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PERTURBATION OF l^1-COPIES AND MEASURE CONVERGENCE IN PREDUALS OF VON NEUMANN ALGEBRAS

HERMANN PFITZNER

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Abstract. Let L^1 be the predual of a von Neumann algebra with a finite faithful normal trace. We show that a bounded sequence in L^1 converges to 0 in measure if and only if each of its subsequences admits another subsequence which converges to 0 in norm or spans l^1 *almost isometrically*. Furthermore we give a quantitative version of an essentially known result concerning the perturbation of a sequence spanning l^1 isomorphically in the dual of a C^*-algebra.

Keywords: *Asymptotically isometric ℓ^1-copies, measure topology, von Neumann preduals.*

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1. INTRODUCTION; MAIN RESULTS

The present article deals with convergence in probability in L^1-spaces from a functional analytic point of view. The L^1-spaces in question are the preduals of von Neumann algebras with finite faithful normal traces. To consider an easy example we look at the commutative case: Let (Ω, Σ, μ) be a finite measure space, let (f_n) be a bounded sequence in $L^1(\Omega, \Sigma, \mu)$. If (appropriately chosen representatives of) the f_n have pairwise disjoint supports then clearly (f_n) converges to 0 in measure. From the functional analytic point of view such a sequence, up to normalization, is the canonical basis of an isometric copy of l^1. If one perturbes (f_n) by a norm null sequence (g_n) then (f_n+g_n) still μ-converges to 0 and spans l^1 almost isometrically (in a sense to be made precise below in Section 2). It has been known ([11], Theorem 2; see also [20], Theorem 3, Remark 6bis) for quite a time that these are essentially the only examples of μ-null sequences.

Theorem 1.1 contains the analogous statement for the predual of a von Neumann algebra with finite faithful normal trace. (For notation and definitions see Section 2.)

THEOREM 1.1. *Let* (x_n) *be a bounded sequence in* $L^1(\mathcal{N},\tau) = \mathcal{N}_*$ *where* $(\mathcal{N},\tau)$ *is a von Neumann algebra with a finite normal faithful trace* τ. *Then the following assertions are equivalent:*

(i) $x_n \xrightarrow{\tau} 0$.

(ii) *For each subsequence* (x_{n_k}) *of* (x_n) *there are a subsequence* $(x_{n_{k_l}})$ *and a sequence* (y_l) *of pairwise orthogonal elements of* $L^1(\mathcal{N},\tau)$ *such that* $\|x_{n_{k_l}} - y_l\|_1 \to 0$.

(iii) *For each subsequence* (x_{n_k}) *of* (x_n) *there is a subsequence* $(x_{n_{k_l}})$ *which tends to* 0 *in* $\|\cdot\|_1$ *or spans* l^1 *almost isometrically.*

(iv) *For each subsequence* (x_{n_k}) *of* (x_n) *there is a subsequence* $(x_{n_{k_l}})$ *which tends to* 0 *in* $\|\cdot\|_1$ *or spans* l^1 *asymptotically.*

The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) *hold also for unbounded sequences* (x_n)*, the implications* (iii) $\Rightarrow$ (ii), (i) *do not.*

Implication (i) $\Rightarrow$ (ii) has already appeared as a special case of a result of Sukochev ([21], Proposition 2.2). The other nontrivial implication (iii) $\Rightarrow$ (ii) follows immediately from Theorem 1.2 which holds for the predual of any von Neumann algebra and is of independent interest:

THEOREM 1.2. *Let* $\mathcal{N}$ *be an arbitrary von Neumann algebra and* (φ_m) *a bounded sequence in its predual* $\mathcal{N}_*$. *If* (φ_m) *spans* l^1 *almost isometrically then there are a subsequence* (φ_{m_l}) *of* (φ_m) *and a sequence* $(\widetilde{\varphi}_l)$ *of pairwise orthogonal functionals in* $\mathcal{N}_*$ *such that* $\|\varphi_{m_l} - \widetilde{\varphi}_l\| \to 0$ *as* $l \to \infty$.

This amounts to saying that there are pairwise orthogonal projections s_l *and pairwise orthogonal projections* t_l *in* $\mathcal{N}$ *such that* $\|\varphi_{m_l} - t_l\varphi_{m_l}s_l\| \to 0$ *as* $l \to \infty$.

It is natural to ask what can be improved in Theorem 1.2 if one replaces the predual of the von Neumann algebra by the dual of a C^*-algebra. At the time of this writing this is not clear. What we have is

PROPOSITION 1.3. *Let* (φ_m) *be a bounded sequence that spans* l^1 *almost isometrically in the dual of an arbitrary* C^**-algebra* A. *Then, given* $\varepsilon > 0$*, there are a subsequence* (φ_{m_n})*, pairwise orthogonal positive normalized elements* a_n *and pairwise orthogonal positive normalized elements* b_n *in* A *such that* $\|\varphi_{m_n} - b_n\varphi_{m_n}a_n\| < \varepsilon$ *for all* $n \in \mathbb{N}$.

For a more detailed discussion see Section 6.

As to the organization of the paper, after recalling some notation and definitions in the next section we gather some auxiliary results in Section 3 in order to prove Theorem 1.2 in Section 4. In Section 5 we prove Theorem 1.1 for the sake of completeness although, as already mentioned, it follows essentially from [21], Proposition 2.2 and Theorem 1.2. In Section 6 perturbations of l^1-copies in the dual of C^*-algebras are considered and Proposition 1.3 is proved.

2. NOTATION; DEFINITIONS

Let (x_n) be a sequence of nonzero elements in a Banach space X.

We say that (x_n) *spans* l^1 *r-isomorphically* or just *isomorphically* if there exists $r > 0$ (trivially $r \leqslant 1$) such that $r\Big(\sum\limits_{n=1}^{\infty} |\alpha_n|\Big) \leqslant \Big\|\sum\limits_{n=1}^{\infty} \alpha_n \frac{x_n}{\|x_n\|}\Big\| \leqslant \sum\limits_{n=1}^{\infty} |\alpha_n|$ for all scalars α_n (the second inequality being trivial).

We say that (x_n) *spans* l^1 *almost isometrically* if there is a sequence (δ_m) in $[0, 1[$ tending to 0 such that $(1-\delta_m) \sum\limits_{n=m}^{\infty} |\alpha_n| \leqslant \Big\|\sum\limits_{n=m}^{\infty} \alpha_n \frac{x_n}{\|x_n\|}\Big\| \leqslant \sum\limits_{n=m}^{\infty} |\alpha_n|$ for all $m \in \mathbb{N}$.

Trivially the property of spanning l^1 almost isometrically passes to subsequences. Recall that James' distortion theorem (see [10] or [4]) for l^1 says that every isomorphic copy of l^1 contains an almost isometric copy of l^1. To be more precise, let $r > 0$, $[0, 1[\ni \delta_n \to 0$, and let (x_n) be a normalized basis spanning l^1 r-isomorphically. Then it follows from the proof of [10] that there is a sequence (λ_i) of scalars and a sequence (F_n) of pairwise disjoint finite subsets of $\mathbb{N}$ such that $(1-\delta_m) \sum\limits_{n=m}^{\infty} |\alpha_n| \leqslant \Big\|\sum\limits_{n=m}^{\infty} \alpha_n y_n\Big\| \leqslant \sum\limits_{n=m}^{\infty} |\alpha_n|$ for all scalars α_n and all $m \in \mathbb{N}$ where $y_n = \sum\limits_{i \in F_n} \lambda_i x_i$ and where $\sum\limits_{i \in F_n} |\lambda_i| \leqslant \frac{1}{r}$ for all $n \in \mathbb{N}$.

Finally (x_n) is said *to span* l^1 *asymptotically isometrically* or just *to span* l^1 *asymptotically* if there is a sequence (δ_n) in $[0, 1[$ tending to 0 such that

$$\sum_{n=1}^{\infty} (1-\delta_n)|\alpha_n| \leqslant \Big\|\sum_{n=1}^{\infty} \alpha_n \frac{x_n}{\|x_n\|}\Big\| \leqslant \sum_{n=1}^{\infty} |\alpha_n|$$

for all scalars α_n. We say that a Banach space is isomorphic (respectively almost isometric, respectively asymptotically isometric) to l^1 if it has a basis with the corresponding property. Clearly a sequence spanning l^1 asymptotically spans l^1 almost isometrically. The main result of [6] states that the converse does not hold because there are almost isometric copies of l^1 which do not contain l^1 asymptotically. However, it follows from [19] that this cannot happen in the predual of a von Neumann algebra because each sequence spanning l^1 almost isometrically in a von Neumann predual contains a subsequence spanning l^1 asymptotically (cf. (iii) $\Rightarrow$ (iv) in the proof of Theorem 1.1). Note that the present definitions of almost and asymptotically isometric differ slightly from those in [6] and [19] by the term $x_n/\|x_n\|$ but that, of course, for normalized sequences the definitions are the same. Note also the technical detail that because of this term one might have $\|x_n\| \to 0$ for a sequence spanning l^1 isomorphically (or almost or asymptotically isometrically) whereas sequences that are equivalent to the canonical l^1-basis ([4], p. 43) are uniformly bounded away from 0.

The dual of a Banach space X is denoted by X'. We work with complex scalars. Two elements a, b of a C^*-algebra are called orthogonal — $a \perp b$ in symbols — if $ab^* = 0 = a^*b$.

Let $\mathcal{N}$ be a von Neumann algebra, $a \in \mathcal{N}$, $\varphi \in \mathcal{N}_*$. Then $a\varphi$ denotes the normal functional $\mathcal{N} \ni x \mapsto \varphi(xa)$ and φa denotes the normal functional $\mathcal{N} \ni x \mapsto \varphi(ax)$. Two elements $\varphi, \psi \in \mathcal{N}_*$ of the predual of $\mathcal{N}$ are called orthogonal

— $\varphi\perp\psi$ in symbols — if they have orthogonal right and orthogonal left support projections.

We recall the polar decomposition of a functional $\varphi \in \mathcal{N}_*$, cf. [22], III.4.2: There exist a partial isometry u in $\mathcal{N}$ and a positive functional $|\varphi|$ in $\mathcal{N}_*$ such that $\varphi = u|\varphi|$, uu^* is the left support projection of φ which equals the support projection of $|\varphi^*|$, and u^*u is the right support projection of φ which equals the support projection of $|\varphi|$; finally $\varphi(u^*) = \|\varphi\|$ and $\|\varphi\| = \|\,|\varphi|\,\|$. If arbitrary $\varphi \in \mathcal{N}_*$ and $x \in \mathcal{N}$ are such that $x^*(\varphi) = \|\varphi\|$ then $\varphi = x|\varphi|$ and $|\varphi| = x^*\varphi$ (see the proof of [22], III.4.2, or compare with (3.4), (3.5) of Lemma 3.3 below). It is known that for positive φ, ψ the condition $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$ is equivalent to $\varphi\perp\psi$ ([22], III.4.2). It seems to be well known that an analogous equivalence holds without the positivity assumption. For lack of suitable reference we provide a proof:

Lemma 2.1. *Let $\mathcal{N}$ be a von Neumann algebra and let $\varphi, \psi \in \mathcal{N}_*$ be two normalized functionals. Then the following assertions are equivalent:*

(i) *The linear span of φ and ψ is isometrically isomorphic to the two-dimensional l_2^1, more specifically $\|\alpha\varphi + \beta\psi\| = |\alpha| + |\beta|$ for all $\alpha, \beta \in \mathbb{C}$.*

(ii) $\|\varphi - \psi\| = \|\varphi\| + \|\psi\| = \|\varphi + \psi\|$.

