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ALGEBRAS OF SINGULAR INTEGRAL OPERATORS WITH *PC* COEFFICIENTS IN REARRANGEMENT-INVARIANT SPACES WITH MUCKENHOUPT WEIGHTS

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ABSTRACT. In this paper we extend results on Fredholmness of singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces $X(\Gamma)$ with nontrivial Boyd indices α_X, β_X ([22]) to the weighted case. Suppose a weight w belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$. We prove that these conditions guarantee the boundedness of the Cauchy singular integral operator S in the weighted rearrangement-invariant space $X(\Gamma, w)$. Under a "disintegration condition" we construct a symbol calculus for the Banach algebra generated by singular integral operators with matrix-valued piecewise continuous coefficients and get a formula for the index of an arbitrary operator from this algebra. We give nontrivial examples of spaces, for which this "disintegration condition" is satisfied. One of such spaces is a Lebesgue space with a general Muckenhoupt weight over an arbitrary Carleson curve.

KEYWORDS: *Singular integral operator, Fredholmness, rearrangement-invariant space, Muckenhoupt weight.*

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1. INTRODUCTION

The study of singular integral operators (SIO's) with piecewise continuous (PC) coefficients in Lebesgue spaces $L^p(\Gamma, w)$ with power (Khvedelidze) weights w over Lyapunov curves Γ had been started in the fifties by B.V. Khvedelidze and was continued in the sixties by H. Widom, I.B. Simonenko, I. Gohberg and N. Krupnik, and others. The history of this topic and corresponding references can be found, e.g., in [4], [14], and [25]. The main result of the Fredholm theory for SIO's with PC coefficients can be formulated in the geometric language as following.

The local spectra of SIO's with PC coefficients have the shape of circular arcs depending on the exponents of Lebesgue spaces and power weights.

About ten years ago I.M. Spitkovsky considered SIO's with PC coefficients in Lebesgue spaces $L^p(\Gamma, w)$ with arbitrary Muckenhoupt weights w over smooth curves Γ ([36]). In that case the spectra have the shape of horns depending on the exponent p of the space and on the indices of powerlikeness (in terminology of [4]) of Muckenhoupt weights. In the middle of nineties A. Böttcher and Yu.I. Karlovich had accomplished the Fredholm theory for the algebra of SIO's with PC coefficients in Lebesgue spaces with general Muckenhoupt weights over arbitrary Carleson curves ([4]). In this general case the local spectra have the shape of so-called leaves, which are "massive" simply connected sets. So, if we consider a general Muckenhoupt weight or an arbitrary Carleson curve instead of a Khvedelidze weight or a Lyapunov curve, then we get massive local spectra of SIO's with PC coefficients.

The main tools of investigation in [4] are local principles, techniques of the Wiener-Hopf factorization and a theory of submultiplicative functions associated with curves and weights. Note that there is another approach to studying SIO's in weighted Lebesgue spaces. This approach is based on the application of the Mellin transform and techniques of pseudodifferential and limit operators (see [5], [6], [33] and also [4], Section 10.6). With the help of these methods one can study SIO's when the coefficients, the Carleson curve and the Muckenhoupt weight are slowly oscillating. But, unfortunately, these methods do not allow yet to consider the general case of arbitrary Carleson curves and Muckenhoupt weights.

The passage from Lebesgue spaces to more general rearrangement-invariant spaces (briefly r.i. spaces) also evokes the appearance of massive spectra for SIO's. Orlicz spaces are the brightest nontrivial example of r.i. spaces. Note that the scale of Orlicz spaces contains the Lebesgue spaces. In [17] and [20], the author showed that in the case of arbitrary reflexive Orlicz spaces over Lyapunov curves the local spectra are horns depending on the interpolation characteristics of the spaces (the Boyd indices). In the case of logarithmic Carleson curves these horns metamorphose into spiralic horns depending on the Boyd indices as well as the spirality indices of curves ([18]).

Recently the author found a so-called disintegration condition connecting the Boyd indices of reflexive Orlicz spaces and the spirality indices ([19]). This condition implies that the local spectra have the shape of logarithmic leaves. The results of [19] were extended to the case of reflexive r.i. spaces of fundamental type with nontrivial Boyd indices ([22]). Note that the results of [19] and [22] generalize the earlier results of the author ([18], the case of arbitrary reflexive Orlicz spaces over logarithmic Carleson curves) and the results of A. Böttcher and Yu.I. Karlovich ([3], the case of Lebesgue spaces over arbitrary Carleson curves).

As we can see from the above mentioned results, general r.i. spaces, or general Muckenhoupt weights, or general Carleson curves lead to massive local spectra of SIO's with PC coefficients. In this paper we consider these three factors together. More precisely, we study the Fredholmness of SIO's with PC coefficients in weighted rearrangement-invariant spaces (briefly w.r.i. spaces) over Carleson curves. In this paper we extend results and ideas of [4] and [22] and continue the investigation of SIO's in w.r.i. spaces, which was started in [23].

The paper is organized as follows. Section 2 contains necessary preliminaries on w.r.i. spaces, the Boyd indices α_X, β_X ([7], [8]) and the Zippin indices p_X, q_X

([37]) of r.i. spaces. In Subsection 2.2 we formulate necessary conditions for the boundedness of the Cauchy singular integral operator S in w.r.i. spaces $X(\Gamma, w)$.

In Section 3 we consider regular and submultiplicative functions associated with spaces, curves, and weights. In Subsections 3.1 and 3.2 we formulate definitions of spirality indices of the curve ([4], Section 1.6) and indices of powerlikeness of the weight ([4], Section 3.6). Also we give examples of curves and weights with distinct indices. In Subsection 3.3 we study four indicator functions α_t, β_t (see [4], Section 3.5) and α_t^*, β_t^* (cf. [23], Subsection 7.2). Their properties and relations between them follow from the results of [22], Section 5, [23], Section 2 and [4], Chapters 1 and 3. In Subsection 3.4 we formulate some "disintegration condition" of indicator functions:

$$\alpha_t^*(\operatorname{Im}\gamma) = \alpha_X + \alpha_t(\operatorname{Im}\gamma), \quad \beta_t^*(\operatorname{Im}\gamma) = \beta_X + \beta_t(\operatorname{Im}\gamma)$$

for every $\gamma \in \mathbb{C}$ such that a weight $|(\tau - t)^\gamma| w(\tau)$ belongs to a local analogue $A_X(\Gamma, t)$ of Muckenhoupt's type class. In the nonweighted case this condition follows from the disintegration condition given in [19] and [22].

In Section 4 we investigate singular integral operators associated with the Riemann boundary value problem. In Subsection 4.1 we formulate two general theorems which are the main tools for further studies of the SIO's with PC coefficients: an analogue of Simonenko's factorization theorem (see [34], [35]) and a local principle. In Subsection 4.2 we formulate necessary conditions for the factorizability of local representatives $g_{t,\gamma}$ of PC coefficients in terms of the indicator functions α_t^* and β_t^*. In Subsection 4.3 we prove that if a weight w belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$, then the operator S is bounded in a w.r.i. space $X(\Gamma, w)$. For such weights we obtain sufficient conditions for the factorizability of $g_{t,\gamma}$ in terms of the indicator functions α_t, β_t and Boyd indices. In Subsection 4.4, using results of Subsections 4.1–4.3, we get a Fredholm criterion for SIO's with PC coefficients under the disintegration condition in a reflexive w.r.i space generated by an r.i. space of fundamental type $X(\Gamma)$ with nontrivial Boyd indices α_X, β_X and a weight w belonging to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$. In Subsection 4.5 we reformulate this result in the geometric language, that is, in terms of essential spectrums and leaves.

In Section 5 we construct a symbol calculus for the Banach algebra $\mathcal{U}$ of SIO's with matrix-valued piecewise continuous coefficients using the results of Section 4. The symbol calculus is obtained with the help of the Allan-Douglas local principle (see, e.g., [4], Chapter 8) and the two projections theorems (see [11], [15] and also [4], Chapter 8). Finally, we give a formula for the index of an arbitrary operator $A \in \mathcal{U}$ in terms of its symbol.

2. SPACES, CURVES, AND WEIGHTS

2.1. Weighted rearrangement-invariant spaces. For a general discussion of r.i. spaces, see C. Bennett and R. Sharpley ([1]), S.G. Krein, Ju.I. Petunin, and E.M. Semenov ([28]), J. Lindenstrauss and L. Tzafriri ([30]). All basic facts used are collected in [22], Sections 1 and 2.

