard triangularization* if

(i) $\operatorname{Lat}\mathcal{S}$ contains a continuous maximal chain, say $\mathcal{C}$,

(ii) each member of $\mathcal{C}$ is a standard subspace.

EXAMPLE 3.10. For $t \in [0,1]$, the multiplication operator $M : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ defined by $(Mf)(t) = tf(t)$ is a nonnegative operator. For any $\alpha \in [0,1]$, define $\mathcal{M}_\alpha = \{f \in \mathcal{L}^2[0,1] : f(t) = 0, \forall t \geqslant \alpha\}$. Then $\{\mathcal{M}_\alpha : \alpha \in [0,1]\}$ is a maximal subspace chain which is continuous and consists of standard invariant subspaces for M. Thus M has a continuous standard triangularization and since $M = M^*$, so does M^*.

EXAMPLE 3.11. Let $\mathcal{H} = \mathcal{L}^2[0,1] \oplus \mathcal{L}^2[0,1]$ and define $E : \mathcal{H} \to \mathcal{H}$ by

$$E = \begin{pmatrix} M & M \\ I - M & I - M \end{pmatrix},$$

where M is the preceding multiplication operator and $I - M : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ is the multiplication operator by $1 - t$. Then E is a nonnegative idempotent and

$$E^* = \begin{pmatrix} M & I - M \\ M & I - M \end{pmatrix}.$$

Let $\mathcal{N}_\alpha = \mathcal{M}_\alpha \oplus \mathcal{M}_\alpha$ for $\alpha \in [0,1]$. Then using the fact that $\{\mathcal{M}_\alpha : \alpha \in [0,1]\}$ is maximal, it is not hard to prove that $\{\mathcal{N}_\alpha : \alpha \in [0,1]\}$ is a maximal subspace chain in $\mathcal{H}$ which is continuous. Also it consists of standard invariant subspaces for E and E^*. Thus E and E^* have a simultaneous continuous standard triangularization.

LEMMA 3.12. *Let A in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and B in $\mathcal{B}(\mathcal{L}^2(\mathcal{Y}))$ be nonzero, nonnegative operators such that neither $\ker A$ nor $\ker B^*$ has a nonzero, nonnegative vector. If $S : \mathcal{L}^2(\mathcal{Y}) \to \mathcal{L}^2(\mathcal{X})$ is a nonnegative operator such that $ASB = 0$, then $S = 0$.*

Proof. Suppose SB is nonzero. Then $SB : \mathcal{L}^2(\mathcal{Y}) \to \mathcal{L}^2(\mathcal{X})$ is a nonzero, nonnegative operator. Therefore, there exists a nonzero, nonnegative vector f in $\mathcal{L}^2(\mathcal{Y})$ such that SBf is nonzero, nonnegative. Write $g = SBf$. Then $Ag = 0$ which implies that $\ker A$ has a nonzero, nonnegative vector, a contradiction. Therefore, we must have $SB = 0$ which gives that $B^*S^* = 0$. If S is nonzero, S^* is nonzero but then $\ker B^*$ has a nonzero, nonnegative vector, a contradiction. Hence, we have $S = 0$. ∎

THEOREM 3.13. (a) *Let A be a nonnegative idempotent on $\mathcal{L}^2(\mathcal{X})$ with rank r which is full.*

(i) *If r is finite, then there exists a decomposition*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r)$$

with respect to which

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix},$$

where each $A_i : \mathcal{L}^2(\mathcal{X}_i) \longrightarrow \mathcal{L}^2(\mathcal{X}_i)$ is a positive idempotent of rank one.

(ii) *If $r = \infty$, then with respect to some direct sum decomposition*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{Y}_1) \oplus \mathcal{L}^2(\mathcal{Y}_2), \qquad A = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},$$

where E and F have the following descriptions: If $E \neq 0$, then $\mathcal{L}^2(\mathcal{Y}_1) = \bigoplus_{i=1}^{N} \mathcal{L}^2(\mathcal{Z}_i)$ for some $N \leqslant \infty$, where $\mathcal{L}^2(\mathcal{Z}_i)$ are standard subspaces of $\mathcal{L}^2(\mathcal{X})$ which are reducing under A, and $E : \mathcal{L}^2(\mathcal{Y}_1) \to \mathcal{L}^2(\mathcal{Y}_1)$ has the block diagonal form

$$\begin{pmatrix} E_1 & & & & \\ & E_2 & & & \\ & & \ddots & & \\ & & & E_i & \\ & & & & \ddots \end{pmatrix}$$

with each $E_i : \mathcal{L}^2(\mathcal{Z}_i) \to \mathcal{L}^2(\mathcal{Z}_i)$ being a positive idempotent of rank one.

If $F \neq 0$, then F and F^* have a simultaneous continuous standard triangularization.

(b) *In general, if A is not full, then there exists a decomposition of $\mathcal{L}^2(\mathcal{X})$, say*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \mathcal{L}^2(\mathcal{W}_3),$$

where $\mathcal{L}^2(\mathcal{W}_i)$ $(i = 1, 2, 3)$ are standard invariant subspaces of $\mathcal{L}^2(\mathcal{X})$ such that with respect to this decomposition

$$A = \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix},$$

where $E : \mathcal{L}^2(\mathcal{W}_2) \to \mathcal{L}^2(\mathcal{W}_2)$ is an idempotent of the form in (i) *or* (ii) *according as rank of A is finite or infinite.*

Proof. (a) (i) When r is finite, we prove the result by induction on r. If $r = 1$, then Lemma 3.6 applies. Let $r > 1$, then we know that A is decomposable and therefore, there exists a Borel subset $U \subseteq \mathcal{X}$ with $\mu(U) \cdot \mu(U^{\mathrm{c}}) > 0$ such that with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^{\mathrm{c}}), \qquad A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix},$$

where with no loss of generality, we can assume that A_1 and A_2 are nonzero. Now $A^2 = A$ implies that $A_1 X + X A_2 = X$. Then $A_1 X A_2 = 0$. Since A is full, $\ker A_1$ and $\ker A_2^*$ have no nonzero, nonnegative vector. Therefore, by Lemma 3.12, $X = 0$. Thus

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Since A_1 and A_2 are nonzero, their ranks are less than r and both are full because A is full. Hence induction applies and we obtain the desired result.