(iii) $\|\,|\varphi| - |\psi|\,\| = \|\varphi\| + \|\psi\| = \|\,|\varphi^*| - |\psi^*|\,\|$.

(iv) $|\varphi|\perp|\psi|$ *and* $|\varphi^*|\perp|\psi^*|$.

(iv′) $\varphi\perp\psi$.

Proof. (iv) $\Leftrightarrow$ (iv′) is immediate from the definition of orthogonality and from the above mentioned facts of the polar decomposition.

(iii) $\Rightarrow$ (iv) By the first (the second) equality of (iii) and by what has been said before the statement of the lemma, the right (the left) support projections of φ and ψ are orthogonal.

(iv) $\Rightarrow$ (i) is elementary: If $\varphi = u|\varphi|$, $\psi = v|\psi|$ are the polar decompositions of φ and ψ then $uu^*\perp vv^*$ and $u^*u\perp v^*v$ by hypothesis. Hence $u\perp v$, and u and v span the two-dimensional l_2^∞ because $\|\alpha u + \beta v\|^2 = \|\,|\alpha|^2u^*u + |\beta|^2v^*v\| = \max(|\alpha|^2, |\beta|^2)$. Since u, v act like biorthogonal functionals on φ, ψ we get (i) by duality.

(i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) Let $w, y \in \mathcal{N}$ be the partial isometries of the polar decomposition of $\varphi + \psi$ and $\varphi - \psi$ that is

$$\varphi + \psi = w|\varphi + \psi|, \quad \varphi - \psi = y|\varphi - \psi|.$$

Then

$$(\varphi + \psi)(w^*) = \|\varphi + \psi\| = \|\varphi\| + \|\psi\|, \tag{2.1}$$

$$(\varphi - \psi)(y^*) = \|\varphi - \psi\| = \|\varphi\| + \|\psi\|, \tag{2.2}$$

by hypothesis. Since $|\varphi(w^*)| \leqslant \|\varphi\|$, $|\psi(w^*)| \leqslant \|\psi\|$ it follows from (2.1) that $\varphi(w^*) = \|\varphi\|$ and $\psi(w^*) = \|\psi\|$ hence

$$\varphi = w|\varphi| \quad \text{and} \quad \psi = w|\psi|. \tag{2.3}$$

Thus

$$\begin{aligned}\|\,|\varphi| - |\psi|\,\| &\geqslant (|\varphi| - |\psi|)(y^*w) = (w|\varphi| - w|\psi|)(y^*)\\ &\overset{(2.3)}{=} (\varphi - \psi)(y^*) \overset{(2.2)}{=} \|\varphi\| + \|\psi\|,\end{aligned}$$

whence the first equality of (iii). Since the involution is isometric, the second equality of (iii) is obtained analogously. ∎

We recall some basic facts on the definition of noncommutative L^p spaces associated to (semi-)finite von Neumann algebras. Let τ be a semifinite faithful normal trace on a von Neumann algebra $\mathcal{N}$. The set $I = \{x \in \mathcal{N} : \tau(|x|) < \infty\}$ is an ideal in $\mathcal{N}$, can be normed by $x \mapsto \tau(|x|) =: \|x\|_1$ and its Banach space completion is denoted by $L^1 = L^1(\mathcal{N}, \tau)$. Then L^1 is isometrically isomorphic to the predual $\mathcal{N}_*$ via the map $L^1 \ni x \mapsto \varphi_x \in \mathcal{N}_*$ where $\varphi_x(y) = \tau(xy)$ for $y \in \mathcal{N}$ and where τ is understood as the (well-defined) extension of τ from I to L^1 ([22], V.2.18). In particular, the multiplication on $\mathcal{N} \times I$ can be extended to $\mathcal{N} \times L^1$, the map $x \mapsto \varphi_x$ respects orthogonality and one has $|\tau(xy)| \leqslant \|x\|_1\|y\|_\infty$ for $x \in L^1$, $y \in L^\infty = L^\infty(\mathcal{N}, \tau) := \mathcal{N}$. More generally one can define $L^p(\mathcal{N}, \tau)$-spaces, $1 \leqslant p < \infty$, as the sets of those $x \in L^0$ for which $\|x\|_p := \tau(|x|^p)^{1/p} < \infty$ where $L^0 = L^0(\mathcal{N}, \tau)$ is the space of τ-measurable densely defined (in general unbounded) operators affiliated with $\mathcal{N}$ and where τ is understood as the extension of τ from $\mathcal{N}$ to L^0. On L^0 one defines the measure topology as the translation invariant topology in which the sets $\{x \in L^0 : \exists p \in \mathcal{N}_{\mathrm{proj}} : xp \in \mathcal{N}, \|xp\|_\infty \leqslant \varepsilon, \tau(p^\perp) \leqslant \delta\}$, $\varepsilon, \delta > 0$, form a base of the zero neighborhoods. ($\mathcal{N}_{\mathrm{proj}}$ denotes the set of projections of $\mathcal{N}$.) In this topology, L^0 becomes a (well-defined) metrizable complete Hausdorff topological vector *-algebra and all L^p embed injectively in L^0. In particular, sum and product are well-defined in L^0. All this can be found for example in [15], [23], Chapter 1 or [24].

In the sequel we will suppose τ to be faithful, normal, and finite not only semifinite. (Of course, in this case we have $I = \mathcal{N}$ in the last paragraph.)

If a sequence (x_n) in L^0 converges to $x \in L^0$ with respect to the measure topology this is denoted by $x_n \xrightarrow{\tau} x$. In this context Chebyshev's inequality reads $\tau(\chi_{]\varepsilon,\infty[}(|x|)) \leqslant \tau(\frac{1}{\varepsilon}|x|) = \frac{1}{\varepsilon}\|x\|_1$ for $x \in L^1$ — which means in particular that the norm topology is finer than the measure topology induced by τ — and from [9], A48, we know that in accordance with the commutative case, $x_n \xrightarrow{\tau} 0$ if and only if $\tau(\chi_{]\varepsilon,\infty[}(|x_n|)) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

Basic properties and definitions which are not explained here can be found in [4], or in [13] and [14] for Banach spaces, and in [17] or [22] for C^*-algebras.

3. SOME AUXILIARY RESULTS

Let us first state an easy lemma which says that almost isometric and asymptotically isometric l^1-copies are stable with respect to perturbations by norm null sequences.

LEMMA 3.1. *Let (x_n), (y_n) be two sequences in a Banach space X such that* $\inf \|x_n\| > 0$, $\|y_n\| \to 0$ *and* $x_n + y_n \neq 0$.

If (x_n) spans l^1 almost isometrically then so does $(x_n + y_n)$.

If (x_n) spans l^1 asymptotically then so does $(x_n + y_n)$.

Proof. Suppose that (x_n) spans l^1 almost isometrically. For all scalar sequences (α_n) one has:

$$\begin{aligned}
&\Big\|\sum_{n=m}^{\infty}\alpha_n\frac{x_n+y_n}{\|x_n+y_n\|}\Big\| \\
&\geqslant \Big\|\sum_{n=m}^{\infty}\alpha_n\frac{x_n}{\|x_n\|}\Big\| - \Big\|\sum_{n=m}^{\infty}\alpha_n\Big(1-\frac{\|x_n\|}{\|x_n+y_n\|}\Big)\frac{x_n}{\|x_n\|}\Big\| - \Big\|\sum_{n=m}^{\infty}\alpha_n\frac{y_n}{\|x_n+y_n\|}\Big\| \\
&\geqslant \Big((1-\delta_m)\sum_{n=m}^{\infty}|\alpha_n|\Big) - \Big(\sup_{n\geqslant m}\Big|1-\frac{\|x_n\|}{\|x_n+y_n\|}\Big|\sum_{n=m}^{\infty}|\alpha_n|\Big) \\
&\qquad - \Big(\sup_{n\geqslant m}\frac{\|y_n\|}{\|x_n+y_n\|}\sum_{n=m}^{\infty}|\alpha_n|\Big) = (1-\delta'_m)\sum_{n=m}^{\infty}|\alpha_n|,
\end{aligned}$$

where $\delta'_m = \delta_m + \sup_{n\geqslant m}|1-\frac{\|x_n\|}{\|x_n+y_n\|}| + \sup_{n\geqslant m}\frac{\|y_n\|}{\|x_n+y_n\|} \to 0$ as $m\to\infty$. Hence (x_n+y_n) spans l^1 almost isometrically. The asymptotic case is proved similarly. ∎

Lemmas 3.2–3.4 seem to be known and are proved mainly for lack of suitable reference. (In part they overlap with [18], Lemmas 3–5].)

LEMMA 3.2. *Let A be a C^*-algebra, ω a positive functional on A and a, b elements of the unit ball of A. Then:*

$$\|a\omega-\omega\| \leqslant (2\|\omega\|)^{1/2}\,|\,\|\omega\|-\omega(a)|^{1/2}, \tag{3.1}$$

$$\|\omega a-\omega\| \leqslant (2\|\omega\|)^{1/2}\,|\,\|\omega\|-\omega(a)|^{1/2}, \tag{3.2}$$

$$\|b\omega a-\omega\| \leqslant (2\|\omega\|)^{1/2}\,(|\,\|\omega\|-\omega(a)|^{1/2}+|\,\|\omega\|-\omega(b)|^{1/2}). \tag{3.3}$$

Proof. Let $x\in A$ and $\|x\|\leqslant 1$. Set $\gamma=\|\omega\|-\omega(a)$, thus $\omega(a^*)=\|\omega\|-\overline{\gamma}$. Without loss of generality we assume $\|\omega\|=1$. The inequality of Cauchy-Schwarz yields

$$\begin{aligned}
|\omega(x)-a\omega(x)|^2 = |\omega(x(1-a))|^2 &\leqslant \omega(xx^*)\,\omega((1-a)^*(1-a)) \\
&\leqslant \omega((1-a)^*(1-a)) = \omega(1-a)-\omega(a^*-a^*a) \\
&= \gamma-(1-\overline{\gamma})+\omega(a^*a) \leqslant 2\mathrm{Re}\,\gamma \leqslant 2|1-\omega(a)|
\end{aligned}$$

whence (3.1); (3.2) follows analogously; (3.3) follows from (3.1), (3.2) and from $\|\omega-b\omega a\| \leqslant \|\omega-b\omega\|+\|b(\omega-\omega a)\| \leqslant \|\omega-b\omega\|+\|\omega-\omega a\|$. ∎

LEMMA 3.3. *Let A be a C^*-algebra, φ a functional on A and a, b in the unit ball of A. Then:*

$$\|\varphi-a|\varphi|\,\| \leqslant (2\|\varphi\|)^{1/2}\,|\,\|\varphi\|-\varphi^*(a)|^{1/2}, \tag{3.4}$$

$$\|\,|\varphi|-a\varphi\| \leqslant (2\|\varphi\|)^{1/2}\,|\,\|\varphi\|-\varphi(a)|^{1/2}, \tag{3.5}$$

$$\|b\varphi a-\varphi\| \leqslant (2\|\varphi\|)^{1/2}\,\big(|\,\|\varphi\|-|\varphi|(a)|+|\,\|\varphi\|-|\varphi^*|(b)|^{1/2}\big). \tag{3.6}$$

Proof. Let $\varphi=u|\varphi|$ be the polar decomposition of φ. Then the polar decomposition of φ^* is $\varphi^*=u^*|\varphi^*|$ (cf. the proof of III.4.2 in [22]), we have $\varphi=|\varphi^*|u$, $|\varphi|=u^*\varphi=|\varphi|^*=\varphi^*u$. Without loss of generality we assume $\|\varphi\|=1$.