Let Γ be a Jordan (i.e., homeomorphic to a circle) rectifiable curve with the Lebesgue length measure $|\mathrm{d}\tau|$. Let $X(\Gamma)$ be an r.i. space over Γ and $X'(\Gamma)$ its associate space. A function $w : \Gamma \to [0, \infty]$ is referred to as a weight if w is measurable and the set $w^{-1}(\{0, \infty\})$ has measure zero. Let $X(\Gamma, w)$ be the set of all measurable functions f such that $fw \in X(\Gamma)$, which is equipped with the norm

$$\|f\|_{X(\Gamma,w)} := \|fw\|_{X(\Gamma)}.$$

Such a normed space $X(\Gamma, w)$ is called a *weighted rearrangement-invariant space* (or, briefly, w.r.i. space). It is not difficult to see that if $w \in X(\Gamma)$ and $w^{-1} \in X'(\Gamma)$, then $X(\Gamma, w)$ is a Banach function space (for the definition, see [1], Section 1.1), and its associate space is the Banach function space $X'(\Gamma, w^{-1})$ with the norm $\|f\|_{X'(\Gamma,w^{-1})} = \|fw^{-1}\|_{X'(\Gamma)}$. From the Hölder inequality for Banach function spaces it follows that if $w \in X(\Gamma)$ and $w^{-1} \in X'(\Gamma)$, then

$$L^\infty(\Gamma) \subset X(\Gamma, w) \subset L^1(\Gamma). \tag{2.1}$$

2.2. Generalization of the Muckenhoupt condition. The Cauchy singular integral of a function $f \in L^1(\Gamma)$ is defined by

$$(S\varphi)(t) := \lim_{R\to 0} \frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma\setminus\Gamma(t,R)} \frac{f(\tau)\,\mathrm{d}\tau}{\tau - t}, \quad t \in \Gamma, \tag{2.2}$$

where $\Gamma(t, R) := \{\tau \in \Gamma : |t-\tau| < R\}$ is the portion of the curve Γ from the disk of radius R centered at the point $t \in \Gamma$. A nice discussion of the problem concerning the existence of the Cauchy singular integral is in Dynkin's survey ([9]). Recall that the Cauchy singular integral exists for almost all $t \in \Gamma$. In this subsection we formulate necessary condition for the boundedness of the operator S in weighted rearrangement-invariant spaces.

Fix $t \in \Gamma$. For a weight $w : \Gamma \to [0, \infty]$, put

$$B_{t,R}(w) := \frac{1}{R}\|w\chi_{\Gamma(t,R)}\|_{X(\Gamma)}\|w^{-1}\chi_{\Gamma(t,R)}\|_{X'(\Gamma)},$$

where $\chi_{\Gamma(t,R)}$ is the characteristic function of the portion $\Gamma(t, R)$. Consider the following classes of weights (cf. [2]):

$$A_X(\Gamma, t) := \Big\{w : \sup_{R>0} B_{t,R}(w) < \infty\Big\}, \quad A_X(\Gamma) := \Big\{w : \sup_{t\in\Gamma}\sup_{R>0} B_{t,R}(w) < \infty\Big\}.$$

Obviously, $A_X(\Gamma) \subset A_X(\Gamma, t)$ for each $t \in \Gamma$. If $X(\Gamma)$ is a Lebesgue space $L^p(\Gamma), 1 < p < \infty$, then $A_X(\Gamma)$ is the Muckenhoupt class $A_p(\Gamma)$, i.e., the class of weights w such that

$$\sup_{t\in\Gamma}\sup_{R>0}\left(\frac{1}{R}\int\limits_{\Gamma(t,R)} w^p(\tau)|\,\mathrm{d}\tau|\right)^{1/p}\left(\frac{1}{R}\int\limits_{\Gamma(t,R)} w^{-q}(\tau)|\,\mathrm{d}\tau|\right)^{1/q} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

For a detailed discussion of Muckenhoupt weights on curves, see, e.g., [4]. For different generalizations of the Muckenhoupt class $A_p(\Gamma)$ in settings of Orlicz and r.i. spaces, see, e.g., [2], [12], [26].

Using Hölder's inequality, it is easy to see that $w \in A_X(\Gamma, t)$ implies

$$C_{\Gamma,t} := \sup_{R>0} \frac{|\Gamma(t,R)|}{R} < \infty. \tag{2.3}$$

We say that Γ is a *locally Carleson curve at the point* $t \in \Gamma$, if (2.3) holds. Analogously, if $w \in A_X(\Gamma)$, then

$$C_\Gamma := \sup_{t\in\Gamma} C_{\Gamma,t} < \infty. \tag{2.4}$$

A rectifiable curve Γ is said to be *Carleson curve* if (2.4) is satisfied. In this case the constant $C_{\Gamma,t}$ (respectively, C_Γ) is referred to as the local Carleson constant at the point t (respectively, the (global) Carleson constant).

Theorem 2.1. ([22], Theorem 3.2, Lemma 3.3) *If the Cauchy singular integral generates a bounded linear operator S in a w.r.i. space $X(\Gamma, w)$, then $w \in X(\Gamma)$ and $w^{-1} \in X'(\Gamma)$. Moreover, $w \in A_X(\Gamma)$.*

It is well known that for Lebesgue spaces $L^p(\Gamma)$, $1 < p < \infty$, the reverse implication is also true (see, e.g., [4], Theorem 4.15). So, the operator S is bounded in a weighted Lebesgue space $L^p(\Gamma, w)$, $1 < p < \infty$, if and only if w belongs to the Muckenhoupt class $A_p(\Gamma)$.

We need the following useful property of weights from the introduced generalization of the Muckenhoupt class.

Lemma 2.2. (i) *If $w_1, w_2 \in A_X(\Gamma, t)$, then for every $\theta \in [0,1]$ we have $w_1^\theta w_2^{1-\theta} \in A_X(\Gamma, t)$.*

(ii) *If $w_1, w_2 \in A_X(\Gamma)$, then for every $\theta \in [0,1]$ we have $w_1^\theta w_2^{1-\theta} \in A_X(\Gamma)$.*

Proof. (i) Suppose $w_1, w_2 \in A_X(\Gamma, t)$, then $w_1, w_2 \in X(\Gamma)$ and $w_1^{-1}, w_2^{-1} \in X'(\Gamma)$. From [28], Section 2.2, Property 6 it follows that for every $\theta \in [0,1]$ and $R > 0$,

$$\|w_1^\theta w_2^{1-\theta} \chi_{\Gamma(t,R)}\|_{X(\Gamma)} \leqslant \|w_1 \chi_{\Gamma(t,R)}\|_{X(\Gamma)}^\theta \|w_2 \chi_{\Gamma(t,R)}\|_{X(\Gamma)}^{1-\theta},$$
$$\|w_1^{-\theta} w_2^{-(1-\theta)} \chi_{\Gamma(t,R)}\|_{X'(\Gamma)} \leqslant \|w_1^{-1} \chi_{\Gamma(t,R)}\|_{X'(\Gamma)}^\theta \|w_2^{-1} \chi_{\Gamma(t,R)}\|_{X'(\Gamma)}^{1-\theta}.$$

Consequently, $B_{t,R}\big(w_1^\theta w_2^{1-\theta}\big) \leqslant \big[B_{t,R}(w_1)\big]^\theta \big[B_{t,R}(w_2)\big]^{1-\theta}$ for $\theta \in [0,1]$ and $R > 0$. From the latter inequality we obtain $w_1^\theta w_2^{1-\theta} \in A_X(\Gamma, t)$. Statement (ii) is proved analogously. ∎

2.3. Submultiplicative functions. Following [4], Section 1.4, we say that a function $\varrho : (0,\infty) \to (0,\infty]$ is *regular* if it is bounded in some open neighborhood of the unity. A function $\varrho : (0,\infty) \to (0,\infty]$ is said to be *submultiplicative* if $\varrho(x_1 x_2) \leqslant \varrho(x_1)\varrho(x_2)$ for all $x_1, x_2 \in (0,\infty)$. It is easy to show that if ϱ is regular and submultiplicative, then ϱ is bounded away from zero in some open neighborhood of the point 1. Moreover, in this case $\varrho(x)$ is finite for all $x \in (0,\infty)$. Given a regular submultiplicative function $\varrho : (0,\infty) \to (0,\infty)$, one defines

$$\alpha(\varrho) := \sup_{x\in(0,1)} \frac{\log \varrho(x)}{\log x}, \quad \beta(\varrho) := \inf_{x\in(1,\infty)} \frac{\log \varrho(x)}{\log x}.$$

Clearly, $-\infty < \alpha(\varrho)$ and $\beta(\varrho) < +\infty$. The quantities $\alpha(\varrho)$ and $\beta(\varrho)$ are called the *lower* and *upper indices* of the regular submultiplicative function ϱ, respectively.

Theorem 2.3. (see, e.g., [28], Chapter 2, Theorem 1.3 or [4], Theorem 1.13) *If $\varrho : (0,\infty) \to (0,\infty)$ is regular and submultiplicative, then*

$$\alpha(\varrho) = \lim_{x\to 0} \frac{\log \varrho(x)}{\log x}, \quad \beta(\varrho) = \lim_{x\to\infty} \frac{\log \varrho(x)}{\log x},$$

and $-\infty < \alpha(\varrho) \leqslant \beta(\varrho) < +\infty$.

2.4. Boyd and Zippin indices. The idea of using indices of some submultiplicative functions for the description of properties of Orlicz spaces goes back to W. Matuszewska and W. Orlicz, 1960. Matuszewska-Orlicz indices were generalized by D.W. Boyd and M. Zippin to the case of rearrangement-invariant spaces (for the history and precise references, see [31]).

