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POWERS OF R-DIAGONAL ELEMENTS

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Abstract. We prove that if (a, b) is an R-diagonal pair in some non-commutative probability space (A, φ) then (a^p, b^p) is R-diagonal too and we compute the determining series $f_{(a^p,b^p)}$ in terms of the distribution of ab. We give estimates of the upper and lower bounds of the support of free multiplicative convolution of probability measures compactly supported on $[0, \infty[$, and use the results to give norm estimates of powers of R-diagonal elements in finite von Neumann algebras. Finally we compute norms, distributions and R-transforms related to powers of the circular element.

Keywords: *Free probability theory, R-diagonal elements, operator algebras.*

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1. INTRODUCTION AND PRELIMINARIES

In the setup of Free Probability Theory we study certain random variables. By a non-commutative probability space (A, φ) we mean a unital algebra A (over the complex numbers) equipped with a unital functional φ. If A is a von Neumann algebra and φ is normal we call (A, φ) *a non-commutative W^*-probability space.* We write $(\mathcal{M}, \tau)$ for a non-commutative W^*-probability space with a faithful normal tracial state τ. Elements in A are called *random variables*, and the distribution μ_a of a random variable in (A, φ) is the linear functional $\mu_a : \mathbb{C}[X] \to \mathbb{C}$ determined by $\mu_a(P) = \varphi(P(a))$ for all P in $\mathbb{C}[X]$. We refer to [16] for the basic facts of Free Probability Theory and record here for easy reference what we need in this paper.

If a is a self-adjoint element in a non-commutative W^*-probability space there exists a unique compactly supported probability measure (also denoted μ_a) such that

$$\varphi(a^p) = \int_{\mathbb{R}} t^p \,\mathrm{d}\mu_a(t), \quad p \in \mathbb{N}.$$

In this case $\operatorname{supp}\mu_a \subseteq \operatorname{sp} a$. We often view these measures as distributions in the sense of [16]. In the following we introduce the R- and S-transforms of distributions, and the definitions carry over to measures.

The S-transform $\mathcal{S}_\mu$ of a distribution with non-vanishing first moment is defined as a formal power series in the following way (cf. [16]): define the moment series ψ_μ as

$$\psi_\mu(z) = \sum_{n=1}^{\infty} \mu(X^n) z^n$$

and let χ_μ denote the unique inverse formal power series (with respect to composition) of ψ_μ. This series is of the form

$$\chi_\mu(z) = \sum_{n=1}^{\infty} \alpha_n z^n$$

where $\alpha_1 = \mu(X)^{-1} \neq 0$. Then we define

$$\mathcal{S}_\mu(z) = \frac{z+1}{z}\chi_\mu(z) = (z+1)\sum_{n=1}^{\infty} \alpha_n z^{n-1}$$

as a formal power series. (Note that $\mathcal{S}_\mu(0) = \mu(X)^{-1}$.) The S-transform converts multiplicative free convolution into multiplication of formal power series in the following way: $\mathcal{S}_{\mu \boxtimes \nu} = \mathcal{S}_\mu \cdot \mathcal{S}_\nu$ whenever μ and ν are distributions with non-vanishing first moments.

If μ is a compactly supported probability measure on $[0,\infty[$ we let $m(\mu) = \min \operatorname{supp}\mu$, $r = r(\mu) = \max \operatorname{supp}\mu$ and we can view μ as the distribution of a positive element in a suitably chosen non-commutative von Neumann probability space. We have then

$$\psi_\mu(z) = \int_{\mathbb{R}} \frac{zs}{1-zs} \,\mathrm{d}\mu(s)$$

hence ψ_μ is analytic on $\{z \in \mathbb{C} \mid z^{-1} \notin \operatorname{supp}\mu\}$, and χ_μ, $\mathcal{S}_\mu$ are analytic in a neighbourhood of ψ_μ $([0, 1/r[)$, cf. [4]. We denote the moments of μ by $\mu(X^p) = \int_{\mathbb{R}} t^p \,\mathrm{d}\mu(t)$ for every natural number p.

Pringsheim's theorem shows that $1/r$ is a non-removable singularity for ψ_μ and the behaviour of ψ_μ near $1/r$ can be classified into one of the following three cases:

(i) ψ_μ is unbounded near $1/r : \psi_\mu(t) \to \infty$ as $t \to 1/r-$;

(ii) ψ_μ is bounded and ψ'_μ is unbounded near $1/r$: $\psi'_\mu(t) \to \infty$ as $t \to 1/r-$ and $\lim\limits_{t\to 1/r-} \psi_\mu(t)$ exists and is finite;

(iii) ψ_μ and ψ'_μ are bounded near $1/r$: $\lim\limits_{t\to 1/r-} \psi_\mu(t)$ and $\lim\limits_{t\to 1/r-} \psi'_\mu(t)$ exist and are finite.

This makes it possible to determine r in terms of the function χ_μ:

(i) if χ_μ is analytic in a neighbourhood of $[0,\infty[$ and $\chi'_\mu > 0$ on $[0,\infty[$ then $1/r = \lim\limits_{y\to\infty} \chi_\mu(y)$;

(ii) if χ_μ is analytic in a neighbourhood of $[0, y_0[$, $\chi'_\mu > 0$ on $[0, y_0[$ and y_0 is the largest number with these properties then $1/r = \lim\limits_{y\to y_0-} \chi_\mu(y)$.

Since χ_μ is increasing we can estimate $r : 1/r \geqslant \lim\limits_{y\to y_0-} \chi_\mu(y)$ whenever χ_μ is analytic on a neighbourhood of $[0, y_0[$ and $\chi'_\mu > 0$ on $]0, y_0[$.

By $V(\mu)$ we denote the variance of the measure $\mu : V(\mu) = \mu(X^2) - \mu(X)^2$, and if $\mu(X) > 0$ we can bound $r(\mu)$ from below:

$$\mu(X^2) = \int_{\mathbb{R}} x^2 \,\mathrm{d}\mu(x) \leqslant r(\mu)\mu(X)$$

hence $V(\mu)/\mu(X) + \mu(X) \leqslant r(\mu)$.