(ii) If r is infinite, A is certainly decomposable. Let $\mathcal{C}$ be a maximal chain in $\operatorname{Lat}' A$. Our first claim is that each gap in the chain is reducing for A. Let

$\mathcal{N} \ominus \mathcal{M}$ be a gap where $\mathcal{M} \subset \mathcal{N}$ in $\mathcal{C}$. Consider the block triangularization of A with respect to the following decomposition of $\mathcal{L}^2(\mathcal{X})$,

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{M} \oplus (\mathcal{N} \ominus \mathcal{M}) \oplus (\mathcal{L}^2(\mathcal{X}) \ominus \mathcal{N}),$$

being

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}.$$

If we first regard A as the 2×2 block matrix $\begin{pmatrix} A_0 & X_0 \\ 0 & B_0 \end{pmatrix}$, where $A_0 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, $X_0 = \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}$, $B_0 = A_{33}$; then, as shown in part (i), the fullness of A gives $X_0 = 0$, i.e., $A_{13} = 0 = A_{23}$.

By a similar argument, we can show that $A_{12} = 0 = A_{13}$. Therefore

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}.$$

This shows that $\mathcal{N} \ominus \mathcal{M}$ is reducing, which proves our claim.

Also, the maximality of $\mathcal{C}$ implies that the compression of A to each gap, if nonzero, must be an indecomposable (and thus positive) idempotent of rank one. Further, because of separability of $\mathcal{L}^2(\mathcal{X})$, there can only be countably many reducing gaps. Thus, after a permutation of basis, we can obtain a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{Y}_1) \oplus \mathcal{L}^2(\mathcal{Y}_2)$$

with respect to $A = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$, where $\mathcal{L}^2(\mathcal{Y}_1) = \bigoplus_{i=1}^{N} \mathcal{L}^2(\mathcal{Z}_i)$, $\{\mathcal{L}^2(\mathcal{Z}_i)\}_{i=1}^{N}, N \leqslant \infty$ being a collection of reducing subspaces of A and $E : \mathcal{L}^2(\mathcal{Y}_1) \to \mathcal{L}^2(\mathcal{Y}_1)$ has the block diagonal form as mentioned in the statement of the theorem. The fullness of A makes both $\mathcal{L}^2(\mathcal{Y}_1)$ and $\mathcal{L}^2(\mathcal{Y}_2)$ reducing standard subspaces. Further, since all the gaps have been absorbed in $\mathcal{L}^2(\mathcal{Y}_1)$, the operators F and F^* are continuously triangularizable and since this triangularization results from a maximal chain of standard subspaces, we can say that F and F^* have a simultaneous continuous standard triangularization.

(b) Here, we consider the general case when A is not full.

Suppose $\mathcal{A}$ is the collection of all nonzero, nonnegative vectors in $\ker A$. By Lemma 2.4, we can find a minimal subset G in $\mathcal{X}$, defined up to a null set, on whose complement all the vectors in $\mathcal{A}$ vanish. This gives the existence of a vector f in $\mathcal{A}$ such that $G = \operatorname{supp} f$ and so $A \equiv 0$ on $\mathcal{L}^2(G)$.

Similarly, we can find a set G^* of positive measure such that $A^* \equiv 0$ on $\mathcal{L}^2(G^*)$. Then, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(G) \oplus \mathcal{L}^2(\mathcal{X} \setminus (G \cup G^*)) \oplus \mathcal{L}^2(G^*),$$

A has the representation $\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}$, where $E^2 = E$, $X = XE$, $Y = EY$ and $Z = XEY$.

Renaming $\mathcal{L}^2(G) = \mathcal{L}^2(\mathcal{W}_1)$, $\mathcal{L}^2(\mathcal{X} \setminus (G \cup G^*)) = \mathcal{L}^2(\mathcal{W}_2)$ and $\mathcal{L}^2(G^*) = \mathcal{L}^2(\mathcal{W}_3)$, we obtain the representation of A as described in part (b) of the theorem. Also, these equations show that E is full and hence it is of the form described in part (a) of the theorem. ∎

From a single nonnegative idempotent, we now move on to analyze a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with more than one element in it. As in the discrete case, we shall find that if a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with rank of each member being > 1 has even a single member of finite rank, it is decomposable.

THEOREM 3.14. *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ having at least one element of finite rank and with* $\operatorname{rank}(S) > 1$ *for all S in $\mathcal{S}$. Then $\mathcal{S}$ is decomposable.*

Proof. Let $m = \min\{\operatorname{rank}(S) : S \in \mathcal{S}\}$; then $m > 1$. Let $\mathcal{J}$ be the set of all elements of rank m in $\mathcal{S}$. For any $S \in \mathcal{S}$ and $J \in \mathcal{J}$, $\operatorname{rank}(SJ) = \operatorname{rank}(JS) = m$ which implies that $SJ, JS \in \mathcal{J}$. Thus $\mathcal{J}$ is a nonzero ideal of $\mathcal{S}$.

Now $\mathcal{S}$ is decomposable if and only if $\mathcal{J}$ is decomposable. Therefore, we can assume with no loss of generality that $\mathcal{S} = \mathcal{J}$ so that $\mathcal{S}$ has constant rank m.

Select a $P \in \mathcal{S}$. Let S be an arbitrary element of $\mathcal{S}$ and consider PSP. This is an idempotent whose range is contained in the range of P and whose null space contains the null space of P and since $\operatorname{rank}(PSP) = m = \operatorname{rank}(P)$, we have $PSP = P$. Thus $P\mathcal{S}P = \{P\}$.

Since $m > 1$, by Theorem 3.13, we can find a Borel subset U of $\mathcal{X}$ with positive measure, such that with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(U) \oplus \mathcal{L}^2(U^{\mathrm{c}}), \tag{3.1}$$

P has the matrix representaion $\begin{pmatrix} P_1 & X \\ 0 & P_2 \end{pmatrix}$, where both P_1 and P_2 are nonzero.

Pick an arbitrary S in $\mathcal{S}$ and let its matrix representation with respect to (3.1) be $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$. Then $PSP = P$ implies that $P_2 S_{21} P_1 = 0$. By Proposition 2.7, there exist Borel subsets E, F in U and U^{c} respectively having positive measures such that

$$\langle S_{21}\chi_E, \chi_F\rangle = 0.$$

Finally, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(E) \oplus \mathcal{L}^2(F) \oplus \mathcal{L}^2(G),$$

where $G = (U \setminus E) \cup (U^{\mathrm{c}} \setminus F)$, every $S \in \mathcal{S}$ has the following matrix representation

$$S = \begin{pmatrix} S'_{11} & S'_{12} & S'_{13} \\ 0 & S'_{22} & S'_{23} \\ S'_{31} & S'_{32} & S'_{33} \end{pmatrix}.$$

This shows that $\langle S\chi_E, \chi_F\rangle = 0$ for all $S \in \mathcal{S}$. Hence, by Lemma 2.5, $\mathcal{S}$ is decomposable. ∎

THEOREM 3.15. *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ such that* $\operatorname{rank}(S) > 1$ *for all S in $\mathcal{S}$ and $\mathcal{S}$ has at least one element of finite rank. Then any maximal standard block triangularization of $\mathcal{S}$ has the property that the compression of $\mathcal{S}$ to each nonzero gap constitutes a nonnegative band with at least one element of rank one in it.*

Proof. Same as in the finite-dimensional case ([5]). ∎

In the Theorem 3.14, we saw that the decomposability of a band in which every member has rank > 1 and which has at least one finite-rank member reduced to the decomposability of a constant-rank band. The most pertinent question to be asked after this is:

QUESTION 3.16. Is every constant-rank nonnegative band decomposable?