Let $X(\Gamma)$ be an r.i. space over Γ generated by a rearrangement-invariant function norm ρ. By the Luxemburg representation theorem ([1], Chapter 2, Theorem 4.10), there is a unique rearrangement-invariant function norm $\overline{\rho}$ over $[0,|\Gamma|]$ with the Lebesgue measure m such that $\rho(f) = \overline{\rho}(f^*)$ for all non-negative and a.e. finite measurable functions f defined on Γ. Here f^* denotes the non-increasing rearrangement of f (see e.g. [1], p. 39). The r.i. space over the measure space $([0,|\Gamma|], m)$ generated by $\overline{\rho}$ is called the *Luxemburg representation* of $X(\Gamma)$ and is denoted by $\overline{X}$. For each $x > 0$, let E_x denote the dilation operator defined on $\overline{X}$ by

$$(E_x f)(t) := \begin{cases} f(xt), & xt \in [0,|\Gamma|] \\ 0, & xt \notin [0,|\Gamma|] \end{cases}, \quad t \in [0,|\Gamma|].$$

For every $x > 0$, the operator $E_{1/x}$ is bounded on $\overline{X}$ ([1], p. 165), and we denote its norm by

$$h_X(x) := \|E_{1/x}\|_{\mathcal{B}(\overline{X})},$$

where $\mathcal{B}(\overline{X})$ is the Banach algebra of the bounded linear operators on $\overline{X}$.

For each $t \in [0,|\Gamma|]$, let E be a measurable subset of Γ with $|E| = t$, χ_E the characteristic function of E, and let

$$\varphi_X(t) := \|\chi_E\|_{X(\Gamma)}.$$

The function φ_X so defined is called the *fundamental function* of $X(\Gamma)$. Given the fundamental function φ_X of $X(\Gamma)$, put

$$M_X(x) := \limsup_{t\to 0} \frac{\varphi_X(xt)}{\varphi_X(t)}, \quad x \in (0,\infty).$$

The functions h_X, M_X are non-decreasing, regular and submultiplicative (see [1], Section 3.5 and [31], Section 4). The indices of the submultiplicative function h_X

are called the *Boyd indices of the r.i. space* X ([7], [8]), and the indices of the submultiplicative function M_X are called the *Zippin (or fundamental) indices of the r.i. space* X ([37]). The Boyd and Zippin indices of the r.i. space X will be denoted by

$$\alpha_X := \alpha(h_X), \quad \beta_X := \beta(h_X); \quad p_X := \alpha(M_X), \quad q_X := \beta(M_X),$$

respectively. Generally, one can prove (see, e.g., [31], Section 4) that

$$0 \leqslant \alpha_X \leqslant p_X \leqslant q_X \leqslant \beta_X \leqslant 1. \tag{2.5}$$

An r.i. space $X(\Gamma)$ is said to be of *fundamental type* ([10]) if its Boyd and Zippin indices coincide:

$$\alpha_X = p_X, \quad \beta_X = q_X.$$

For the Lebesgue spaces $L^p(\Gamma), 1 \leqslant p \leqslant \infty$, all indices are equal to $1/p$. Less trivial examples of r.i. spaces of fundamental type are Orlicz spaces ([10] and [31]).

Recall the definition of Orlicz spaces (see, e.g., [1], [27], [31]). A convex and continuous function $\Phi : [0, \infty) \to [0, \infty)$, for which $\Phi(0) = 0$, $\Phi(t) > 0$ for $t > 0$, and

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0,$$

is called an N-*function.* For a measurable function $f : \Gamma \to \mathbb{C}$ define the functional

$$N_\Phi(f) := \int_\Gamma \Phi(|f(\tau)|)\, |d\tau|.$$

The set of all measurable functions f, for which there exists a $\lambda = \lambda(f) > 0$ such that $N_\Phi(f/\lambda) < \infty$, is called the *Orlicz space.* This space is denoted by $L^\Phi(\Gamma)$.

EXAMPLE 2.4. Let $k > 0$ and $p > 1 + \sqrt{2}k$. Set

$$m := \left[\frac{1}{2\pi} \log \frac{k\sqrt{2}}{p - 1 - k\sqrt{2}}\right] + 1,$$

where $[r]$ denotes the integral part of $r \in \mathbb{R}$. The N-function

$$\Phi(t) := \begin{cases} t^p, & t \in [0, \exp(\exp(2\pi m))], \\ t^{p+k\sin(\log\log t)}, & t \in (\exp(\exp(2\pi m)), \infty), \end{cases}$$

generates the Orlicz space $L^\Phi(\Gamma)$ with the Boyd indices

$$\alpha_{L^\Phi} = \frac{1}{p + \sqrt{2}k}, \quad \beta_{L^\Phi} = \frac{1}{p - \sqrt{2}k}.$$

This is a modification of an example given by K. Lindberg ([29]; see also [32], Chapter 11).

We will say that the Boyd indices are nontrivial if

$$0 < \alpha_X \quad \text{and} \quad \beta_X < 1.$$

In the case of Orlicz spaces these inequalities are equivalent to the reflexivity of the space (see, e.g., [10], Theorem 2.2, [18], Theorem 2.4 and [31]). Note that there are r.i. spaces for which the Boyd and Zippin indices do not coincide, that is, there exist r.i. spaces of non-fundamental type (see [31] and the references therein).

3. SUBMULTIPLICATIVE FUNCTIONS ASSOCIATED WITH SPACES, CURVES, AND WEIGHTS

3.1. SPIRALITY INDICES. In this subsection we mainly follow [4], Chapter 1. Let Γ be a Jordan rectifiable curve. Fix $t \in \Gamma$. We then have

$$\tau - t = |\tau - t| e^{\mathrm{i} \arg(\tau - t)}, \quad \tau \in \Gamma \setminus \{t\},$$

and $\arg(\tau - t)$ may be chosen to be a continuous function of $\tau \in \Gamma \setminus \{t\}$.

Let $d_t = \max_{\tau \in \Gamma} |\tau - t|$ and $R \in (0, d_t]$. For a continuous function $\psi : \Gamma \setminus \{t\} \to (0, \infty)$, define the function (see [4], Chapter 1):

$$(W_t \psi)(x) := \limsup_{R \to 0} \frac{\max\limits_{\substack{\tau \in \Gamma \\ |\tau - t| = xR}} \psi(\tau)}{\min\limits_{\substack{\tau \in \Gamma \\ |\tau - t| = R}} \psi(\tau)}, \quad x \in (0, \infty).$$

Consider the continuous function $\eta_t : \Gamma \setminus \{t\} \to (0, \infty)$ defined by $\eta_t(\tau) := e^{-\arg(\tau - t)}$. Using the local Carleson constant $C_{\Gamma,t}$ instead of the Carleson constant C_Γ we obtain the following local version of [4], Lemmas 1.15–1.17.

LEMMA 3.1. *If Γ is a locally Carleson curve at $t \in \Gamma$, then the function $W_t \eta_t$ is regular and submultiplicative.*

Under the assumptions of Lemma 3.1, in view of Theorem 2.3, there exist the spirality indices of the curve Γ at the point $t \in \Gamma$:

$$\delta_t^- := \alpha(W_t \eta_t), \quad \delta_t^+ := \beta(W_t \eta_t).$$

Let Γ be a locally Carleson curve at $t \in \Gamma$ and

$$\arg(\tau - t) = -\delta_t \log |\tau - t| + O(1), \quad \tau \to t \tag{3.1}$$

where $\delta_t \in \mathbb{R}$. One can prove (see [4], Chapter 1) that in this case $\delta_t^- = \delta_t^+ = \delta_t$. A Carleson curve Γ is said to be a *logarithmic Carleson curve* if it satisfies (3.1) at each point $t \in \Gamma$. The simplest examples of logarithmic Carleson curves are piecewise smooth curves with corners and cusps. For these curves, $\delta_t \equiv 0$.

EXAMPLE 3.2. (see [4], Section 1.6) Define arcs Γ_1 and Γ_2 by

$$\Gamma_1 := \{t\} \cup \{\tau \in \mathbb{C} : \tau = t + r e^{\mathrm{i}\varphi(r)}, \ 0 < r \leqslant 1\},$$
$$\Gamma_2 := \{t\} \cup \{\tau \in \mathbb{C} : \tau = t + r e^{\mathrm{i}(\varphi(r) + b(r))}, \ 0 < r \leqslant 1\},$$

where $\varphi(r) = h(\log(-\log r))(-\log r), h(x) = \delta + \mu \sin \lambda x$ with $\delta, \mu, \lambda, x \in \mathbb{R}$, the function b satisfies the following conditions: $0 < b(r) < 2\pi$ for $r \in (0, 1)$ and $b \in C(0, 1] \cap C^1(0, 1)$, the function $r b'(r)$ is bounded on $(0, 1)$. Then the curve $\Gamma = \Gamma_1 \cup \Gamma_2$ has the following spirality indices at t:

$$\delta_t^- = \delta - |\mu| \sqrt{\lambda^2 + 1}, \quad \delta_t^+ = \delta + |\mu| \sqrt{\lambda^2 + 1}.$$

3.2. INDICES OF POWERLIKENESS. Let $w : \Gamma \to [0,\infty]$ be a weight such that $\log w \in L^1(\Gamma(t,R))$ for every $R \in (0,d_t]$. For every $x \in (0,\infty)$, consider the function (see [4], Chapter 3):

$$(V_t w)(x) := \limsup_{R\to 0} \exp\biggl(\frac{1}{|\Gamma(t,xR)|}\int_{\Gamma(t,xR)} \log w(\tau)|\,\mathrm{d}\tau| - \frac{1}{|\Gamma(t,R)|}\int_{\Gamma(t,R)} \log w(\tau)|\,\mathrm{d}\tau|\biggr).$$

LEMMA 3.3. *If $w \in A_X(\Gamma,t)$, then the function $V_t w$ is regular and submultiplicative.*

This statement follows from [23], Lemma 1.6 (a) and [4], Lemmas 3.2 (a) and 3.5 (a).