Inequality (3.5) follows from

$$\| a\varphi - |\varphi|\, \| = \| au|\varphi| - |\varphi|\, \| \overset{(3.1)}{\leqslant} |2(1 - |\varphi|(au))|^{1/2} = |2(1 - \varphi(a))|^{1/2}.$$

Replacing φ by φ^* we have $\| a\varphi^* - |\varphi^*|\, \| \leqslant |2(1 - \varphi^*(a))|^{1/2}$ whence (3.4) by $\| a|\varphi| - \varphi \| = \| (a\varphi^* - |\varphi^*|)u \| \leqslant \| a\varphi^* - |\varphi^*|\, \|$. (3.6) follows from

$$\begin{aligned} \| \varphi - b\varphi a \| &\leqslant \| \varphi - b\varphi \| + \| b\varphi - b\varphi a \| \\ &= \|\, |\varphi^*|u - b|\varphi^*|u \| + \| bu|\varphi| - bu|\varphi|a \| \\ &\leqslant \|\, |\varphi^*| - b|\varphi^*|\, \| + \|\, |\varphi| - |\varphi|a \| \\ &\overset{(3.1)(3.2)}{\leqslant} (2\|\varphi\|)^{1/2}\, (|\, \|\varphi\| - |\varphi^*|(b)|^{1/2} + |\, \|\varphi\| - |\varphi|(a)|^{1/2}). \end{aligned}$$

Lemma 3.4. *Let $\mathcal{N}$ be a von Neumann algebra with predual $\mathcal{N}_*$. If a functional σ in the unit ball of $\mathcal{N}_*$, projections $r, l \in \mathcal{N}$ and a number $\beta \in]0,1[$ are such that $r(|\sigma|) \geqslant 1 - \beta$ and $l(|\sigma^*|) \geqslant 1 - \beta$ then $\| \sigma - \tau \| < 6\sqrt{\beta}$ where $\tau = \frac{l\sigma r}{\| l\sigma r \|}$.*

Proof. $\| l\sigma r - \sigma \| \leqslant 2\sqrt{2\beta}$ by (3.6) and $\left\| \frac{l\sigma r}{\| l\sigma r\|} - l\sigma r \right\| = \frac{1 - \| l\sigma r \|}{\| l\sigma r \|} \| l\sigma r \| = 1 - \| l\sigma r \| \leqslant \beta + \|\sigma\| - \| l\sigma r \|$ (because $\|\sigma\| \geqslant 1 - \beta$) $\cdots \leqslant \beta + \| \sigma - l\sigma r \| \leqslant \beta + 2\sqrt{2\beta}$ thus $\| \sigma - \tau \| < 6\sqrt{\beta}$. ∎

We recall some more definitions and notation. Let A be a C^*-algebra. By A_+ (respectively A_{sa}) we denote the positive (respectively selfadjoint) part of A. A projection $p \in A''$ is called open if it is the limit of an increasing net of positive elements of A ([17], 3.11, [22]). If $p \in A''$ is open then $B'' = pA''p$ where $B = pA''p \cap A$ is a hereditary subalgebra. A projection $q \in A''$ is called closed if there is an open projection $p \in A''$ such that $q = p^{\mathrm{c}}$ where p^{c} denotes the complement $1 - p$ of p. (This makes sense also if A is not unital because one always has $1 \in A''$.) By definition the closure $\overline{p}$ of a projection $p \in A''$ is the infimum of all closed projections majorizing p. χ_M denotes the characteristic function of a set M. By functional calculus, $\chi_{]\varepsilon,1]}(x)$ (respectively $\chi_{[\varepsilon,1]}(x)$) is an open (respectively closed) projection in A'' if $1 > \varepsilon > 0$, $x \in A$, $0 \leqslant x \leqslant 1$, because $\chi_{]\varepsilon,1]}$ (respectively $\chi_{[\varepsilon,1]}$) is the pointwise limit of an increasing (respectively decreasing) sequence of continuous functions on $[0,1]$ (cf. [1], II.3, if A is unital). [As to $\chi_{]\varepsilon,1]}(x)$ this is easy but as to $\chi_{[\varepsilon,1]}(x)$ a bit more attention must be paid to the case where A is not unital; in this case one works with the unitisation $\widetilde{A}$ of A and with a result of Akemann and Pederson ([2], [17], 3.11.9). We sketch this for the sake of completeness (and for lack of due reference): Showing the closedness of $\chi_{[\varepsilon,1]}(x)$ amounts to showing that $\chi_{[0,\varepsilon[}(x)$ is open. Since $\chi_{[0,\varepsilon[}$ is the pointwise limit of an increasing sequence of continuous functions on $[0,1]$ we have that $\chi_{[0,\varepsilon[}(x) \in (\widetilde{A}_{\mathrm{sa}})^m$ where $(\widetilde{A}_{\mathrm{sa}})^m$ denotes the set of those elements in the enveloping von Neumann algebra of A (identified with A'') which are limits of increasing nets of elements of $\widetilde{A}_{\mathrm{sa}}$. Now, $\chi_{[0,\varepsilon[}(x)$ is open because $\chi_{[0,\varepsilon[}(x) \in (\widetilde{A}_+)^m$ by [17], 3.11.9.

The following Lemma 3.5 is a natural generalisation of Lemma 3.5 from [18].

LEMMA 3.5. *For each $\varepsilon > 0$ and each $n \in \mathbb{N}$ there is $\delta = \delta(n, \varepsilon) > 0$ with the following property.*

Let A be a C^*-algebra. If there are functionals $\varphi_1, \ldots, \varphi_n$ in the unit ball of A' and open projections $s, t \in A''$ such that

$$(3.7) \qquad (1-\delta) \sum_1^n |\alpha_k| \leqslant \Big\| \sum_1^n \alpha_k \, t\varphi_k s \Big\| \leqslant \sum_1^n |\alpha_k| \quad \forall (\alpha_k) \subset \mathbb{C}$$

then there are open projections $p_1, \ldots, p_n \in sA''s$ with pairwise orthogonal closures in $sA''s$ and open projections $q_1, \ldots, q_n \in tA''t$ with pairwise orthogonal closures in $tA''t$ such that

$$(3.8) \qquad p_k(|\varphi_k|) > 1 - \varepsilon,$$

$$(3.9) \qquad q_k(|\varphi_k^*|) > 1 - \varepsilon,$$

for $k = 1, \ldots, n$.

In particular the φ_k are close to normalized orthogonal elements ψ_k on sAt in the sense that $\|\varphi_k - \psi_k\| < 6\sqrt{\varepsilon}$ where $\psi_k = q_k\varphi_k p_k / \|q_k\varphi_k p_k\|$ are normalized and pairwise orthogonal with left (right) supports majorized by t (by s).

Proof. Recall that A'' always contains the unit as an open projection. Therefore the assumption $s = t = 1$ below in part (a) makes sense also when A is not unital. Note also that the last assertion of the lemma (concerning the ψ_k) is immediate from Lemma 3.4 and from (3.8), (3.9).

(a) First we suppose $s = t = 1$ and deal only with the special case:

(a1) of positive functionals φ_k.

Let $\varepsilon > 0$. For $n = 1$ choose an $x \geqslant 0$ in the unit ball of A such that $\varphi_1(x) > 1 - \varepsilon$ and set $p_1 = q_1 = \chi_{]0,1]}(x)$, $\delta(1, \varepsilon) = \varepsilon$.

Suppose now that the assertion holds true (for positive functionals, for $s = t = 1$ and) for some $n \in \mathbb{N}$. By hypothesis on n we choose $\delta_n = \delta(n, \varepsilon)$. We choose $\delta_{n+1} > 0$ such that

$$\delta_{n+1} + (32n\delta_{n+1})^{1/2} < \delta_n \quad \text{and} \quad 4n\delta_{n+1} < \varepsilon.$$

Consider positive functionals φ_k, $k = 1, \ldots, n+1$, in the unit ball of A such that $\Big\| \sum_1^{n+1} \alpha_k \varphi_k \Big\| \geqslant (1 - \delta_{n+1}) \sum_1^{n+1} |\alpha_k|$. Set $\sigma = \frac{1}{n} \sum_1^n \varphi_k$ and $\tau = \varphi_{n+1}$. Then $(1 - \delta_{n+1})(|\alpha| + |\beta|) \leqslant \|\alpha\sigma + \beta\tau\| \leqslant |\alpha| + |\beta|$ for all scalars α, β. In particular $\|\sigma - \tau\| \geqslant 2(1 - \delta_{n+1})$. There is a selfadjoint normalized element $x \in A$ such that $(\sigma - \tau)(x) > 2(1 - 2\delta_{n+1})$. Decompose $x = x^+ + x^-$ in its negative and positive parts. Then $(\sigma - \tau)(x) = (\sigma(x^+) + \tau(x^-)) - (\sigma(x^-) + \tau(x^+)) > 2(1 - 2\delta_{n+1})$ whence, since $\tau(x^-) \leqslant 1$,

$$(3.10) \qquad \sigma(x^+) > 2(1 - 2\delta_{n+1}) - \tau(x^-) \geqslant 1 - 4\delta_{n+1}$$

and similarly

$$(3.11) \qquad \varphi_{n+1}(x^-) > 1 - 4\delta_{n+1}.$$

Together with $\varphi_k(x^+) \leqslant 1$ we obtain that

$$(3.12) \qquad \varphi_k(x^+) > 1 - 4n\delta_{n+1} \quad \text{for all } k = 1, \ldots, n$$

because otherwise one would have $n\sigma(x^+) \leqslant 1 - 4n\delta_{n+1} + (n-1) = n(1-4\delta_{n+1})$ in contrast to (3.10). If $0 \leqslant a \leqslant 1$, $a \in A$, then by functional calculus $\chi_{]\eta,1]}(a) \to \chi_{]0,1]}(a)$ in the w^*-topology of A'' as $0 < \eta \to 0$. Furthermore, $\chi_{]0,1]}(a) \geqslant a$. Thus there is $\eta > 0$ such that by (3.12) and (3.11) the projections $p = \chi_{]\eta,1]}(x^+)$, $p_{n+1} = \chi_{]\eta,1]}(x^-)$ satisfy

$$p(\varphi_k) > 1 - 4n\delta_{n+1} \quad \text{for } k = 1, \dots, n \tag{3.13}$$

and

$$p_{n+1}(\varphi_{n+1}) > 1 - 4\delta_{n+1} > 1 - \varepsilon. \tag{3.14}$$

By functional calculus the projections p and p_{n+1} are open and orthogonal. Since $\eta > 0$ they have orthogonal closures because their closures are majorized by the orthogonal closed projections $\chi_{[\eta,1]}(x^+)$ and $\chi_{[\eta,1]}(x^-)$.

$B = pA''p \cap A \subset A$ is a hereditary subalgebra of A. This explains the equality sign in the following formula:

$$\Big\|\sum_1^n \alpha_k \varphi_{k|B}\Big\|_B = \Big\|\sum_1^n \alpha_k (p\varphi_k p)\Big\| \geqslant \Big\|\sum_1^n \alpha_k \varphi_k\Big\| - \Big\|\sum_1^n \alpha_k(\varphi_k - p\varphi_k p)\Big\|$$
$$\overset{(3.3)(3.13)}{>} (1-\delta_{n+1})\sum_1^n |\alpha_k| - \sqrt{32n\delta_{n+1}}\sum_1^n |\alpha_k|$$
$$> (1-\delta_n)\sum_1^n |\alpha_k|.$$

By induction hypothesis applied to B and to $\varphi_{k|B}$ one gets n open projections $p_1, \dots, p_n \in B''$ with pairwise orthogonal closures in B'' — whence in A'' — such that (3.8) holds for $k = 1, \dots, n$. For $k = n+1$, (3.8) holds by (3.14). We have that $p_k \perp p_{n+1}$ for $k = 1, \dots, n$ because $p_k \leqslant p \perp p_{n+1}$. Furthermore, (3.9) holds with $q_k = p_k$ because we have supposed $\varphi_k \geqslant 0$. This proves the case where $s = t = 1$ for positive functionals φ_k.