The answer to the question above is in the negative if the rank is one. A simple example is the band $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$. If the constant rank is greater than one, then we know that the band is decomposable ([5]). This completes our analysis of the problem in finite dimensions. For a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, we have seen in the proof of Theorem 3.14 that with constant finite rank greater than one, the band is decomposable. Now, the natural question which occurs is whether a constant infinite-rank nonnegative band is decomposable? The answer is a resounding no as we illustrate through a counter example in $\mathcal{B}(l^2)$.

EXAMPLE 3.17. There exists an indecomposable nonnegative band in $\mathcal{B}(l^2)$ in which every member has infinite rank.

Proof. For each integral i, define an operator S_i as follows

$$S_i = \begin{pmatrix} T_i & & \\ & T_i & \\ & & \ddots \end{pmatrix},$$

where T_i is a $2^i \times 2^i$ block with each entry equal to $1/2^i$. Let $\mathcal{S} = \{S_0, S_1, S_2, \ldots\}$. It is easily verified that for $i \leqslant j$, $S_iS_j = S_j$ and $S_jS_i = S_j$. Thus $\mathcal{S}$ is a nonnegative band where each S_i is of infinite rank. Suppose $\mathcal{S}$ has a common zero entry, say $(S_i)_{\alpha\beta} = 0$ for all $S_i \in \mathcal{S}$. Now, we can find i and $ji \leqslant j$ such that $\alpha \leqslant 2^i$ and $\beta \leqslant 2^j$. But then S_j will have the entry $(S_i)_{\alpha\beta}$ in its first diagonal block T_j which is positive. Thus $\mathcal{S}$ cannot have a common zero entry and hence is indecomposable. ∎

4. THE STRUCTURE OF NONNEGATIVE, CONSTANT FINITE-RANK BANDS

We saw in the previous section that constant-rank bands play a significant role in ascertaining the decomposability of nonnegative bands. It would be therefore interesting to study their structure completely which will be our task in this section. It is a generalization of the same in the finite-dimensional case. We already know that an infinite-rank nonnegative band may not be decomposable; therefore we shall restrict ourselves to nonnegative bands with constant finite rank.

Lemma 4.1. *If $\mathcal{S}$ is a band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ of nonnegative operators with constant finite rank r, then $\mathcal{S}$ has a standard block triangularization with r nonzero diagonal blocks, each block constituting an indecomposable band of rank-one operators. Furthermore, no two consecutive diagonal blocks are zero. Therefore, if k is the total number of diagonal blocks, then $k \leqslant 2r+1$.*

Proof. The proof runs exactly on the same lines as for the finite-dimensional case ([5]). ∎

Lemma 4.2. *Let $\mathcal{S}$ be a nonnegative full band of rank-one operators. Then $\mathcal{S}$ is indecomposable.*

Proof. Same as for the finite-dimensional case ([5]). ∎

Theorem 4.3. *Let $\mathcal{S}$ be a band of nonnegative operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with constant finite rank r.*

(i) *If $\mathcal{S}$ is full, then there exists a decomposition*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r),$$

with respect to which every member S of $\mathcal{S}$ is of the form $\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix}$,

where each $\mathcal{S}_i = \{S_i \in \mathcal{L}^2(\mathcal{X}_i) : S \in \mathcal{S}\}$ is an indecomposable band of rank-one operators.

(ii) *In general, there exists a decomposition*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1') \oplus \mathcal{L}^2(\mathcal{X}_2') \oplus \mathcal{L}^2(\mathcal{X}_3'),$$

with respect to which every member S of $\mathcal{S}$ is of the form $\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix}$,

where X, Y are nonnegative operators on suitable spaces. Furthermore, the diagonal blocks in $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ constitute a band of the form in case (i).

Proof. (i) The proof is exactly as in the finite-dimensional case ([5]).

(ii) Now, let us consider the general case. Suppose $\mathcal{A}$ is the collection of all the nonzero, nonnegative vectors in $\ker \mathcal{S}$. Just as in the proof of Theorem 3.13 (b), we can find sets G and G^* of positive measure such that $\mathcal{S} \equiv 0$ on $\mathcal{L}^2(G)$ and $\mathcal{S}^* \equiv 0$ on $\mathcal{L}^2(G^*)$.

Then, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(G) \oplus \mathcal{L}^2(\mathcal{X} \setminus (G \cup G^*)) \oplus \mathcal{L}^2(G^*),$$

every member S in $\mathcal{S}$ has the form $\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}$, where $E^2 = E$, $X = XE$, $Y = EY$, and $Z = XEY$.

These equations show that the set $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ of the middle diagonal blocks is such that neither $\mathcal{S}_0$ nor $\mathcal{S}_0^*$ have any nonzero, nonnegative vectors in their null spaces and thus $\mathcal{S}_0$ is of the form in part (i) of the theorem. ∎

REMARK 4.4. If in the statement of the theorem above, $\mathcal{S}$ is taken to be a maximal band, then it is readily observed that the bands $\mathcal{S}_i$ must be maximal. In part (ii), $\mathcal{S}_0$ and the collection of all X, Y are maximal too.

In Theorem 4.7, we prove the converse of part (i) of the preceding theorem to obtain a characterization of maximal, nonnegative, constant-rank bands which are full. This will require a couple of lemmas, which may also be of independent interest.

LEMMA 4.5. *Let $\mathcal{S}$ be a nonnegative, indecomposable semigroup in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and f be a nonzero, nonnegative vector in $\mathcal{L}^2(\mathcal{X})$. Let $\mathcal{A}$ be the set of all nonnegative linear combinations of the members of $\{Sf : S \in \mathcal{S}\}$. Then $\overline{\mathcal{A}}$ contains a positive vector in $\mathcal{L}^2(\mathcal{X})$.*

Proof. Since $\mathcal{L}^2(\mathcal{X})$ is separable, so is the set $\mathcal{S}f$. Therefore, let $\mathcal{M} = \{S_1 f, S_2 f, \dots\}$ be a countable dense subset of $\mathcal{S}f$, where $S_1 f, S_2 f, \dots$ are the chosen representatives of the equivalence classes of functions in $\mathcal{M}$. Write $U = \bigcup_i \operatorname{supp} S_i f$. Then the function g defined by

$$g = \frac{S_1 f}{\|S_1 f\|} + \frac{1}{2}\frac{S_2 f}{\|S_2 f\|} + \frac{1}{2^2}\frac{S_3 f}{\|S_3 f\|} + \cdots$$

is a nonegative vector in $\overline{\mathcal{A}}$ with support U; in other words, $g > 0$ in $\mathcal{L}^2(U)$. We shall prove that g is the desired positive vector in $\mathcal{L}^2(\mathcal{X})$, for which we need to show that $U = \mathcal{X}$ (up to a null set).

