J. OPERATOR THEORY
48(2002), 15–40

OPERATORS REPRESENTABLE AS MULTIPLICATION-CONDITIONAL EXPECTATION OPERATORS

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Communicated by Şerban Strătilă

ABSTRACT. In this paper a unified approach is presented to the study of some classes of operators, such as kernel operators and partial integral operators, between ideals of measurable functions. In particular it is shown that if the underlying measure spaces are non-atomic, then the kernel operators and partial integral operators are mutually disjoint, and these operators are disjoint to all weighted composition operators. Moreover, if the ideals of measurable functions are Banach function spaces satisfying appropriate conditions on the norms, then the partial integral operators are disjoint to all positive compact operators.

KEYWORDS: *Conditional expectation, kernel operator, partial integral operator, Riesz space.*

MSC (2000): 47B38, 47B65.

1. INTRODUCTION

In recent years several authors have studied the class of so-called partial integral operators (see e.g., [2], [4], [3], [8], [11] and the forthcoming monograph [5]). These operators arise in some areas of analysis and applications (see the references in the above mentioned papers). Such partial integral operators act between spaces of functions and are of the form

$$(Tf)(x_1, x_2) = \int_{X_1} k(x_1, x_2, z) f(z, x_2)\, \mathrm{d}\mu_1(z),$$

where f is a scalar measurable function on a product space $X_1 \times X_2$ and the kernel k is defined on the product space $X_1 \times X_2 \times X_1$. For precise definitions and other types of partial integral operators see Example 5.2 in the present paper. In appearance such operators bear resemblance to the well-known kernel operators (integral operators), which are of the form

$$(Kf)(x) = \int_Y k(x,y)f(y)\,\mathrm{d}\nu(y).$$

It turns out that on the one hand these classes of operators have a number of properties in common (see e.g. Corollary 5.4), and on the other hand they behave quite differently (see Corollaries 6.8 and 6.12). In this paper we propose the common framework of so-called multiplication-conditional expectation (MCE-)representable operators (see Definition 4.1) to study these classes of operators, including some other types of operators (such as weighted composition operators) as well. The main results in the present paper deal with the structure of the spaces of such operators (e.g. majorization results) and with the mutual relation between several classes of operators (e.g. Corollary 6.8).

The natural domain and range spaces of the above mentioned operators are ideals of measurable functions, in particular Banach function spaces (such as L_p-spaces, Orlicz spaces, etc.). This will be the setting in which we will study these operators. The concepts and theory of vector lattices (Riesz spaces) are very useful for this and we will use it extensively.

In Section 2 we fix some notation and gather some properties of Riesz homomorphisms and conditional expectation operators which will be used in the sequel. In Section 3 we study the properties so-called multiplication-conditional expectation (MCE) operators. In particular it will be shown that such operators form a band in the space of all order bounded operators. These results provide a basic technique from which the order properties of kernel operators and partial integral operators will be derived. In Section 4 we consider operators factorizing through MCE-operators and Riesz homomorphisms (in an appropriate way). We call such operators MCE-representable. It will be shown that this class of operators includes a number of interesting special cases such as kernel operators, partial integral operators and Riesz homomorphisms. This provides a common framework for the study of these operators and their mutual relations. The kernel operators and partial integral operators are special cases of τ-kernel operators, which are discussed in Section 5. These τ-kernel operators are of the form

$$(Kf)(x) = \int_Z k(x,z)f(\tau(x,z))\,\mathrm{d}\lambda(z),$$

where $\tau : X \times Z \to Y$ is a measurable null-preserving mapping. For a fixed map τ such operators form a band in the space of all order bounded operators between two ideals of measurable functions (see Theorem 5.3 for the details).

In the final Section 6 we obtain a general theorem providing sufficient conditions for the disjointness of two classes of MCE-representable operators. This result has a number of consequences. In particular it will follow that, assuming that the underlying measure spaces are non-atomic, the kernel operators and partial integral operators are disjoint, that the different types of partial integral operators are mutually disjoint, and that all these operators are disjoint to all Riesz homomorphisms. Moreover it is shown (see Example 6.10) that on an L_p-space over a non-atomic separable space, there exist uncountably many disjoint bands of partial integral operators.

2. PRELIMINARIES

We assume the reader to be familiar with the basic concepts of the theory of vector lattices (Riesz spaces) and operators on Riesz spaces. For unexplained terminology and notations we refer to the books [1], [13], [14] and [19]. Let E be a Riesz space. An ideal $L \subset E$ is a linear subspace with the property that $|g| \leqslant |f|$ with $f \in L$ and $g \in E$ implies $g \in L$. The ideal B is called a *band in* E if it follows from $0 \leqslant f_\alpha \uparrow f \in E$, $f_\alpha \in B$, that $f \in B$. We denote the set of order bounded linear operators from the Riesz space E into the Riesz space F by $\mathfrak{L}_{\mathrm{b}}(E, F)$. If F is Dedekind complete, $\mathfrak{L}_{\mathrm{b}}(E, F)$ is a Dedekind complete Riesz space. All Riesz spaces considered in this paper are Dedekind complete. The band of order continuous linear operators is denoted by $\mathfrak{L}_{\mathrm{n}}(E, F)$. The kernel and range of a linear operator $T : E \to F$ will be denoted by $\ker(T)$ and $\operatorname{ran}(T)$ respectively. The null ideal N_T of $T \in \mathfrak{L}_{\mathrm{b}}(E, F)$ is defined by $N_T = \{x \in E : |T||x| = 0\}$. The disjoint complement of N_T in E is called the *carrier of* T and is denoted by C_T. We note that N_T is a band if $T \in \mathfrak{L}_{\mathrm{n}}(E, F)$, and then $E = N_T \oplus C_T$.

We shall mainly be concerned with ideals of measurable functions. Let (Y, Λ, ν) be a σ-finite measure space. The Riesz space of all real valued Λ-measurable functions on Y, with the usual identification of ν-a.e. equal functions, is denoted by $L^0(Y, \Lambda, \nu)$. By $M^+(Y, \Lambda, \nu)$ we denote the collection of all (equivalence classes of) Λ-measurable functions into $[0, \infty]$. Let L be an ideal in $L^0(Y, \Lambda, \nu)$. The set $Z \in \Lambda$ is called an *L-zero set* if every $f \in L$ vanishes ν-a.e on Z. There exists (modulo ν-null sets) a maximal L-zero set Z_1 in Λ and the set $Y_1 = Y \setminus Z_1$ is called the *carrier* of the ideal L. There exists a sequence $A_n \uparrow Y_1$ in Λ such that $\nu(A_n) < \infty$ and $\mathbb{1}_{A_n} \in L$ for all $n \in \mathbb{N}$ (see Theorem 86.2 from [19]). Clearly, the carrier of L is equal to Y, if and only if L is order dense in $L^0(Y, \Lambda, \nu)$.

We now insert some remarks concerning Riesz homomorphisms and Boolean homomorphisms, which will be used in the sequel.

Proposition 2.1. *Let (Y, Λ, ν) and (X, Σ, μ) be σ-finite measure spaces. Suppose that φ is a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$. Then:*

(i) *φ is order continuous;*

(ii) *if φ is interval preserving,* $\operatorname{ran}(\varphi)$ *is a band in $L^0(X, \Sigma, \mu)$;*

(iii) *if φ is surjective, then it is interval preserving; in particular, if $L \subset L^0(Y, \Lambda, \nu)$ is an ideal, then $\varphi(L)$ is an ideal in $L^0(X, \Sigma, \mu)$.*

If $L \subset L^0(Y, \Lambda, \nu)$ and $M \subset L^0(X, \Sigma, \mu)$ are order dense ideals and if $\varphi : L \to M$ is an order continuous Riesz homomorphism, then it can be extended uniquely to a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$ (which by (i) *is order continuous).*

Proof. (i) Since any order convergent sequence in $L^0(Y, \Lambda, \nu)$ is uniformly convergent (see e.g. Section 71 of [13]), it follows that φ is automatically σ-order continuous and hence order continuous, as the spaces involved are order separable.