(a2) For the case of arbitrary functionals (but still with $s = t = 1$) suppose that the lemma is false. Then there are $\varepsilon > 0$, $n \in \mathbb{N}$, a sequence (A_i) of C^*-algebras, and $\varphi_{k,i}$ in the unit ball of A_i for $k = 1, \dots, n$ such that for each $i \in \mathbb{N}$,

$$\Big(1 - \frac{1}{i}\Big)\sum_{k=1}^n |\alpha_k| < \Big\|\sum_{k=1}^n \alpha_k \varphi_{k,i}\Big\| \leqslant \sum_{k=1}^n |\alpha_k| \quad \forall\, (\alpha_k) \subset \mathbb{C}, \tag{3.15}$$

but for each $i \in \mathbb{N}$ the $\varphi_{k,i}$ are far from orthogonal functionals, more precisely

$$\min_{k \leqslant n} p_{k,i}(|\varphi_{k,i}|) \leqslant 1 - \varepsilon \quad \text{or} \quad \min_{k \leqslant n} q_{k,i}(|\varphi_{k,i}^*|) \leqslant 1 - \varepsilon \tag{3.16}$$

for all sequences $(p_{k,i})_{k=1}^n$ and $(q_{k,i})_{k=1}^n$ of open projections with orthogonal closures in A_i''.

We recall some basic facts on ultraproducts (see e.g. [8]). If $\mathcal{U}$ is an ultrafilter on an index set I the ultraproduct $X = (X_i)/\mathcal{U}$ of a family $(X_i)_{i \in I}$ of Banach spaces is defined as the quotient $l^\infty(X_i)/c_0(X_i)$ where $l^\infty(X_i) = \{(x_i)_{i \in I} : \|(x_i)\| = \sup \|x_i\| < \infty\}$ and $c_0(X_i) = \{(x_i)_{i \in I} \in l^\infty(X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. With the quotient norm X becomes a Banach space. By $[x_i]_{\mathcal{U}}$ we denote the equivalence class represented by $(x_i)_{i\in I} \in l^\infty(X_i)$. One has $\|[x_i]_\mathcal{U}\| = \lim_\mathcal{U} \|x_i\|$ independently of the representative of $[x_i]_\mathcal{U}$. The ultraproduct $(X_i')/\mathcal{U}$ of the duals can be identified isometrically with a closed subspace of the dual X' via $[x_i']_\mathcal{U}([x_i]_\mathcal{U}) = \lim_\mathcal{U} x_i'(x_i)$. An ultraproduct $A = (A_i)/\mathcal{U}$ of a family of C^*-algebras A_i is canonically a C^*-algebra with pointwise multiplication and involution because in this case the null space $c_0(X_i)$ is an ideal in $l^\infty(A_i)$.

Let now $I = \mathbb{N}$ and set $A = (A_i)/\mathcal{U}$. For each element $\Psi \in A'$ of the form $\Psi = [\psi_i]_\mathcal{U}$ we have $|\Psi| = [|\psi_i|]_\mathcal{U}$ and $|\Psi^*| = [|\psi_i^*|]_\mathcal{U}$. [To see this choose $a = [a_i]_\mathcal{U}$ in the unit ball of A such that $\|a_i\| = 1$ and $\Psi(a) = \lim_\mathcal{U} \psi_i(a_i) = \lim_\mathcal{U} \|\psi_i\|$. Then $|\Psi| = a\Psi = [a_i\psi_i]_\mathcal{U}$ and $\|\,|\psi_i| - a_i\psi_i\| \leqslant (2\,\|\psi_i\|\,|\,\|\psi_i\| - \psi_i(a_i)|)^{1/2} \overset{\mathcal{U}}{\to} 0$ by (3.5) of Lemma 3.3. Hence $|\Psi| = [|\psi_i|]_\mathcal{U}$. For $|\Psi^*| = [|\psi_i^*|]_\mathcal{U}$ the proof is analogous.]

By (3.15) the n functionals $\Phi_k = [\varphi_{k,i}]_\mathcal{U}$ span l_n^1 isometrically. Therefore, by Lemma 2.1 the n functionals $|\Phi_k| = [|\varphi_{k,i}|]_\mathcal{U}$ span l_n^1 isometrically, too, and so do the n functionals $|\Phi_k^*| = [|\varphi_{k,i}^*|]_\mathcal{U}$. This means that there is a sequence (δ_i) of positive numbers tending to 0 (along $\mathcal{U}$) such that

$$\Big(1-\delta_i\Big)\sum_{k=1}^n |\alpha_k| \leqslant \Big\|\sum_{k=1}^n \alpha_k|\varphi_{k,i}|\Big\| \leqslant \sum_{k=1}^n |\alpha_k| \quad \forall\,(\alpha_k)\subset\mathbb{C},\ i\in\mathbb{N};$$
$$\Big(1-\delta_i\Big)\sum_{k=1}^n |\alpha_k| \leqslant \Big\|\sum_{k=1}^n \alpha_k|\varphi_{k,i}^*|\Big\| \leqslant \sum_{k=1}^n |\alpha_k| \quad \forall\,(\alpha_k)\subset\mathbb{C},\ i\in\mathbb{N}.$$

By part (a1) we choose $\delta_0 = \delta(n, \varepsilon/2)$. There is an infinite subset $J \subset \mathbb{N}$ such that $\delta_i \leqslant \delta_0$ for all $i \in J$. Thus, by part (a1), we obtain, for each $i \in J$, two finite sequences $(p_{k,i})_{k=1}^n$, $(q_{k,i})_{k=1}^n$ of open projections with pairwise orthogonal closures in A_i'' such that

$$p_{k,i}(|\varphi_{k,i}|) > 1 - \frac{\varepsilon}{2}, \quad q_{k,i}(|\varphi_{k,i}^*|) > 1 - \frac{\varepsilon}{2}$$

for all $k = 1, \dots, n$ and all $i \in J$. This contradicts (3.16) and thus proves the lemma for the case where $s = t = 1$.

(b) Now we turn to the general case of arbitrary open projections $s, t \in A''$.

By part (a) we choose $\delta' = \delta(n, \varepsilon/2)$. Further we choose $\delta'' > 0$ and $\delta''' > 0$ such that

$$2\sqrt{\delta''} + \delta'' \leqslant \delta', \quad \delta''' < \frac{\varepsilon}{2}, \quad 4\sqrt{\delta'''} < \delta''.$$

Finally, we choose $\delta = \delta(n, (\delta'''/6)^2)$ by part (a) assuming in addition that $\delta \leqslant \delta'''$. Suppose now that (3.7) holds for this just defined δ.

Since (3.7) remains valid if $t\varphi_k s$ is replaced by φ_k we can apply what has been proved in part (a) in order to get normalized pairwise orthogonal $\widetilde{\varphi}_k \in A'$ such that

$$\|\varphi_k - \widetilde{\varphi}_k\| \leqslant 6\sqrt{(\delta'''/6)^2} = \delta''' < \frac{\varepsilon}{2} \quad \text{for } k = 1, \dots, n. \tag{3.17}$$

The $|\widetilde{\varphi}_k|$ are normalized and, by Lemma 2.1, orthogonal and so are the $|\widetilde{\varphi}_k^*|$. Thus

$$\Big\|\sum_1^n \alpha_k|\widetilde{\varphi}_k|\,\Big\| = \sum_1^n |\alpha_k| \quad \text{and} \quad \Big\|\sum_1^n \alpha_k|\widetilde{\varphi}_k^*|\,\Big\| = \sum_1^n |\alpha_k| \tag{3.18}$$

for all scalars $\alpha_k \in \mathbb{C}$. Let $\widetilde{\varphi}_k = u_k|\widetilde{\varphi}_k|$ be the polar decomposition. Then

$$\begin{aligned} 1-2\delta''' &\leqslant 1-(\delta+\delta''') \\ &\overset{(3.17)}{\leqslant} 1-\delta-(\|t\varphi_k s\|-\|t\widetilde{\varphi}_k s\|) \\ &\overset{(3.7)}{\leqslant} \|t\varphi_k s\|-(\|t\varphi_k s\|-\|t\widetilde{\varphi}_k s\|)=\|t\widetilde{\varphi}_k s\| \\ &= \|tu_k|\widetilde{\varphi}_k|s\| = \|\,|(tu_k|\widetilde{\varphi}_k|s)|\,\| = s(|(tu_k|\widetilde{\varphi}_k|s)|) \leqslant s(|\widetilde{\varphi}_k|), \end{aligned}$$

where the last inequality follows from [22], III.4.9. Analogously $t(|\widetilde{\varphi}_k^*|) \geqslant 1-2\delta'''$. Hence by (3.6) of Lemma 3.3,

$$\|t\widetilde{\varphi}_k s-\widetilde{\varphi}_k\| \leqslant 4\sqrt{\delta'''} < \delta'' \quad \text{and} \quad \|s\widetilde{\varphi}_k^* t-\widetilde{\varphi}_k\| \leqslant 4\sqrt{\delta'''} < \delta''.$$

Recall that the absolute value is norm continuous on preduals of von Neumann algebras and that, more precisely, $\|\,|\sigma|-|\tau|\,\| \leqslant 2\sqrt{\|\sigma-\tau\|}+\|\sigma-\tau\|$ for any pair of elements σ,τ in the predual of a von Neumann algebra (see the proof of [22], III.4.10 or see [12] for an improvement). Thus

$$\|\,|t\widetilde{\varphi}_k s|-|\widetilde{\varphi}_k|\,\| \leqslant 2\sqrt{\delta''}+\delta'' \leqslant \delta', \quad \|\,|s\widetilde{\varphi}_k^* t|-|\widetilde{\varphi}_k^*|\,\| \leqslant \delta'.$$

In view of (3.18) we get the first inequalities of (3.19) and (3.20) (the second ones being trivial):

$$(3.19)\qquad (1-\delta')\sum_1^n|\alpha_k| \leqslant \Big\|\sum_1^n \alpha_k\,|t\widetilde{\varphi}_k s|\Big\| \leqslant \sum_1^n |\alpha_k| \quad \forall\,(\alpha_k)\subset\mathbb{C},$$

$$(3.20)\qquad (1-\delta')\sum_1^n|\alpha_k| \leqslant \Big\|\sum_1^n \alpha_k\,|s\widetilde{\varphi}_k^* t|\Big\| \leqslant \sum_1^n |\alpha_k| \quad \forall\,(\alpha_k)\subset\mathbb{C}.$$

