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TOEPLITZ OPERATORS ON DIRICHLET-TYPE SPACES

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Abstract. In this paper, variants of the classical Toeplitz operators on H^2 are studied. A characterization is obtained for the bounded, harmonic symbols giving rise to a bounded Toeplitz operator on a Dirichlet-type space. The relationship between the characterizing condition and multipliers of the holomorphic and harmonic Dirichlet spaces is examined.

Keywords: *Toeplitz operator, Dirichlet space, Dirichlet-type spaces.*

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1. INTRODUCTION

In this paper, we study operators of the type $f \mapsto P(\varphi f)$ on Dirichlet-type spaces $D(\mu)$ (see Definition 2.2), where φ is a function on $\mathbb{D}$ or $\partial\mathbb{D}$ and P is a projection. These operators are variants of the classical Toeplitz operators on H^2, and will be referred to as Toeplitz operators. The function φ is called the *symbol* of the operator $f \mapsto P(\varphi f)$, which will be denoted T_φ.

In Section 2, two kinds of Dirichlet-type spaces are defined, and some of their properties are given. In Section 3, one sort of Toeplitz operator is examined. A characterization is obtained for the bounded, harmonic symbols for which this operator is bounded on the Dirichlet space. The characterizing condition is compared with D. Stegenga's characterization of the multipliers of the Dirichlet space. In Section 4, another sort of Toeplitz operator is examined, and its boundedness on Dirichlet-type spaces is characterized. Connections with multipliers of Dirichlet-type spaces are obtained.

2. DIRICHLET-TYPE SPACES

DEFINITION 2.1. The *Bergman space* L^2_a is the subspace of holomorphic functions in $L^2(\mathbb{D})$ (with respect to normalized Lebesgue measure). The *Dirichlet space* D consists of those holomorphic functions f on $\mathbb{D}$ having $f' \in L^2_a$; the norm is given by $\|f\|^2_D = \|f\|^2_{H^2} + \|f'\|^2_{L^2_a}$. The quantity $\|f'\|^2_{L^2_a} = \int_{\mathbb{D}} |f'|^2 \,\mathrm{d}A = \sum n|\widehat{f}(n)|^2$ is called the *Dirichlet integral* of f, denoted $D(f)$. The formula for the Dirichlet integral in terms of the power-series coefficients of f makes it clear that $D \subset H^2$. There is also a formula, due to J. Douglas ([3]), in terms of integrals over $\partial\mathbb{D}$:

$$\int_{\mathbb{D}} |f'|^2 \,\mathrm{d}A = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \left| \frac{f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}t})}{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{\mathrm{i}t}} \right|^2 \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\theta}{2\pi}. \tag{2.1}$$

The inner integral is the *local Dirichlet integral* of f at $\mathrm{e}^{\mathrm{i}\theta}$, denoted $D_{\mathrm{e}^{\mathrm{i}\theta}}(f)$, and can be regarded as a function on $\partial\mathbb{D}$.

DEFINITION 2.2. Let μ be a finite, positive, Borel measure on $\partial\mathbb{D}$. The *Dirichlet-type space* $D(\mu)$ is the set of holomorphic functions on $\mathbb{D}$ having a local Dirichlet integral that is integrable with respect to μ. Equation (2.1) says that $D = D(\frac{\mathrm{d}\theta}{2\pi})$. The norm is given by $\|f\|^2_\mu = \|f\|^2_{H^2} + \int D_\lambda(f) \,\mathrm{d}\mu(\lambda)$. That $D(\mu) \subset H^2$ is shown in [5] (also see Corollary 2.7).

The properties of a Toeplitz operator can depend both on its symbol and on the projection P used in the definition of the operator. There are several possible projections that can be used to define Toeplitz operators on $D(\mu)$.

EXAMPLE 2.3. Let L^2_a be the Bergman space, a subspace of $L^2(\mathbb{D})$. Let P_B be the orthogonal projection of $L^2(\mathbb{D})$ onto L^2_a, known as the *Bergman projection*. It can be expressed as an integral operator, or in terms of reproducing kernels:

$$(P_B f)(z) = \int f(w) \frac{1}{(1 - z\overline{w})^2} \,\mathrm{d}A(w) = \langle f, k^B_z \rangle_{L^2(\mathbb{D})}, \tag{2.2}$$

where $\mathrm{d}A$ denotes normalized Lebesgue measure on $\mathbb{D}$.

If φ is a function on $\mathbb{D}$ such that $\varphi D(\mu) \subset L^2(\mathbb{D})$, then a Toeplitz operator T_φ can be defined on $D(\mu)$ by $T_\varphi f = P_B(\varphi f)$.