For a measure μ we let μ^{-1} denote the image measure of μ induced by the reciprocal map $x \mapsto 1/x$, and let μ_{sq} denote the image measure induced by the squaring function $\mathrm{sq} : z \mapsto z^2$. Note that if μ is supported on $]0,\infty[$ then $r(\mu^{-1}) = m(\mu)^{-1}$.

The R-transform $\mathcal{R}_\mu$ of a distribution μ was introduced by Voiculescu in [14] (see also [16]) as a formal power series obtained in the following way: Define

$$G_\mu(z) = \sum_{n=0}^{\infty} \mu(X^n) z^{-n-1}$$

as a formal Laurent series. (The symbol G_μ will be referred to as the Cauchy transform of μ.) Then G_μ is invertible with respect to composition and the inverse G_μ^{-1} is of the form

$$G_\mu^{-1}(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n$$

and $\mathcal{R}_\mu$ is defined to be the power series part of G_μ^{-1}:

$$\mathcal{R}_\mu(z) = \sum_{n=0}^{\infty} \alpha_n z^n.$$

The R-transform converts additive free convolution of distributions into addition of formal power series: $\mathcal{R}_{\mu \boxplus \nu} = \mathcal{R}_\mu + \mathcal{R}_\nu$ for all distributions μ and ν.

In [7] the (1-dimensional) R-transform was generalized to multidimensional distributions $\mu : \mathbb{C}\langle X_i \mid i \in I\rangle \to \mathbb{C}$ and the R-transform of μ is then denoted R_μ. In the 1-dimensional case we have the relation $R_\mu(z) = z\mathcal{R}_\mu(z)$.

The circular element c (of norm 2) was introduced in [15] and the polar decomposition $c = uh$ was determined: u and h are $*$-free, u is a Haar unitary (every non-trivial moment of u is 0) and h is quarter circular (of radius 2):

$$\mathrm{d}\mu_h = \frac{1}{\pi}\sqrt{4 - x^2} \cdot 1_{[0,2]}(x)\,\mathrm{d}x.$$

A model for the circular element is the following: Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{\xi_1, \xi_2\}$, let $\mathcal{T}(\mathcal{H}) = \mathbb{C}\,\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ be the full Fock space of $\mathcal{H}$ and let l_1, l_2 be the creation operators of ξ_1 and ξ_2 respectively. Then $c = (l_1 + l_2^*)/\sqrt{2}$ is a circular element in the finite non-commutative W^*-probability space $(W^*(c, c^*), \langle \cdot\,\Omega, \Omega\rangle)$.

The R-transforms of the $*$-distributions of c and u have similar forms, cf. [10]:

$$R_{\mu_{(c,c^*)}}(z_1, z_2) = z_1 z_2 + z_2 z_1, \tag{1.1}$$

$$R_{\mu_{(u,u^*)}}(z_1, z_2) = \sum_{n=1}^{\infty}(-1)^{n-1}C_{n-1}(z_1 z_2)^n + \sum_{n=1}^{\infty}(-1)^{n-1}C_{n-1}(z_2 z_1)^n. \tag{1.2}$$

The numbers C_n are the Catalan numbers (cf. [2] and [6]): $C_n^{(p)} = \binom{pn}{n-1}/n$ is the n'th Fuss-Catalan number of parameter p and $C_n = C_n^{(2)} = \binom{2n}{n-1}/n$, $n, p \in \mathbb{N}$. For convenience we let $C_0 = 1$. In [10] Nica and Speicher (see also [8] for a more general definition) introduced the class of R-diagonal pairs in non-commutative probability spaces as those pairs (a, b) whose R-transform is of the form

$$R_{\mu_{(a,b)}}(z_1, z_2) = \sum_{n=1}^{\infty} \alpha_n (z_1 z_2)^n + \sum_{n=1}^{\infty} \alpha_n (z_2 z_1)^n$$

where $\alpha_n \in \mathbb{C}$. The series $f_{\mu_{(a,b)}}(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ is called the determining series of $R_{\mu_{(a,b)}}$. An R-diagonal element is an element a (in a non-commutative $*$-probability space) such that (a, a^*) is an R-diagonal pair.

The determining series $f_{\mu_{(a,b)}}$ can be obtained from the moment series $\psi_{\mu_{ab}}$ using the $\circledast$-operation on formal power series, cf. [10]. In dimension 1 the operation $\circledast$ satisfies (and can be defined by) $R_{\mu_{ab}} = R_{\mu_a} \circledast R_{\mu_b}$ for arbitrary free random variables a and b in some non-commutative probability space. Then $R_{\mu_a} = \psi_{\mu_a} \circledast \text{Möb}$ where Möb is the series

$$\text{Möb}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} C_{n-1} z^n,$$

and $f_{\mu_{(a,b)}} = R_{\mu_{ab}} \circledast \text{Möb}$. (If a and b are random variables in a tracial non-commutative probability space then $R_{\mu_{ab}} = R_{\mu_{ba}}$.) The series $\zeta(z) = \sum_{n=1}^{\infty} z^n$ is the inverse to Möb with respect to $\circledast$: $\text{Möb} \circledast \zeta(z) = \zeta \circledast \text{Möb}(z) = z$.

In the n-dimensional case we let M_μ denote the moment series of an n-dimensional distribution $\mu : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}$:

$$M_\mu(z_1, \dots, z_n) = \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k=1}^{n} \mu(X_{i_1} \cdots X_{i_k}) z_{i_1} \cdots z_{i_k}.$$

Then $R_\mu = M_\mu \circledast \text{Möb}_n$, $M_\mu = R_\mu \circledast \zeta_n$, where Möb_n and ζ_n are n-dimensional analogues of Möb and ζ respectively, and likewise for $\circledast$, cf. [10].

We often simplify the notation and write $R_{(a,b)}$ in place of $R_{\mu_{(a,b)}}$ etc.

If a is a random variable in a non-commutative C^*-probability space the R-transform of μ_a is analytic in a neighbourhood of 0, cf. [4].

The paper is organized as follows. In Section 2 we show that any power of any R-diagonal element is R-diagonal. In Section 3 we derive some estimates on the radius of the support of the free multiplicative convolution power of a measure compactly supported in $[0, \infty[$. In Section 4 we compute distributions and R-series related to powers of the circular element.