Note that the support projection of $|t\widetilde{\varphi}_k s|$ (respectively of $|s\widetilde{\varphi}_k^* t|$) is majorized by s (respectively by t) and that $sA''s\cap A$ and $tA''t\cap A$ are hereditary subalgebras of A. Therefore

$$\Big\|\sum_1^n \alpha_k\,|t\widetilde{\varphi}_k s|\Big\| = \Big\|\sum_1^n \alpha_k\,|(t\widetilde{\varphi}_k s)_{|sA''s\cap A}|\Big\|_{sA''s\cap A}$$

in (3.19) and likewise

$$\Big\|\sum_1^n \alpha_k\,|t\widetilde{\varphi}_k^* s|\Big\| = \Big\|\sum_1^n \alpha_k\,|(t\widetilde{\varphi}_k^* s)_{|tA''t\cap A}|\Big\|_{tA''t\cap A}$$

in (3.20). Now we apply part (a) to $sA''s\cap A$ and to $tA''t\cap A$. The choice of δ' yields the desired open projections $p_k \in (sA''s\cap A)'' = sA''s$ and $q_k \in tA''t$ satisfying the analogues of (3.8) and (3.9); more precisely they satisfy

$$p_k(|\widetilde{\varphi}_k|) > 1-\frac{\varepsilon}{2}, \quad q_k(|\widetilde{\varphi}_k^*|) > 1-\frac{\varepsilon}{2},$$

for $k=1,\dots,n$. Together with (3.17) this gives (3.8) and (3.9). ∎

Corollary 3.6. *For each $\varepsilon > 0$ and each $n \in \mathbb{N}$ there is $\delta = \delta(n, \varepsilon) > 0$ with the following property.*

Let $\mathcal{N}$ be a von Neumann algebra. If there are functionals $\varphi_1, \ldots, \varphi_n$ in the unit ball of $\mathcal{N}_*$ and (arbitrary) projections $s, t \in \mathcal{N}$ such that

$$(1-\delta)\sum_1^n |\alpha_k| \leqslant \Big\|\sum_1^n \alpha_k\, t\varphi_k s\Big\| \leqslant \sum_1^n |\alpha_k| \quad \forall\, (\alpha_k) \subset \mathbb{C}, \tag{3.21}$$

then there are pairwise orthogonal projections $p_1, \ldots, p_n \in s\mathcal{N}s$ and pairwise orthogonal projections $q_1, \ldots, q_n \in t\mathcal{N}t$ such that

$$\|\varphi_k - \psi_k\| < \varepsilon \quad \textit{for } k = 1, \ldots, n$$

where $\psi_k = q_k\varphi_k p_k/\|q_k\varphi_k p_k\|$ (or $\psi_k = \varphi_k p_k/\|\varphi_k p_k\|$ or $\psi_k = q_k\varphi_k/\|q_k\varphi_k\|$).

Proof. For $s = t = 1$ the assertion is immediate from Lemma 3.5 and Lemma 3.4. For arbitrary projections $s, t \in \mathcal{N}$ we proceed as in part (b) of the proof of Lemma 3.5 in order to show that (3.21) can be replaced by (3.19) and (3.20) and to apply this to the subalgebras $s\mathcal{N}s$ and $t\mathcal{N}t$. ∎

4. PROOF OF THEOREM 1.2

Without loss of generality we assume that $\|\varphi_m\| = 1$ for all $m \in \mathbb{N}$. Let (η_n) be a sequence of positive numbers such that $\sum \eta_n$ converges.

By induction on $n = 1, 2, \ldots$ we construct an increasing sequence (m_n) in $\mathbb{N}$, functionals $\psi_{m_k}^{(n)} \in \mathcal{N}_*$ for $k = 1, \ldots, n$, such that for all $n \in \mathbb{N}$:

$$|\psi_{m_k}^{(n)}| \perp |\psi_{m_l}^{(n)}|, \qquad k, l = 1, \ldots, n,\ k \neq l, \tag{4.1}$$

$$\|\psi_{m_k}^{(n)}\| = 1, \qquad k = 1, \ldots, n \tag{4.2}$$

$$\|\psi_{m_k}^{(n)} - \psi_{m_k}^{(n-1)}\| < \eta_n, \quad k = 1, \ldots, n-1, \tag{4.3}$$

$$\|\psi_{m_n}^{(n)} - \varphi_{m_n}\| < \eta_n. \tag{4.4}$$

For $n = 1$ one may simply set $\psi_{m_1}^{(1)} = \varphi_1$; ((4.1), $n = 1$) and ((4.3), $n = 1$) are void, ((4.2), $n = 1$) and ((4.4), $n = 1$) are trivial.

Induction step $n \mapsto n + 1$.

Suppose that m_k and $\psi_{m_k}^{(n)}$ have been constructed for $k = 1, \ldots, n$ according to (4.1)–(4.4).

Choose $\delta_1 = \delta(n, \eta_{n+1}/2) > 0$ according to Corollary 3.6 such that furthermore $\delta_1 < \eta_{n+1}/2$. Let $j \in \mathbb{N}$ be such that $(2/j)^{1/2} < \delta_1$. Now, again according to Corollary 3.6, choose $\delta_0 = \delta(nj, \eta_{n+1})$.

Since (φ_m) spans l^1 almost isometrically there is an index $m_0 > m_n$ such that $(\varphi_m)_{m \geqslant m_0}$ spans l^1 $(1 - \delta_0)$-isomorphically. By Corollary 3.6 (with $s = t = 1$, $\delta = \delta_0$) we find a finite set $N \subset \mathbb{N}$ of cardinality nj (for example $N = \{m_0 + 1, \ldots, m_0 + nj\}$), a finite sequence of orthogonal projections $(p_m)_{m \in N}$ in $\mathcal{N}$ such that

$$\Big\|\varphi_m - \frac{\varphi_m p_m}{\|\varphi_m p_m\|}\Big\| < \eta_{n+1} \quad \forall\, m \in N. \tag{4.5}$$

Set $\varphi = \sum_{k=1}^{n} |\psi_{m_k}^{(n)}|$; φ is positive. We have $\Big(\sum_{m\in N} p_m\Big)(\varphi) \leqslant \|\varphi\| \leqslant n$. Thus there is an index $m_{n+1} \in N$ such that $0 \leqslant p_{m_{n+1}}(\varphi) \leqslant 1/j$ and $0 \leqslant p_{m_{n+1}}(|\psi_{m_k}^{(n)}|) \leqslant 1/j$ for $k = 1, \ldots, n$. We set $s = 1 - p_{m_{n+1}}$ and define $\widetilde{\psi}_{m_k}^{(n+1)} = \psi_{m_k}^{(n)} s$ for $k = 1, \ldots, n$ and

$$\psi_{m_{n+1}}^{(n+1)} = \frac{\varphi_{m_{n+1}} p_{m_{n+1}}}{\|\varphi_{m_{n+1}} p_{m_{n+1}}\|}.$$

Then ((4.2), $n+1$) holds for $k = n+1$ and ((4.4), $n+1$) holds by (4.5). We have $s(|\psi_{m_k}^{(n)}|) = \|\psi_{m_k}^{(n)}\| - p_{m_{n+1}}(|\psi_{m_k}^{(n)}|) \geqslant 1 - 1/j$ by (4.2). From this and (3.6) one gets that

$$\|\widetilde{\psi}_{m_k}^{(n+1)} - \psi_{m_k}^{(n)}\| \leqslant (2/j)^{1/2} < \delta_1 < \frac{\eta_{n+1}}{2}, \quad k = 1, \ldots, n. \tag{4.6}$$

Thus, up to δ_1 the $\widetilde{\psi}_{m_k}^{(n+1)}$ are near to an isometric copy of l_n^1 because

$$\begin{aligned}
\sum_{k=1}^{n} |\alpha_k| &\geqslant \Big\|\sum_{k=1}^{n} \alpha_k \widetilde{\psi}_{m_k}^{(n+1)}\Big\| = \Big\|\sum_{k=1}^{n} \alpha_k \widetilde{\psi}_{m_k}^{(n+1)} s\Big\| \\
&\geqslant \Big\|\sum_{k=1}^{n} \alpha_k \psi_{m_k}^{(n)}\Big\| - \Big\|\sum_{k=1}^{n} \alpha_k (\widetilde{\psi}_{m_k}^{(n+1)} - \psi_{m_k}^{(n)})\Big\| \\
&\overset{(4.6)}{\geqslant} \Big\|\sum_{k=1}^{n} \alpha_k \psi_{m_k}^{(n)}\Big\| - (2/j)^{1/2} \sum_{k=1}^{n} |\alpha_k| \\
&\overset{(4.1)(4.2)}{=} \big(1 - (2/j)^{1/2}\big) \sum_{k=1}^{n} |\alpha_k| \\
&> (1 - \delta_1) \sum_{k=1}^{n} |\alpha_k|.
\end{aligned}$$

It remains to apply Corollary 3.6 another time (with $t = 1$, $\delta = \delta_1$) in order to get small normalized orthogonal perturbations $\psi_{m_k}^{(n+1)}$ — whence ((4.2), $n+1$) for $k \leqslant n$ — of the $\widetilde{\psi}_{m_k}^{(n+1)}$ whose right supports are majorized by s and thus orthogonal to the right support of $\psi_{m_k}^{(n+1)}$ such that $\|\psi_{m_k}^{(n+1)} - \widetilde{\psi}_{m_k}^{(n+1)}\| < \eta_{n+1}/2$ for $k = 1, \ldots, n$. Together with (4.6) this gives ((4.3), $n+1$). Finally one verifies ((4.1), $n+1$) by observing that the support projections of the $|\psi_{m_k}^{(n+1)}|$ are the right supports of the $\psi_{m_k}^{(n+1)}$. This ends the induction.

By construction, $(\psi_{m_k}^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence for each k because $\|\psi_{m_k}^{(n)} - \psi_{m_k}^{(i)}\| \leqslant \sum_{l=i+1}^{n} \eta_l \to 0$ as $n > i \to \infty$. Let $\psi_k = \lim_n \psi_{m_k}^{(n)}$ be its limit. Then $\|\psi_k - \varphi_{m_k}\| \leqslant \|\varphi_{m_k} - \psi_{m_k}^{(k)}\| + \|\psi_{m_k}^{(k)} - \lim_n \psi_{m_k}^{(n)}\| \leqslant \eta_k + \sum_{l=k+1}^{\infty} \eta_l \to 0$ as $k \to \infty$. The ψ_k have pairwise orthogonal right supports because by continuity of the absolute value ([22], III.4.10), if $k \neq l$ one has

$$\| \, |\psi_k| - |\psi_l| \, \| = \lim_{n\to\infty} \| \, |\psi^{(n)}_{m_k}| - |\psi^{(n)}_{m_l}| \, \|$$

$$\overset{(4.1)}{=} \lim_{n\to\infty} \| \, |\psi^{(n)}_{m_k}| \, \| + \| \, |\psi^{(n)}_{m_l}| \, \| = \|\psi_k\| + \|\psi_l\|.$$

So far we have proved that if (φ_m) spans l^1 almost isometrically then there is a subsequence (φ_{m_k}) and there are pairwise orthogonal projections $s_k \in \mathcal{N}$ (namely the right support projections of the ψ_k) such that $\|\varphi_{m_k} - \varphi_{m_k} s_k\| \leqslant \|\varphi_{m_k} - \psi_k\| + \|\psi_k s_k - \varphi_{m_k} s_k\| \leqslant 2\|\varphi_{m_k} - \psi_k\| \to 0$. Since $(\varphi^*_{m_k})$ spans l^1 almost isometrically, too, there are pairwise orthogonal projections $t_l \in \mathcal{N}$ such that $\|\varphi^*_{m_{k_l}} - \varphi^*_{m_{k_l}} t_l\| \to 0$ for an appropriate sequence (m_{k_l}) in $\mathbb{N}$. Set $\widetilde{\varphi}_l = t_l \varphi_{m_{k_l}} s_{k_l}$. Then $\|\varphi_{m_{k_l}} - \widetilde{\varphi}_l\| \leqslant \|\varphi_{m_{k_l}} - \varphi_{m_{k_l}} s_{k_l}\| + \|(\varphi_{m_{k_l}} - t_l \varphi_{m_{k_l}}) s_{k_l}\| \leqslant \|\varphi_{m_{k_l}} - \varphi_{m_{k_l}} s_{k_l}\| + \|\varphi^*_{m_{k_l}} - \varphi^*_{m_{k_l}} t_l\| \to 0$.