By construction, $g = 0$ a.e. on U^{c}. This implies that $S_i f = 0$ a.e. on U^{c} for every i, since each $S_i f$ is nonnegative. By the density of $\mathcal{M}$ in $\mathcal{S}f$, $Sf = 0$ a.e. on U^{c} for every $S \in \mathcal{S}$, and thus

$$(4.1)\quad Sg = \frac{SS_1 f}{\|S_1 f\|} + \frac{1}{2}\frac{SS_2 f}{\|S_2 f\|} + \frac{1}{2^2}\frac{SS_3 f}{\|S_3 f\|} + \cdots = 0 \quad \text{a.e. on } U^{\mathrm{c}} \text{ for every } S \in \mathcal{S}.$$

Our claim is that $\mathcal{L}^2(U)$ is invariant under $\mathcal{S}$. Since $\mathcal{S}$ is indecomposable, this will prove that $\mathcal{L}^2(U) = \mathcal{L}^2(\mathcal{X})$. We prove this considering two possibilities: (i) g is bounded below on U, and (ii) g is not bounded below on U.

In case (i), there exists a nonnegative, nonzero scalar α such that $g(x) \geqslant \alpha$ a.e. on U. Let $E = \{x \in U : g(x) \geqslant \alpha\}$, then $\mu(E^{\mathrm{c}} \cap U) = 0$. Also

$$g(x) \geqslant \alpha\chi_E(x) \quad \text{for all } x \in U, \text{ i.e. } \chi_E \leqslant \frac{1}{\alpha} g.$$

For any $S \in \mathcal{S}$, $S\chi_E \leqslant \frac{1}{\alpha} Sg$. Using (4.1), we obtain $S\chi_E = 0$ a.e. on U^{c} for all $S \in \mathcal{S}$, i.e., $\mathcal{L}^2(U)$ is invariant under $\mathcal{S}$.

If g is not bounded below on U, we can write U as a disjoint union of the sets U_n, where

$$U_n = \left\{ x \in U : \frac{1}{n+1} < g(x) \leqslant \frac{1}{n} \right\}.$$

Now, g is bounded below on each U_n. Just as in case (i), we obtain $\langle S\chi_{U_n}, \chi_{U^c}\rangle = 0$ for all $S \in \mathcal{S}$. But $\chi_U = \Sigma\chi_{U_n}$. This will give $\langle S\chi_U, \chi_{U^c}\rangle = 0$ and we are in case (i). ∎

LEMMA 4.6. *Suppose $\mathcal{S}$ is a direct sum of r nonnegative, indecomposable semigroups $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$ so that each member of $\mathcal{S}$ has a block diagonal representation* $\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix}$*, where $S_i \in \mathcal{S}_i$, $i = 1, \ldots, r$ with respect to some decomposition of $\mathcal{L}^2(\mathcal{X})$, say*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r).$$

Then every $\mathcal{M} \in \mathrm{Lat}'\,\mathcal{S}$ is of the form $\mathcal{M} = \bigoplus_{i=1}^{r} \varepsilon_i \mathcal{L}^2(\mathcal{X}_i)$, where each ε_i is either 0 or 1.

Proof. Obviously, $\mathcal{L}^2(\mathcal{X}_i) \in \mathrm{Lat}'\,\mathcal{S}$ for every $i = 1, \ldots, r$. Further, each $\mathcal{S}_i$ being indecomposable, $\mathcal{L}^2(\mathcal{X}_i)$ is a minimal standard subspace in $\mathrm{Lat}'\,\mathcal{S}$ in the sense that $\mathcal{S}$ has no nonzero standard invariant subspace properly contained in it. Now let $\mathcal{M} \in \mathrm{Lat}'\,\mathcal{S}$, where $\mathcal{M} = \mathcal{L}^2(U)$ for some Borel subset U of $\mathcal{X}$ of positive measure. We first show that if a nonzero, nonnegative f is in $\mathcal{M}$ such that $\operatorname{supp} f = \mathcal{X}_i$ for some i, then $\mathcal{L}^2(\mathcal{X}_i) \subseteq \mathcal{M}$. Suppose $\mu(U^c \cap \mathcal{X}_i) > 0$. Now $f \in \mathcal{M}$ implies that $f = 0$ a.e. on U^c, and, in particular, $f = 0$ a.e. on $U^c \cap \mathcal{X}_i$ which is contained in $\mathcal{X}_i$ i.e., f is zero a.e on a subset of $\mathcal{X}_i$ of positive measure which is not possible as $\operatorname{supp} f = \mathcal{X}_i$. Therefore, we must have $\mu(U^c \cap \mathcal{X}_i) = 0$ and this proves that $\mathcal{X}_i \subseteq U$ up to a null set, and we are done.

Next, observe that we can write

$$\mathcal{M} = \mathcal{L}^2(U_1) \oplus \cdots \oplus \mathcal{L}^2(U_r),$$

where $U_i = U \cap \mathcal{X}_i$. Let $f_i = \chi_{U_i}$; then the vector $f = (0\ \ldots\ f_i\ \ldots\ 0)^{\mathrm{T}} \in \mathcal{M}$ and by our assumption $\mathcal{S}f \in \mathcal{M}$ where $\mathcal{S}f = \{(0\ \ldots\ Sf_i\ \ldots\ 0)^{\mathrm{T}} : S \in \mathcal{S}_i\}$. Define $\varepsilon_i = \begin{cases} 0 & \text{if } f_i \text{ is zero,} \\ 1 & \text{if } f_i \text{ is nonzero.} \end{cases}$ To complete the proof, we must show that whenever $\varepsilon_i = 1$, we have $\mathcal{L}^2(\mathcal{X}_i) \subseteq \mathcal{M}$. Now $\mathcal{S}_i$ is a band acting on $\mathcal{L}^2(\mathcal{X}_i)$ and $f_i \in \mathcal{L}^2(U_i)$. By Lemma 3.14, we obtain a positive vector, say g_i, in $\mathcal{L}^2(\mathcal{X})$ which is also a limit of nonnegative linear combinations of the members of $\{\mathcal{S}_i f_i\}$. Consider the vector $g = (0\ \ldots\ g_i\ \ldots\ 0)^{\mathrm{T}}$. Then $g \in \mathcal{M}$ and $\operatorname{supp} g = \mathcal{X}_i$. Therefore, by what we have proved above, we obtain $\mathcal{L}^2(\mathcal{X}_i) \subseteq \mathcal{M}$. ∎