EXAMPLE 2.4. The Hardy space H^2 can be identified with a subspace of $L^2(\partial\mathbb{D})$, with radial limits transforming an H^2 function on $\mathbb{D}$ to its boundary function, and the Poisson integral doing the reverse. The orthogonal projection of $L^2(\partial\mathbb{D})$ onto $H^2(\mathbb{D})$ is known as the *Szegö projection*, and will be denoted P_H. Like P_B, the Szegö projection can be expressed as an integral operator or in terms of reproducing kernels:

$$(P_H f)(z) = \int f(\mathrm{e}^{\mathrm{i}\theta}) \frac{1}{1 - z\mathrm{e}^{-\mathrm{i}\theta}} \frac{\mathrm{d}\theta}{2\pi} = \langle f, k^{H^2}_z \rangle_{L^2(\partial\mathbb{D})}.$$

Since $D(\mu) \subset H^2$, every element of $D(\mu)$ has a boundary function defined almost everywhere on $\partial\mathbb{D}$. So if φ is a function on $\partial\mathbb{D}$ such that $\varphi D(\mu) \subset L^2(\partial\mathbb{D})$, a Toeplitz operator T_φ can be defined on $D(\mu)$ by $T_\varphi f = P_H(\varphi f)$.

Before proceeding with the next example, a harmonic analogue of $D(\mu)$ will be defined.

In the sequel, if ν is a measure on $\partial\mathbb{D}$, then $P\nu$ denotes the *Poisson integral* of ν, the integral with respect to ν of the Poisson kernel: $(P\nu)(z) = \int \frac{1-|z|^2}{|z-\lambda|^2}\,\mathrm{d}\nu(\lambda)$. Note that the Poisson kernel itself is the Poisson integral of the point mass δ_λ. If g is a function on $\partial\mathbb{D}$, then Pg denotes the Poisson integral of the measure $g\frac{\mathrm{d}\theta}{2\pi}$.

If $\lambda \in \partial\mathbb{D}$ and δ_λ denotes the point mass at λ, then from Definition 2.2 it follows that $f \in D(\delta_\lambda)$ iff f has a finite local Dirichlet integral at λ. The following criterion of S. Richter and C. Sundberg ([6]) for $D_\lambda(f)$ to be finite will be useful:

Proposition 2.5. *Let $\lambda \in \partial\mathbb{D}$, f a function on $\mathbb{D}$. Then $f \in D(\delta_\lambda)$ iff $f = \alpha + (z-\lambda)f_\lambda$ for some constant α and function $f_\lambda \in H^2$. If this is the case, then α is the radial limit $f(\lambda)$ of f at λ, and $D_\lambda(f) = \|f_\lambda\|_2^2$.*

Remark 2.6. It is shown in [6] that in fact, if $f \in D(\delta_\lambda)$ then $f(z) \to f(\lambda)$ as z approaches λ within any disc tangent to $\partial\mathbb{D}$ at λ. Also, P. Chernoff ([2]) showed that if $D_\lambda(f) < \infty$, then the Fourier series of f at λ converges to $f(\lambda)$.

Corollary 2.7. *$D(\mu) \subset H^2$.*

Proof. Let $f \in D(\mu)$. Since $\int D_\lambda(f)\,\mathrm{d}\mu(\lambda)$ is finite, there is at least one λ such that $f \in D(\delta_\lambda)$. For any such λ, by the proposition there are $\alpha \in \mathbb{C}$ and $f_\lambda \in H^2$ such that $f = \alpha + (z-\lambda)f_\lambda$. Therefore $f \in H^2$. ∎

The following analogue of the Douglas formula (2.1) for the Dirichlet integral will be used, and is proved by Richter and Sundberg in [6].

Proposition 2.8. *If $f \in H^2$, then*

$$\int_{\partial\mathbb{D}} D_\lambda(f)\,\mathrm{d}\mu(\lambda) = \int_{\mathbb{D}} |f'|^2 P\mu\,\mathrm{d}A. \tag{2.3}$$

Like the Douglas formula, equation (2.3) can be extended to harmonic functions.

Proposition 2.9. *Let f be a harmonic function on $\mathbb{D}$ of the form $f = f_+ + f_-$, where $f_+, \overline{f_-} \in D(\mu)$ and $f_-(0) = 0$. Then*

$$\int_{\partial\mathbb{D}} D_\lambda(f)\,\mathrm{d}\mu(\lambda) = \int_{\mathbb{D}} \left(\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2\right) P\mu\,\mathrm{d}A = \int_{\mathbb{D}} (|f_+'|^2 + |\overline{f_-}'|^2) P\mu\,\mathrm{d}A. \tag{2.4}$$

Proof. For $\lambda \in \partial\mathbb{D}$ and functions $g, h \in D(\delta_\lambda)$, define

$$D_\lambda(g,h) = \int_{\partial\mathbb{D}} \frac{g(\lambda) - g(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}} \overline{\left(\frac{h(\lambda) - h(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}}\right)} \frac{\mathrm{d}t}{2\pi}.$$

Then $D_\lambda(\cdot\,,\cdot)$ is a sesquilinear form, and $D_\lambda(g) = D_\lambda(g,g)$. Hence

$$D_\lambda(f) = D_\lambda(f_+ + f_-, f_+ + f_-) = D_\lambda(f_+) + 2\mathrm{Re}\, D_\lambda(f_+, f_-) + D_\lambda(f_-). \tag{2.5}$$

Since $D_\lambda(f_-) = D_\lambda(\overline{f_-})$, the proposition will follow by integrating (2.5) with respect to μ and applying Proposition 2.8, once it is shown that $D_\lambda(f_+, f_-) = 0$ for $[\mu]$-almost every λ.