The second statement of the theorem is trivial by the definiton of the $\widetilde{\varphi}_l$. This ends the proof. ∎

From Remark 5.1 (2) after the proof of Theorem 1.1 at the end of the next section it follows that Theorem 1.2 does not hold for unbounded sequences (φ_m).

5. PROOF OF THEOREM 1.1

(i) ⇒ (ii). Let (x_{n_k}) be a subsequence of (x_n). If (x_{n_k}) contains a sequence $(x_{n_{k_l}})$ such that $x_{n_{k_l}} = 0$ for all $l \in \mathbb{N}$ then we simply choose $y_l = 0$ for $l \in \mathbb{N}$. Otherwise we may (pass to another subsequence and) suppose that $\|x_{n_k}\|_1 \neq 0$ for all $k \in \mathbb{N}$. By norm density of $L^1 \cap L^\infty$ in L^1 and the fact that the norm topology is finer than the measure topology we may suppose without loss of generality that $0 \neq \|x_{n_k}\|_\infty < \infty$ for all $k \in \mathbb{N}$. We set $\varepsilon_l = 2^{-l}/\tau(1)$ for $l \in \mathbb{N}$.

By induction over $l \in \mathbb{N}$ we construct a strictly increasing subsequence (n_{k_l}) of (n_k), projections $p_l \in \mathcal{N}$ and positive numbers δ_l such that for all $l \in \mathbb{N}$

$$\tau(p_l) < \delta_l \quad \text{where } p_l = \chi_{]\varepsilon_l,\infty[}(|x_{n_{k_l}}|) \tag{5.1}$$

and where

$$\delta_l = \frac{2^{-l}}{\max\limits_{1\leqslant m\leqslant l-1} \|x_{n_{k_m}}\|_\infty}, \quad \text{if } l \geqslant 2. \tag{5.2}$$

For $l = 1$ we choose $n_{k_1} = n_1$ and any $\delta_1 > \tau(p_1)$. For the induction step $l \mapsto l+1$ we suppose n_{k_m}, p_m, and δ_m to be constructed for $m = 1, \ldots, l$, we define δ_{l+1} by (5.2) and choose $n_{k_{l+1}}$ such that

$$\tau(\chi_{]\varepsilon_{l+1},\infty[}(|x_{n_{k_{l+1}}}|) < \delta_{l+1}$$

which is possible because $x_n \xrightarrow{\tau} 0$. We define p_{l+1} by (5.1). This settles ((5.1), $l+1$) and ends the induction.

By (5.2) we have

$$\delta_{l+1+r} = \frac{2^{-(l+1+r)}}{\max\limits_{m\leqslant l+r} \|x_{n_{k_m}}\|_\infty} \leqslant 2^{-(r+1)} \frac{2^{-l}}{\|x_{n_{k_l}}\|_\infty}$$

for $r \in \mathbb{N} \cup \{0\}$ which gives

$$\sum_{m \geqslant l+1} \delta_m = \sum_{r \geqslant 0} \delta_{l+1+r} \leqslant \frac{2^{-l}}{\|x_{n_{k_l}}\|_\infty}. \tag{5.3}$$

Put $q_l = 1 - \bigvee_{m \geqslant l+1} p_m$ and $\widetilde{y}_l = x_{n_{k_l}}(p_l \wedge q_l)$. By construction the $\widetilde{y}_l$ have pairwise orthogonal right support projections and their left support projections are majorized by the ones of the $x_{n_{k_l}}$. We show that $\|x_{n_{k_l}} - \widetilde{y}_l\|_1 \to 0$. In order to save indices we use the abbreviations $x = x_{n_{k_l}}$, $p = p_l$, $q = q_l$, $\widetilde{y} = \widetilde{y}_l$ until the end of formula (5.4):

$$\begin{aligned}
\|x - \widetilde{y}\|_1 &\leqslant \|x - xp\|_1 + \|xp - \widetilde{y}\|_1 = \|x(1-p)\|_1 + \|x(p - (p \wedge q))\|_1 \\
&\leqslant \|x(1-p)\|_\infty \, \tau(1) + \|x\|_\infty \, \tau(p - (p \wedge q)) \\
&\overset{(*)}{=} \| \, |x| \, \chi_{[0,\varepsilon_l]}(|x|)\|_\infty \, \tau(1) + \|x\|_\infty \, \tau\big((p \vee q) - q\big) \\
&\leqslant \varepsilon_l \, \tau(1) + \|x\|_\infty \, \tau(1-q) \leqslant \varepsilon_l \, \tau(1) + \|x\|_\infty \Big(\sum_{m \geqslant l+1} \tau(p_m) \Big) \\
&\overset{(5.1)(5.3)}{\leqslant} 2^{-(l-1)}.
\end{aligned} \tag{5.4}$$

For $(*)$ we used that $p - (p \wedge q)$ and $(p \vee q) - q$ are equivalent projections for any two projections p, q ([22], V.1.6) hence $\tau(p - (p \wedge q)) = \tau((p \vee q) - q)$.

So far we have proved that given a τ-null subsequence (x_{n_k}) there are $x_{n_{k_l}}$ and there are $\widetilde{y}_l$ whose right supports are orthogonal and whose left supports are majorized by the left supports of the $x_{n_{k_l}}$ such that $\|x_{n_{k_l}} - \widetilde{y}_l\|_1 \to 0$. In particular, $\widetilde{y}_l \overset{\tau}{\longrightarrow} 0$ whence $\widetilde{y}_l^* \overset{\tau}{\longrightarrow} 0$. Thus we can apply the same reasoning (up to passing to appropriate subsequences) in order to find perturbations y_l^* of the $\widetilde{y}_l^*$ which have both orthogonal right and orthogonal left supports such that $\|\widetilde{y}_l - y_l\|_1 = \|\widetilde{y}_l^* - y_l^*\|_1 \to 0$ hence $\|x_{n_{k_l}} - y_l\|_1 \to 0$. This ends the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) Since τ is finite and the y_l are pairwise orthogonal we have that $y_l \overset{\tau}{\longrightarrow} 0$. And $\|x_{n_{k_l}} - y_l\| \to 0$ entails $x_{n_{k_l}} - y_l \overset{\tau}{\longrightarrow} 0$ hence $x_{n_{k_l}} \overset{\tau}{\longrightarrow} 0$. Thus each subsequence of (x_n) contains a subsequence which converges to 0 in measure whence $x_n \overset{\tau}{\longrightarrow} 0$.

(ii) $\Rightarrow$ (iii) follows from Lemma 3.1: Suppose (ii) holds and $\inf \|x_{n_k}\|_1 > 0$ for a subsequence (x_{n_k}) of (x_n). Then by (ii), there are orthogonal y_l and there is $(x_{n_{k_l}})$ such that $\|x_{n_{k_l}} - y_l\|_1 \to 0$. One may suppose that $\inf \|y_l\|_1 > 0$ hence (y_l) spans l^1 isometrically. Thus by Lemma 3.1, the sequence $(x_{n_{k_l}}) = (y_l + (x_{n_{k_l}} - y_l))$ spans l^1 almost isometrically.

(iii) $\Rightarrow$ (iv): Von Neumann preduals are L-embedded spaces ([7], IV.1.1), thus by [19] each sequence spanning l^1 almost isometrically admits a subsequence spanning l^1 asymptotically.

(iv) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) follows from Theorem 1.2.

See the following Remark 5.1 (2) for an example which shows that in general (iii) does not imply (i), (ii) for unbounded sequences (x_n). ∎

REMARKS 5.1. (1) As an illustration of how to get an orthogonal subsequence consider the sequence $x_n = n\,1_{[0,\,1/n]}$ in $L^1([0,1])$. One may take, for example, $y_l = x_{n_l} 1_{]1/n_{l+1},1]} = n_l 1_{]1/n_{l+1},1/n_l]}$ where $n_l = 2^{(2^l)}$.

(2) In general (iii) does not imply (i), neither (ii), if the sequence (x_n) is unbounded. Take the bounded sequence $x_n = n^2 1_{[1/n+1,\,1/n[} + \frac{1}{n}$ in $L^1([0,1])$. It converges to zero in measure and does not contain a norm null sequence. Hence by (i)$\Rightarrow$(iii) an appropriate subsequence (x_{n_k}) spans l^1 almost isometrically. Thus the unbounded sequence $(n_k^2 x_{n_k})$ satisfies (iii) but not (i). It cannot satisfy (ii) either because (ii) $\Leftrightarrow$ (i) holds also for unbounded sequences. This means in particular that Theorem 1.2 does not hold for unbounded sequences (φ_m).

(3) A few straightforward modifications show that (i) $\Leftrightarrow$ (ii) holds accordingly also for $L^p(\mathcal{N},\tau)$, $1 \leqslant p < \infty$. (Cf. [21].)

6. l^1-COPIES IN THE DUAL OF C^*-ALGEBRAS; PROOF OF PROPOSITION 1.3

The proof of the main result of [18] gives the following: Let $(\varphi_m) \subset A'$ be a bounded sequence of selfadjoint functionals on a C^*-algebra A, let $\varepsilon > 0$. If (φ_m) spans l^1 r-isomorphically $(0 < r < 1)$ then there is a subsequence (φ_{m_n}) and there is a sequence (x_n) of pairwise orthogonal normalized selfadjoint elements of A such that $\varphi_{m_n}(x_n) > (1-\varepsilon)r\|\varphi_{m_n}\|$. This amounts to saying that $|\varphi_{m_n}|(|x_n|) > (1-\varepsilon)r\|\varphi_{m_n}\|$. (To see this it is enough to decompose both φ_{m_n} and x_n in their positive and negative parts.) This is Lemma 6.3 for selfadjoint φ_m with $a_n = b_n = |x_n|$ and with the better factor r instead of r^2 in (6.2) and (6.3).

With Lemma 3.5 at one's disposal, the proof of Lemma 6.3 — and thus of Proposition 1.3 — is a straightforward modification of [18] and gives a kind of quantitative version of [18] which holds for arbitrary functionals, not only selfadjoint ones. (We give the entire proof of Proposition 1.3 not only for the sake of completeness but also because it is quite lengthy; therefore the usual argument "The details are left to the reader" would be somewhat misleading.) Yet, it does not complete the subject "perturbations of l^1-copies in C^*-algebras" as at least two questions remain open.

QUESTION 6.1. Is it necessary in Theorem 1.2 or in Proposition 1.3 to pass to subsequences?

In the commutative case a result of Dor ([5]) shows that, if $\mathcal{N}_* = L^1([0,1])$ contains a $(1-\delta)$-isomorphic copy of l^1 then the whole canonical basis of this copy can be perturbed in norm so to span l^1 isometrically with the perturbation smaller than δ' and $\delta' \to 0$ as $\delta \to 0$. Furthermore Arazy ([3]) proved that if the predual of an arbitrary von Neumann algebra $\mathcal{N}$ contains a $(1-\delta)$-copy of l^1 then the whole copy is complemented by a projection whose norm is majorized by $1+\delta'$ — a result which has recently been generalized by N. Ozawa ([16]) to the category of operator spaces.