(ii) The assumption in (ii) implies that $G := \operatorname{ran}(\varphi)$ is an ideal in $L^0(X, \Sigma, \mu)$. Since $L^0(Y, \Lambda, \nu)$ is laterally complete and φ is order continuous, G is laterally complete. We next observe that for every positive $w \in G$, we have $B_w := \{w\}^{\mathrm{dd}} \subset G$. Indeed, if $0 \leqslant f \in B_w$, let

$$f_n := f\mathbb{1}_{E_n} \quad \text{with } E_n := \{t \in X : nw(t) < f(t) \leqslant (n+1)w(t)\}$$

for $n = 0, 1, 2, \ldots$. Then $\{f_n\}$ is a disjoint system in G and so $\sup_n f_n$ belongs to G. Since this supremum is equal to f in $L^0(X, \Sigma, \mu)$ we have $f \in G$ and the claim is proved. Now let $\{w_\alpha\}$ be a maximal disjoint system in G. Then $w = \sup w_\alpha \in G$ and the band generated by w is contained in G. On the other hand, if $0 \leqslant v \in G$, we write $v = v_1 + v_2$ with $v_1 \in B_w$ and $v_2 \in B_w^{\mathrm{d}}$. Since v_2 is disjoint to every w_α, and since $\{w_\alpha\}$ is a maximal disjoint system in G, we have $v_2 = 0$. Thus, $G \subset B_w$ and so $G = B_w$ is a band in $L^0(X, \Sigma, \mu)$.

(iii) Let φ be surjective, and let $0 \leqslant g \leqslant \varphi(u)$ for some $0 \leqslant u \in L^0(Y, \Lambda, \nu)$. Let $w \in L^0(Y, \Lambda, \nu)$ be such that $\varphi(w) = g$. Then $0 \leqslant w^+ \wedge u \leqslant u$ and $\varphi(w^+ \wedge u) = g$.

The last assertion follows from the fact that $L^0(X, \Sigma, \mu)$ is laterally complete and a well known extension theorem (see Theorem 2.7.20 in [1]). ∎

For a Riesz homomorphism $\varphi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$, we denote by N_φ and C_φ its null-ideal and carrier respectively, i.e.,

$$N_\varphi = \{f \in L^0(Y, \Lambda, \nu) : \varphi(f) = 0\} \quad \text{and} \quad C_\varphi = N_\varphi^{\mathrm{d}}.$$

Since φ is order continuous, N_φ is a band an so $L^0(Y, \Lambda, \nu) = C_\varphi \oplus N_\varphi$. If $Y_1 \in \Lambda$ is the carrier of C_φ, then

$$C_\varphi = \{f \mathbb{1}_{Y_1} : f \in L^0(Y, \Lambda, \nu)\} = L^0(Y, \Lambda_{Y_1}, \nu),$$

where $\Lambda_{Y_1} = \{A \cap Y_1 : A \in \Lambda\}$. Observe that the restriction of φ to $L^0(Y_1, \Lambda_{Y_1}, \nu)$ is a Riesz isomorphism into $L^0(X, \Sigma, \mu)$.

Suppose that $\varphi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$ is a Riesz homomorphism such that $\varphi(\mathbb{1})$ is an almost everywhere strictly positive function on X. If $A \in \Lambda$, then $\varphi(\mathbb{1}_A) = \varphi(\mathbb{1})\mathbb{1}_B$ for some $B \in \Sigma$ which is uniquely determined modulo μ-null sets by A. We denote the measure algebras of (Y, Λ, ν) and (X, Σ, μ) by Λ_ν and Σ_μ respectively and the equivalence class in either of the algebras to which a measurable set C belongs by $\dot{C}$. Putting $\widehat{\varphi}(\dot{A}) = \dot{B}$ it follows that $\widehat{\varphi} : \Lambda_\nu \to \Sigma_\mu$ is an order continuous Boolean homomorphism satisfying $\widehat{\varphi}(\dot{Y}) = \dot{X}$. Let

$$\Sigma_\varphi = \{B \in \Sigma : \dot{B} \in \widehat{\varphi}(\Lambda_\nu)\}. \tag{2.1}$$

Then Σ_φ is a sub-σ-algebra of Σ and with slight abuse of notation we will write $\Sigma_\varphi = \widehat{\varphi}(\Lambda)$ (usually, we will not distinguish between the σ-algebra Λ and the measure algebra Λ_ν). It is easy to see that φ is interval preserving as a mapping from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma_\varphi, \mu)$ and so if follows via Proposition 2.1 (ii) that $\varphi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma_\varphi, \mu)$ is surjective.

Conversely, if $\sigma : \Lambda_\nu \to \Sigma_\mu$ is an order continuous Boolean homomorphism with $\sigma(\dot{Y}) = \dot{X}$, then there exists a unique Riesz homomorphism $\varphi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$ with $\varphi(\mathbb{1}) = \mathbb{1}$ such that $\widehat{\varphi} = \sigma$. Note that the condition $\varphi(\mathbb{1}) = \mathbb{1}$ is equivalent to the multiplicativity of φ.

Now assume that $\tau : X \to Y$ is a (Σ, Λ)-measurable mapping which is null preserving (i.e., if $B \in \Lambda$ and $\nu(B) = 0$ then $\mu(\tau^{-1}(B)) = 0$). The mapping $B \mapsto \tau^{-1}(B)$ defines an order continuous Boolean homomorphism $\tau_* : \Lambda_\nu \to \Sigma_\mu$ with $\tau_*(\dot{Y}) = \dot{X}$. We denote by φ_τ the associated Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$, i.e., $\widehat{\varphi}_\tau = \tau_*$ and $\varphi_\tau(\mathbb{1}) = \mathbb{1}$. It is easily verified that $(\varphi_\tau f)(x) = f(\tau x)$ μ-a.e. on X for all $f \in L^0(Y, \Lambda, \nu)$.

Let $L \subset L^0(Y, \Lambda, \nu)$ be an order dense ideal with order continuous dual $L_n^\sim$. As usual we identify $L_n^\sim$ with an ideal L' of functions in $L^0(Y, \Lambda, \nu)$ and we will assume that L' is again an order dense ideal (which is always the case if L is a Banach function space; see Theorem 112.1 from [19] or Theorem 2.6.4 from [14]). Equivalent to this assumption is that $L_n^\sim$ separates the points of L. The duality relation between L and L' is given by $\langle f, g\rangle = \int_Y fg \,\mathrm{d}\nu$ for $f \in L$ and $g \in L'$ (see Section 86 of [19]). Let $T \in \mathfrak{L}_\mathrm{n}(L, M)$ with L and M ideals of functions in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$ respectively. We define its order continuous adjoint $T' : M' \to L'$ by $\langle g, T'f\rangle = \langle Tg, f\rangle$ for all $f \in M'$ and $g \in L$ (see Section 97 of [19]). Then $T' \in \mathfrak{L}_\mathrm{n}(M', L')$. In the next lemma we gather some results relating to the adjoints of homomorphisms.