2. POWERS OF R-DIAGONAL PAIRS

In the sequel we let $M'(\mathbb{R})$ denote the set of symmetric compactly supported probability measures on $\mathbb{R}$, and let $r_j(\mu)$ denote the j'th coefficient in R_μ:

$$R_\mu(z) = \sum_{j=1}^{\infty} r_j(\mu) z^j.$$

Proposition 5.2 in [11] shows that if $\mu \in M'(\mathbb{R})$ then $r_{2j-1}(\mu) = 0$ for all j in $\mathbb{N}$.

LEMMA 2.1. *The sets*

$$\big\{(\mu(X^2),\dots,\mu(X^{2n}))\;\big|\;\mu\in M'(\mathbb{R})\big\},\quad \big\{(r_2(\mu),\dots,r_{2n}(\mu))\;\big|\;\mu\in M'(\mathbb{R})\big\}$$

have interior points for all natural numbers n.

Proof. The measure $\mu=\frac{1}{2}1_{[-1,1]}(x)\,\mathrm{d}x$ is symmetric and the even moments are $\mu(X^{2k})=(2k+1)^{-1}$ ($k\in\mathbb{N}$). Fix a natural number n and put

$$B_1=\Big\{(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n\;\Big|\;\sum_{j=1}^{n}|\alpha_j|<1\Big\}.$$

Let P_{2j} denote the normalized Legendre polynomial of order $2j$, and for $(\alpha_1,\dots,\alpha_n)\in B_1$ we define

$$\mathrm{d}\mu_{(\alpha_1,\dots,\alpha_n)}=\frac{1}{2}\big(P_0(x)+\alpha_1P_2(x)+\cdots+\alpha_nP_{2n}(x)\big)\cdot 1_{[-1,1]}(x)\,\mathrm{d}x.$$

Observe that P_{2j} is an even polynomial, $|P_{2j}(x)|\leqslant 1$ for $|x|\leqslant 1$ hence

$$\Big|\sum_{j=1}^{n}\alpha_jP_{2j}(x)\Big|\leqslant 1$$

on $[-1,1]$. The orthogonality properties of the sequence $(P_{2j})_{j\in\mathbb{N}}$ implies that

$$\int_{-1}^{1}x^{2i}P_{2j}(x)\,\mathrm{d}x=0,\quad \int_{-1}^{1}x^{2j}P_{2j}(x)\,\mathrm{d}x\neq 0$$

whenever $i=0,\dots,j-1$, $j\in\mathbb{N}$. In particular it follows that $\mu_{(\alpha_1,\dots,\alpha_n)}\in M'(\mathbb{R})$ (for $(\alpha_1,\dots,\alpha_n)\in B_1$). Then

$$\mu_{(\alpha_1,\dots,\alpha_n)}(X^{2k})=\frac{1}{2k+1}+\frac{1}{2}\sum_{j=1}^{n}\alpha_j\int_{-1}^{1}x^{2k}P_{2j}(x)\,\mathrm{d}x=\mu(X^{2k})+\sum_{j=1}^{n}b_{kj}\alpha_j,$$

where $b_{kj}=\frac{1}{2}\int_{-1}^{1}x^{2k}P_{2j}(x)\,\mathrm{d}x$. Especially $b_{jj}\neq 0$ ($j=1,\dots,n$) and $b_{kj}=0$ if $k<j$. Then

$$(2.1)\quad \begin{pmatrix}\mu_{(\alpha_1,\dots,\alpha_n)}(X^2)\\ \vdots\\ \mu_{(\alpha_1,\dots,\alpha_n)}(X^{2n})\end{pmatrix}-\begin{pmatrix}\mu(X^2)\\ \vdots\\ \mu(X^{2n})\end{pmatrix}=\begin{pmatrix}b_{11}&\cdots&0\\ \vdots&\ddots&\vdots\\ b_{n1}&\cdots&b_{nn}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}=B\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$$

where $B = (b_{ij})_{i,j=1}^n$. Since B is invertible Equation (2.1) shows that $(\mu(X^2), \ldots, \mu(X^{2n}))$ is an interior point in $\{(\nu(X^2), \ldots, \nu(X^{2n})) \mid \nu \in M'(\mathbb{R})\}$.

There exist (universal) continuous functions $F, G : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\begin{pmatrix} r_2(\nu) \\ \vdots \\ r_{2n}(\nu) \end{pmatrix} = F \begin{pmatrix} \nu(X^2) \\ \vdots \\ \nu(X^{2n}) \end{pmatrix}, \qquad \begin{pmatrix} \nu(X^2) \\ \vdots \\ \nu(X^{2n}) \end{pmatrix} = G \begin{pmatrix} r_2(\nu) \\ \vdots \\ r_{2n}(\nu) \end{pmatrix}, \qquad \nu \in M'(\mathbb{R})$$

and $F \circ G = \mathrm{id}_{\mathbb{C}^n} = G \circ F$. In particular F is open and $\{(r_2(\nu), \ldots, r_{2n}(\nu)) \mid \nu \in M'(\mathbb{R})\}$ has an interior point. ∎

REMARK. Let $(\mathcal{M}, \tau) = L(\mathbb{Z}_2) * \mathop{\ast}\limits_{\mu \in M'(\mathbb{R})} (L^\infty(\mu), \int \cdot \, \mathrm{d}\mu)$. Then $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal trace τ. Let a be a generating unitary in $L(\mathbb{Z}_2)$. If T is an arbitrary element in some finite non-commutative W^*-probability space it follows from Corollary 3.2 in [5] that the element $a \cdot \mathrm{id}_{L^\infty(\tilde{\mu}_{|T|})}$ is an R-diagonal element in $\mathcal{M}$ and that it has the same $*$-distribution as T. Thus $\mathcal{M}$ contains a representative of every R-diagonal element. Let

$$S_n = \left\{ (\alpha_1, \ldots, \alpha_n) \;\middle|\; \exists T \in \mathcal{M} : R_{(T,T^*)}(z_1, z_2) = \sum_{j=1}^{\infty} \alpha_j (z_1 z_2)^j + \sum_{j=1}^{\infty} \alpha_j (z_2 z_1)^j \right\}.$$

It follows from Proposition 5.2 in [11] that

$$S_n = \{(r_2(\mu), \ldots, r_{2n}(\mu)) \mid \mu \in M'(\mathbb{R})\},$$

thus S_n has an interior point according to Lemma 2.1.