Question 6.2. Can the (m_n), (a_n) and (b_n) in Proposition 1.3 be arranged such that $\|\varphi_{m_n} - b_n\varphi_{m_n}a_n\| \to 0$ as $n \to \infty$?

[Let us sketch in passing why this would generalize Lemma 6.3. If $(\varphi_m) \subset A'$ is normalized and spans l^1 r-isomorphically then by James' distortion theorem there are blocks $\psi_n = \sum_{i\in F_n} \lambda_i\varphi_i$ spanning l^1 almost isometrically such that $\sum_{i\in F_n} |\lambda_i| \leqslant 1/r$. Now, if there are appropriate $a_n, b_n \in A$ such that $\|\psi_n - b_n\psi_n a_n\| \to 0$ (after passing, if necessary, to an appropriate subsequence of (ψ_n)), one deduces that $|\psi_n|(a_n) \to \lim \|\psi_n\| = 1$ that is there are ε_n such that $0 < \varepsilon_n \to 0$ and $|\psi_n|(a_n) > \sqrt{1-\varepsilon_n}$. Thus Lemma 6.3 holds because for each n there is $i_n \in F_n$ such that $|\varphi_{i_n}|(a_n) > (1-\varepsilon_n)r^2$ because otherwise by [22], III.4.7 one would have the contradiction

$$\sqrt{1-\varepsilon_n} < |\psi_n|(a_n) = \Big|\sum_{i\in F_n} \lambda_i\varphi_i\Big|(a_n) \leqslant \Big(\sum_{F_n} |\lambda_i|\,\|\varphi_i\|\Big)^{1/2}\Big(\sum_{F_n} |\lambda_i|\,(|\varphi_i|(a_n^2))\Big)^{1/2}$$
$$\leqslant \frac{1}{r}\Big(\max_{F_n} |\varphi_i|(a_n^2)\Big)^{1/2} \leqslant \frac{1}{r}\Big(\max_{F_n} |\varphi_i|(a_n)\Big)^{1/2} \leqslant \sqrt{1-\varepsilon_n}.$$

Similarly one obtains $|\varphi_{i_n}^*|(b_n) > (1-\varepsilon_n)r^2$.]

Proposition 1.3 follows immediately from Lemma 6.3 (and (3.6) of Lemma 3.3) with $s = 1 = t$. The technical part concerning s, t is added because it might be usefull for answering the second question just mentioned above.

Lemma 6.3. *Let A be a C^*-algebra (unital or not), $r > 0$, let (φ_m) be a normalized sequence in A' spanning l^1 r-isomorphically that is such that*

$$r\sum |\alpha_m| \leqslant \Big\|\sum \alpha_m\varphi_m\Big\| \leqslant \sum |\alpha_m| \quad \forall\,(\alpha_m) \subset \mathbb{C}. \tag{6.1}$$

Then, given $\varepsilon > 0$, there are a sequence (m_n) in $\mathbb{N}$ and a sequence (a_n) of pairwise orthogonal positive normalized elements in A and another sequence (b_n) of pairwise orthogonal positive normalized elements in A such that

$$|\varphi_{m_n}|(a_n) > (1-\varepsilon)r^2, \tag{6.2}$$
$$|\varphi_{m_n}^*|(b_n) > (1-\varepsilon)r^2, \tag{6.3}$$

for each $n \in \mathbb{N}$.

Moreover, if s and t are open projections in A'' such that s (respectively t) majorizes the right (respectively left) supports of all φ_m (that is $t\varphi_m s = \varphi_m$ for all $m \in \mathbb{N}$) then one can obtain in addition that $a_n \in sA''s$ and $b_n \in tA''t$.

Moreover, if the φ_m are selfadjoint such that $s\varphi_m s = \varphi_m$ for an open projection $s \in A''$ then one can obtain in addition $|\varphi_{m_n}|(a_n) = |\varphi_{m_n}^*|(a_n) > (1-\varepsilon)r$ instead of (6.2) *and* (6.3) *with $a_n \in sA''s$.*

Proof. The last statement concerning selfadjoint functionals follows from the discussion in the beginning of this section.

For the other statements it is enough to construct a sequence (p_n) of orthogonal open projections in $sA''s$ such that

$$|\varphi_{m_n}|(p_n) > (1-\varepsilon)r^2 \tag{6.4}$$

for an appropriate subsequence (φ_{m_n}) because then, by the definition of open projections, for all $n \in \mathbb{N}$ positive elements $a_n \leqslant p_n$ can be choosen so to be pairwise orthogonal (since the p_n are) and so to satisfy (6.2); finally, since (6.1) remains valid if φ_n^* is substituted for φ_n the same reasoning that leads to (6.2) shows the existence of a sequence (b_n) in $tA''t$ as desired in (6.3).

Let $0 < \varepsilon < 1$ and choose a sequence (ε_n) of positive numbers such that $\sum \varepsilon_n = \varepsilon$ and $\varepsilon_n \leqslant 3/4$ for all $n \in \mathbb{N}$.

By induction over $n = 1, 2, \dots$ we construct a sequence (p_n) of open projections in $sA''s$, a sequence of indices (m_n), a decreasing sequence (N_n) of infinite subsets of $\mathbb{N}$, i.e. $\cdots \subset N_{n+1} \subset N_n \subset \cdots \subset N_1 \subset N_0 = \mathbb{N}$, such that we have for all $n \in \mathbb{N}$:

$$\overline{p_n} \in sA''s, \tag{6.5}$$

$$\overline{p_i}\,\overline{p_n} = 0 \qquad \forall\, i < n, \tag{6.6}$$

$$\overline{p_n}(|\varphi_m|) < \frac{1}{72} r^2 \varepsilon_n^4 \qquad \forall\, m \in N_n, \tag{6.7}$$

$$p_n(|\varphi_{m_n}|) > r^2 \Big(1 - \sum_1^n \varepsilon_i\Big), \tag{6.8}$$

$$m_n \in N_{n-1}. \tag{6.9}$$

We start the induction with $n = 1$.

Choose $j_1 \in \mathbb{N}$ with $1/j_1 < r^2 \varepsilon_1^4/72$. For j_1 and $\varepsilon_1/4$ Lemma 3.5 yields a number $\delta_1 = \delta_1(j_1, \varepsilon_1/2) > 0$. By James' distortion theorem applied to (6.1) there are pairwise disjoint finite sets $F_k^{(1)} \subset N_0 = \mathbb{N}$, a finite sequence $(\lambda_i^{(1)})_{i \in F_k^{(1)}} \subset \mathbb{C}$ and functionals $\tau_k^{(1)} = \sum_{i \in F_k^{(1)}} \lambda_i^{(1)} \varphi_i$ for $k \in \mathbb{N}$, such that

$$\sum_{F_k^{(1)}} |\lambda_i^{(1)}| \leqslant \frac{1}{r}, \tag{6.10}$$

$$(1 - \delta_1) \sum_{k \geqslant 1} |\alpha_k| \leqslant \Big\| \sum_{k \geqslant 1} \alpha_k \tau_k^{(1)} \Big\| \leqslant \sum_{k \geqslant 1} |\alpha_k| \quad \forall\, (\alpha_k)_{k \in N_0} \subset \mathbb{C}. \tag{6.11}$$

By Lemma 3.5 and the choice of δ_1 there are pairwise orthogonal open projections $p_k^{(1)} \in sA''s$, $k \leqslant j_1$, such that

$$p_k^{(1)}(|\tau_k^{(1)}|) > 1 - \frac{\varepsilon_1}{2} \quad \forall\, k \leqslant j_1, \tag{6.12}$$

and since the projections can be chosen to have orthogonal closures in $sA''s$ we have

$$\Big(\sum_1^{j_1} \overline{p_k^{(1)}}\Big)(|\varphi_m|) \leqslant 1 \quad \forall\, m \in N_0.$$

Therefore there exist a $k_1 \leqslant j_1$ and an infinite set $N_1 \subset N_0$ such that

$$\overline{p_{k_1}^{(1)}}(|\varphi_m|) \leqslant \frac{1}{j_1} < \frac{r^2 \varepsilon_1^4}{72} \quad \forall\, m \in N_1. \tag{6.13}$$

Set $p_1 = p_{k_1}^{(1)}$, $\tau_1 = \tau_{k_1}^{(1)}$, $F_1 = F_{k_1}^{(1)}$. Then (6.5) holds for $n = 1$. Now we infer that

$$p_1(|\tau_1|) \overset{(6.12)}{>} 1 - \frac{\varepsilon_1}{2} > \sqrt{1-\varepsilon_1},$$

which in turn yields the existence of an index $m_1 \in F_1 \subset N_0$ as desired in (6.8) and (6.9) for $n = 1$, because otherwise we would have

$$\begin{aligned} p_1(|\tau_1|) &= p_1\Big(\Big|\sum_{F_1} \lambda_i^{(1)}\varphi_i\Big|\Big) \\ &\overset{(*)}{\leqslant} \Big(\sum_{F_1} \|\lambda_i^{(1)}\varphi_i\|\Big)^{1/2}\Big(\sum_{F_1} p_1\big(|\lambda_i^{(1)}\varphi_i|\big)\Big)^{1/2} \\ &\leqslant \Big(\sum_{F_1} |\lambda_i^{(1)}|\Big)^{1/2}\Big(\max_{i\in F_1} p_1(|\varphi_i|)\sum_{F_1} |\lambda_i^{(1)}|\Big)^{1/2} \\ &\overset{(6.10)}{\leqslant} \frac{1}{r}\big(r^2(1-\varepsilon_1)\big)^{1/2} = \sqrt{1-\varepsilon_1}. \end{aligned}$$

Here inequality $(*)$ can be proved like III.4.7 in [22]. For ((6.6), $n = 1$) nothing needs to be proved. Inequality ((6.7), $n = 1$) corresponds to (6.13). The first induction step is done.

Induction step $n \to n + 1$:

Suppose p_k, N_k, m_k to be constructed for $k \leqslant n$ according to (6.6)–(6.9).

Since the $\overline{p_k}$ are orthogonal in $sA''s$, $\sum_1^n \overline{p_k}$ is closed by [1], Theorem II.7. Therefore $s_n = s - \sum_1^n \overline{p_k} \in sA''s$ is open. Set $\widetilde{\varphi}_m = \varphi_m s_n$ for $m \in N_n$.