LEMMA 2.2. *Let (X, Σ, μ) and (Y, Λ, ν) be σ-finite measure spaces and let $L \subseteq L^0(Y, \Lambda, \nu)$ and $M \subseteq L^0(X, \Sigma, \mu)$ be order dense ideals for which L' and M' are order dense ideals as well. Let $\varphi : L \to M$ be an order continuous interval preserving Riesz homomorphism. Then the adjoint $\varphi' : M' \to L'$ is an order continuous interval preserving Riesz homomorphism as well and φ' extends uniquely to an order continuous interval preserving Riesz homomorphism $\varphi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)$. Moreover,*

(i) *if $\varphi(L)$ is order dense in $L^0(X, \Sigma, \mu)$, then φ' is injective;*

(ii) *if φ is injective, then $\varphi'(\mathbb{1}_X)$ is strictly positive and $\varphi'(M')$ is order dense in L';*

(iii) *if φ is injective and $\varphi(L)$ is a band in M then $\varphi' : M' \to L'$ is surjective.*

Proof. It follows from Theorem 7.7 and Theorem 7.8 of [1], that φ' is an interval preserving Riesz homomorphism. By Proposition 2.1, φ' can be extended to an order continuous Riesz homomorphism $\varphi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)$, which is easily seen to be interval preserving.

(i) Now assume that $\varphi(L)$ is order dense in $L^0(X, \Sigma, \mu)$. Let $0 \leqslant g \in M'$ be such that $\varphi'(g) = 0$. Then

$$\langle \varphi(u), g\rangle = \langle u, \varphi'(g)\rangle = 0, \quad \forall\, 0 \leqslant u \in L,$$

and so $g = 0$. Hence, φ' is injective.

(ii) Now assume that φ is injective. To show that $\varphi'(\mathbb{1}_X)$ is strictly positive, take $0 \leqslant g \in L$ such that $g \wedge \varphi'(\mathbb{1}_X) = 0$. Take $X_n \in \Sigma$ such that $X_n \uparrow X$ and $\mathbb{1}_{X_n} \in M'$ for all n. Then

$$\int_{X_n} \varphi(g)\,\mathrm{d}\mu = \langle \varphi(g), \mathbb{1}_{X_n}\rangle = \langle g, \varphi'(\mathbb{1}_{X_n})\rangle = 0$$

for all n, and so $\int_X \varphi(g)\,\mathrm{d}\mu = 0$. This implies that $\varphi(g) = 0$ μ-a.e. on X. Hence $g = 0$, which shows that $\varphi'(\mathbb{1}_X)$ is strictly positive, as L is order dense in $L^0(Y, \Lambda, \nu)$.

In order to see that $\varphi'(M')$ is order dense in L', let $0 < h \in L'$. Since $\varphi'(\mathbb{1}_X)$ is strictly positive, $0 < \varphi'(\mathbb{1}_X) \wedge h \leqslant \varphi'(\mathbb{1}_X)$ and as φ' is interval preserving, there exists $0 < f \in L^0(X, \Sigma, \mu)$ such that $0 < f \leqslant \mathbb{1}_X$ and $\varphi'(f) = \varphi'(\mathbb{1}_X) \wedge h$. But M' is order dense in $L^0(X, \Sigma, \mu)$ and so for some $g \in M'$, we have $0 < g \leqslant f$. By injectivity, $0 < \varphi'(g) \leqslant \varphi'(f) \leqslant h$ and it follows that $\varphi'(M')$ is order dense in L'.

(iii) Assume now that φ is injective and that $\varphi(L)$ is a band in M. By (ii), $\varphi'(\mathbb{1}_X)$ is strictly positive and since φ' is interval preserving, it follows that $\varphi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)$ is surjective. Let $X_1 \in \Sigma$ be the carrier of $\varphi(L)$. By hypothesis, $\varphi(L) = \{f\mathbb{1}_{X_1} : f \in M\}$. Furthermore it is easy to see that $\varphi'(g) = 0$ for all $g \in L^0(X, \Sigma, \mu)$ such that $g = 0$ on X_1.

Now let $0 < h \in L'$ be given. Then $h = \varphi'(g)$ for some $0 \leqslant g \in L^0(X, \Sigma, \mu)$, and we may assume that $g = 0$ on $X \setminus X_1$. It remains to show that $0 \leqslant g \in M'$, i.e., that $\int_X gf \, \mathrm{d}\mu < \infty$ for all $0 \leqslant f \in M$. To this end, take $0 \leqslant f \in M$. Then $f\mathbb{1}_{X_1} \in \varphi(L)$, so $f\mathbb{1}_{X_1} = \varphi(u)$ for some $0 \leqslant u \in L$. Let $0 \leqslant g_n \in M'$ be such that $0 \leqslant g_n \uparrow g$. We find that

$$\begin{aligned}\int_X gf \, \mathrm{d}\mu &= \int_X g(f\mathbb{1}_{X_1}) \, \mathrm{d}\mu = \int_X g\varphi(u) \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X g_n\varphi(u) \, \mathrm{d}\mu \\ &= \lim_{n\to\infty} \int_Y \varphi'(g_n)u \, \mathrm{d}\nu = \int_Y \varphi'(g)u \, \mathrm{d}\nu = \int_Y hu \, \mathrm{d}\nu < \infty,\end{aligned}$$

which shows that $0 \leqslant g \in M'$. ∎

The notion of conditional expectation plays an important role throughout the paper and so we recall the definition and some elementary properties for the reader's convenience. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space (i.e., $\mathbb{P}(\Omega) = 1$) and let $\mathfrak{G}$ be a sub-σ-algebra of $\mathfrak{F}$. For $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$, we denote by $\mathbb{E}(f \mid \mathfrak{G})$ the ($\mathbb{P}$-a.e.) unique $\mathfrak{G}$-measurable function with the property that

$$\int_A \mathbb{E}(f \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} = \int_A f \, \mathrm{d}\mathbb{P} \quad \text{for all } A \in \mathfrak{G}.$$

The existence of $\mathbb{E}(f \mid \mathfrak{G})$ is a consequence of the Radon-Nikodym theorem. The function $\mathbb{E}(f \mid \mathfrak{G})$ is called the *conditional expectation* of f with respect to $\mathfrak{G}$. If $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ and $g \in L^\infty(\Omega, \mathfrak{G}, \mathbb{P})$, then $\mathbb{E}(gf \mid \mathfrak{G}) = g\mathbb{E}(f \mid \mathfrak{G})$. For a proof, and also for properties of $\mathbb{E}(\cdot \mid \mathfrak{G})$ used without proof in this paper, we refer to [15], [7] and [17].

The conditional expectation $\mathbb{E}(\cdot \mid \mathfrak{G})$ can be extended from a mapping of $L^1(\Omega, \mathfrak{F}, \mathbb{P})$ into itself, to a mapping from $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ into itself. Indeed, if $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$, then $\mathbb{E}(f \mid \mathfrak{G}) \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ is defined by

$$\mathbb{E}(f \mid \mathfrak{G}) = \sup_n \mathbb{E}(f_n \mid \mathfrak{G}),$$

where $0 \leqslant f_n \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ with $n = 1, 2, \ldots$ satisfy $0 \leqslant f_n \uparrow f$ $\mathbb{P}$-a.e. (see [15]). For a proof of properties (i) to (vi) in the next proposition we refer to Lemma I-2-9 from [15].

Proposition 2.3. (i) $\mathbb{E}(\alpha f + \beta g \mid \mathfrak{G}) = \alpha\mathbb{E}(f \mid \mathfrak{G}) + \beta\mathbb{E}(g \mid \mathfrak{G})$ *for all* $f, g \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *and for all* $0 \leqslant \alpha, \beta \in \mathbb{R}$.