Since $f_+, \overline{f_-} \in D(\mu)$, both belong to $D(\delta_\lambda)$ for $[\mu]$-almost every λ; fix such a λ. By Proposition 2.8, choose $g_+, g_- \in H^2$ such that $f_+ = f_+(\lambda) + (z - \lambda)g_+$ and $\overline{f_-} = \overline{f_-}(\lambda) + (z - \lambda)g_-$. Then

$$\begin{aligned} D_\lambda(f_+, f_-) &= \int_{\partial\mathbb{D}} \frac{f_+(\lambda) - f_+(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}} \overline{\left(\frac{f_-(\lambda) - f_-(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}}\right)} \frac{\mathrm{d}t}{2\pi} \\ &= \int_{\partial\mathbb{D}} \frac{f_+(\lambda) - f_+(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}} \frac{\overline{f_-}(\lambda) - \overline{f_-}(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}} \frac{\lambda - \mathrm{e}^{\mathrm{i}t}}{\overline{\lambda} - \mathrm{e}^{-\mathrm{i}t}} \frac{\mathrm{d}t}{2\pi} \\ &= \int_{\partial\mathbb{D}} g_+(\mathrm{e}^{\mathrm{i}t}) g_-(\mathrm{e}^{\mathrm{i}t})(-\lambda \mathrm{e}^{\mathrm{i}t}) \frac{\mathrm{d}t}{2\pi} = 0. \quad \blacksquare \end{aligned}$$

DEFINITION 2.10. The *harmonic Dirichlet-type space* $\mathcal{D}(\mu)$ is the set of functions $f \in L^2(\partial\mathbb{D})$ such that $D_\lambda(f)$ is integrable with respect to μ. For such an f, the harmonic extension $f(z) = (Pf)(z)$ to $\mathbb{D}$ satisfies (2.4); in the usual way, elements of $\mathcal{D}(\mu)$ can be regarded as functions on $\partial\mathbb{D}$ or as functions on $\mathbb{D}$. Define the norm by $\|f\|^2 = \int D_\lambda(f)\,\mathrm{d}\mu(\lambda) + \|f\|^2_{L^2(\partial\mathbb{D})}$.

PROPOSITION 2.11. *$\mathcal{D}(\mu)$ is a reproducing-kernel Hilbert space containing $D(\mu)$ as a closed subspace.*

Proof. Suppose $f \in \mathcal{D}(\mu)$; write $f = f_+ + f_-$, with $f_+, \overline{f_-} \in D(\mu)$ and $f_-(0) = 0$. If $w \in \mathbb{D}$, then by the existence of H^2 reproducing kernels,

$$\begin{aligned} |f(w)| = |f_+(w) + f_-(w)| &\leqslant |f_+(w)| + |\overline{f_-}(w)| \leqslant \|k_w^{H^2}\|_{H^2}(\|f_+\|_{H^2} + \|\overline{f_-}\|_{H^2}) \\ &\leqslant C\|f\|_{L^2(\partial\mathbb{D})} \leqslant C\|f\|_{\mathcal{D}(\mu)}. \end{aligned}$$

Thus, the functional of evaluation at w is bounded on $\mathcal{D}(\mu)$, as was to be proved.

Similarly, for $k \in \mathbb{N}$

$$|\widehat{f}(-k)| = |\widehat{f_-}(-k)| = |\widehat{\overline{f_-}}(k)| = |\overline{f_-}^{(k)}(0)|/k! \leqslant C\|\overline{f_-}\|_{H^2} \leqslant C\|f\|_{\mathcal{D}(\mu)}.$$

Therefore if $\{f_n\}$ is a sequence in $D(\mu)$ converging in $\mathcal{D}(\mu)$ to f, then $\widehat{f}(-k) = 0$ for all $k \in \mathbb{N}$. Thus $D(\mu)$ is closed in $\mathcal{D}(\mu)$. ∎

EXAMPLE 2.12. If φ is a function on $\partial\mathbb{D}$ such that $\varphi D(\mu) \subset \mathcal{D}(\mu)$, then a Toeplitz operator T_φ can be defined on $D(\mu)$ by $T_\varphi f = P_\mu(\varphi f)$, where P_μ is the orthogonal projection of $\mathcal{D}(\mu)$ onto $D(\mu)$.