Under the assumptions of Lemma 3.3, in view of Theorem 2.3, for the weight w, there exist the indices of powerlikeness:

$$\mu_t := \alpha(V_t w), \quad \nu_t := \beta(V_t w).$$

Clearly, for the power weight $w(\tau) = |\tau - t|^{\lambda_t}$, indices of powerlikeness equal $\mu_t = \nu_t = \lambda_t$.

EXAMPLE 3.4. Let Γ be the curve as in Example 3.2. Define a weight w on the curve Γ by $w(\tau) = \mathrm{e}^{v(|\tau-t|)}$ where $v(r) = g(\log(-\log r))(-\log r)$, $g(x) = \lambda + \varepsilon\sin(\eta x)$, and $\lambda,\varepsilon,\eta,x \in \mathbb{R}$. Then indices of powerlikeness of w at $t \in \Gamma$ equal

$$\mu_t = \lambda - |\varepsilon|\sqrt{\eta^2+1}, \quad \nu_t = \lambda + |\varepsilon|\sqrt{\eta^2+1}.$$

Moreover, if $1 < p \leqslant q < \infty$ and $0 < 1/q + \mu_t \leqslant 1/p + \nu_t < 1$, then w belongs to the Muckenhoupt classes $A_p(\Gamma)$ and $A_q(\Gamma)$.

This result follows from [4], Examples 3.24–3.28.

3.3. INDICATOR FUNCTIONS. Let Γ be a locally Carleson curve at $t \in \Gamma$. Put $\eta_t(\tau) := \mathrm{e}^{-\arg(\tau-t)}$, where $\tau \in \Gamma \setminus \{t\}$. From [4], Lemmas 1.15, 1.16 and Proposition 3.1 we see that the function $W_t\eta_t^x$ is regular and submultiplicative and

$$\begin{aligned} \alpha_t^0(x) &:= \alpha(W_t\eta_t^x) = \min\{\delta_t^- x, \delta_t^+ x\}, \\ \beta_t^0(x) &:= \beta(W_t\eta_t^x) = \max\{\delta_t^- x, \delta_t^+ x\}. \end{aligned} \tag{3.2}$$

Consider the portion of the curve Γ in the annulus $\Delta(t,R) := \Gamma(t,R) \setminus \Gamma(t,R/2)$. Let $w : \Gamma \to [0,\infty]$ be a weight such that $w\chi_{\Delta(t,R)} \in X(\Gamma)$ and $w^{-1}\chi_{\Delta(t,R)} \in X'(\Gamma)$ for every $R \in (0,d_t]$. Define the following function (see [22]):

$$(Q_t w)(x) := \limsup_{R\to 0} \frac{\|w\chi_{\Delta(t,xR)}\|_{X(\Gamma)}\|w^{-1}\chi_{\Delta(t,R)}\|_{X'(\Gamma)}}{|\Delta(t,R)|}, \quad x \in (0,\infty).$$

For a complex number $\gamma \in \mathbb{C}$, we define a continuous function $\varphi_{t,\gamma}$ on $\Gamma \setminus \{t\}$ by

$$\varphi_{t,\gamma}(\tau) := |(\tau-t)^\gamma| = |\tau-t|^{\operatorname{Re}\gamma}\mathrm{e}^{-\operatorname{Im}\gamma\arg(\tau-t)} = |\tau-t|^{\operatorname{Re}\gamma}(\eta_t(\tau))^{\operatorname{Im}\gamma}.$$

Suppose $w \in A_X(\Gamma,t)$. By analogy with [22], Lemma 7.2 and taking into account [22], Lemma 5.2, one can prove that for every $\gamma \in \mathbb{C}$ the function $Q_t(\varphi_{t,\gamma} w)$ is regular and submultiplicative,

$$\begin{aligned}\alpha(Q_t(\varphi_{t,\gamma} w)) &= \operatorname{Re}\gamma + \alpha(Q_t(\eta_t^{\operatorname{Im}\gamma} w)),\\ \beta(Q_t(\varphi_{t,\gamma} w)) &= \operatorname{Re}\gamma + \beta(Q_t(\eta_t^{\operatorname{Im}\gamma} w)).\end{aligned} \tag{3.3}$$

From [23], Lemma 1.6 (a) and [4], Corollary 3.18 it follows that for every $\gamma \in \mathbb{C}$ the function $V_t(\varphi_{t,\gamma} w)$ is regular and submultiplicative,

$$\begin{aligned}\alpha(V_t(\varphi_{t,\gamma} w)) &= \operatorname{Re}\gamma + \alpha(V_t(\eta_t^{\operatorname{Im}\gamma} w)),\\ \beta(V_t(\varphi_{t,\gamma} w)) &= \operatorname{Re}\gamma + \beta(V_t(\eta_t^{\operatorname{Im}\gamma} w)).\end{aligned} \tag{3.4}$$

Hence, in view of Theorem 2.3, the following indicator functions are well-defined for every $x \in \mathbb{R}$:

$$\begin{aligned}\alpha_t(x) &= \alpha(V_t(\eta_t^x w)), \quad \beta_t(x) = \beta(V_t(\eta_t^x w)),\\ \alpha_t^*(x) &= \alpha(Q_t(\eta_t^x w)), \quad \beta_t^*(x) = \beta(Q_t(\eta_t^x w)).\end{aligned}$$

LEMMA 3.5. *For every $x, y \in \mathbb{R}$ we have:*

(i) $\alpha_t(x) + \alpha_t^0(y) \leqslant \alpha_t(x+y) \leqslant \min\{\alpha_t(x) + \beta_t^0(y), \beta_t(x) + \alpha_t^0(y)\}$,
$\beta_t(x) + \beta_t^0(y) \geqslant \beta_t(x+y) \geqslant \max\{\alpha_t(x) + \beta_t^0(y), \beta_t(x) + \alpha_t^0(y)\}$;

(ii) $\alpha_t^*(x) + \alpha_t^0(y) \leqslant \alpha_t^*(x+y) \leqslant \min\{\alpha_t^*(x) + \beta_t^0(y), \beta_t^*(x) + \alpha_t^0(y)\}$,
$\beta_t^*(x) + \beta_t^0(y) \geqslant \beta_t^*(x+y) \geqslant \max\{\alpha_t^*(x) + \beta_t^0(y), \beta_t^*(x) + \alpha_t^0(y)\}$;

(iii) $\alpha_t^*(x) \leqslant \min\{p_X + \beta_t(x), q_X + \alpha_t(x)\}$,
$\beta_t^*(x) \geqslant \max\{p_X + \beta_t(x), q_X + \alpha_t(x)\}$;

(iv) $p_X + \mu_t \leqslant \alpha_t^*(0) \leqslant \beta_t^*(0) \leqslant q_X + \nu_t$.

Proof. Statement (i) follows from [23], Lemma 1.6 (a) and [4], Lemma 3.17 which is applied for the weights $w := \eta_t^x w$ and $\varphi := \eta_t^y$. Statement (ii) follows from [22], Lemmas 7.3(c) and 5.2. Statements (iii) and (iv) follow from Theorems 2.6 and 2.7 of [23], respectively, and from [22], Lemma 5.2. ∎

LEMMA 3.6. *The functions α_t and α_t^* are concave, the functions β_t and β_t^* are convex. In particular, these four functions are continuous on the whole $\mathbb{R}$.*

Proof. The concavity of α_t and the convexity of β_t is proved in [4], Proposition 3.20. The concavity of α_t^* and the convexity of β_t^* is proved by analogy. Here we essentially use [28], Section 2.2, Property 6 and argue as in the proof of Lemma 2.2. ∎

3.4. DISINTEGRATION CONDITION. Given $w \in A_X(\Gamma,t)$ we define the indicator set at $t \in \Gamma$ by

$$N_t := \{\gamma \in \mathbb{C} : \varphi_{t,\gamma} w \in A_X(\Gamma,t)\}. \tag{3.5}$$

Obviously, the indicator set is nonempty. From Lemma 2.2 it follows that the set N_t is convex. We say that the indicator functions α_t, β_t and α_t^*, β_t^* satisfy *the disintegration condition* (cf. [23], Section 7.2), if for every $\gamma \in N_t$,

$$\alpha_t^*(\operatorname{Im}\gamma) = \alpha_X + \alpha_t(\operatorname{Im}\gamma), \quad \beta_t^*(\operatorname{Im}\gamma) = \beta_X + \beta_t(\operatorname{Im}\gamma). \tag{3.6}$$

If $w = 1$, then from [4], Proposition 3.23 it follows that $\alpha_t(x) = \alpha_t^0(x)$ and $\beta_t(x) = \beta_t^0(x)$ for every $x \in \mathbb{R}$. Hence, in the case $w = 1$ the disintegration condition from [19] and [22] implies the disintegration condition considered here.