(ii) $0 \leqslant f \leqslant g$ *in* $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *implies that* $0 \leqslant \mathbb{E}(f \mid \mathfrak{G}) \leqslant \mathbb{E}(g \mid \mathfrak{G})$.

(iii) $0 \leqslant f_n \uparrow f$ $\mathbb{P}$*-a.e. implies that* $0 \leqslant \mathbb{E}(f_n \mid \mathfrak{G}) \uparrow \mathbb{E}(f \mid \mathfrak{G})$ $\mathbb{P}$*-a.e.*

(iv) $\mathbb{E}(gf \mid \mathfrak{G}) = g\mathbb{E}(f \mid \mathfrak{G})$ *for all* $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *and all* $g \in M^+(\Omega, \mathfrak{G}, \mathbb{P})$.

(v) *If* $g \in M^+(\Omega, \mathfrak{G}, \mathbb{P})$ *and* $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$*, then* $\int_A g \, d\mathbb{P} = \int_A f \, d\mathbb{P}$ *for all* $A \in \mathfrak{G}$ *if and only if* $g = \mathbb{E}(f \mid \mathfrak{G})$ $\mathbb{P}$*-a.e.*

(vi) *If* $\mathfrak{G} \subset \mathfrak{H}$ *are sub-*σ*-algebras of* $\mathfrak{F}$*, then* $\mathbb{E}(f \mid \mathfrak{G}) = \mathbb{E}(\mathbb{E}(f \mid \mathfrak{H}) \mid \mathfrak{G})$ *for every* $0 \leqslant f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$.

(vii) *If* $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *is such that* $\mathbb{E}(f \mid \mathfrak{G}) \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$*, then also* $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$.

To see that property (vii) holds, note that $\mathbb{E}(f \mid \mathfrak{G}) \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ implies that there exists a sequence $(\Omega_n) \subset \mathfrak{G}$ such that $\Omega_n \uparrow \Omega$ and $\int_{\Omega_n} \mathbb{E}(f \mid \mathfrak{G}) \, d\mathbb{P} < \infty$. Hence, $\int_{\Omega_n} f \, d\mathbb{P} < \infty$ which implies (vii).

Observe that the converse of (vii) does not hold in general. For example, if $\Omega = [0, 1]$ and $\mathbb{P}$ is Lebesgue measure, let $\mathfrak{G} = \{\emptyset, [0, 1/2], [1/2, 1], [0, 1]\}$ and let $f(x) = 1/x$, $0 \leqslant x \leqslant 1$ then $\mathbb{E}(f \mid \mathfrak{G}) = \infty$ on $[0, 1/2]$. We therefore define the domain $\operatorname{dom} \mathbb{E}(\cdot \mid \mathfrak{G})$ of $\mathbb{E}(\cdot \mid \mathfrak{G})$ by

$$\operatorname{dom} \mathbb{E}(\cdot \mid \mathfrak{G}) := \{f \in L^0(\Omega, \mathfrak{F}, \mathbb{P}) : \mathbb{E}(|f| \mid \mathfrak{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P})\}.$$

Clearly, $\operatorname{dom} \mathbb{E}(\cdot \mid \mathfrak{G})$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ which contains $L^1(\Omega, \mathfrak{F}, \mathbb{P})$ and is therefore order dense in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$. For $f \in \operatorname{dom} \mathbb{E}(\cdot \mid \mathfrak{G})$, we define:

$$\mathbb{E}(f \mid \mathfrak{G}) := \mathbb{E}(f^+ \mid \mathfrak{G}) - \mathbb{E}(f^- \mid \mathfrak{G}).$$

This defines a positive linear operator

$$\mathbb{E}(\cdot \mid \mathfrak{G}) : \operatorname{dom} \mathbb{E}(\cdot \mid \mathfrak{G}) \to L^0(\Omega, \mathfrak{G}, \mathbb{P}) \subset L^0(\Omega, \mathfrak{F}, \mathbb{P}).$$

EXAMPLE 2.4. Let $(\Omega_1, \mathfrak{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathfrak{F}_2, \mathbb{P}_2)$ be two probability spaces, and let $\Omega = \Omega_1 \times \Omega_2$, $\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$, and $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$. Let $\mathfrak{G} := \{A \times \Omega_2 : A \in \mathfrak{F}_1\}$, then $\mathfrak{G}$ is a sub-σ-algebra of $\mathfrak{F}$. Note that an $\mathfrak{F}$-measurable function $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $\mathfrak{G}$-measurable if and only if $g(x_1, x_2) = \widehat{g}(x_1)$ for some $\mathfrak{F}_1$-measurable function $\widehat{g}$ on Ω_1 (i.e., if and only if g is independent of x_2). For $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ we now have:

$$\mathbb{E}(f \mid \mathfrak{G})(x_1, x_2) = \int_{\Omega_2} f(x_1, y) \, d\mathbb{P}_2(y) \quad \mathbb{P}\text{-a.e. on } \Omega.$$

The way in which $\mathbb{E}(\cdot \mid \mathfrak{G})$ is extended to $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ implies that if $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ then $f \in \operatorname{dom} \mathbb{E}(\cdot \mid \mathfrak{G})$ if and only if

$$\int_{\Omega_2} |f(x_1, y)| \, d\mathbb{P}_2(y) < \infty \quad \mathbb{P}\text{-a.e. on } \Omega.$$

In this case, $\mathbb{E}(f \mid \mathfrak{G})(x_1, x_2) = \int_{\Omega_2} f(x_1, y) \, d\mathbb{P}_2(y)$ $\mathbb{P}$-a.e. on Ω. Note also that the condition $\int_{\Omega_2} |f(x_1, y)| \, d\mathbb{P}_2(y) < \infty$ $\mathbb{P}$-a.e. on Ω is equivalent to $\int_{\Omega_2} |f(x_1, y)| \, d\mathbb{P}_2(y) < \infty$ $\mathbb{P}_1$-a.e on Ω_1.

PROPOSITION 2.5. *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and* $\mathfrak{G} \subset \mathfrak{F}$ *a sub-*σ*-algebra.*

(i) *If* $f \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$ *and* $g \in L^0(\Omega, \mathfrak{G}, \mathbb{P})$, *then* $gf \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$ *and* $\mathbb{E}(gf \mid \mathfrak{G}) = g\mathbb{E}(f \mid \mathfrak{G})$.

(ii) *If* $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$, *then* $f \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$ *if and only if there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *in* $\mathfrak{G}$ *such that* $A_n \uparrow \Omega$ *and*

$$\int_{A_n} |f| \, \mathrm{d}\mathbb{P} < \infty \quad \forall \, n = 1, 2, \ldots .$$

Moreover, if $f \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$ *then, for all* $A \in \mathfrak{G}$ *with* $\int_A |f| \, \mathrm{d}\mathbb{P} < \infty$,

$$\int_A \mathbb{E}(f \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} = \int_A f \, \mathrm{d}\mathbb{P}.$$

Proof. (i) From Proposition 2.3(iv) it follows that

$$\mathbb{E}(|gf| \mid \mathfrak{G}) = |g|\mathbb{E}(|f| \mid \mathfrak{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P}),$$

which implies that $gf \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$. Writing $gf = (g^+ - g^-)(f^+ - f^-)$ and using Proposition 2.3 again, it follows that $\mathbb{E}(gf \mid \mathfrak{G}) = g\mathbb{E}(f \mid \mathfrak{G})$.