There are advantages to using each of the projections in Examples 2.3, 2.4, and 2.12. The Bergman projection can be used for the largest collection of symbols, as the requirement that $\varphi D(\mu) \subset L^2(\mathbb{D})$ is the weakest requirement among the three. Using the Szegö projection has the advantage of giving rise to the best-understood sort of Toeplitz operator. The theory of Toeplitz operators is most often studied in settings where the range of the projection is the domain of the operator; such is the case if $P = P_\mu$.

See R. Rochberg and Z. Wu ([7]) for results concerning a type of Toeplitz operator on D different from that defined in Section 1.

3. BERGMAN TOEPLITZ OPERATORS ON D

The problem to be studied in this section is to determine the symbols φ for which the Toeplitz operator $T_\varphi f = P_B(\varphi f)$ is bounded on D. It will be assumed that φ is a bounded, harmonic function on $\mathbb{D}$.

The Bergman projection is one of a family of projections of $L^2(\mathbb{D})$ onto L^2_a. For $\alpha > -1$, define the operator P_α by:

$$(P_\alpha f)(z) = (\alpha + 1) \int \frac{(1 - |w|^2)^\alpha}{(1 - z\overline{w})^{\alpha+2}} f(w)\, \mathrm{d}A(w).$$

Clearly $P_0 = P_B$. If $1 \leqslant p < \infty$ and $p(\alpha + 1) > 1$, then P_α is bounded on $L^p(\mathbb{D})$ and fixes the holomorphic functions in $L^p(\mathbb{D})$, as shown in [10], Section 4.2.

The main result of this section hinges on the following lemma.

LEMMA 3.1. *For $f \in D$, $(T_\varphi f)' = \frac{\partial \varphi}{\partial z} f + P_1(\varphi f')$.*

Note that if $\frac{\partial\varphi}{\partial z} f \in L^2(\mathbb{D})$, then the right side of the equation is $P_1\big(\frac{\partial}{\partial z}(\varphi f)\big)$. Thus the lemma says that in a restricted sense, differentiation intertwines P_B and P_1.

Proof. First, the lemma will be verified in the case of $\varphi(z) = \overline{z}^m$ and $f(z) = z^n$:

$$(T_\varphi f)(z) = P_B(\varphi f)(z) = \int \frac{\overline{w}^m w^n}{(1 - z\overline{w})^2}\, \mathrm{d}A(w) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{r^m \mathrm{e}^{-\mathrm{i}m\theta} r^n \mathrm{e}^{\mathrm{i}n\theta}}{(1 - zr\mathrm{e}^{-\mathrm{i}\theta})^2} r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \frac{1}{\pi} \int_0^1 r^{m+n+1} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}(n-m+2)\theta}}{(\mathrm{e}^{\mathrm{i}\theta} - rz)^2}\, \mathrm{d}\theta\, \mathrm{d}r = \frac{1}{\pi \mathrm{i}} \int_0^1 r^{m+n+1} \int_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta - rz)^2}\, \mathrm{d}\zeta\, \mathrm{d}r.$$

A residue calculation shows that the contour integral $\int_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta - rz)^2}\, \mathrm{d}\zeta$ is zero if $n - m + 1 \leqslant 0$, and is otherwise $2\pi\mathrm{i}(n - m + 1)(rz)^{n-m}$. Hence

$$(T_\varphi f)(z) = 2(n - m + 1) z^{n-m} \int r^{2n+1}\, \mathrm{d}r = \frac{n - m + 1}{n + 1} z^{n-m} \tag{3.1}$$

if $n - m \geqslant 0$ and zero otherwise.

The derivative of $T_\varphi f(z)$ is to be compared with:

$$P_1(\varphi f')(z) = 2 \int \frac{1 - |w|^2}{(1 - z\overline{w})^3} n\overline{w}^m w^{n-1}\, \mathrm{d}A(w)$$

$$= \frac{2n}{\pi \mathrm{i}} \int_0^1 (1 - r^2) r^{m+n} \int_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta - rz)^3}\, \mathrm{d}\zeta\, \mathrm{d}r.$$

Since $\int\limits_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta - rz)^3}\,\mathrm{d}\zeta = \pi\mathrm{i}(n-m+1)(n-m)(rz)^{n-m-1}$ if $n-m \geqslant 1$ and is zero otherwise,

$$\begin{aligned} P_1(\varphi f')(z) &= 2n(n-m+1)(n-m)z^{n-m-1}\int (r^{2n-1} - r^{2n+1})\,\mathrm{d}r \\ &= \frac{(n-m+1)(n-m)}{n+1} z^{n-m-1} \end{aligned}$$

for $n-m \geqslant 1$ and is otherwise zero. Comparing this with the derivative of the right side of (3.1), we see that the lemma holds in this case.

Now let φ be any bounded, harmonic function on $\mathbb{D}$, and f any element of D. Define φ_+ by $\varphi_+(z) = \sum\limits_{n=0}^{\infty} \widehat{\varphi}(n)z^n$; let $\varphi_- = \varphi - \varphi_+$. Both φ_+ and φ_- belong to $L^2(\mathbb{D})$, but they need not be bounded functions.