THEOREM 2.2. *Let (a, b) be an R-diagonal pair in a non-commutative probability space (A, φ), and let $p \in \mathbb{N}$. Then (a^p, b^p) is an R-diagonal pair with determining series*

$$f_{(a^p,b^p)} = R_{ab}^{\circledast p} \circledast \text{Möb}.$$

In particular $R_{a^p b^p} = R_{ab}^{\circledast p}$.

Proof. Let $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \{1, 2\}$. We first assume that $(i_1, \ldots, i_n)$ denotes an index off the diagonal, i.e., $(i_1, \ldots, i_n) \neq (1, 2, \ldots, 1, 2)$ and $(i_1, \ldots, i_n) \neq (2, 1, \ldots, 2, 1)$. We note the existence of a universal polynomial P such that

$$[\mathrm{coef}(i_1, \ldots, i_n)] R_{(a^p,b^p)} = P\big([\mathrm{coef}(1,2)] R_{(a,b)}, \ldots, [\mathrm{coef}(\underbrace{1, 2, \ldots, 1, 2}_{2pn})] R_{(a,b)}\big)$$

whenever (a, b) is an R-diagonal pair in some non-commutative probability space. Indeed, if $R \in \mathbb{C}\langle z_1, z_2\rangle$ is of the form

$$R(z_1, z_2) = \sum_{j=1}^{\infty} \alpha_j (z_1 z_2)^j + \sum_{j=1}^{\infty} \alpha_j (z_2 z_1)^j$$

we define $M, M^{(p)}, R^{(p)} \in \mathbb{C}\langle z_1, z_2\rangle$ by

$$M = R \boxast \zeta_2, \quad M^{(p)}(z_1, z_2) = M(z_1^p, z_2^p), \quad R^{(p)} = M^{(p)} \boxast \text{Möb}_2,$$

and it follows that there exists a polynomial P such that $[\text{coef}(i_1, \ldots, i_n)]R^{(p)} = P(\alpha_1, \ldots, \alpha_{pn})$. If a and b are random variables in a non-commutative probability space (A, φ) and (a, b) is an R-diagonal pair with determining series $f_{(a,b)}(z) = \sum\limits_{j=1}^{\infty} \alpha_j z^j$, then

$$[\text{coef}(i_1, \ldots, i_n)]R_{(a^p, b^p)} = P(\alpha_1, \ldots, \alpha_{pn}).$$

If T is an R-diagonal element in $(\mathcal{M}, \tau)$ it follows from Proposition 3.10 in [5] that T^p is R-diagonal. This implies that $P(\alpha_1, \ldots, \alpha_{pn}) = 0$ whenever $(\alpha_1, \ldots, \alpha_{pn}) \in S_{pn}$. Since S_{pn} has an interior point this implies that $P = 0$ hence $[\text{coef}(i_1, \ldots, i_n)]R_{(a^p, b^p)} = 0$.

We next assume that $(i_1, \ldots, i_n) = (\underbrace{1, 2, \ldots, 1, 2}_{n})$. Then a symmetry argument reveals that

$$[\text{coef}(\underbrace{1, 2, \ldots, 1, 2}_{n})]R_{(a^p, b^p)} = [\text{coef}(\underbrace{2, 1, \ldots, 2, 1}_{n})]R_{(a^p, b^p)}.$$

Since n and $i_1, \ldots, i_n$ are arbitrary we conclude that (a^p, b^p) is R-diagonal.

Finally we compute the determining series $f_{(a^p, b^p)}$ for (a^p, b^p). Note that for given n there exists a universal polynomial P such that

$$[\text{coef}(n)]f_{(a^p, b^p)} - [\text{coef}(n)]R_{ab}^{\boxast p} \boxast \text{Möb} = P(\alpha_1, \ldots, \alpha_{pn})$$

where $\alpha_j = [\text{coef}(j)]f_{(a,b)}$. If (T, T^*) is an R-diagonal pair in $(\mathcal{M}, \tau)$ then $f_{(T^p, (T^p)^*)} = R_{T^p(T^p)^*} \boxast \text{Möb} = R_{\mu_{TT^*}^{\boxtimes p}} \boxast \text{Möb} = R_{\mu_{TT^*}^{\boxast p}} \boxast \text{Möb}$ (cf. Proposition 3.10 in [5]) whence $P(S_{pn}) = \{0\}$. We conclude that $P = 0$ and thus $f_{(a^p, b^p)} = R_{ab}^{\boxast p} \boxast \text{Möb}$. ∎

3. NORM-ESTIMATES OF POWERS AND PRODUCTS OF R-DIAGONAL ELEMENTS

THEOREM 3.1. *Let p be a natural number, let $\mu, \mu_1, \dots, \mu_p$ be compactly supported probability measures on $[0, \infty[$. Then:*

(i) $r(\mu^{\boxtimes p}) \leqslant \mathrm{e} p\, r(\mu)\mu(X)^{p-1}$;

(ii) *if $r(\mu_j) > 0$ for all $j = 1, \dots, p$ then*

$$r(\mu_1 \boxtimes \cdots \boxtimes \mu_p) \leqslant \mathrm{e} p \max_{j=1,\dots,p} \frac{r(\mu_j)}{\mu_j(X)} \cdot \mu_1(X) \cdots \mu_p(X);$$

(iii) $r(\mu^{\boxtimes p}) \geqslant \mu(X)^p + pV(\mu)\mu(X)^{p-2}$;

(iv) *if $r(\mu_j) > 0$ for all $j = 1, \dots, p$ then*

$$r(\mu_1 \boxtimes \cdots \boxtimes \mu_p) \geqslant \mu_1(X) \cdots \mu_p(X) \Big(1 + \sum_{j=1}^{p} \frac{V(\mu_j)}{\mu_j(X)^2} \Big).$$

Proof. The idea of the proof of (i) is to find an interval $]0, y_0[$ on which $\chi_{\mu^{\boxtimes p}}$ is analytic and $\chi'_{\mu^{\boxtimes p}} > 0$. Then $r(\mu^{\boxtimes p})^{-1} \geqslant \lim_{y \to y_0 -} \chi_{\mu^{\boxtimes p}}(y)$.