LEMMA 3.7. *If the indicator functions α_t, β_t and α_t^*, β_t^* satisfy the disintegration condition, then $\alpha_X = p_X$ and $\beta_X = q_X$.*

Proof. Since $w \in A_X(\Gamma, t)$, conclude that $0 \in N_t$ due to (3.5). From Lemma 3.5 (iv) and the disintegration condition we obtain

$$p_X + \mu_t \leqslant \alpha_t^*(0) = \alpha_X + \alpha_t(0) = \alpha_X + \mu_t,$$
$$q_X + \nu_t \geqslant \beta_t^*(0) = \beta_X + \beta_t(0) = \beta_X + \nu_t.$$

Hence, $p_X \leqslant \alpha_X \leqslant \beta_X \leqslant q_X$. This and (2.5) imply $\alpha_X = p_X$ and $\beta_X = q_X$. ∎

LEMMA 3.8. *If $X(\Gamma)$ is an r.i. space of fundamental type and one of the following two conditions is fulfilled:*
(i) $p_X = q_X$,
(ii) $\alpha_t(\operatorname{Im}\gamma) = \beta_t(\operatorname{Im}\gamma)$ *for every* $\gamma \in N_t$,
then the indicator functions satisfy the disintegration condition.

Proof. Due to [23], Theorem 2.7 and taking into account [22], Lemma 5.2, we have for every $\gamma \in N_t$,

$$p_X + \alpha(V_t(\varphi_{t,\gamma}w)) \leqslant \alpha(Q_t(\varphi_{t,\gamma}w)) \leqslant \beta(Q_t(\varphi_{t,\gamma}w)) \leqslant q_X + \beta(V_t(\varphi_{t,\gamma}w)). \tag{3.7}$$

From (3.4), (3.3) and (3.7) we get

$$p_X + \alpha_t(\operatorname{Im}\gamma) \leqslant \alpha_t^*(\operatorname{Im},\gamma) \leqslant \beta_t^*(\operatorname{Im}\gamma) \leqslant q_X + \beta_t(\operatorname{Im}\gamma).$$

The latter inequalities and Lemma 3.5 (iii) give

$$\begin{aligned} p_X + \alpha_t(\operatorname{Im}\gamma) \leqslant \alpha_t^*(\operatorname{Im}\gamma) \leqslant \min\{p_X + \beta_t(\operatorname{Im}\gamma), q_X + \alpha_t(\operatorname{Im}\gamma)\}, \\ q_X + \beta_t(\operatorname{Im}\gamma) \geqslant \beta_t^*(\operatorname{Im}\gamma) \geqslant \max\{p_X + \beta_t(\operatorname{Im}\gamma), q_X + \alpha_t(\operatorname{Im}\gamma)\}. \end{aligned} \tag{3.8}$$

If one of the conditions (i) or (ii) holds, then inequalities (3.8) become equalities, that is, the indicator functions satisfy the disintegration condition. ∎

LEMMA 3.9. *If $\delta_t^- = \delta_t^+ =: \delta_t$ and $\mu_t = \nu_t =: \lambda_t$, then*

$$\alpha_t(x) = \beta_t(x) = \lambda_t + \delta_t x \quad \textit{for all } x \in \mathbb{R}. \tag{3.9}$$

Proof. From (3.2) we obtain $\alpha_t^0(x) = \beta_t^0(x) = \delta_t x$ for all $x \in \mathbb{R}$. Hence, taking into account $\mu_t = \alpha_t(0)$ and $\nu_t = \beta_t(0)$, from Lemma 3.5 (i) we get

$$\alpha_t(x) = \mu_t + \alpha_t^0(x), \quad \beta_t(x) = \nu_t + \beta_t^0(x), \quad x \in \mathbb{R}.$$

Since $\mu_t = \nu_t$, the latter equalities imply (3.9). ∎

In view of Lemmas 3.8 and 3.9, with the help of Examples 2.4, 3.2, and 3.4, one can construct nontrivial examples of w.r.i. spaces satisfying the disintegration condition. In particular, the disintegration condition is satisfied for Lebesgue spaces $L^p(\Gamma, w)$, $1 < p < \infty$, with general Muckenhoupt weights over arbitrary Carleson curves or for reflexive Orlicz spaces $L^\Phi(\Gamma, w)$ with powerlike weights (i.e., with weights for which $\mu_t = \nu_t$ at every $t \in \Gamma$) over logarithmic Carleson curves.

4. SINGULAR INTEGRAL OPERATORS

4.1. SINGULAR INTEGRAL OPERATORS WITH MEASURABLE COEFFICIENTS. Let Γ be a Jordan Carleson curve. Assume the Cauchy singular integral generates the bounded linear operator in a w.r.i. space $X(\Gamma, w)$. Due to Theorem 2.1, $w \in X(\Gamma)$ and $w^{-1} \in X'(\Gamma)$. Hence, $X(\Gamma, w)$ is a Banach function space. Suppose that this space is reflexive. In that case $S^2 = I$ ([23], Lemma 3.2). Hence, the operators $P_\pm := (I \pm S)/2$ are bounded projections in the reflexive w.r.i. space $X(\Gamma, w)$.

Define the following subspaces

$$X_+(\Gamma, w) := P_+X(\Gamma, w), \quad X_-(\Gamma, w) := P_-X(\Gamma, w) \dot{+} \mathbb{C}.$$

We denote by D^+ and D^- the bounded and unbounded component of $\mathbb{C} \setminus \Gamma$, respectively. Without loss of generality we will always assume that $0 \in D^+$. To check whether a function belongs to $X_\pm(\Gamma, w)$, the following result is often useful.

LEMMA 4.1. *Suppose the functions $f_\pm$ are analytic in $D^\pm$ and continuous on $D^\pm \cup \Gamma$ with the possible exception of finitely many points $t_1, \dots, t_m \in \Gamma$. Suppose that $f_\pm|\Gamma \in X(\Gamma, w)$ and that $f_\pm$ admits the estimate*

$$|f_\pm(z)| \leqslant M|z - t_k|^{-\mu}, \quad k = 1, \dots, m$$

with some $M > 0$ and $\mu > 0$ for all $z \in D^\pm$ sufficiently close to t_k. Then $f_\pm|\Gamma \in X_\pm(\Gamma, w)$.

Lemma 4.1 can be obtained literally from [4], Lemma 6.10 if we replace $L^p(\Gamma, w)$ and $L^p_\pm(\Gamma, w)$ by $X(\Gamma, w)$ and $X_\pm(\Gamma, w)$, respectively.

We say that a function $a \in L^\infty(\Gamma)$ admits a factorization in $X(\Gamma, w)$ if $a^{-1} \in L^\infty(\Gamma)$ and a can be written in the form

$$a(t) = a_-(t)t^\kappa a_+(t) \quad \text{a.e. on } \Gamma, \tag{4.1}$$

where $\kappa \in \mathbb{Z}$,

(i) $a_- \in X_-(\Gamma, w)$, $a_-^{-1} \in X'_-(\Gamma, w^{-1})$, $a_+ \in X'_+(\Gamma, w^{-1})$, $a_+^{-1} \in X_+(\Gamma, w)$;

(ii) the operator $a_+^{-1}Sa_+I$ is bounded in $X(\Gamma, w)$.

One can prove that the number κ is uniquely determined.

Let $a \in L^\infty(\Gamma)$. In view of (2.1), the operator aI is bounded in $X(\Gamma, w)$. Consider a singular integral operator R_a defined in $X(\Gamma, w)$ by the formula

$$R_a = aP_+ + P_-.$$

THEOREM 4.2. (see [23], Theorem 3.5) *A function $a \in L^\infty(\Gamma)$ admits a factorization (4.1) in a reflexive w.r.i. space $X(\Gamma, w)$ if and only if the operator R_a is Fredholm in $X(\Gamma, w)$. If R_a is Fredholm, then its index is equal to $-\kappa$.*

This theorem goes back to I.B. Simonenko ([34], [35]) in the case of Lebesgue spaces. For more about this topic we refer to [4], Section 6.12, [14], Section 8.3 in the case of weighted Lebesgue spaces and to [18], Theorem 5.6, [22], Theorem 6.10 in the case of reflexive Orlicz and r.i. spaces, respectively. Since the set of all rational functions without poles on Γ is dense in the spaces $X(\Gamma, w)$ and $(X(\Gamma, w))^* = X'(\Gamma, w^{-1})$ ([23], Lemma 1.4), in the weighted case the proof is developed by analogy.