Since φ is bounded and $f \in D \subset L^2$, the sum $\varphi \sum \widehat{f}(n)z^n$ converges in L^2 norm. Then since P_B is bounded on L^2,

$$\begin{aligned} P_B(\varphi f) &= \sum \widehat{f}(n)P_B(\varphi z^n) = \sum \widehat{f}(n)(\varphi_+ z^n + P_B(\varphi_- z^n)) \\ &= \varphi_+ f + \sum_{n=0}^{\infty} \widehat{f}(n) \sum_{m=1}^{\infty} \widehat{\varphi}(-m)P_B(\overline{z}^m z^n). \end{aligned}$$

Since φ is harmonic, $\varphi_+' = \frac{\partial\varphi}{\partial z}$; hence

$$P_B(\varphi f)' = \frac{\partial\varphi}{\partial z} f + \varphi_+ f' + \sum_{n=0}^{\infty} \widehat{f}(n) \sum_{m=1}^{\infty} \widehat{\varphi}(-m)P_B(\overline{z}^m z^n)'. \tag{3.2}$$

Similarly,

$$\begin{aligned} P_1(\varphi f') &= \sum n\widehat{f}(n)P_1(z^{n-1}\varphi) \\ &= \sum_{n=1}^{\infty} n\widehat{f}(n)(z^{n-1}\varphi_+ + \sum_{m=1}^{\infty} \widehat{\varphi}(-m)P_1(\overline{z}^m z^{n-1})) \\ &= \varphi_+ f' + \sum_{n=1}^{\infty} \widehat{f}(n) \sum_{m=1}^{\infty} \widehat{\varphi}(-m)P_1(n\overline{z}^m z^{n-1}). \end{aligned} \tag{3.3}$$

Since $P_B(\overline{z}^m z^n)' = P_1(n\overline{z}^m z^{n-1})$ for each m and n, the lemma follows by comparing (3.3) with (3.2), and observing that the $n = 0$ term of the sum in (3.2) is zero, since $P_B(\overline{z}^m) = 0$ for all $m \geqslant 1$. ∎

THEOREM 3.2. *Let φ be a bounded, harmonic function on $\mathbb{D}$. Then the Toeplitz operator T_φ is bounded on D iff*

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2\,\mathrm{d}A \leqslant C\|f\|_D^2 \tag{3.4}$$

for all $f \in D$, for some constant C not depending on f.

Proof. Suppose that (3.4) holds. Then since P_1 is bounded on L^2 and φ is a bounded function,

$$D(T_\varphi f) = \int |(T_\varphi f)'|^2 \,\mathrm{d}A = \int \left|\frac{\partial\varphi}{\partial z} f + P_1(\varphi f')\right|^2 \mathrm{d}A$$
$$\leqslant 2\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2 \,\mathrm{d}A + 2\|P_1\|^2\|\varphi\|_\infty^2 \|f'\|_{L^2}^2$$
$$\leqslant 2(C + \|P_1\|^2\|\varphi\|_\infty^2)\|f\|_D^2.$$

Also,

$$|(T_\varphi f)(0)| = \left|\int \varphi f \,\mathrm{d}A\right| \leqslant \|\varphi\|_\infty \|f\|_{L^2} \leqslant \|\varphi\|_\infty\|f\|_D.$$

Since

$$\|T_\varphi f\|_{H^2}^2 \leqslant |(T_\varphi f)(0)|^2 + D(T_\varphi f),$$

it follows that T_φ is bounded on D.

Conversely, suppose that T_φ is bounded. Then by the lemma,

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2 \,\mathrm{d}A \leqslant 2\|(T_\varphi f)'\|_2^2 + 2\|P_1(\varphi f')\|_2^2 \leqslant 2\|T_\varphi f\|_D^2 + 2\|P_1\|^2\|\varphi\|_\infty^2\|f'\|_2^2$$
$$\leqslant 2\|T_\varphi\|^2\|f\|_D^2 + 2\|P_1\|^2\|\varphi\|_\infty^2\|f\|_D^2 = C\|f\|_D^2. \quad ∎$$

The condition of Theorem 3.2 is equivalent to that of $\frac{\partial\varphi}{\partial z}$ being a multiplier of D into L_a^2; that is, $\frac{\partial\varphi}{\partial z}D \subset L_a^2$. The condition also says that $\left|\frac{\partial\varphi}{\partial z}\right|^2 \mathrm{d}A$ is a *D-Carleson measure* on $\mathbb{D}$. Compare with the following theorem of D. Stegenga ([9]):

THEOREM 3.3. *A function g is a multiplier of D (into itself) iff $g \in H^\infty$ and $|g'|^2\,\mathrm{d}A$ is a D-Carleson measure.*

Stegenga also gives a geometric characterization of D-Carleson measures in [9].