The statement holds trivially for $p = 1$ and for $\mu = \delta_0$ so let p be a natural number greater than 2 and assume that $\mu \neq \delta_0$. Then

$$\chi_{\mu^{\boxtimes p}}(z) = \Big(\frac{z+1}{z} \Big)^{p-1} \chi_\mu(z)^p \tag{3.1}$$

and it follows that $\chi_{\mu^{\boxtimes p}}$ is analytic in a neighbourhood of $]0, y_0[$ if χ_μ is analytic in a neighbourhood of $]0, y_0[$. It follows from (3.1) that $\chi'_{\mu^{\boxtimes p}} > 0$ on $]0, y_0[$ if and only if

$$\frac{p-1}{p} < y(1+y) \frac{\chi'_\mu(y)}{\chi_\mu(y)}$$

for all y in $]0, y_0[$. Inserting $y = \psi_\mu(t)$ we infer that $\chi'_{\mu^{\boxtimes p}} > 0$ on $]0, y_0[$ if and only if

$$\frac{p-1}{p} < \frac{\psi_\mu(t)(1+\psi_\mu(t))}{t\psi'_\mu(t)} \tag{3.2}$$

for all t in $]0, t_0[$ where $t_0 = \lim_{y \to y_0 -} \chi_\mu(y)$.

Choose $t_0 = (p\, r(\mu))^{-1}$. Using the integral formula for ψ_μ we estimate:

$$\frac{\psi_\mu(t)(1+\psi_\mu(t))}{t\psi'_\mu(t)} > \frac{\psi_\mu(t)}{\int_{\mathbb{R}} \frac{ts}{(1-ts)^2} \,\mathrm{d}\mu(s)} \geqslant 1 - tr(\mu)$$

hence (3.2) holds for all t in $]0,t_0[$. Then ψ_μ is analytic in a neighbourhood of $]0,t_0]$ hence χ_μ is analytic in a neighbourhood of $[0,y_0]$ and we can estimate:

$$\frac{\psi_\mu(t_0)}{t_0} = \int_{\mathbb{R}} \frac{s}{1-st_0}\,\mathrm{d}\mu(s) \leqslant \frac{\mu(X)}{1-r(\mu)t_0},$$

$$\chi_{\mu^{\boxtimes p}}(\psi_\mu(t_0)) \geqslant (\psi_\mu(t_0)+1)^{p-1}\Big(\frac{1-r(\mu)t_0}{\mu(X)}\Big)^{p-1} t_0 \geqslant \frac{1}{p\,r(\mu)\mu(X)^{p-1}}(1-p^{-1})^{p-1}$$

whence

$$r(\mu^{\boxtimes p}) \leqslant p\,r(\mu)\mu(X)^{p-1}\Big(1+\frac{1}{p-1}\Big)^{p-1} \leqslant \mathrm{e}p\,r(\mu)\mu(X)^{p-1},$$

which shows (i).

Put $\alpha = (\mu_1(X)\cdots\mu_p(X))^{1/p}$ and let ν_j $(j=1,\dots,p)$ be the image measure $(\mu_j)_{z\mapsto \alpha z/\mu_j(X)}$. Then $\nu_1\boxtimes\cdots\boxtimes\nu_p = \mu_1\boxtimes\cdots\boxtimes\mu_p$ and $\nu_1(X)=\cdots=\nu_p(X)=\alpha$. Due to the foregoing analysis we have that

$$\frac{p-1}{p} < y(y+1)\frac{\chi'_{\nu_j}(y)}{\chi_{\nu_j}(y)} \tag{3.3}$$

for all y in $]0,y_0^{(j)}[$ where $y_0^{(j)} = \psi_{\nu_j}\big((p\,r(\nu_j))^{-1}\big)$. Put $r = \max_{j=1,\dots,p} r(\nu_j)$ and $t_0 = \min_{j=1,\dots,p}(p\,r(\nu_j))^{-1} = (p\,r)^{-1}$. Then (3.3) holds on $]0,y_{0,j}[$ where $y_{0,j} = \psi_{\nu_j}(t_0)$ hence the estimate (3.3) holds on $]0,y_0[$ where $y_0 = \min_{j=1,\dots,p} y_{0,j}$. We assume without loss of generality that $y_0 = y_{0,1}$. Note that $\chi_{\nu_j}(y_0) \leqslant \chi_{\nu_j}(y_{0,j}) = \chi_{\nu_j}(\psi_{\nu_j}(t_0)) = t_0$. Then

$$(p-1)\frac{1}{y(y+1)} < p\min_{j=1,\dots,p}\frac{\chi'_{\nu_j}(y)}{\chi_{\nu_j}(y)}$$

for all y in $]0,y_0[$ and we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}y}\log\chi_{\nu_1\boxtimes\cdots\boxtimes\nu_p}(y) = \sum_{j=1}^{p}\frac{\chi'_{\nu_j}(y)}{\chi_{\nu_j}(y)} - (p-1)\frac{1}{y(y+1)} > 0$$

for all y in $]0,y_0[$. Thus $\chi'_{\nu_1\boxtimes\cdots\boxtimes\nu_p} > 0$ on $]0,y_0[$, and since $p\geqslant 2$ each χ_{ν_j} is analytic on a neighbourhood of $]0,y_0]$, hence

$$\begin{aligned}
r(\nu_1\boxtimes\cdots\boxtimes\nu_p)^{-1} &\geqslant \chi_{\nu_1\boxtimes\cdots\boxtimes\nu_p}(y_0) = \Big(\frac{y_0+1}{y_0}\Big)^{p-1}\chi_{\nu_1}(y_0)\cdots\chi_{\nu_p}(y_0)\\
&\geqslant \chi_{\nu_1}(\psi_{\nu_1}(t_0))\prod_{j=2}^{p}\frac{\chi_{\nu_j}(y_0)}{y_0} = t_0\prod_{j=2}^{p}\frac{\chi_{\nu_j}(y_0)}{\psi_{\nu_j}(\chi_{\nu_j}(y_0))}\\
&\geqslant t_0\prod_{j=2}^{p}\frac{1-r(\nu_j)\chi_{\nu_j}(y_0)}{\nu_j(X)} \geqslant t_0\prod_{j=2}^{p}\frac{1-rt_0}{\nu_j(X)} = t_0\Big(1-\frac{1}{p}\Big)^{p-1}\prod_{j=2}^{p}\frac{1}{\nu_j(X)}.
\end{aligned}$$