Theorem 4.7. *A direct sum of r maximal, indecomposable, nonnegative rank-one bands is a maximal band of constant rank r.*

Proof. For $r = 1$, the result is obvious. Therefore let $r > 1$. Suppose S_1, $S_2, \dots, S_r$ are r maximal, indecomposable, nonnegative rank-one bands and consider their direct sum. Every member S of $\mathcal{S}$ is of the form $\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix}$, where $S_i \in \mathcal{S}_i$, $i = 1, 2, \dots, r$. Also suppose that this representation of the members of $\mathcal{S}$ is with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r),$$

where $\mathcal{X}_1, \dots, \mathcal{X}_r$ are Borel subsets of $\mathcal{X}$ of positive measure.

If $\mathcal{S}$ is not maximal, then let $\mathcal{S}'$ be a band properly containing $\mathcal{S}$ and having constant rank r. Now observe that $\mathcal{S}$ is a full band. Therefore, $\mathcal{S}'$ is full too. By part (i) of Theorem 4.3, $\mathcal{S}'$ is a direct sum of r rank-one, indecomposable, nonnegative bands, say, $\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_r$. Now, $\operatorname{Lat}' \mathcal{S}' \subseteq \operatorname{Lat}' \mathcal{S}$. By the previous lemma, the cardinality of both $\operatorname{Lat}' \mathcal{S}$ and $\operatorname{Lat}' \mathcal{S}'$ is the same which is 2^r. Therefore, we must have $\operatorname{Lat}' \mathcal{S} = \operatorname{Lat}' \mathcal{S}'$. Thus we can rearrange the spaces $\mathcal{L}^2(\mathcal{X}_i)$ in the direct sum above to obtain a new decomposition of $\mathcal{L}^2(\mathcal{X})$ so that $\mathcal{S}_i \subseteq \mathcal{S}'_i$. But since the bands $\mathcal{S}_i$ are maximal, we have $\mathcal{S}'_i = \mathcal{S}_i$ for each i. Hence $\mathcal{S}$ is maximal. ∎

Theorem 4.3 and the Remark 4.4 can be combined to give the following characterization of maximal nonnegative bands of constant finite rank.

Theorem 4.8. *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ of constant finite rank r.*

(i) *If $\mathcal{S}$ is full, then $\mathcal{S}$ is maximal if and only if*

$$\mathcal{S} = \left\{ \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{pmatrix} : S_i \in \mathcal{S}_i,\ i = 1, 2, \dots, r \right\},$$

where $\mathcal{S}_i$ is a maximal rank-one indecomposable band for each i.

(ii) *In general, if $\mathcal{S}$ is maximal, then*

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix} : E \in \mathcal{S}_0,\ X \in \mathcal{X},\ Y \in \mathcal{Y} \right\},$$

where $\mathcal{S}_0$ is a direct sum as in part (i) *and $\mathcal{X}, \mathcal{Y}$ are the entire sets of nonnegative operators on appropriate spaces.*

We shall see in Theorem 5.6 in the next section that in special cases, a nonnegative band with constant infinite rank is decomposable.

5. SOME CONDITIONS LEADING TO DECOMPOSABILITY OF INFINITE-RANK, NONNEGATIVE BANDS

DEFINITION 5.1. Suppose $\{\mathcal{M}_i\}_{i\in\mathcal{I}}$ and $\{\mathcal{N}_j\}_{j\in\mathcal{J}}$ are collections of mutually orthogonal subspaces of $\mathcal{L}^2(\mathcal{X})$ whose direct sum equals $\mathcal{L}^2(\mathcal{X})$. Then $\{\mathcal{M}_i\}_i$ is said to be a *refinement* of $\{\mathcal{N}_j\}_j$ if each $\mathcal{N}_j$ can be expressed as a direct sum of a (finite or infinite) subcollection of $\{\mathcal{M}_i\}_i$.

In the definition above, $\{\mathcal{N}_j\}_j$ is called a *coarsening* of $\{\mathcal{M}_i\}_i$.

DEFINITION 5.2. A nonnegative operator A in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ will be called *nondegenerate* if A is full and there is no continuous part in any maximal chain in $\operatorname{Lat}' A$.

LEMMA 5.3. *Let A be a full nonnegative idempotent in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, and $\mathcal{C}$ any maximal chain in* $\operatorname{Lat}' A$. *Then there cannot be any nontrivial gaps in $\mathcal{C}$ with corresponding compressions of A equal to zero. If A is nondegenerate, then it can be expressed as a direct sum of countably many positive idempotents of rank one.*

Proof. Let A and $\mathcal{C}$ be as described in the statement. It was shown in the proof of Theorem 3.13 that each nontrivial gap in $\mathcal{C}$ is a reducing subspace for A and thus the compression to any such gap cannot be zero for this will contradict the fullness of A. In fact each nonzero compression to a gap is a positive idempotent of rank one. Again by Theorem 3.13, if A is nondegenerate, then it is a direct sum of positive idempotents of rank one which are countable because of the separability of $\mathcal{L}^2(\mathcal{X})$. ∎

LEMMA 5.4. *If A, B are positive operators on $\mathcal{L}^2(\mathcal{X})$ and S is a nonzero, nonnegative operator on $\mathcal{L}^2(\mathcal{X})$, then ASB is positive.*

Proof. Let f be a nonzero, nonnegative vector in $\mathcal{L}^2(\mathcal{X})$. Since B is positive, $Bf > 0$. Also, S being nonzero and nonnegative, $SBf \neq 0$ (by Proposition 2.1). Thus $0 \neq SBf \geqslant 0$ because $S \geqslant 0$. But A is positive. Therefore, $A(SBf) > 0$ which implies that ASB is positive. ∎

LEMMA 5.5. *Let A be a nondegenerate idempotent on $\mathcal{L}^2(\mathcal{X})$ such that with respect to some decomposition*

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \mathcal{L}^2(\mathcal{X}_3) \oplus \cdots, \qquad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each A_{ij} is either zero or positive. Then A has a block diagonalization with positive diagonal blocks with respect to some decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \cdots,$$

where the collection $\{\mathcal{L}^2(\mathcal{W}_i)\}_i$ is a coarsening of the collection $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$.