Since $\frac{\partial\varphi}{\partial z} = \varphi_+'$, from Theorems 3.2 and 3.3 it follows that if T_φ is bounded on D and φ_+ is a bounded function, then φ_+ is a multiplier of D. However, it is possible for T_φ to be bounded without φ_+ being bounded:

EXAMPLE 3.4. Define the function g on $\mathbb{D}$ by $g(z) = \sum \frac{z^n}{n \log n \log\log n}$. Since $\sum n|\widehat{g}(n)|^2$ is finite, $g \in D$. Since $D \subset$ BMOA, the space of analytic functions having bounded mean oscillation on $\partial\mathbb{D}$ (see [8]), it follows from Fefferman's Theorem that we can choose a bounded, harmonic function φ such that $g = P_B\varphi = \varphi_+$. Since g is unbounded, φ_+ is not a multiplier of D. However, by a result of S. Axler and A. Shields ([1]), g' is a multiplier of D into L_a^2. Therefore T_φ is bounded on D.

4. HARDY TOEPLITZ OPERATORS ON $D(\mu)$

Let μ be a positive, finite Borel measure on $\partial\mathbb{D}$. In this section, the symbols $\varphi \in L^\infty(\partial\mathbb{D})$ for which the Toeplitz operator $T_\varphi f = P_H(\varphi f)$ is bounded on $D(\mu)$ will be determined.

Remark 4.1. Recall that $\|f\|_\mu^2 = \|f\|_2^2 + \int D_\lambda(f)\,\mathrm{d}\mu(\lambda)$. Since the projection P_H has norm one as an operator on $L^2(\partial\mathbb{D})$,

$$\|T_\varphi f\|_2 = \|P_H(\varphi f)\|_2 \leqslant \|\varphi f\|_2 \leqslant \|\varphi\|_{L^\infty(\partial\mathbb{D})}\|f\|_2 \leqslant \|\varphi\|_\infty \|f\|_\mu.$$

Therefore T_φ is bounded on $D(\mu)$ iff $\int D_\lambda(T_\varphi f)\,\mathrm{d}\mu(\lambda) \leqslant C\|f\|_\mu^2$ for $f \in D(\mu)$ and some C not depending on f.

Fix $f \in D(\mu)$. Then $D_\lambda(f) < \infty$ for $[\mu]$-almost every $\lambda \in \partial\mathbb{D}$. For each such λ define $f_\lambda \in H^2$ as in Proposition 2.5.

Lemma 4.2. $\int D_\lambda(T_\varphi((z-\lambda)f_\lambda))\,\mathrm{d}\mu(\lambda) \leqslant \|\varphi\|_\infty^2 \|f\|_\mu^2$.

Proof. The result hinges on a commutation relation obtained by following $T_\varphi((z-\lambda)f_\lambda)$ by $T_{\overline{z}}$, and using composition properties of H^2 Toeplitz operators:

$$T_{\overline{z}}T_\varphi((z-\lambda)f_\lambda) = T_{\overline{z}}T_\varphi T_{z-\lambda}f_\lambda = T_{\overline{z}\varphi(z-\lambda)}f_\lambda = T_{(1-\overline{z}\lambda)\varphi}f_\lambda = T_{\overline{z}}(z-\lambda)T_\varphi f_\lambda.$$

Subtracting the end from the beginning, we see that if $g = T_\varphi((z-\lambda)f_\lambda) - (z-\lambda)T_\varphi f_\lambda$, then $T_{\overline{z}}g = 0$. Since

$$T_{\overline{z}}g = P_H\left(\mathrm{e}^{-\mathrm{i}\theta}\sum_{n=0}^{\infty}\widehat{g}(n)\mathrm{e}^{\mathrm{i}n\theta}\right) = P_H\left(\sum_{n=-1}^{\infty}\widehat{g}(n+1)\mathrm{e}^{\mathrm{i}n\theta}\right) = \sum_{n=0}^{\infty}\widehat{g}(n+1)\mathrm{e}^{\mathrm{i}n\theta} = 0,$$

it follows that $\widehat{g}(n+1) = 0$ for all $n \geqslant 0$. Thus g is constant, say with constant value α. Hence

$$T_\varphi((z-\lambda)f_\lambda) = \alpha + (z-\lambda)T_\varphi f_\lambda.$$

Then by Proposition 2.5, $T_\varphi((z-\lambda)f_\lambda) \in D(\delta_\lambda)$, and

$$D_\lambda(T_\varphi((z-\lambda)f_\lambda)) = \|T_\varphi f_\lambda\|_2^2 \leqslant \|\varphi\|_\infty^2\|f_\lambda\|_2^2 = \|\varphi\|_\infty^2 D_\lambda(f).$$

Therefore

$$\int D_\lambda(T_\varphi((z-\lambda)f_\lambda))\,\mathrm{d}\mu(\lambda) \leqslant \|\varphi\|_\infty^2 \int D_\lambda(f)\,\mathrm{d}\mu(\lambda) \leqslant \|\varphi\|_\infty^2\|f\|_\mu^2. \quad \blacksquare$$

Remark 4.3. For each $\lambda \in \partial\mathbb{D}$, $D_\lambda(\cdot)^{1/2}$ is a seminorm on $D(\delta_\lambda)$, and hence satisfies the triangle inequality. Thus it follows from the lemma and the previous remark that T_φ is bounded on $D(\mu)$ iff $\int D_\lambda(T_\varphi(f(\lambda)))\,\mathrm{d}\mu(\lambda) \leqslant C\|f\|_\mu^2$, for some C not depending on f.