Therefore

$$r(\mu_1 \boxtimes \cdots \boxtimes \mu_p) = r(\nu_1 \boxtimes \cdots \boxtimes \nu_p) \leqslant pr\Big(1 - \frac{1}{p-1}\Big)^{p-1} \alpha^{p-1}$$
$$\leqslant \mathrm{e}p \max_{j=1,\ldots,p} \frac{\alpha r(\mu_j)}{\mu_j(X)} \alpha^{p-1} = \mathrm{e}p \max_{j=1,\ldots,p} \frac{r(\mu_j)}{\mu_j(X)} \cdot \mu_1(X) \cdots \mu_p(X)$$

and this proves (ii).

To prove (iv) we first note that if μ is a measure with non-vanishing first moment the power series of $\mathcal{S}_\mu$ is

$$\mathcal{S}_\mu(z) = \frac{1}{\mu(X)} - \frac{V(\mu)}{\mu(X)^3} z + \mathrm{O}(z^2)$$

as $z \to 0$. This implies that

$$\mathcal{S}_{\mu_1 \boxtimes \cdots \boxtimes \mu_p}(z) = \prod_{j=1}^{p} \frac{1}{\mu_j(X)} \Big(1 - \frac{V(\mu_j)}{\mu_j(X)^2} z + \mathrm{O}(z^2)\Big)$$
$$= \frac{1}{\mu_1(X) \cdots \mu_p(X)} \Big(1 - \sum_{j=1}^{p} \frac{V(\mu_j)}{\mu_j(X)^2} z\Big) + \mathrm{O}(z^2)$$

hence

$$\frac{V(\mu_1 \boxtimes \cdots \boxtimes \mu_p)}{\mu_1 \boxtimes \cdots \boxtimes \mu_p(X)^3} = \frac{1}{\mu_1(X) \cdots \mu_p(X)} \sum_{j=1}^{p} \frac{V(\mu_j)}{\mu_j(X)^2},$$

and thus

$$r(\mu_1 \boxtimes \cdots \boxtimes \mu_p) \geqslant \frac{V(\mu_1 \boxtimes \cdots \boxtimes \mu_p)}{\mu_1 \boxtimes \cdots \boxtimes \mu_p(X)} + \mu_1 \boxtimes \cdots \boxtimes \mu_p(X)$$
$$= \mu_1(X) \cdots \mu_p(X) \Big(1 + \sum_{j=1}^{p} \frac{V(\mu_j)}{\mu_j(X)^2}\Big).$$

This shows (iv).

If μ is a Dirac measure then $r(\mu^{\boxtimes p}) = \mu(X)^p$, $V(\mu) = 0$ and (iii) is fulfilled. If μ is not a Dirac measure then (iii) follows from (iv). ∎

Corollary 3.2. *Let $p \in \mathbb{N}$, $T, T_1, \ldots, T_p$ be R-diagonal elements in a non-commutative probability space $(\mathcal{M}, \tau)$ with a faithful normal tracial state. Then*

$$\|T^p\| \leqslant \sqrt{\mathrm{e}p}\, \|T\|\, \|T\|_2^{p-1}$$

for every natural number p, and if $T_1, \ldots, T_p$ are $*$-free and $T_1, \ldots, T_p \neq 0$ then

$$\|T_1 \cdots T_p\| \leqslant \sqrt{\mathrm{e}p} \max_{j=1,\ldots,p} \frac{\|T_j\|}{\|T_j\|_2} \cdot \|T_1\|_2 \cdots \|T_p\|_2.$$

Proof. It follows from Propositions 3.6 and 3.10 in [5] that T^p and $T_1 \cdots T_p$ are R-diagonal and that $\mu_{|T^p|^2} = \mu_{|T|^2}^{\boxtimes p}$, $\mu_{|T_1 \cdots T_p|^2} = \mu_{|T_1|^2} \boxtimes \cdots \boxtimes \mu_{|T_p|^2}$. Then we estimate:

$$\begin{aligned} r(\mu_{|T_1|^2} \boxtimes \cdots \boxtimes \mu_{|T_p|^2}) &\leqslant \mathrm{e}p \max_{j=1,\ldots,p} \frac{r(\mu_{|T_j|^2})}{\mu_{|T_j|^2}(X)} \cdot \mu_{|T_1|^2}(X) \cdots \mu_{|T_p|^2}(X) \\ &= \mathrm{e}p \max_{j=1,\ldots,p} \frac{\|T_j\|^2}{\|T_j\|_2^2} \cdot \|T_1\|_2^2 \cdots \|T_p\|_2^2 \end{aligned}$$

and the conclusion follows. ∎

If μ is not a Dirac measure Theorem 3.1 (i) shows that $r(\mu^{\boxtimes p}) = \mathrm{O}(p\mu(X)^p)$ and (iii) shows that this is the best asymptotic estimate. Thus $\|T^p\| = \mathrm{O}(\sqrt{p}\,\|T\|_2^p)$ is the best asymptotic estimate for the norm of an R-diagonal element (compare to the estimate $\|T^p\| \leqslant (1+p)\|T\|\,\|T\|_2^{p-1}$ obtained in Corollary 4.2 in [5]).