(ii) Assume first that there exists $A_n \in \mathfrak{G}$ such that $A_n \uparrow \Omega$ and $\int_{A_n} |f| \, \mathrm{d}\mathbb{P} < \infty$ for all $n = 1, 2, \ldots$. By Proposition 2.3 (v) it follows that

$$\int_{A_n} \mathbb{E}(|f| \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} = \int_{A_n} |f| \, \mathrm{d}\mathbb{P} < \infty \quad \text{for all } n = 1, 2, \ldots .$$

Hence $\mathbb{E}(|f| \mid \mathfrak{G}) < \infty$ $\mathbb{P}$-a.e. on A_n, for all $n = 1, 2, \ldots$. Since $A_n \uparrow \Omega$, this implies that $\mathbb{E}(|f| \mid \mathfrak{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P})$, i.e., $f \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$.

Now assume that $f \in \operatorname{dom}\mathbb{E}(\cdot \mid \mathfrak{G})$. Then $\mathbb{E}(|f| \mid \mathfrak{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P})$ and so there is a sequence $A_n \in \mathfrak{G}$,$n = 1, 2, \ldots$ with $A_n \uparrow \Omega$ and

$$\int_{A_n} |f| \, \mathrm{d}\mathbb{P} = \int_{A_n} \mathbb{E}(|f| \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} < \infty, \quad n = 1, 2, \ldots .$$

To prove the last statement, take $A \in \mathfrak{G}$ such that $\int_A |f| \, \mathrm{d}\mathbb{P} < \infty$. Then

$$\int_A \mathbb{E}(|f| \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} = \int_A |f| \, \mathrm{d}\mathbb{P} < \infty.$$

Since $\mathbb{E}(f^+ \mid \mathfrak{G}),\ \mathbb{E}(f^- \mid \mathfrak{G}) \leqslant \mathbb{E}(|f| \mid \mathfrak{G})$, it follows that $\mathbb{E}(f \mid \mathfrak{G})$ is integrable over A and that

$$\begin{aligned} \int_A \mathbb{E}(f \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} &= \int_A \mathbb{E}(f^+ \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} - \int_A \mathbb{E}(f^- \mid \mathfrak{G}) \, \mathrm{d}\mathbb{P} \\ &= \int_A f^+ \, \mathrm{d}\mathbb{P} - \int_A f^- \, \mathrm{d}\mathbb{P} = \int_A f \, \mathrm{d}\mathbb{P}. \quad \blacksquare \end{aligned}$$

3. MULTIPLICATION-CONDITIONAL EXPECTATION OPERATORS ON IDEALS

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let $\mathfrak{F}_L$ and $\mathfrak{F}_M$ be sub-σ-algebras of $\mathfrak{F}$. Let $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be ideals, both with carriers Ω. Set

$$\mathfrak{M}(L, M) = \{m \in L^0(\Omega, \mathfrak{F}, \mathbb{P}) : \mathbb{E}(|mf| \,|\, \mathfrak{F}_M) \in M \,\forall f \in L\}.$$

Since $M \subseteq L^0(X, \mathfrak{F}, \mathbb{P})$, we have $mf \in \mathrm{dom}\mathbb{E}(\cdot \,|\, \mathfrak{F}_M)$ for all $m \in \mathfrak{M}(L, M)$ and $f \in L$.

For $m \in \mathfrak{M}(L, M)$ we define $S_m : L \to M$ by

$$S_m f := \mathbb{E}(mf \,|\, \mathfrak{F}_M) \quad \forall f \in L.$$

Since $\mathrm{dom}\,\mathbb{E}(\cdot \,|\, \mathfrak{F}_M)$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ and since M is an ideal in $L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ it follows immediately that $\mathfrak{M}(L, M)$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ as well (but it may happen that $\mathfrak{M}(L, M) = \{0\}$).

From the definition of $\mathfrak{M}(L, M)$ it follows that S_m is a well-defined linear operator. If $0 \leqslant m \in \mathfrak{M}(L, M)$ then $S_m \geqslant 0$. If $m \in \mathfrak{M}(L, M)$ then

$$|S_m f| = |\mathbb{E}(mf \,|\, \mathfrak{F}_M)| \leqslant \mathbb{E}(|m| \cdot |f| \,|\, \mathfrak{F}_M) = S_{|m|}|f|,$$

so S_m is order bounded and $|S_m| \leqslant S_{|m|}$. Moreover, it is clear that S_m is order continuous.

Define

$$\mathfrak{L}_{\mathfrak{M}}(L, M) := \{S_m : m \in \mathfrak{M}(L, M)\}.$$

Then $\mathfrak{L}_{\mathfrak{M}}(L, M) \subset \mathfrak{L}_n(L, M)$ is a linear subspace. We will show in the sequel that it is actually a band. To this end we first prove:

PROPOSITION 3.1. *Let $\mathfrak{F}_0 = \sigma(\mathfrak{F}_M, \mathfrak{F}_L)$ be the σ-algebra generated by $\mathfrak{F}_M$ and $\mathfrak{F}_L$. Let*

$$\mathfrak{M}_0(L, M) := L^0(\Omega, \mathfrak{F}_0, \mathbb{P}) \cap \mathfrak{M}(L, M).$$

Then, for the corresponding spaces of operators, we have

$$\mathfrak{L}_{\mathfrak{M}}(L, M) = \mathfrak{L}_{\mathfrak{M}_0}(L, M).$$

Proof. It is sufficient to show that for each $0 \leqslant m \in \mathfrak{M}(L, M)$ there exists $0 \leqslant m_0 \in \mathfrak{M}_0(L, M)$ such that $S_m f = S_{m_0} f$, $\forall\, 0 \leqslant f \in L$. Let $0 \leqslant m \in \mathfrak{M}(L, M)$ be given, and define

$$m_0 := \mathbb{E}(m \,|\, \mathfrak{F}_0) \in M^+(\Omega, \mathfrak{F}_0, \mathbb{P}).$$

For $0 \leqslant f \in L$ we have $S_m f = \mathbb{E}(mf \,|\, \mathfrak{F}_M) = \mathbb{E}(\mathbb{E}(mf \,|\, \mathfrak{F}_0) \,|\, \mathfrak{F}_M)$, and since $0 \leqslant f \in L \subset L^0(\Omega, \mathfrak{F}_0, \mathbb{P})$ it follows from Proposition 2.3 (iv) that

$$\mathbb{E}(mf \,|\, \mathfrak{F}_0) = \mathbb{E}(m \,|\, \mathfrak{F}_0) f = m_0 f.$$

Hence, $S_m f = \mathbb{E}(m_0 f \,|\, \mathfrak{F}_M)$ for all $0 \leqslant f \in L$. In particular, $\mathbb{E}(m_0 f \,|\, \mathfrak{F}_M) \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ and so $m_0 f \in L^0(\Omega, \mathfrak{F}_0, \mathbb{P})$ for all $0 \leqslant f \in L$. Since the carrier of L is Ω, this implies that $0 \leqslant m_0 \in L^0(\Omega, \mathfrak{F}_0, \mathbb{P})$. It is now clear that $0 \leqslant m_0 \in \mathfrak{M}_0(L, M)$ and $S_m f = S_{m_0} f$ for all $0 \leqslant f \in L$. ∎

The above shows that, for the study of the space of operators $\mathfrak{L}_{\mathfrak{M}}(L,M)$ we may assume, without loss of generality, that $\mathfrak{F}=\sigma(\mathfrak{F}_M,\mathfrak{F}_L)$.

Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a probability space and suppose that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two sub-algebras of $\mathfrak{F}$ such that $\mathfrak{F}=\sigma(\mathfrak{F}_1,\mathfrak{F}_2)$. Let

$$\Gamma=\{A\cap B : A\in\mathfrak{F}_1, B\in\mathfrak{F}_2\}.$$

Since $\mathfrak{F}_1,\mathfrak{F}_2\subset\Gamma$, we have $\sigma(\Gamma)=\mathfrak{F}$ and Γ is a semi-ring (see [18]). Consequently, $\mathfrak{F}$ is the monotone class generated by the finite disjoint unions of sets in Γ. Therefore, if $f\in L^1(\Omega,\mathfrak{F},\mathbb{P})$ such that $\int_G f\,\mathrm{d}\mathbb{P}\geqslant 0$ for all $G\in\Gamma$, it follows that $\int_E f\,\mathrm{d}\mathbb{P}\geqslant 0$ for all $E\in\mathfrak{F}$ because the set $\mathfrak{M}:=\{C\in\mathfrak{F}:\int_C f\,\mathrm{d}\mathbb{P}\geqslant 0\}$ is a monotone class which contains all finite disjoint unions of sets in Γ. Hence, if $\int_{A\cap B} f\,\mathrm{d}\mathbb{P}\geqslant 0$ for all $A\in\mathfrak{F}_1$ and $B\in\mathfrak{F}_2$ then $f\geqslant 0$ $\mathbb{P}$-a.e. on Ω (this can also be proved using the method of proof of Theorem 4, Section 18 in [18]).

Lemma 3.2. *Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a probability space, let $L\subset L^0(\Omega,\mathfrak{F}_L,\mathbb{P})$ and $M\subset L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ be order dense ideals of measurable functions and suppose that $\mathfrak{F}=\sigma(\mathfrak{F}_L,\mathfrak{F}_M)$. For $m\in\mathfrak{M}(L,M)$ we have:*

(i) *if $S_m\geqslant 0$, then $m\geqslant 0$ $\mathbb{P}$-a.e. on Ω;*

(ii) *if $S_m=0$, then $m=0$ $\mathbb{P}$-a.e. on Ω.*

Proof. (i) Take $A_0\in\mathfrak{F}_L$ such that $\mathbb{1}_{A_0}\in L$. Now take $B_0\in\mathfrak{F}_M$ such that $\int_{B_0}|m\mathbb{1}_{A_0}|\,\mathrm{d}\mathbb{P}<\infty$. We will first show that $m\geqslant 0$ $\mathbb{P}$-a.e. on $A_0\cap B_0$.

For $A\in\mathfrak{F}_L,\ A\subset A_0$ and $B\in\mathfrak{F}_M,\ B\subset B_0$ we have

$$\int_B |m\mathbb{1}_A|\,\mathrm{d}\mathbb{P}<\infty,$$

and so by Proposition 2.5 (ii)

$$\int_B m\mathbb{1}_A\,\mathrm{d}\mathbb{P}=\int_B \mathbb{E}(m\mathbb{1}_A\,|\,\mathfrak{F}_M)\,\mathrm{d}\mathbb{P}=\int_B S_m(\mathbb{1}_A)\,\mathrm{d}\mathbb{P}\geqslant 0.$$

This shows that $\int_{A\cap B} m\,\mathrm{d}\mathbb{P}\geqslant 0$ for all such A and B. Now define the following σ-algebras of subsets of $A_0\cap B_0$:

$$\begin{aligned}\mathfrak{F}_M^0&=\{A_0\cap B : B\in\mathfrak{F}_M, B\subseteq B_0\},\\ \mathfrak{F}_L^0&=\{A\cap B_0 : A\in\mathfrak{F}_L, A\subseteq A_0\},\\ \mathfrak{F}^0&=\{C\in\mathfrak{F} : C\subseteq A_0\cap B_0\}.\end{aligned}$$

It is then easily seen that $\sigma(\mathfrak{F}_M^0,\mathfrak{F}_L^0)=\mathfrak{F}^0$ and we have $\int_{A\cap B} m\,\mathrm{d}\mathbb{P}\geqslant 0$ for all $A\in\mathfrak{F}_L^0$ and $B\in\mathfrak{F}_M^0$. By our remark preceding the proposition it follows that $\int_C m\,\mathrm{d}\mathbb{P}\geqslant 0$ for all $C\in\mathfrak{F}^0$, i.e., $m\geqslant 0$ $\mathbb{P}$-a.e on $A_0\cap B_0$.

Keep $A_0\in\mathfrak{F}_L$ with $\mathbb{1}_{A_0}\in L$ fixed. Since $m\mathbb{1}_{A_0}\in\operatorname{dom}\mathbb{E}(\cdot\,|\,\mathfrak{F}_M)$, it follows from Proposition 2.5 (ii) that there exists a sequence $B_n\in\mathfrak{F}_M$, $n=1,2,\ldots$ such that $B_n \uparrow \Omega$ and $\int\limits_{B_n} |m\mathbb{1}_{A_0}|\,\mathrm{d}\mathbb{P} < \infty$. From the first part of the proof it follows that $m \geqslant 0$ $\mathbb{P}$-a.e. on $A_0 \cap B_n$, $\forall\, n = 1, 2, \ldots$. Since $B_n \uparrow \Omega$, this implies that $m \geqslant 0$ $\mathbb{P}$-a.e. on A_0.

Finally, since the carrier of L is Ω, there exists a sequence $A_n \in \mathfrak{F}_L$, $n = 1, 2, \ldots$ such that $A_n \uparrow \Omega$ and $\mathbb{1}_{A_n} \in L$. From the above we know that $m \geqslant 0$ $\mathbb{P}$-a.e. on each A_n, and hence $m \geqslant 0$ $\mathbb{P}$-a.e. on Ω.

(ii) This follows immediately from (i). ∎

LEMMA 3.3. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be ideals of measurable functions. If $S_\alpha \in \mathfrak{L}_{\mathfrak{M}}(L, M)$ is an upwards directed net and if $0 \leqslant S_\alpha \uparrow S$ in $\mathfrak{L}_{\mathrm{b}}(L, M)$ then $S \in \mathfrak{L}_{\mathfrak{M}}(L, M)$.*

Proof. Without loss of generality we may assume that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$. Write $S_\alpha = S_{m_\alpha}$ with $m_\alpha \in \mathfrak{M}(L, M)$. By the preceding proposition we have that $0 \leqslant m_a \uparrow$ in $M^+(\Omega, \mathfrak{F}, \mathbb{P})$. Then $m := \sup\limits_\alpha m_\alpha$ exists in $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ and there exists a sub-sequence (m_n) of (m_α) such that $m_n \uparrow m$ (see Lemma 94.4 from [19]). For each $0 \leqslant f \in L$ we have

$$Sf \geqslant S_{m_n} f = \mathbb{E}(m_n f \,|\, \mathfrak{F}_M) \uparrow \mathbb{E}(mf \,|\, \mathfrak{F}_M) \in M^+(\Omega, \mathfrak{F}_M, \mathbb{P}).$$

Now, $Sf \in M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ implies $\mathbb{E}(mf \,|\, \mathfrak{F}_M) \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ and consequently, $mf \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$. Since L has carrier Ω, this implies that $m \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ and hence $m \in \mathfrak{M}(L, M)$. By Proposition 2.3 (iii), $m_n \uparrow m$ in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ implies that $S_{m_n} \uparrow S_m$ in $\mathfrak{L}_{\mathrm{b}}(L, M)$ and hence $S_m \leqslant S$. On the other hand, $m \geqslant m_\alpha$ and so $S_m \geqslant S_{m_\alpha}$ for all α and thus $S_m \geqslant S$. Hence, $S = S_m \in \mathfrak{L}_{\mathfrak{M}}(L, M)$. ∎