Two functions $a, b \in L^\infty(\Gamma)$ are said to be locally equivalent at a point $t \in \Gamma$ if

$$\inf\{\|(a-b)c\|_\infty : c \in C(\Gamma),\, c(t) = 1\} = 0.$$

THEOREM 4.3. (see [23], Theorem 3.6) *Let $a \in L^\infty(\Gamma)$ and suppose for each $t \in \Gamma$ we are given a function $a_t \in L^\infty(\Gamma)$ which is locally equivalent to a at t. If the operators R_{a_t} are Fredholm in $X(\Gamma, w)$ for all $t \in \Gamma$, then R_a is also Fredholm in $X(\Gamma, w)$.*

In the case of Lebesgue spaces this theorem is known as Simonenko's local principle. Since the operator $aS - SaI$ is compact in the w.r.i. space $X(\Gamma, w)$ for every continuous function a ([23], Lemma 3.1), this local principle can be obtained from the Gohberg-Krupnik local principle (see, e.g., [14], Chapter 6) as in [4], Theorem 6.30.

4.2. FACTORIZATION OF LOCAL REPRESENTATIVES: NECESSITY. We denote by $PC(\Gamma)$ the Banach algebra of all piecewise continuous functions on the curve Γ: a function $a \in L^\infty(\Gamma)$ belongs to $PC(\Gamma)$ if and only if the finite one-sided limits

$$a(t \pm 0) := \lim_{\tau \to t \pm 0} a(\tau)$$

exist for every $t \in \Gamma$. Let $GL^\infty(\Gamma)$ denote the set of all invertible functions in $L^\infty(\Gamma)$, i.e., the set of all functions $a \in L^\infty(\Gamma)$ for which $\operatorname{ess\,inf}\{|a(t)| : t \in \Gamma\} > 0$.

Fix $t \in \Gamma$. For a function $a \in PC(\Gamma) \cap GL^\infty(\Gamma)$ we construct a "canonical" function $g_{t,\gamma}$ which is locally equivalent to a at the point $t \in \Gamma$. The interior and exterior of the unit circle can be conformally mapped onto D^+ and D^- of Γ, respectively, so that the point 1 is mapped to t, and the points $0 \in D^+$ and $\infty \in D^-$ remain fixed. Let Λ_0 and Λ_∞ denote the images of $[0,1]$ and $[1,\infty) \cup \{\infty\}$ under this map. The curve $\Lambda_0 \cup \Lambda_\infty$ joins 0 to ∞ and meets Γ at exactly one point, namely t. Let $\arg z$ be a continuous branch of argument in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. For $\gamma \in \mathbb{C}$ define the function $z^\gamma := |z|^\gamma e^{i\gamma \arg z}$, where $z \in \mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. Clearly, z^γ is an analytic function in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. The restriction of z^γ to $\Gamma \setminus \{t\}$ will be denoted $g_{t,\gamma}$. Obviously, $g_{t,\gamma}$ is continuous and nonzero on $\Gamma \setminus \{t\}$. Since $a(t \pm 0) \neq 0$, we can define $\gamma \in \mathbb{C}$ by formulas

$$\operatorname{Re}\gamma := \frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)}, \quad \operatorname{Im}\gamma := -\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right|, \tag{4.2}$$

where we can take any value of $\arg\big(a(t-0)/a(t+0)\big)$, which implies that any two choices of $\operatorname{Re}\gamma$ differ by an integer only. Clearly, there is a constant $c \in \mathbb{C} \setminus \{0\}$ such that $a(t \pm 0) = c g_{t,\gamma}(t \pm 0)$, which means that a is locally equivalent to $c g_{t,\gamma}$ at the point t.

THEOREM 4.4. (see [23], Theorem 4.1) *Suppose the Cauchy singular integral generates the bounded linear operator in a reflexive w.r.i. space $X(\Gamma, w)$. If the function $g_{t,\gamma}$ admits a factorization in the space $X(\Gamma, w)$, then*

$$-\operatorname{Re}\gamma + \theta\alpha_t^*(-\operatorname{Im}\gamma) + (1-\theta)\beta_t^*(-\operatorname{Im}\gamma) \notin \mathbb{Z}$$

for all $\theta \in [0,1]$. Moreover, there is an $l \in \mathbb{Z}$ such that $\varphi_{t,l-\gamma} w \in A_X(\Gamma)$.

4.3. FACTORIZATION OF LOCAL REPRESENTATIVES: SUFFICIENCY. We start this subsection with sufficient conditions for the boundedness of the Cauchy singular integral operator in weighted rearrangement-invariant spaces.

Theorem 4.5. *Let $X(\Gamma)$ be an r.i. space with nontrivial Boyd indices α_X, β_X. If a weight w belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$, then the operator S is bounded in the w.r.i. space $X(\Gamma, w)$.*

Proof. Due to [4], Theorem 2.31, there are p and q such that

$$1 < q < \frac{1}{\beta_X} \leqslant \frac{1}{\alpha_X} < p < \infty, \tag{4.3}$$

$w \in A_p(\Gamma)$ and $w \in A_q(\Gamma)$. By [4], Theorem 4.15, the operator S is bounded in the weighted Lebesgue spaces $L^p(\Gamma, w)$ and $L^q(\Gamma, w)$. In that case the operator $A := wSw^{-1}I$ is bounded in the Lebesgue spaces $L^p(\Gamma)$ and $L^q(\Gamma)$. Taking into account (4.3), by the Boyd interpolation theorem ([7]), the operator A is bounded in the r.i. space $X(\Gamma)$. Consequently, the operator S is bounded in the w.r.i. space $X(\Gamma, w)$. ∎

Lemma 4.1 and the following result are the keys for finding sufficient conditions for a factorizability of local representatives.

Lemma 4.6. *Let $X(\Gamma)$ be an r.i. space with nontrivial Boyd indices α_X, β_X. Suppose a weight w belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$. If*

$$0 < \alpha_X + \operatorname{Re}\gamma + \alpha_t(\operatorname{Im}\gamma) \leqslant \beta_X + \operatorname{Re}\gamma + \beta_t(\operatorname{Im}\gamma) < 1, \tag{4.4}$$

then the operator $\varphi_{t,\gamma} S \varphi_{t,\gamma}^{-1} I$ is bounded in the w.r.i. spaces $X(\Gamma, w)$.

Proof. If (4.4) holds, then

$$0 < \alpha_X + \operatorname{Re}\gamma + \alpha_t(\operatorname{Im}\gamma) \leqslant \alpha_X + \operatorname{Re}\gamma + \beta_t(\operatorname{Im}\gamma) < 1.$$

In that case, by [4], Theorem 3.21, the weight $\varphi_{t,\gamma} w$ belongs to the Muckenhoupt class $A_{\frac{1}{\alpha_X}}(\Gamma)$. Analogously, $\varphi_{t,\gamma} w \in A_{\frac{1}{\beta_X}}(\Gamma)$. Due to Theorem 4.5, the operator S is bounded in the space $X(\Gamma, \varphi_{t,\gamma} w)$. Hence, the operator $\varphi_{t,\gamma} S \varphi_{t,\gamma}^{-1} I$ is bounded in $X(\Gamma, w)$. ∎

Now we are in a position to prove the main result of this subsection.

Theorem 4.7. *Let $X(\Gamma)$ be an r.i. space with nontrivial Boyd indices α_X, β_X. Suppose a weight $w : \Gamma \to [0, \infty]$ belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$. Suppose the w.r.i. space $X(\Gamma, w)$ is reflexive. If for all $\theta \in [0, 1]$,*

$$\kappa_t(\theta) := -\operatorname{Re}\gamma + \theta\big(\alpha_X + \alpha_t(-\operatorname{Im}\gamma)\big) + (1-\theta)\big(\beta_X + \beta_t(-\operatorname{Im}\gamma)\big) \notin \mathbb{Z}$$

then the integral part $[\kappa_t(\theta)] =: -k$ does not depend on θ, and

$$g_{t,\gamma}(\tau) = \Big(1 - \frac{t}{\tau}\Big)^{k-\gamma} \tau^k (\tau - t)^{\gamma - k}, \quad \tau \in \Gamma \setminus \{t\}, \tag{4.5}$$

is a factorization of $g_{t,\gamma}$ in the space $X(\Gamma, w)$.