Proof. If $\operatorname{rank}(A) = 1$, then the fullness of A implies that A is positive and therefore $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$ itself is the required coarsening. Therefore, assume that $\operatorname{rank}(A) > 1$ in which case A is decomposable. Thus it has a nontrivial invariant standard subspace, say $\mathcal{L}^2(\mathcal{Y})$, where $\mathcal{Y}$ is a Borel subset of $\mathcal{X}$ such that $\mu(\mathcal{Y})\cdot$

$\mu(\mathcal{Y}^{\mathrm{c}}) > 0$. We can assume, with no loss of generality, that the sets $\mathcal{X}_i$ are disjoint so that $\mathcal{X} = \dot{\bigcup\limits_i} \mathcal{X}_i$. Now we can write $\mathcal{Y} = \mathcal{Y}_1 \,\dot{\cup}\, \mathcal{Y}_2 \,\dot{\cup} \cdots$, where $\mathcal{Y}_i = \mathcal{Y} \cap \mathcal{X}_i$.

Let $J = \{j \in \mathbb{N} : \mu(\mathcal{Y}_j) > 0\}$. Then J is nonempty, for otherwise $\mathcal{L}^2(\mathcal{Y}) = \{0\}$. We rearrange $\{\mathcal{X}_i\}$ to obtain

$$\mathcal{X} = \Big(\dot{\bigcup_{j \in J}} \mathcal{X}_j \Big) \cup \Big(\dot{\bigcup_{j \notin J}} \mathcal{X}_j \Big).$$

Suppose $A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2\Big(\dot{\bigcup_{j \in J}} \mathcal{X}_j \Big) \oplus \mathcal{L}^2\Big(\dot{\bigcup_{j \notin J}} \mathcal{X}_j \Big). \tag{5.1}$$

We shall prove that $G = 0$. Clearly, any vector in $\mathcal{L}^2(\mathcal{Y})$ is of the form $\begin{pmatrix} f \\ 0 \end{pmatrix}^{\mathrm{T}}$, for some $f \in \mathcal{L}^2\Big(\dot{\bigcup\limits_{j \in J}} \mathcal{X}_j \Big)$ with respect to (5.1). Since for each $i \in J$, $\mu(\mathcal{Y}_i) > 0$, we can select a nonzero, nonnegative function f_i in $\mathcal{L}^2(\mathcal{X}_i)$ with $\operatorname{supp} f_i = \mathcal{Y}_i$ such that $\begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}^{\mathrm{T}}$ is a vector in $\mathcal{L}^2\Big(\dot{\bigcup\limits_{j \in J}} \mathcal{X}_j \Big) = \bigoplus\limits_{j \in J} \mathcal{L}^2(\mathcal{X}_j)$. Write $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}^{\mathrm{T}}$. Now

$$A \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} Ef \\ Gf \end{pmatrix} \in \mathcal{L}^2(\mathcal{Y})$$

by the invariance of $\mathcal{L}^2(\mathcal{Y})$. The form of vectors in $\mathcal{L}^2(\mathcal{Y})$ gives that $Gf = 0$. Let $\begin{pmatrix} G_{11} & G_{12} & \cdots \\ G_{21} & G_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$ be the block matrix form of $G : \bigoplus\limits_{j \in J} \mathcal{L}^2(\mathcal{X}_j) \to \bigoplus\limits_{j \notin J} \mathcal{L}^2(\mathcal{X}_j)$. Now $Gf = 0$ implies that $G_{i1} f_1 + G_{i2} f_2 + \cdots = 0$ for each $i = 1, 2, \ldots$, which by nonnegativity of G_{ij} further implies that $G_{ij} f_j = 0$ for each $i, j = 1, 2, \ldots$. If G_{ij} is nonzero for some (i, j), then it is positive and f_j being nonzero, nonnegative, we shall obtain $G_{ij} f_j > 0$ which is not true. Therefore $G_{ij} = 0$ for every i, j and hence $G = 0$. Thus $A = \begin{pmatrix} E & F \\ 0 & H \end{pmatrix}$. This shows that $\bigoplus\limits_{j \in J} \mathcal{L}^2(\mathcal{X}_j)$ is invariant under A. Since A is full, we have $F = 0$. We now claim that $\mathcal{L}^2(\mathcal{Y}) = \bigoplus\limits_{j \in J} \mathcal{L}^2(\mathcal{X}_j)$. Working with the same f as above, we have

$$A \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} Ef \\ 0 \end{pmatrix} \in \mathcal{L}^2(\mathcal{Y}).$$

Suppose $E : \bigoplus\limits_{j \in J} \mathcal{L}^2(\mathcal{X}_j) \to \bigoplus\limits_{j \in J} \mathcal{L}^2(\mathcal{X}_j)$ has the block matrix form $\begin{pmatrix} E_{11} & E_{12} & \cdots \\ E_{21} & E_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$.

Then

$$Ef = \begin{pmatrix} E_{11} & E_{12} & \cdots \\ E_{21} & E_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} E_{11}f_1 + E_{12}f_2 + \cdots \\ E_{21}f_1 + E_{22}f_2 + \cdots \\ \vdots \end{pmatrix}.$$

Since A is full, each of its rows contains at least one positive block. This coupled with the fact that each f_i is a nonzero, nonnegative function in $\mathcal{L}^2(\mathcal{X}_i)$, implies that each component of Ef is a positive function in $\mathcal{L}^2(\mathcal{X}_i)$; in other words, $\operatorname{supp} Ef = \dot{\bigcup\limits_{j\in J}} \mathcal{X}_j$. But $Ef \in \mathcal{L}^2(\mathcal{Y})$. Therefore, we must have

$$\mathcal{L}^2(\mathcal{Y}) = \mathcal{L}^2\Big(\dot{\bigcup_{j\in J}} \mathcal{X}_j\Big) = \bigoplus_{j\in J} \mathcal{L}^2(\mathcal{X}_j).$$

As $\mathcal{L}^2(\mathcal{Y})$ is nontrivial, $\bigoplus\limits_{j\notin J} \mathcal{L}^2(\mathcal{X}_j)$ is nontrivial i.e., J is a proper subset of $\mathbb{N}$.

Since A is nondegenerate, by Lemma 5.3 there exists a decomposition of $\mathcal{L}^2(\mathcal{X})$, say

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \cdots,$$

with respect to which A has a block diagonal form $\begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix}$, where each $A_i : \mathcal{L}^2(\mathcal{W}_i) \to \mathcal{L}^2(\mathcal{W}_i)$ is a positive idempotent of rank one. Clearly, each $\mathcal{L}^2(\mathcal{W}_i)$ is a standard subspace invariant under A. Therefore, by what we have proved above, each $\mathcal{L}^2(\mathcal{W}_i)$ is a direct sum of a subcollection of $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$. Hence $\{\mathcal{L}^2(\mathcal{W}_i)\}_i$ is a coarsening of $\{\mathcal{L}^2(\mathcal{X}_i)\}_i$ such that with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}_1) \oplus \mathcal{L}^2(\mathcal{W}_2) \oplus \cdots,$$

A has a block diagonalization with positive diagonal blocks. ∎

Theorem 5.9 below answers Question 3.16 affirmatively under the additional hypothesis of finiteness; Example 3.17 shows the necessity of this hypothesis. But we first consider a finite, nonnegative infinite-rank band whose members are nondegenerate and prove that under this special condition of nondegeneracy, the band has a block diagonalization.