LEMMA 3.4. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ ideals of measurable functions. If $S \in \mathfrak{L}_{\mathrm{n}}(L, M)$ such that $0 \leqslant S \leqslant S_m$ for some $m \in \mathfrak{M}(L, M)$, then $S \in \mathfrak{L}_{\mathfrak{M}}(L, M)$.*

Proof. We first prove the proposition under the additional assumption that $\mathfrak{F}_L = \mathfrak{F}$. Then L is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ and the operator S_m is interval preserving (i.e., it has the Maharam property). Indeed, if $f \in L$ and $0 \leqslant g \leqslant S_m f$ in M, define the function σ to be equal to $g/S_m f$ if $S_m f \neq 0$ and 0 otherwise. Since $0 \leqslant \sigma \leqslant 1$ and since $\sigma \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ we have $0 \leqslant \sigma f \leqslant f$ in L and $S_m(\sigma f) = \mathbb{E}(m\sigma f \,|\, \mathfrak{F}_M) = \sigma \mathbb{E}(mf \,|\, \mathfrak{F}_M) = \sigma S_m f = g$. It follows from the Luxemburg-Schep Radon-Nikodym theorem (see [12]) that $S = S_m \pi$ for some $0 \leqslant \pi \leqslant I$ in the centre $Z(F)$ of F. Now π is multiplication by some function $0 \leqslant p \leqslant 1$ on Ω (see [19] Example 141.3) and, since $\mathfrak{M}(L, M)$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$, it follows that $S = S_{pm} \in \mathfrak{L}_{\mathfrak{M}}(L, M)$.

For general L, let $I(L)$ be the ideal generated in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ by L. It is easily verified that $\mathfrak{M}(I(L), M) = \mathfrak{M}(L, M)$. Define $\overline{S}_m : I(L) \to M$ by $\overline{S}_m f = \mathbb{E}(mf \,|\, \mathfrak{F}_M)$ for all $f \in I(L)$. Then $\overline{S}_m$ is an extension of S_m and $\overline{S}_m \in \mathfrak{L}_{\mathfrak{M}}(I(L), M)$. It follows from the Kantorovich extension theorem (see [1] Theorem I.2.2), that there exists an extension $0 \leqslant \overline{S} \in \mathfrak{L}_{\mathrm{b}}(I(L), M)$ of S such that $0 \leqslant \overline{S} \leqslant \overline{S}_m$. By the first part of the proof $\overline{S} \in \mathfrak{L}_{\mathfrak{M}}(I(L), M)$. Since $\mathfrak{M}(I(L), M) = \mathfrak{M}(L, M)$, the restriction of $\overline{S}$ to L belongs to $\mathfrak{L}_{\mathfrak{M}}(L, M)$, i.e., $S \in \mathfrak{L}_{\mathfrak{M}}(L, M)$. This completes the proof. ∎

Proposition 3.5. *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and* $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ *and* $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ *be ideals of measurable functions. Then* $\mathfrak{L}_{\mathfrak{M}}(L, M)$ *is a band in* $\mathfrak{L}_{\mathrm{n}}(L, M)$*. Moreover, assuming that* $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$*, the mapping* $m \to S_m$ *is a Riesz isomorphism from the ideal* $\mathfrak{M}(L, M)$ *onto* $\mathfrak{L}_{\mathfrak{M}}(L, M)$*.*

Proof. First we show that $\mathfrak{L}_{\mathfrak{M}}(L, M)$ is a Riesz subspace of $\mathfrak{L}_{\mathrm{n}}(L, M)$. Take $S_m \in \mathfrak{L}_{\mathfrak{M}}(L, M)$. As observed at the beginning of this section, $|S_m| \leqslant S_{|m|}$. By the preceding lemma, this implies that $|S_m| \in \mathfrak{L}_{\mathfrak{M}}(L, M)$. Hence, $\mathfrak{L}_{\mathfrak{M}}(L, M)$ is a Riesz subspace. Using the above lemma once more, we see that $\mathfrak{L}_{\mathfrak{M}}(L, M)$ is actually an ideal in $\mathfrak{L}_{\mathrm{n}}(L, M)$. Moreover, Lemma 3.3 yields that $\mathfrak{L}_{\mathfrak{M}}(L, M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L, M)$. Assuming that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$, it follows from Lemma 3.2 that the mapping $m \mapsto S_m$ is a bi-positive bijection from $\mathfrak{M}(L, M)$ onto $\mathfrak{L}_{\mathfrak{M}}(L, M)$, consequently this mapping is a Riesz isomorphism. ∎

4. MCE-REPRESENTABLE OPERATORS

In this section we study operators which factorize through MCE-operators. It turns out that this class of operators includes a number of interesting special cases such as kernel operators, partial integral operators and Riesz homomorphisms.

Let (Y, Λ, ν) be a σ-finite measure space and let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. Assume that L is an ideal in $L^0(Y, \Lambda, \nu)$, with carrier equal to Y. Let φ_L be a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ such that $\varphi_L(\mathbb{1})$ is $\mathbb{P}$-a.e. strictly positive on Ω. Let

$$L_\Omega := \varphi_L(L) \subseteq L^0(\Omega, \mathfrak{F}, \mathbb{P})$$

and let

$$\mathfrak{F}_L = \widehat{\varphi}_L(\Lambda),$$

with the notation as defined after equation (2.1) of Section 2. As we remarked in Section 2, $\mathrm{ran}(\varphi_L) = L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and hence, by Proposition 2.1 (iii), L_Ω is an ideal in $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ which is order dense as $\varphi_L(\mathbb{1})$ is $\mathbb{P}$-a.e. strictly positive. So the carrier of L_Ω is equal to Ω. We will denote the restriction of φ_L to L again by φ_L.

Definition 4.1. Let (X, Σ, μ) and (Y, Λ, ν) be σ-finite measure spaces and let $L \subseteq L^0(Y, \Lambda, \nu)$ and $M \subseteq L^0(X, \Sigma, \mu)$ be ideals with carriers Y and X respectively. A linear operator $T : L \to M$ is called *Multiplication-Conditional Expectation* (MCE-) *representable* if there exist:

(i) a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$;

(ii) a Riesz homomorphism $\varphi_L : L^0(Y, \Lambda, \nu) \to L^0(\Omega, \mathfrak{F}, \mathbb{P})$ with $\varphi_L(\mathbb{1})$ $\mathbb{P}$-a.e. strictly positive on Ω;

(iii) a sub-σ-algebra $\mathfrak{F}_M \subset \mathfrak{F}$ and an interval preserving Riesz isomorphism $\psi_M : L^0(\Omega, \mathfrak{F}_M, \mathbb{P}) \to L^0(X, \Sigma, \mu)$;

(iv) a function $m \in \mathfrak{M}(L_\Omega, M_\Omega)$, such that:

$$T = \psi_M S_m \varphi_L, \tag{4.1}$$

where $L_\Omega = \varphi_L(L)$, $M_\Omega = \psi_M^{-1} M$ and $S_m : L_\Omega \to M_\Omega$ is given by

$$S_m f = \mathbb{E}(mf \,|\, \mathfrak{F}_M).$$

We call $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$ a *representation triple for the operator* T and m a Φ-*kernel of* T. We say that Φ is *minimal* if $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$.