Proof. Obviously, $[\kappa_t(\theta)]$ is independent of θ. By the definition of $k = -[\kappa_t(\theta)]$,

$$-k < -\operatorname{Re}\gamma + \theta\big(\alpha_X + \alpha_t(-\operatorname{Im}\gamma)\big) + (1-\theta)\big(\beta_X + \beta_t(-\operatorname{Im}\gamma)\big) < 1 - k$$

for all $\theta \in [0,1]$. Hence,

$$0 < \mathrm{Re}\,(k-\gamma)+\theta\big(\alpha_X+\alpha_t(\mathrm{Im}\,(k-\gamma))\big) \leqslant \mathrm{Re}\,(k-\gamma)+(1-\theta)\big(\beta_X+\beta_t(\mathrm{Im}\,(k-\gamma))\big) < 1$$

for all $\theta \in [0,1]$. Consequently,

$$0 < \mathrm{Re}\,(k-\gamma) + \alpha_X + \alpha_t(\mathrm{Im}\,(k-\gamma)) \leqslant \mathrm{Re}\,(k-\gamma) + \beta_X + \beta_t(\mathrm{Im}\,(k-\gamma)) < 1.$$

By Lemma 4.6, the operator $\varphi_{t,k-\gamma}S\varphi_{t,\gamma-k}I$ is bounded in the space $X(\Gamma,w)$. In view of Theorem 2.1, the weight $\varphi_{t,k-\gamma}w$ belongs to the class $A_X(\Gamma)$. In that case $\varphi_{t,k-\gamma}(\tau) = |(\tau-t)^{k-\gamma}| \in X(\Gamma,w)$. By analogy with [4], Lemma 7.1 and with the help of Lemma 4.1, one can prove that $(\tau-t)^{k-\gamma} \in X_+(\Gamma,w)$, $(\tau-t)^{\gamma-k} \in X'_+(\Gamma,w^{-1})$, $(1-t/\tau)^{k-\gamma} \in X_-(\Gamma,w)$, $(1-t/\tau)^{\gamma-k} \in X'_-(\Gamma,w^{-1})$. Thus, (4.5) is a factorization of $g_{t,\gamma}$ in the space $X(\Gamma,w)$. ∎

4.4. FREDHOLM CRITERION. Further we will suppose the following. Let Γ be a Jordan Carleson curve and $X(\Gamma)$ be an r.i. space with nontrivial Boyd indices α_X, β_X. Suppose a weight $w : \Gamma \to [0,\infty]$ belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$. In that case, in view of Theorem 4.5, the operator S is bounded in the w.r.i. space $X(\Gamma,w)$. Suppose that this space is reflexive. Due to Theorem 2.1, $w \in A_X(\Gamma) \subset A_X(\Gamma,t)$ for every $t \in \Gamma$. Hence, the indicator functions α_t, β_t and α_t^*, β_t^* are well-defined for every $t \in \Gamma$. Let the indicator functions α_t, β_t and α_t^*, β_t^* satisfy the disintegration condition at every point $t \in \Gamma$.

THEOREM 4.8. *The operator R_a, where $a \in PC(\Gamma)$, is Fredholm in the space $X(\Gamma,w)$ if and only if $a \in GL^\infty(\Gamma)$ and*

$$\begin{aligned}\kappa_t(\theta) := -\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)} + \theta\left(\alpha_X + \alpha_t\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right)\right)\\ + (1-\theta)\left(\beta_X + \beta_t\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right)\right) \notin \mathbb{Z}\end{aligned} \tag{4.6}$$

for all $t \in \Gamma$ and all $\theta \in [0,1]$.

Proof. Let $a \in PC(\Gamma) \cap GL^\infty(\Gamma)$. For every $t \in \Gamma$, define $\gamma = \gamma_t \in \mathbb{C}$ by (4.2). In that case the function a is locally equivalent to the function $a_t = c_t g_{t,\gamma_t}$ at the point t, where $c_t \in \mathbb{C} \setminus \{0\}$ is some constant.

NECESSITY. If the operator R_a is Fredholm, then as in the proof of necessity from [22], Theorem 7.8 and taking into account Theorem 4.4, one can prove that

$$\kappa_t(\theta) = -\mathrm{Re}\,\gamma_t + \theta\alpha_t^*(-\mathrm{Im}\,\gamma_t) + (1-\theta)\beta_t^*(-\mathrm{Im}\,\gamma_t) \notin \mathbb{Z} \tag{4.7}$$

for every $\theta \in [0,1]$ and every $t \in \Gamma$. Besides, for every $t \in \Gamma$ there is an $l \in \mathbb{Z}$ such that $\varphi_{t,l-\gamma_t}w \in A_X(\Gamma)$. In that case $l - \gamma_t \in N_t$ (see (3.5)). Note that $\mathrm{Im}\,(l-\gamma_t) = -\mathrm{Im}\,\gamma_t$. Hence, from (4.7) and the disintegration condition of the indicator functions it follows that (4.6) holds for every $\theta \in [0,1]$.

SUFFICIENCY. From (4.2) and (4.6) it follows that

$$\kappa_t(\theta) = -\mathrm{Re}\,\gamma_t + \theta\big(\alpha_X + \alpha_t(-\mathrm{Im}\,\gamma_t)\big) + (1-\theta)\big(\beta_X + \beta_t(-\mathrm{Im}\,\gamma_t)\big) \notin \mathbb{Z}$$

for all $t \in \Gamma$ and all $\theta \in [0,1]$. By Theorem 4.7, the function g_{t,γ_t} admits a factorization in the space $X(\Gamma,w)$ for every $t \in \Gamma$. Clearly, the function a_t also admits a factorization for every $t \in \Gamma$. Due to Theorem 4.2, the operators R_{a_t}, where $t \in \Gamma$, are Fredholm in the space $X(\Gamma,w)$. By Theorem 4.3, the operator R_a is Fredholm too. ∎

4.5. LEAVES AND ESSENTIAL SPECTRUM. Following [4], Section 7.3, we describe the essential spectrum of the operator R_a. Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $\alpha(x) \leqslant \beta(x)$ for every $x \in \mathbb{R}$. Consider the set

$$Y(\alpha,\beta) := \big\{\gamma = x + \mathrm{i}y \in \mathbb{C} : \alpha(x) \leqslant y \leqslant \beta(x)\big\}.$$

Given $z_1, z_2 \in \mathbb{C}$, put

$$\mathcal{L}(z_1,z_2;\alpha,\beta) := \{z_1,z_2\} \cup \big\{\xi = M_{z_1,z_2}(\mathrm{e}^{2\pi\gamma}) : \gamma \in Y(\alpha,\beta)\big\},$$

where $M_{z_1,z_2}(\xi) := (z_2\xi - z_1)/(\xi - 1)$ is the Möbius transform. The set $\mathcal{L}(z_1,z_2;\alpha,\beta)$ is called the *leaf between z_1 and z_2*.

Fix $t \in \Gamma$ and consider the leaf generated by the indicator functions $\alpha_X + \alpha_t$ and $\beta_X + \beta_t$. From [4], Theorem 3.31 it follows that $Y(\alpha_X + \alpha_t, \beta_X + \beta_t)$ is a simply connected set which contains points with arbitrary real parts. Hence, the set $\big\{\mathrm{e}^{2\pi\gamma} : \gamma \in Y(\alpha_X + \alpha_t, \beta_X + \beta_t)\big\}$ is simply connected and contains points arbitrarily close to the origin and to infinity. The Möbius transform M_{z_1,z_2} maps 0 to z_1 and ∞ to z_2. Consequently, the leaf $\mathcal{L}(z_1,z_2;\alpha_X + \alpha_t, \beta_X + \beta_t)$ is a simply connected set containing z_1 and z_2.

For $a \in PC(\Gamma)$, denote by $\mathcal{R}(a)$ the essential range of a, that is, the set

$$\mathcal{R}(a) := \bigcup_{t\in\Gamma}\{a(t-0), a(t+0)\} = \bigcup_{t\in\Gamma\setminus J_a}\{a(t)\} \cup \bigcup_{t\in J_a}\{a(t-0), a(t+0)\}$$

where J_a is the set of all points of Γ at which a has a jump (as we know, this set is at most countable).

The essential spectrum of an operator A in a Banach space E is the set of all numbers $\lambda \in \mathbb{C}$ for which the operator $A - \lambda I$ is not Fredholm in E. The essential spectrum of the operator A is denoted by $\mathrm{sp}_{\mathrm{ess}}\, A$.

The main goal of this subsection is to reformulate the Fredholm criterion for R_a, where $a \in PC(\Gamma)$, which is contained in Theorem 4.8, in geometric language, that is, in terms of the essential range of a and leaves filled in between the endpoints of jumps.

THEOREM 4.9. *Suppose the following five conditions are fulfilled:*

(i) Γ *is a Jordan Carleson curve;*

(ii) $X(\Gamma)$ *is an r.i. space with nontrivial Boyd indices;*

(iii) *a weight w belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X}}(\Gamma)$ and $A_{\frac{1}{\beta_X}}(\Gamma)$;*

(iv) *the w.r.i. space $X(\Gamma, w)$ is reflexive;*

(v) *the indicator functions α_t, β_t and α_t^*, β_t^* satisfy the disintegration condition* (3.6) *at every point $t \in \Gamma$.*

Then the essential spectrum of the operator R_a, where $a \in PC(\Gamma)$, is given by

$$\mathrm{sp}_{\mathrm{ess}} R_a = \mathcal{R}(a) \cup \bigcup_{t\in J_a} \mathcal{L}(a(t-0), a(t+0); \alpha_X + \alpha_t, \beta_X + \beta_t) \cup \{1\}.$$

This theorem is proved by analogy with [4], Theorem 7.4.