CLAIM 6.4. The normalized functionals $\Big(\frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|}\Big)_{m\in N_n}$ form an l^1-basis with

$$\begin{aligned} r\Big(1-\sum_1^n \varepsilon_i^2\Big)^{1/2} \sum_{m\in N_n} |\alpha_m| &\leqslant \Big\| \sum_{m\in N_n} \alpha_m \frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|}\Big\| \\ &\leqslant \sum_{m\in N_n} |\alpha_m| \quad \forall\, (\alpha_m)_{m\in N_n} \subset \mathbb{C}. \end{aligned} \tag{6.14}$$

Set $\eta = \frac{r^2}{72}\sum_1^n \varepsilon_i^4$. Then

$$(s - s_n)(|\varphi_m|) = \Big(\sum_1^n \overline{p_k}\Big)(|\varphi_m|) \overset{(6.7)}{<} \eta \quad \forall\, m \in N_n, \tag{6.15}$$

thus since $s(|\varphi_m|) = \|\varphi_m\| = 1$

$$\begin{aligned} \|\varphi_m s_n - \varphi_m\| &\overset{(3.6)}{\leqslant} |2(\|\varphi_m\| - s_n(|\varphi_m|))|^{1/2} \\ &= \Big(2\sum_1^n \overline{p}_k(|\varphi_m|)\Big)^{1/2} \overset{(6.15)}{<} \sqrt{2\eta} \quad \forall\, m \in N_n; \end{aligned} \tag{6.16}$$

further we note that for all $m \in N_n$

$$\|\widetilde{\varphi}_m\| \leqslant \|\varphi_m\| = 1, \tag{6.17}$$

$$0 \leqslant 1 - \|\widetilde{\varphi}_m\| = \|\varphi_m\| - \|\widetilde{\varphi}_m\| \leqslant \|\varphi_m - \widetilde{\varphi}_m\| \overset{(6.18)}{\leqslant} \sqrt{2\eta} \tag{6.18}$$

$$\|\widetilde{\varphi}_m\| \overset{(6.18)}{\geqslant} 1 - \sqrt{2\eta}, \tag{6.19}$$

hence

$$\begin{aligned} \|\frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|} - \widetilde{\varphi}_m\| &\overset{(6.17)}{\leqslant} \frac{1}{\|\widetilde{\varphi}_m\|} - 1 = \big(1 - \|\widetilde{\varphi}_m\|\big)\frac{1}{\|\widetilde{\varphi}_m\|} \\ &\overset{(6.18)(6.19)}{\leqslant} \frac{\sqrt{2\eta}}{1-\sqrt{2\eta}} \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} \|\frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|} - \varphi_m\| &\leqslant \|\frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|} - \widetilde{\varphi}_m\| + \|\widetilde{\varphi}_m - \varphi_m\| \\ &\overset{(6.20)(6.16)}{\leqslant} \sqrt{2\eta}\Big(1 + \frac{1}{1-\sqrt{2\eta}}\Big) < 3\sqrt{2\eta} \end{aligned} \tag{6.21}$$

because $\varepsilon < 1$, $r \leqslant 1$, thus $\sqrt{2\eta} < 1/2$. Then (6.14) follows from

$$\begin{aligned} \Big\|\sum_{m\in N_n} \alpha_m \frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|}\Big\| &\geqslant \Big\|\sum_{N_n} \alpha_m\varphi_m\Big\| - \Big\|\sum_{N_n} \alpha_m \frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|} - \sum_{N_n} \alpha_m\varphi_m\Big\| \\ &\overset{(6.1)}{\geqslant} r\Big(1 - \sup_{N_n} \|\frac{\widetilde{\varphi}_m}{\|\widetilde{\varphi}_m\|} - \varphi_m\|\Big)\sum_{N_n}|\alpha_m| \\ &\overset{(6.21)}{\geqslant} r(1 - 3\sqrt{2\eta})\sum_{N_n}|\alpha_m| = r\Big(1 - \frac{1}{2}\Big(\sum_1^n \varepsilon_i^4\Big)^{1/2}\Big)\sum_{N_n}|\alpha_m| \\ &> r\Big(1 - \Big(\sum_1^n \varepsilon_i^4\Big)^{1/2}\Big)^{1/2}\sum_{N_n}|\alpha_m| > r\Big(1 - \sum_1^n \varepsilon_i^2\Big)^{1/2}\sum_{m\in N_n}|\alpha_m| \end{aligned}$$

and the Claim 6.4 is established.

Choose a number $j_{n+1} \in \mathbb{N}$ such that $1/j_{n+1} < r^2\varepsilon_{n+1}^2/4$. Further choose a number $\delta_{n+1} = \delta_{n+1}(j_{n+1}, \varepsilon_{n+1}^2/2) > 0$ according to Lemma 3.5. Now we apply James' distortion theorem. By (6.14) there are pairwise disjoint finite sets $F_k^{(n+1)} \subset N_n$, a finite sequence $(\lambda_i^{(n+1)})_{i\in F_k^{(n+1)}} \subset \mathbb{C}$ and functionals $\tau_k^{(n+1)} = \sum_{i\in F_k^{(n+1)}} \lambda_i^{(n+1)} \frac{\widetilde{\varphi}_i}{\|\widetilde{\varphi}_i\|}$ for each $k \in \mathbb{N}$ such that

$$\sum_{i\in F_k^{(n+1)}} |\lambda_i^{(n+1)}| \leqslant \frac{1}{r(1 - \sum_1^n \varepsilon_i^2)^{1/2}} \quad \forall\, k \in \mathbb{N}, \tag{6.22}$$

$$(1 - \delta_{n+1})\sum_{k\geqslant 1}|\alpha_k| \leqslant \Big\|\sum_{k\geqslant 1} \alpha_k\tau_k^{(n+1)}s_n\Big\| \leqslant \sum_{k\geqslant 1}|\alpha_k|. \tag{6.23}$$

By Lemma 3.5, applied to the open projections s_n and 1, to the functionals $\tau_k^{(n+1)} \in A'$, and to (6.23), there exist open projections $p_k^{(n+1)} \in s_n A'' s_n$, $k \leqslant j_{n+1}$, with pairwise orthogonal closures in $s_n A'' s_n$ such that

$$p_k^{(n+1)}(|\tau_k^{(n+1)}|) > 1 - \frac{\varepsilon_{n+1}^2}{2} \tag{6.24}$$

for $k \leqslant j_{n+1}$. Since the projections have orthogonal closures we have

$$\Big(\sum_1^{j_{n+1}} \overline{p_k^{(n+1)}}\Big)(|\varphi_m|) \leqslant 1 \quad \forall\, m \in N_n.$$

Therefore there exist an index $k_{n+1} \leqslant j_{n+1}$ and an infinite subset $N_{n+1} \subset N_n$ such that

$$\overline{p_{k_{n+1}}^{(n+1)}}(|\varphi_m|) \leqslant \frac{1}{j_{n+1}} < \frac{r^2 \varepsilon_{n+1}^4}{72} \quad \forall\, m \in N_{n+1}.$$

Set $p_{n+1} = p_{k_{n+1}}^{(n+1)}$, $\tau_{n+1} = \tau_{k_{n+1}}^{(n+1)}$, $F_{n+1} = F_{k_{n+1}}^{(n+1)}$. Then (6.5), (6.6) and (6.7) hold for $n+1$. Now we infer that

$$p_{n+1}(|\tau_{n+1}|) \overset{(6.24)}{>} 1 - \frac{\varepsilon_{n+1}^2}{2} > \sqrt{1 - \varepsilon_{n+1}^2}, \tag{6.25}$$

hence there is an index $m_{n+1} \in F_{n+1} \subset N_n$ as desired in (6.8) and (6.9) such that

$$p_{n+1}\left(\frac{\widetilde{\varphi}_{m_{n+1}}}{\|\widetilde{\varphi}_{m_{n+1}}\|}\right) > r^2(1 - \varepsilon_{n+1}^2)\Big(1 - \sum_1^n \varepsilon_i^2\Big), \tag{6.26}$$

because otherwise the following estimates would contradict (6.25):

$$\begin{aligned}
p_{n+1}(|\tau_{n+1}|) &= p_{n+1}\Big(\Big|\sum_{F_{n+1}} \lambda_i^{(n+1)} \frac{\widetilde{\varphi}_i}{\|\widetilde{\varphi}_i\|}\Big|\Big) \\
&\overset{(*)}{\leqslant} \Big(\sum_{F_{n+1}} \Big\|\lambda_i^{(n+1)} \frac{\widetilde{\varphi}_i}{\|\widetilde{\varphi}_i\|}\Big\|\Big)^{1/2} \Big(\sum_{F_{n+1}} p_{n+1}\Big(\Big|\lambda_i^{(n+1)} \frac{\widetilde{\varphi}_i}{\|\widetilde{\varphi}_i\|}\Big|\Big)\Big)^{1/2} \\
&\leqslant \Big(\sum_{F_{n+1}} |\lambda_i^{(n+1)}|\Big)^{1/2} \Big(\max_{i \in F_{n+1}} p_{n+1}\Big(\Big|\frac{\widetilde{\varphi}_i}{\|\widetilde{\varphi}_i\|}\Big|\Big) \sum_{F_{n+1}} |\lambda_i^{(n+1)}|\Big)^{1/2} \\
&\overset{(6.22)}{\leqslant} \frac{1}{r(1 - \sum_1^n \varepsilon_i^2)^{1/2}} \Big(r^2(1 - \varepsilon_{n+1}^2)\Big(1 - \sum_1^n \varepsilon_i^2\Big)\Big)^{1/2} = \sqrt{1 - \varepsilon_{n+1}^2}.
\end{aligned}$$

Here inequality $(*)$ can be proved like III.4.7 in [22]. Note that for a functional φ with polar decomposition $\varphi = u|\varphi|$ one has $|\varphi s| = |u\,|\varphi|\,s| \leqslant |\varphi|$ by III.4.9 in [22] which explains inequality $(**)$ below; now ((6.8), $n+1$) follows from

$$\begin{aligned} p_{n+1}(|\varphi_{m_{n+1}}|) &\overset{(**)}{\geqslant} p_{n+1}(|\varphi_{m_{n+1}} s_n|) \\ &\overset{(6.26)}{>} \|\varphi_{m_{n+1}} s_n\| r^2 (1-\varepsilon_{n+1}^2)\Big(1-\sum_1^n \varepsilon_i^2\Big) \\ &\overset{(6.19)}{\geqslant} \Big(1-\sqrt{2\eta}\Big) r^2 (1-\varepsilon_{n+1}^2)\Big(1-\sum_1^n \varepsilon_i^2\Big) \\ &> r^2\Big(1-\sum_1^{n+1}\varepsilon_i\Big) \end{aligned}$$

where the last inequality follows from the following completely elementary estimates:

$$\begin{aligned} &\Big(1-\sqrt{2\eta}\Big)(1-\varepsilon_{n+1}^2)\Big(1-\sum_1^n \varepsilon_i^2\Big) \\ &= \Big(1-\frac{r}{6}\Big(\sum_1^n \varepsilon_i^4\Big)^{1/2}\Big)(1-\varepsilon_{n+1}^2)\Big(1-\sum_1^n \varepsilon_i^2\Big) \\ &> \Big(1-\frac{r}{3}\sum_1^n \varepsilon_i^2\Big)(1-\varepsilon_{n+1}^2)\Big(1-\sum_1^n \varepsilon_i^2\Big) \\ &= \Big(1-\sum_1^{n+1} \varepsilon_i^2\Big) - \frac{r}{3}\Big[1-(1-\varepsilon_{n+1}^2)\sum_1^n \varepsilon_i^2 - \Big(1+\frac{3}{r}\Big)\varepsilon_{n+1}^2\Big]\sum_1^n \varepsilon_i^2 \\ &> \Big(1-\sum_1^{n+1} \varepsilon_i^2\Big) - \frac{r}{3}\sum_1^n \varepsilon_i^2 > \Big(1-\sum_1^{n+1} \varepsilon_i^2\Big) - \frac{1}{3}\sum_1^{n+1} \varepsilon_i^2 \\ &= \Big(1-\sum_1^{n+1} \varepsilon_i\Big) + \sum_1^{n+1} \varepsilon_i\Big(1-\varepsilon_i-\frac{\varepsilon_i}{3}\Big) \geqslant 1-\sum_1^{n+1}\varepsilon_i \end{aligned}$$

since we assumed $\varepsilon_i \leqslant 3/4$ for all $i \in \mathbb{N}$. Thus ((6.8), $n+1$) is proved. This ends the induction and the proof. ∎

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HERMANN PFITZNER
Université d'Orléans
BP 6759
F-45067 Orléans Cedex 2
FRANCE

E-mail: pfitzner@labomath.univ-orleans.fr

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