COROLLARY 3.3. *If p is a natural number, and $\mu, \mu_1, \ldots, \mu_p$ are compactly supported probability measures on $]0, \infty[$, then*

$$\begin{aligned} m(\mu_1 \boxtimes \cdots \boxtimes \mu_p) &\geqslant \frac{1}{\mathrm{e}p} \cdot \frac{\min_{j=1,\ldots,p} m(\mu_j)\mu_j^{-1}(X)}{\mu_1^{-1}(X) \cdots \mu_p^{-1}(X)}, \\ m(\mu^{\boxtimes p}) &\geqslant \frac{1}{\mathrm{e}p} \cdot \frac{m(\mu)}{(\mu^{-1}(X))^{p-1}}. \end{aligned} \tag{3.4}$$

Proof. In the finite non-commutative W^*-probability space $\mathop{\ast}_{j=1}^{p} \big(L^\infty(\mu_j), \int \cdot \, \mathrm{d}\mu_j\big)$ we can find a free family $\{a_1, \ldots, a_p\}$ of positive invertible elements such that the distribution (as a measure) of a_j is μ_j for all j. Then the distribution of $a_1 \cdots a_p$ is $\mu_1 \boxtimes \cdots \boxtimes \mu_p$ and the distribution of $(a_1 \cdots a_p)^{-1}$ is

$$\mu_{(a_1 \cdots a_p)^{-1}} = \mu_{a_p^{-1} \cdots a_1^{-1}} = \mu_{a_p^{-1}} \boxtimes \cdots \boxtimes \mu_{a_1^{-1}} = \mu_1^{-1} \boxtimes \cdots \boxtimes \mu_p^{-1}.$$

Then

$$\begin{aligned} m(\mu_1 \boxtimes \cdots \boxtimes \mu_p)^{-1} &= r\big((\mu_1 \boxtimes \cdots \boxtimes \mu_p)^{-1}\big) = r(\mu_1^{-1} \boxtimes \cdots \boxtimes \mu_p^{-1}) \\ &\leqslant \mathrm{e}p \max_{j=1,\ldots,p} \frac{r(\mu_j^{-1})}{\mu_j^{-1}(X)} \cdot \mu_1^{-1}(X) \cdots \mu_p^{-1}(X) \end{aligned}$$

hence

$$m(\mu_1 \boxtimes \cdots \boxtimes \mu_p) \geqslant \frac{1}{\mathrm{e}p} \cdot \frac{\min_{j=1,\ldots,p} m(\mu_j)\mu_j^{-1}(X)}{\mu_1^{-1}(X) \cdots \mu_p^{-1}(X)}$$

and the conclusion follows. ∎

Let μ be a compactly supported probability measure on $]0,\infty[$. For the free multiplicative convolution power $\mu^{\boxtimes p}$ of μ we have the obvious estimate

$$m(\mu^{\boxtimes p}) \geqslant m(\mu)^p. \tag{3.5}$$

If μ is a Dirac measure then $m(\mu^{\boxtimes p}) = m(\mu)^p$. In the following we assume that μ is not a Dirac measure. Then

$$m(\mu)\mu^{-1}(X) = m(\mu)\int_{\mathbb{R}} t^{-1}\,\mathrm{d}\mu(t) < m(\mu)\int_{\mathbb{R}} m(\mu)^{-1}\,\mathrm{d}\mu(t) = 1$$

and the estimate

$$\frac{m(\mu)^p}{\frac{m(\mu)}{\mathrm{e}p\mu^{-1}(X)}} = \mathrm{e}p\big(m(\mu)\mu^{-1}(X)\big)^{p-1} \to 0 \quad \text{as } p\to\infty$$

shows that the estimate (3.4) is sharper than (3.5) for large p.

4. POWERS OF THE CIRCULAR ELEMENT

In this section we compute the norm of every power of the circular element c. Furthermore we determine the distribution of $|c^p|^2$ and the R-transform of $(c^p,(c^p)^*)$ for any p in $\mathbb{N}$. It is recently shown in [12] that powers of the circular element are R-diagonal elements, and the coefficients in the R-transforms were computed.

PROPOSITION 4.1. *Let c be the circular element and let p be a natural number. Then:*

(i) $\|c^p\|^2 = (p+1)^{p+1}/p^p$;

(ii) *the moments of $|c^p|^2$ are $\mu_{|c^p|^2}(X^n) = C_n^{(p+1)}$ for all natural numbers p and n;*

(iii) *the determining function $f_{(c^p,(c^p)^*)}$ for the R-diagonal element c^p is*

$$f_{(c^p,(c^p)^*)}(z) = \sum_{n=1}^{\infty} C_n^{(p-1)} z^n.$$

for every natural number p greater than 2.

Proof. Let $\mu = \mu_{|c|^2}$. We first note that $\|c^p\|^2 = r(\mu_{|c^p|^2}) = r(\mu^{\boxtimes p})$. It is shown in [5] that $\mathcal{S}_\mu(z) = 1/(z+1)$ hence $\chi_\mu(z) = z/(z+1)^2$ and $\chi_{\mu^{\boxtimes p}}(z) = z/(z+1)^{p+1}$ are analytic in a neighbourhood of $[0,\infty[$. It is straightforward to verify that $\chi'_{\mu^{\boxtimes p}} > 0$ on $]0,p^{-1}[$ and $\chi'_{\mu^{\boxtimes p}}(p^{-1}) = 0$ hence $1/r(\mu^{\boxtimes p}) = \chi_{\mu^{\boxtimes p}}(p^{-1}) = p^p/(p+1)^{p+1}$ and (i) follows.

For n, p natural numbers we find

$$\begin{aligned}\mu_{|c^p|^2}(X^n) &= \text{the coefficient of } z^n \text{ in } \psi_{\mu_{|c^p|^2}}(z) \\ &\overset{(*)}{=} \frac{1}{n}\operatorname{Res}(\chi_{\mu_{|c^p|^2}}(z)^{-n}, z=0) = \frac{1}{n}\operatorname{Res}(\chi_{\mu^{\boxtimes p}}(z)^{-n}, z=0) \\ &= \frac{1}{n}\operatorname{Res}\left(\frac{(z+1)^{(p+1)n}}{z^n}, z=0\right) = \frac{1}{n}\binom{(p+1)n}{n-1} = C_n^{(p+1)}.\end{aligned}$$

At $(*)$ we use Lagranges Inversion Theorem, cf. Section 3.8 in [1], [3]. This shows (ii).