Put $\sigma_t(x) := (\alpha_X + \alpha_t(x) + \beta_X + \beta_t(x))/2$ for $x \in \mathbb{R}$, and for arbitrary $z_1, z_2 \in \mathbb{C}$, consider the leaf $\mathcal{L}_t(z_1,z_2) := \mathcal{L}(z_1,z_2;\sigma_t,\sigma_t)$. Suppose $a \in PC(\Gamma)$. It is easy to see that the set

$$a^{\#} := \mathcal{R}(a) \cup \bigcup_{t\in J_a} \mathcal{L}_t(a(t-0), a(t+0))$$

is a closed, continuous, and naturally oriented curve.

Let γ be a closed continuous oriented curve which does not contain the origin. Denote by $\operatorname{wind}\gamma$ the winding number of γ about the origin. The index of a Fredholm operator A is denoted by $\operatorname{Ind} A$.

THEOREM 4.10. *Suppose the conditions* (i)–(v) *of Theorem* 4.9 *are fulfilled. If an operator R_a, where $a \in PC(\Gamma)$, is Fredholm, then $0 \notin a^{\#}$, and*

$$\operatorname{Ind} R_a = -\operatorname{wind} a^{\#}.$$

This theorem can be proved by analogy with [4], Theorem 7.14.

5. THE ALGEBRA OF SINGULAR INTEGRAL OPERATORS

5.1. SYMBOL CALCULUS FOR SINGULAR INTEGRAL OPERATORS. Let $X^n(\Gamma, w)$ stand for the direct sum of n copies of the reflexive w.r.i. space $X(\Gamma, w)$. We denote by $\mathcal{B} := \mathcal{B}(X^n(\Gamma, w))$ the Banach algebra of all bounded linear operators in $X^n(\Gamma, w)$ and by $\mathcal{K} := \mathcal{K}(X^n(\Gamma, w))$ the two-sided ideal of compact operators in $\mathcal{B}$. Let I be the identity operator and let the operator S be defined in $X^n(\Gamma, w)$ elementwise by the formula (2.2). Let $PC_n(\Gamma)$ denote the set of all $n \times n$ matrix functions with entries in $PC(\Gamma)$. Consider the smallest Banach subalgebra $\mathcal{U}$ of $\mathcal{B}$ containing the operator S and the operators of multiplication by piecewise continuous matrix-valued functions.

As in [18], Lemma 9.1, one can prove that $\mathcal{K}$ is the closed two-sided ideal of $\mathcal{U}$. Therefore, since we can calculate the essential spectrum of the operator R_a, where $a \in PC(\Gamma)$ (in the scalar case!), we can derive a symbol calculus for operators $A \in \mathcal{U}$ (in the matrix case!). The main industrial and now standard tools for obtaining this result are the Allan-Douglas local principle (see [4], Theorem 8.2) and the two projections theorem (see [11], [15], and [4], Sections 8.3 and 8.4). For the details of establishing a symbol calculus, see [4], Section 8.5 and [18].

THEOREM 5.1. *Suppose the conditions* (i)–(v) *of Theorem* 4.9 *are fulfilled. Define the "leaves bundle"*

$$\mathcal{M} := \mathcal{M}_{X(\Gamma,w)} := \bigcup_{t\in\Gamma} \big(\{t\} \times \mathcal{L}(0,1;\alpha_X + \alpha_t, \beta_X + \beta_t)\big).$$

Then

(a) *for each point $(t, \mu) \in \mathcal{M}$, the map*

$$\sigma_{t,\mu} : \{S\} \cup \{aI : a \in PC_n(\Gamma)\} \to \mathbb{C}^{2n\times 2n},$$

given by

$$\sigma_{t,\mu}(S) = \begin{pmatrix} E & \mathrm{O} \\ \mathrm{O} & -E \end{pmatrix}, \tag{5.1}$$

$$\sigma_{t,\mu}(aI) = \begin{pmatrix} a(t+0)\mu + a(t-0)(1-\mu) & (a(t+0)-a(t-0))\sqrt{\mu(1-\mu)} \\ (a(t+0)-a(t-0))\sqrt{\mu(1-\mu)} & a(t+0)(1-\mu)+a(t-0)\mu \end{pmatrix}, \tag{5.2}$$

where O *and E are the zero and identity $n \times n$ matrices, respectively, extends to a Banach algebra homomorphism $\sigma_{t,\mu} : \mathcal{U} \to \mathbb{C}^{2n\times 2n}$ with the property that $\sigma_{t,\mu}(K)$ is the zero $2n \times 2n$ matrix for every compact operator K;*

(b) *an operator* $A \in \mathcal{U}$ *is Fredholm in* $X^n(\Gamma, w)$ *if and only if*

$$\det \sigma_{t,\mu}(A) \neq 0 \quad \textit{for all } (t,\mu) \in \mathcal{M};$$

(c) *the quotient algebra* $\mathcal{U}/\mathcal{K}$ *is inverse closed in the Calkin algebra* $\mathcal{B}/\mathcal{K}$, *i.e., if an arbitrary element* $A+\mathcal{K} \in \mathcal{U}/\mathcal{K}$ *is invertible in* $\mathcal{B}/\mathcal{K}$, *then* $(A+\mathcal{K})^{-1} \in \mathcal{U}/\mathcal{K}$.

We remark that in (5.2) we understand by $\sqrt{\mu(1-\mu)}$ any (complex) number whose square is $\mu(1-\mu)$.

Theorem 5.1 was obtained by I. Gohberg and N. Krupnik in the case of Lebesgue spaces on Lyapunov curves with power weights ([13]) with the help of other methods. This theorem in the case of piecewise smooth curves and general Muckenhoupt weights was got in [11] and [16]. For Lebesgue spaces on arbitrary Carleson curves with general Muckenhoupt weights this theorem was established by A. Böttcher and Yu. I. Karlovich ([4], Chapter 8). For further generalizations to reflexive Orlicz spaces and reflexive r.i. spaces, see [18], [19], [22].

5.2. INDEX FORMULA. Unfortunately, the two projections theorem and the Allan-Douglas local principle do not allow us to calculate the index of operators from $\mathcal{U}$. In this subsection we obtain an index formula for an arbitrary operator $A \in \mathcal{U}$.

The matrix function $\mathcal{A}(t,\mu) = \sigma_{t,\mu}(A)$, $(t,\mu) \in \mathcal{M}$, is said to be the symbol of the operator $A \in \mathcal{U}$. We can write the symbol in the form

$$\mathcal{A}(t,\mu) = \begin{pmatrix} \mathcal{A}_{11}(t,\mu) & \mathcal{A}_{12}(t,\mu) \\ \mathcal{A}_{21}(t,\mu) & \mathcal{A}_{22}(t,\mu) \end{pmatrix}, \quad (t,\mu) \in \mathcal{M},$$

where $\mathcal{A}_{ij}(t,\mu)$ are $n \times n$ matrix functions.

In general, the family of homomorphisms $\sigma_{t,\mu}$ is not uniformly bounded with respect to $(t,\mu) \in \mathcal{M}$. But the functions $\det \mathcal{A}$, $\det \mathcal{A}_{ii}$ $(i = 1,2)$ have this property (see [21], Theorems 2 and 3).

THEOREM 5.2. *Suppose the conditions* (i)–(v) *of Theorem* 4.9 *are fulfilled. If an operator* $A \in \mathcal{U}$ *is Fredholm in* $X^n(\Gamma, w)$, *then the function*

$$A(t,\mu) := \frac{\det \mathcal{A}(t,\mu)}{\det \mathcal{A}_{22}(t,0) \det \mathcal{A}_{22}(t,1)}, \quad (t,\mu) \in \mathcal{M}$$

has the following properties:
(i) $A(t,\mu) \neq 0$ *for all* $(t,\mu) \in \mathcal{M}$;
(ii) $A(\,\cdot\,,0) \in PC(\Gamma)$;
(iii) *the set*

$$A_{\#} := \mathcal{R}(A(\,\cdot\,,0)) \cup \bigcup_{t \in \mathcal{J}_A} \{z = A(t,\mu) : \mu \in \mathcal{L}_t(0,1)\},$$

where $\mathcal{J}_A$ *is the set of all points* $t \in \Gamma$ *at which the function* $A(t,\cdot)$ *is not constant, is a closed, continuous, and naturally oriented curve, which does not contain the origin. In that case*

$$\operatorname{Ind} A = -\operatorname{wind} A_{\#}.$$

The proof is developed by analogy with [21] and [22], Section 8.2 in several steps using the scheme of [13]. As it is pointed out by Yu.I. Karlovich and E. Ramirez de Arellano ([24], p. 464), the index formulas in [4], Section 10.2 and

[22], Section 8.2 are not correct. We need to replace in [4], Section 10.2 and [22], Theorem 8.4 the set of discontinuities of the function $A(\,\cdot\,,t)$ by the greater set $\mathcal{J}_A$, because it is possible a situation when the function $A(\,\cdot\,,t)$ is continuous at some point t_0 but the non-constant function $A(t_0,\cdot\,)$ generates a nontrivial loop

$$\{z = A(t_0,\mu) : \mu \in \mathcal{L}_{t_0}(0,1)\}$$

with the common endpoint $A(t_0,0) = A(t_1,0)$. The origin may lie as inside as well as outside of the domain bounded by this loop. Hence the index of the operator A depends on such loops.

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