THEOREM 5.6. *A nonnnegative finite band in which every member is nondegenerate and has infinite rank is decomposable. Furthermore, it has infinitely many mutually orthogonal standard invariant subspaces whose direct sum is $\mathcal{L}^2(\mathcal{X})$; equivalently, the band is block diagonalizable.*

Proof. Let $\mathcal{S}$ be a band with k elements, say $S_1, S_2, \ldots, S_k$ such that each S_i is nondegenerate and is of infinite rank. Consider S_1. By Lemma 5.3, there is a collection $\{\mathcal{M}_i^{(1)}\}_{i=1}^{\infty}$ of standard subspaces of $\mathcal{L}^2(\mathcal{X})$ such that with respect to

$$\mathcal{L}^2(\mathcal{X}) = \bigoplus_{i=1}^{\infty} \mathcal{M}_i^{(1)}, \qquad S_1 = \begin{pmatrix} S_{11}^{(1)} & & & & \\ & S_{22}^{(1)} & & & \\ & & \ddots & & \\ & & & S_{ii}^{(1)} & \\ & & & & \ddots \end{pmatrix},$$

where each $S_{ii}^{(1)} : \mathcal{M}_i^{(1)} \to \mathcal{M}_i^{(1)}$ is a positive idempotent of rank one.

Next, consider $S_1S_2S_1$ where

$$S_2 = \begin{pmatrix} S_{11}^{(2)} & S_{12}^{(2)} & \cdots \\ S_{21}^{(2)} & S_{22}^{(2)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \bigoplus_{i=1}^{\infty} \mathcal{M}_i^{(1)}$. Then

$$S_1S_2S_1 = \begin{pmatrix} S_{11}^{(1)}S_{11}^{(2)}S_{11}^{(1)} & S_{11}^{(1)}S_{12}^{(2)}S_{22}^{(1)} & \cdots \\ S_{22}^{(1)}S_{21}^{(2)}S_{11}^{(1)} & S_{22}^{(1)}S_{22}^{(2)}S_{22}^{(1)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

By Lemma 5.4, since each $S_{ii}^{(1)}$ is positive, an arbitrary block $S_{jj}^{(1)}S_{jk}^{(2)}S_{kk}^{(1)}$ in $S_1S_2S_1$ is zero or positive according as $S_{jk}^{(2)}$ is zero or nonzero. Now, by hypothesis $S_1S_2S_1$ is nondegenerate. Therefore, by Lemma 5.5, there exists a coarsening of $\{\mathcal{M}_i^{(1)}\}_{i=1}^{\infty}$ which we denote by $\{\mathcal{M}_i^{(2)}\}_{i=1}^{\infty}$ such that with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \bigoplus_{i=1}^{\infty} \mathcal{M}_i^{(2)}$, $S_1S_2S_1$ is a direct sum of positive rank-one idempotents. Since $\{\mathcal{M}_i^{(2)}\}_i$ is a coarsening of $\{\mathcal{M}_i^{(1)}\}_i$, S_1 is a direct sum of idempotents which are full (because each is a direct sum of positive idempotents) with respect to $\mathcal{L}^2(\mathcal{X}) = \bigoplus_i \mathcal{M}_i^{(2)}$. Suppose

$$S_1 = \begin{pmatrix} S'_{11} & & \\ & S'_{22} & \\ & & \ddots \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} S''_{11} & S''_{12} & S''_{13} & \cdots \\ S''_{21} & S''_{22} & S''_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with respect to $\mathcal{L}^2(\mathcal{X}) = \bigoplus_i \mathcal{M}_i^{(2)}$ so that

$$S_1S_2S_1 = \begin{pmatrix} S'_{11}S''_{11}S'_{11} & S'_{11}S''_{12}S'_{22} & \cdots \\ S'_{22}S''_{21}S'_{11} & S'_{22}S''_{22}S'_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then we know that the nondiagonal blocks are zero and the diagonal blocks are positive idempotents. But any nondiagonal block is of the form $S'_{ii}S''_{ij}S'_{jj}$ for $i \neq j$. Since S'_{ii} and S'_{jj} are full, therefore $S'_{ii}S''_{ij}S'_{jj} = 0$ implies that $S''_{ij} = 0$. Thus both S_1 and S_2 are diagonal with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \bigoplus_i \mathcal{M}_i^{(2)}$.

Next, consider $(S_1S_2S_1)S_3(S_1S_2S_1)$. As reasoned above, there exists a coarsening $\{\mathcal{M}_i^{(3)}\}_i$ of $\{\mathcal{M}_i^{(2)}\}_i$ such that with respect to the decomposition $\mathcal{L}^2(\mathcal{X}) = \bigoplus_i \mathcal{M}_i^{(3)}$, S_1, S_2, and S_3 are diagonal. Proceeding like this, after k steps we shall arrive at a direct sum decomposition $\bigoplus_{i=1}^{\infty} \mathcal{M}_i^{(k)}$ of $\mathcal{L}^2(\mathcal{X})$ with respect to which each

S_i has zero nondiagonal blocks . This proves that $\mathcal{S}$ is decomposable, in fact it is block diagonalizable with respect to $\mathcal{L}^2(\mathcal{X}) = \bigoplus_i \mathcal{M}_i^{(k)}$. ∎

Next, we prove that a nonnegative finite band with constant infinite rank is decomposable. For this, we need a couple of lemmas.