Theorem 4.7. *Let $\varphi \in L^\infty(\partial\mathbb{D})$. Then T_φ is bounded on $D(\mu)$ iff*

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 P(|f|^2\mu)\,\mathrm{d}A \leqslant C\|f\|_\mu^2,$$

for $f \in D(\mu)$ and some constant C not depending on f.

If φ satisfies the condition of the theorem, the measure $\left|\frac{\partial\varphi}{\partial z}\right|^2 \mathrm{d}A$ will be called a *μ-Carleson measure.*

Proof. Following the previous remark, we fix $\lambda \in \partial\mathbb{D}$ and calculate the local Dirichlet integral at λ of $T_\varphi(f(\lambda))$:

$$\begin{aligned} D_\lambda(T_\varphi(f(\lambda))) &= \int D_\zeta(T_\varphi(f(\lambda)))\,\mathrm{d}\delta_\lambda(\zeta) = \int |(P_H(\varphi f(\lambda)))'|^2 P\delta_\lambda\,\mathrm{d}A \\ &= |f(\lambda)|^2 \int \left|\frac{\partial\varphi}{\partial z}\right|^2 P\delta_\lambda\,\mathrm{d}A, \end{aligned}$$

where the last occurrence of φ denotes the harmonic extension of φ to $\mathbb{D}$.

Integrating with respect to μ gives

$$\begin{aligned} \int D_\lambda(T_\varphi(f(\lambda)))\,\mathrm{d}\mu(\lambda) &= \int |f(\lambda)|^2 \int \left|\frac{\partial\varphi}{\partial z}\right|^2 \frac{1-|z|^2}{|z-\lambda|^2}\,\mathrm{d}A(z)\,\mathrm{d}\mu(\lambda) \\ &= \int \left|\frac{\partial\varphi}{\partial z}\right|^2 \int |f(\lambda)|^2 \frac{1-|z|^2}{|z-\lambda|^2}\,\mathrm{d}\mu(\lambda)\,\mathrm{d}A(z) \\ &= \int \left|\frac{\partial\varphi}{\partial z}\right|^2 P(|f|^2\mu)\,\mathrm{d}A. \end{aligned}$$

The theorem now follows from the previous remark. ∎

In the case of the Dirichlet space $D = D(\frac{\mathrm{d}\theta}{2\pi})$, the theorem says that the Hardy Toeplitz operator T_φ is bounded on D iff

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 P(|f|^2)\,\mathrm{d}A \leqslant C\|f\|_D^2. \tag{4.1}$$

Compare this with Theorem 3.2, which says that the Bergman Toeplitz operator T_φ is bounded on D iff

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2\,\mathrm{d}A \leqslant C\|f\|_D^2. \tag{4.2}$$

However,

$$\|\,|f|^2 - P(|f|^2)\|_\infty \leqslant C_1\|f\|_{\mathrm{BMO}}^2 \leqslant C_2\|f\|_D^2,$$

the first inequality being due to A. Garsia (see [4], p. 221), the second to Stegenga ([8]). Therefore the two conditions (4.1) and (4.2) are equivalent.

Stegenga's Theorem 3.3 characterizing the multipliers of D can be generalized to the harmonic Dirichlet-type space $\mathcal{D}(\mu)$:

THEOREM 4.8. *A bounded function φ on $\partial\mathbb{D}$ is a multiplier of $\mathcal{D}(\mu)$ iff $\left|\frac{\partial\varphi}{\partial z}\right|^2 \mathrm{d}A$ and $\left|\frac{\partial\varphi}{\partial \bar z}\right|^2 \mathrm{d}A$ are μ-Carleson measures.*

Proof. Suppose that φ is a multiplier of $\mathcal{D}(\mu)$. Since $\|\cdot\|_{L^2(\partial\mathbb{D})} \leqslant \|\cdot\|_{\mathcal{D}(\mu)}$, norm convergence of a sequence in $\mathcal{D}(\mu)$ implies almost-everywhere pointwise convergence on $\partial\mathbb{D}$ of a subsequence. It then follows from the closed-graph theorem that the operator M_φ of multiplication by φ is bounded on $\mathcal{D}(\mu)$.