To prove (iii) we first note that $\mathcal{R}_\mu(z) = (1-z)^{-1}$ hence $R_\mu(z) = \sum_{j=1}^{\infty} z^j = \zeta(z)$. Then

$$R_{\mu_{c^p(c^p)^*}} = R_{\mu_{(c^p)^*c^p}} = R_{\mu^{\boxtimes p}} = \underbrace{R_\mu \circledast \cdots \circledast R_\mu}_{p \text{ terms}} = \zeta^{\circledast p}$$

hence

$$f_{(c^p,(c^p)^*)} = R_{\mu_{c^p(c^p)^*}} \circledast \text{Möb} = \zeta^{\circledast p} \circledast \text{Möb} = \zeta^{\circledast (p-1)}.$$

If $p=2$ then $f_{(c^p,(c^p)^*)}(z) = \zeta(z) = \sum_{n=1}^{\infty} z^n$ and if $p>2$ then

$$f_{(c^p,(c^p)^*)} = \zeta^{\circledast(p-1)} = \zeta \circledast R_{\mu^{\boxtimes(p-2)}} = \zeta \circledast \text{Möb} \circledast \psi_{\mu^{\boxtimes(p-2)}} = \psi_{\mu^{\boxtimes(p-2)}},$$

i.e., $f_{(c^p,(c^p)^*)}(z) = \sum_{n=1}^{\infty} C_n^{(p-1)} z^n$. ∎

In addition the computations show that

$$\zeta^{\circledast p}(z) = \sum_{n=1}^{\infty} C_n^{(p)} z^n \tag{4.1}$$

for every $p \geqslant 2$. Using the combinatorial Fourier Transform invented in [9] we can also compute $\text{Möb}^{\circledast p}$ for any natural number p: let $\mathcal{F}$ denote the Fourier transform on formal power series without constant terms and with non-vanishing first coefficients defined as follows (cf. [9]):

$$z\mathcal{F}f(z) = f^{-1}(z)$$

(here f^{-1} denotes the inverse with respect to composition.) Then $\mathcal{F}(f \circledast g) = \mathcal{F}f \cdot \mathcal{F}g$ for all power series f and g in the domain of $\mathcal{F}$. Also $\mathcal{F}\zeta(z) = 1/(1+z)$ hence $\mathcal{F}(\zeta^{\circledast p})(z) = 1/(1+z)^p$ and $\mathcal{F}(\text{Möb}^{\circledast p})(z) = (1+z)^p$. Thus $\text{Möb}^{\circledast p}(z) = (z(1+$

$z)^p)^{\langle -1\rangle}$, and Lagranges Inversion Formula applies to compute the coefficients in $\text{Möb}^{⊠ p}$: Let $\text{Möb}^{⊠ p}(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, then

$$\alpha_n = \frac{1}{n}\operatorname{Res}\Big(\Big(\frac{1}{z(1+z)^p}\Big)^n, z=0\Big) = \frac{1}{n}\operatorname{Res}\Big(z^{-n}\sum_{j=1}^{\infty}\binom{-pn}{j} z^j, z=0\Big)$$
$$= \frac{1}{n}\binom{-pn}{n-1} =: C_n^{(-p)}$$

and thus

$$\text{Möb}^{⊠ p}(z) = \sum_{n=1}^{\infty} C_n^{(-p)} z^n. \tag{4.2}$$

Following formulas (4.1) and (4.2), we have $\text{Möb} = \zeta^{⊠(-1)}$ and $\text{Möb}^{⊠ p} = \zeta^{⊠(-p)}$ for all natural numbers p.

The coefficients in the series $\zeta^{⊠ p}$ was also computed in [13].

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REFERENCES

1. L. COMTET, *Advanced Combinatorics. The art of finite and infinite expansions*, D. Reidel Publishing Co., Dordrecht 1974.
2. R.L. GRAHAM, D.E. KNUTH, O. PATASHNIK, *Concrete Mathematics*, Addison-Wesley, 1989.
3. U. HAAGERUP, A new upper bound for the complex Grothendieck constant, *Israel J. Math.* **60**(1987), 199–224.
4. U. HAAGERUP, On Voiculescu's R- and S-transforms for free non-commuting random variables, in *Free Probability Theory*, vol. 12, Fields Institute Communications, 1997, pp. 127–148.
5. U. HAAGERUP, F. LARSEN, Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras, *J. Funct. Anal.* **176**(2000), 331–367.
6. P. HILTON, J. PEDERSEN, Catalan numbers, their generalization, and their uses, *Math. Intelligencer* **13**(1991), 64–75.
7. A. NICA, R-transforms of free joint distributions and non-crossing partitions, *J. Funct. Anal.*, **135**(1996), 271–296.
8. A. NICA, D. SLYAKHTENKO, R. SPEICHER, R-diagonal elements and freeness with amalgamation,
9. A. NICA, R. SPEICHER, A "Fourier transform" for multiplicative functions on non-crossing partitions, *J. Alg. Combinat.*, **6**(1997), 141–160.

10. A. NICA, R. SPEICHER, R-diagonal pairs — a common approach to Haar unitaries and circular elements, in *Free Probability Theory*, vol. 12, Fields Institute Communications, 1997, pp. 149–188.
11. A. NICA, R. SPEICHER, Commutators of free random variables, *Duke Math. J.* **92**(1998), 553–592.
12. F. ORAVECZ, On the powers of Voiculescu's circular element, preprint no. 15, Mathematical Institute of the Hungarian Academy of Sciences, 1998.
13. R. SPEICHER, Multiplicative functions on the lattice of non-crossing partitions and free convolution, *Math. Ann.* **298**(1994), 611–628.
14. D.V. VOICULESCU, Addition of certain non-commuting random variables, *J. Funct. Anal.* **66**(1986), 323–346.
15. D.V. VOICULESCU, Circular and semicircular systems and free product factors, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progress in Mathematics, vol. 92, Birkhäuser, Boston 1990, pp. 45–60.
16. D.V. VOICULESCU, K.J. DYKEMA, A. NICA, *Free Random Variables*, Amer. Math. Soc., 1991.

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