LEMMA 5.7. *If a band $\mathcal{S}$ has more than one member, then there exists $P \in \mathcal{S}$ such that $P\mathcal{S}P$ is a proper subset of $\mathcal{S}$.*

Proof. Suppose there is no P in $\mathcal{S}$ satisfying the required condition. Then $P\mathcal{S}P = \mathcal{S}$, $\forall P \in \mathcal{S}$. If $S \in \mathcal{S}$, then $S = PS_1P$ for some $S_1 \in \mathcal{S}$, i.e., $PSP = PS_1P = S$. This further gives that $PSPS = S$ and $SPSP = S$, i.e., $PS = S = SP$ for all P and S in $\mathcal{S}$. Thus $S = P$ for all P, S in $\mathcal{S}$. Hence $\mathcal{S}$ is a singleton which contradicts the hypothesis. Therefore, there exists some $P \in \mathcal{S}$ such that $P\mathcal{S}P$ is properly contained in $\mathcal{S}$. ∎

LEMMA 5.8. *If a collection $\mathcal{S} \subseteq \mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ contains a member P which is a full idempotent such that $P\mathcal{S}P$ is decomposable, then so is $\mathcal{S}$.*

Proof. Since $P\mathcal{S}P$ is decomposable, there exists some decomposition of $\mathcal{L}^2(\mathcal{X})$ with respect to which every member T of $P\mathcal{S}P$ has the block matrix form $\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$. As P is a member of $P\mathcal{S}P$, it also has a block matrix form with respect to this decomposition, say $\begin{pmatrix} P_1 & X \\ 0 & P_2 \end{pmatrix}$. But since P is a full idempotent, by Lemma 3.12, we get $X = 0$. Now for any $S \in \mathcal{S}$, let $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ be the block matrix form of S with respect to the given decomposition. Then

$$PSP = \begin{pmatrix} P_1S_{11}P_1 & P_1S_{12}P_2 \\ P_2S_{21}P_1 & P_2S_{22}P_2 \end{pmatrix}.$$

By decomposability of $P\mathcal{S}P$, we have $P_2S_{21}P_1 = 0$, $\forall S \in \mathcal{S}$. But P_1 and P_2 are full because P is full and therefore, by Lemma 3.12 we get $S_{21} = 0$, $\forall S \in \mathcal{S}$. Hence $\mathcal{S}$ is decomposable. ∎

THEOREM 5.9. *A nonnegative finite band in which every member has infinite rank is decomposable.*

Proof. Let $\mathcal{S}$ be a nonnegative finite band with constant infinite rank. We shall prove the theorem by induction on $|\mathcal{S}|$, the cardinality of $\mathcal{S}$. Suppose $|\mathcal{S}| = n$. Assume that every nonnegative band with constant infinite rank which has cardinality less than n is decomposable.

Consider $\mathcal{S}$. If $\mathcal{S}$ is a singleton, then by Theorem 3.5, it is decomposable. Therefore, assume that $|\mathcal{S}| > 1$. By Lemma 5.7, there exists $P \in \mathcal{S}$ such that $P\mathcal{S}P$ is a proper subset of $\mathcal{S}$. By Theorem 3.13 (b), P has a block matrix form $\begin{pmatrix} 0 & XE & XEY \\ 0 & E & EY \\ 0 & 0 & 0 \end{pmatrix}$ with respect to some decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \mathcal{L}^2(\mathcal{X}_3),$$

where $E : \mathcal{L}^2(\mathcal{X}_2) \to \mathcal{L}^2(\mathcal{X}_2)$ is full. For any $S \in \mathcal{S}$, let $\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$ be its block matrix representation with respect to the above-mentioned decomposition of the space. Then

$$PSP = \begin{pmatrix} 0 & XE(S_{21}X+YS_{31}X+S_{22}+YS_{32})E & XE(S_{21}X+YS_{31}X+S_{22}+YS_{32})EY \\ 0 & E(S_{21}X+YS_{31}X+S_{22}+YS_{32})E & E(S_{21}X+YS_{31}X+S_{22}+YS_{32})EY \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\mathcal{T} = \{E(S_{21}X + YS_{31}X + S_{22} + YS_{32})E : PSP \in P\mathcal{S}P\}.$$

Observe that $\mathcal{T}$ is a nonnegative band such that $|\mathcal{T}| \leqslant |P\mathcal{S}P| < |\mathcal{S}|$. Then by the inductive hypothesis, $\mathcal{T}$ is decomposable. Therefore, there exist Borel subsets E, F of $\mathcal{X}$ with $\mu(E) \cdot \mu(F) > 0$ such that $\langle T\chi_E, \chi_F\rangle = 0$, $\forall T \in \mathcal{T}$. This implies

$$\langle ES_{21}XE\chi_E, \chi_F\rangle + \langle EYS_{31}XE\chi_E, \chi_F\rangle + \langle ES_{22}E\chi_E, \chi_F\rangle + \langle EYS_{32}E\chi_E, \chi_F\rangle = 0,$$

for all $S \in \mathcal{S}$. Since all the operators are nonnegative, this gives $\langle ES_{22}E\chi_E, \chi_F\rangle = 0$, in other words, the collection $\{ES_{22}E : S \in \mathcal{S}\}$ is decomposable. Also this collection contains E which is a full idempotent. Therefore, by Lemma 5.8, the collection

$$\left\{ S_{22} : S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \in \mathcal{S} \right\}$$

is decomposable. Just as in the proof of Theorem 3.14, we conclude that $\mathcal{S}$ is decomposable. ∎

COROLLARY 5.10. *A finitely generated nonnegative band in which every member has infinite rank is decomposable.*

Proof. This is a consequence of the interesting result on abstract bands due to Green and Rees ([4]): every finitely generated band is finite. ∎

COROLLARY 5.11. *Every finitely generated nonnegative infinite-rank band $\mathcal{S}$ has the property that any maximal standard block triangularization of $\mathcal{S}$ is such that the compression of $\mathcal{S}$ to each nonzero gap constitutes a nonnegative finite band with at least one element of rank one in it.*

Proof. Same as in the finite-dimensional case ([5]). ∎

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REFERENCES

1. M.D. CHOI, E.A. NORDGREN, H. RADJAVI, P. ROSENTHAL, Y. ZHONG, Triangularizing semigroups of quasinilpotent operators with nonnegative entries, *Indiana Univ. Math. J.* **42**(1993), 15-25.

2. P. Fillmore, G. Macdonald, M. Radjabalipour, H. Radjavi, Towards a classification of maximal unicellular bands, *Semigroup Forum*, vol. 49, Springer-Verlag, New York 1994, pp. 195–215.
3. P. Fillmore, G. Macdonald, M. Radjabalipour, H. Radjavi, On principal-ideal bands, preprint 1997.
4. J.A. Green, D. Rees, On semigroups in which $x^r = x$, *Proc. Cambridge Philos. Soc.* **48**(1952), 35–40.
5. A. Marwaha, Decomposability and structure of nonnegative bands in $\mathcal{M}_n(\mathbb{R})$, *Linear Algebra Appl.*, to appear.
6. H. Radjavi, A trace condition equivalent to simultaneous triangularizability, *Canad. J. Math.* **38**(1986), 376–386.
7. H. Radjavi, On reducibility of semigroups of compact operators, *Indiana Univ. Math. J.* **39**(1990), 499–515.
8. J.R. Ringrose, *Compact Non-Selfadjoint Operators*, Van Nostrand, Princeton 1971.
9. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1986.
10. Y. Zhong, Functional positivity and invariant subspaces of semigroups of operators, *Houston J. Math.* **2**(1993), 239–261.

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