Let $f \in \mathcal{D}(\mu)$. Then

$$\frac{\varphi(\lambda)f(\lambda) - \varphi(\mathrm{e}^{\mathrm{i}t})f(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}} = f(\lambda)\frac{\varphi(\lambda) - \varphi(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}} + \varphi(\mathrm{e}^{\mathrm{i}t})\frac{f(\lambda) - f(\mathrm{e}^{\mathrm{i}t})}{\lambda - \mathrm{e}^{\mathrm{i}t}}. \tag{4.3}$$

Hence

$$\int |f(\lambda)|^2 D_\lambda(\varphi)\,\mathrm{d}\mu(\lambda) \leqslant 2\|\varphi\|_\infty^2 \int D_\lambda(f)\,\mathrm{d}\mu(\lambda) + 2\int D_\lambda(\varphi f)\,\mathrm{d}\mu(\lambda)$$
$$\leqslant 2(\|\varphi\|_\infty^2 + \|M_\varphi\|^2)\|f\|_{\mathcal{D}(\mu)}^2.$$

Since $\varphi = \varphi \cdot 1 \in \mathcal{D}(\mu)$, by Proposition 2.9

$$\int |f(\lambda)|^2 D_\lambda(\varphi)\,\mathrm{d}\mu(\lambda) = \int |f(\lambda)|^2 \int_{\mathbb{D}} \left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial\bar z}\right|^2\right) P\delta_\lambda\,\mathrm{d}A(z)\,\mathrm{d}\mu(\lambda)$$
$$= \int \left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial\bar z}\right|^2\right) P(|f|^2\mu)\,\mathrm{d}A.$$

Therefore $\left|\frac{\partial\varphi}{\partial z}\right|^2 \mathrm{d}A$ and $\left|\frac{\partial\varphi}{\partial\bar z}\right|^2 \mathrm{d}A$ are μ-Carleson measures.

Conversely, suppose that $\left|\frac{\partial\varphi}{\partial z}\right|^2 \mathrm{d}A$ and $\left|\frac{\partial\varphi}{\partial\bar z}\right|^2 \mathrm{d}A$ are μ-Carleson measures. Since $\frac{\partial\varphi}{\partial z} = \varphi_+'$ and $\overline{\frac{\partial\varphi}{\partial\bar z}} = \overline{\varphi_-}'$, applying the μ-Carleson condition with $f = 1$ gives that $\varphi_+, \overline{\varphi_-} \in D(\mu)$. Thus $\varphi \in \mathcal{D}(\mu)$. Then by (4.3) and Proposition 2.9,

$$\int D_\lambda(\varphi f)\,\mathrm{d}\mu(\lambda) \leqslant 2\int |f(\lambda)|^2 D_\lambda(\varphi)\,\mathrm{d}\mu(\lambda) + 2\|\varphi\|_\infty^2 \int D_\lambda(f)\,\mathrm{d}\mu(\lambda)$$
$$= \int \left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial\bar z}\right|^2\right) P(|f|^2\mu)\,\mathrm{d}A + 2\|\varphi\|_\infty^2 \int D_\lambda(f)\,\mathrm{d}\mu(\lambda)$$
$$\leqslant C\|f\|_{\mathcal{D}(\mu)}^2.$$

Therefore φ is a multiplier of $\mathcal{D}(\mu)$. ∎

COROLLARY 4.9. *A holomorphic function φ on $\mathbb{D}$ is a multiplier of $D(\mu)$ iff φ is bounded and $|\varphi'|^2\,\mathrm{d}A$ is a μ-Carleson measure.*

Proof. Suppose φ is a multiplier of $D(\mu)$. That $|\varphi'|^2\,\mathrm{d}A$ is a μ-Carleson measure follows as in the proof of the theorem, with $\mathcal{D}(\mu)$ replaced with $D(\mu)$, and noting that $\frac{\partial\varphi}{\partial\bar z} = 0$. That φ is bounded follows from the existence of reproducing kernels in $D(\mu)$: as above, M_φ is bounded on $D(\mu)$. Then

$$|\varphi(w)|\,\|k_w\|^2 = |\varphi(w)k_w(w)| = |\langle \varphi k_w, k_w\rangle| \leqslant \|\varphi k_w\|\,\|k_w\| \leqslant \|M_\varphi\|\,\|k_w\|^2;$$

thus $|\varphi|$ is bounded by $\|M_\varphi\|$ on $\mathbb{D}$.

If φ is bounded and $|\varphi'|^2\,\mathrm{d}A$ is a μ-Carleson measure, then since $\frac{\partial\varphi}{\partial\bar z} = 0$ and $\varphi' = \frac{\partial\varphi}{\partial z}$, the theorem gives that φ is a multiplier of $\mathcal{D}(\mu)$. Since φ is holomorphic, φ is a multiplier of $D(\mu)$. ∎

The following connection between bounded Toeplitz operators and multipliers of $\mathcal{D}(\mu)$ is an immediate consequence of Theorems 4.7 and 4.8.

COROLLARY 4.10. *A function $\varphi \in L^\infty(\partial\mathbb{D})$ is a multiplier of $\mathcal{D}(\mu)$ iff T_φ and $T_{\overline{\varphi}}$ are bounded on $D(\mu)$.*

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