We note that M_Ω is an order dense ideal in $L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ since ψ is interval preserving. Also, as we have seen, we may always assume that Φ is minimal and in this case the Φ-kernel m of T is unique. The set of all linear operators $T : L \to M$ which are MCE-representable via a fixed triple Φ will be denoted by $\mathfrak{L}_\Phi(L, M)$ and we will also say that Φ is a representation triple for the class $\mathfrak{L}_\Phi(L, M)$. An operator $T \in \mathfrak{L}_\Phi(L, M)$ will be called Φ-*representable.*

We can represent this in the following commutative diagram

$$\begin{array}{ccc} L^0(Y,\Lambda,\nu) \supset L & \xrightarrow{\ T\ } & M \subset L^0(X,\Sigma,\mu) \\ {\scriptstyle \phi_L}\big\downarrow & & \big\uparrow{\scriptstyle \psi_M} \\ L^0(\Omega,\mathfrak{F}_L,\mathbb{P}) \supset L_\Omega & \xrightarrow[S_m]{} & M_\Omega \subset L^0(\Omega,\mathfrak{F}_M,\mathbb{P}). \end{array}$$

Example 4.2. Let (X, Σ, μ) and (Y, Λ, ν) be σ-finite measure spaces and let $L \subset L^0(Y, \Lambda, \nu)$ and $M \subset L^0(X, \Sigma, \mu)$ be ideals with carriers Y and X respectively. Let $k \in L^0(X \times Y, \Sigma \otimes \Lambda, \mu \otimes \nu)$ satisfy the condition

$$\int_Y |k(\cdot, y)f(y)| \,\mathrm{d}\nu(y) \in M, \quad \text{for all } f \in L.$$

The operator $K : L \to M$ defined by

$$Kf(x) = \int_Y k(x, y)f(y) \,\mathrm{d}\nu(y), \quad \text{for all } f \in L$$

is called an *absolute kernel operator* from L into M (see Chapter 13 from [19]). We will show that it is MCE-representable.

Let $w_1 \in L^1(X, \Sigma, \mu)$ and $w_2 \in L^1(Y, \Lambda, \nu)$ be strictly positive and satisfy $\int_X w_1 \,\mathrm{d}\mu = \int_Y w_2 \,\mathrm{d}\nu = 1$. Taking $\Omega := X \times Y$, $\mathfrak{F} := \Sigma \otimes \Lambda$ and $\mathbb{P} := w_1\mu \otimes w_2\nu$, we have that $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space. Define $(\varphi_L f)(x, y) := f(y)\mathbb{1}_X(x)$ for all $f \in L$ and $\mathfrak{F}_M = \{A \times Y : A \in \Sigma\}$. Then $L^0(\Omega, \mathfrak{F}_M, \mathbb{P}) = \{f \in L^0(\Omega, \mathfrak{F}, \mathbb{P}) : f(x, y) = g(x)\mathbb{1}_Y(y) \text{ for some } g \in L^0(X, \Sigma, \mu)\}$. If $f = g\mathbb{1}_Y \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$, define $\psi_M(f) = g$. Clearly, $\varphi_L(\mathbb{1}) = \mathbb{1}$ and ψ_M is a bijection from $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ onto $L^0(X, \Sigma, \mu)$ and therefore interval preserving. The triple $\Phi := ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$ represents K with Φ-kernel $m := w_2^{-1}k$. This follows immediately from the defining condition of an absolute kernel operator and the observation (see Example 2.4) that

$$\mathbb{E}(mf \,|\, \mathfrak{F}_M)(x, y) = \mathbb{E}(w_2^{-1}k \,|\, \mathfrak{F}_M)(x, y) = \int_Y k(x, z)f(z) \,\mathrm{d}\nu(z)$$

for all $f \in L$. Denoting by $\mathfrak{K}(L, M)$ the ideal of all absolute kernels in $L^0(X \times Y, \Sigma \otimes \lambda, \mu \otimes \nu)$, we see that $k \in \mathfrak{K}(L, M)$ if and only if $w_2^{-1}k \in \mathfrak{M}(L, M)$. These two ideals are therefore Riesz isomorphic. We also note that in this case $\mathfrak{F}_L = \{X \times B : B \in \Lambda\}$.

Example 4.3. Let $L \subset L^0(Y,\Lambda,\nu)$ and $M \subset L^0(X,\Sigma,\mu)$ be ideals of measurable functions on the σ-finite measure spaces (Y,Λ,ν) and (X,Σ,μ) having carriers Y and X respectively. Let $T : L \to M$ be an order continuous Riesz homomorphism. Then T is MCE-representable. To see this, we first extend T to a Riesz homomorphism φ_L from $L^0(Y,\Lambda,\nu)$ into $L^0(X,\Sigma,\mu)$ using Proposition 2.1. Next, let $X_T \in \Sigma$ be such that $\{\varphi_L(\mathbb{1})\}^{\mathrm{dd}} = \{\mathbb{1}_{X_T}\}^{\mathrm{dd}}$. Let w_M be a strictly positive function on X_T such that $\int_{X_T} w_M \,\mathrm{d}\mu = 1$ and set $\Omega := X_T$, $\mathbb{P} := w_M \cdot \mu$, $\mathfrak{F} := \{A \cap X_T : A \in \Sigma\}$ and $\mathfrak{F}_M = \mathfrak{F}$. Let ψ_M be the canonical embedding of $L^0(\Omega,\mathfrak{F},\mathbb{P})$ into $L^0(X,\Sigma,\mu)$. Clearly, φ_L is a Riesz homomorphism from $L^0(Y,\Lambda,\nu)$ into $L^0(\Omega,\mathfrak{F},\mathbb{P})$ with $\varphi_L(\mathbb{1})$ strictly positive on Ω. Take $m := \mathbb{1}$. Then, for all $f \in L$,

$$\psi_M S_m \varphi_L(f) = \mathbb{E}(m\varphi_L(f) \,|\, \mathfrak{F}_M) = \mathbb{E}(T(f) \,|\, \mathfrak{F}) = T(f),$$

since $\mathfrak{F}_M = \mathfrak{F}$. Thus T is MCE-representable.

The results in Section 3 are now applied to yield the following result.

Proposition 4.4. *Let (X,Σ,μ) and (Y,Λ,ν) be σ-finite measure spaces and let $L \subseteq L^0(Y,\Lambda,\nu)$ and $M \subseteq L^0(X,\Sigma,\mu)$ be ideals with carriers Y and X respectively. Let Φ be a representation triple for the class $\mathfrak{L}_\Phi(L,M)$. Then*

(i) *$\mathfrak{L}_\Phi(L,M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L,M)$;*

(ii) *if Φ is minimal, the mapping $m \mapsto \psi_M S_m \varphi_L$ is a Riesz isomorphism from $\mathfrak{M}(L_\Omega,M_\Omega)$ onto $\mathfrak{L}_\Phi(L,M)$.*

Proof. (i) Define $L_1 = L \cap C_{\varphi_L}$, where C_{φ_L} denotes the carrier of φ_L. Since C_{φ_L} is a band in $L^0(Y,\Lambda,\nu)$, it is clear that L_1 is a band in L. Let φ_1 be the restriction of φ_L to L_1. Then φ_1 is a Riesz isomorphism from L_1 onto L_Ω. By Proposition 2.1 (ii), $\mathrm{ran}(\psi_M)$ is a band in $L^0(X,\Sigma,\mu)$ and so $M_1 = M \cap \mathrm{ran}(\psi_M)$ is a band in M. Let ψ_1 denote the restriction of ψ_M to $M_\Omega = \psi_M^{-1}(M)$. Then ψ_1 is a Riesz isomorphism from M_Ω onto M_1. The space $\mathfrak{L}_{\mathrm{n}}(L_1,M_1)$ can be identified with the band in $\mathfrak{L}_{\mathrm{n}}(L,M)$ consisting of all operators $T \in \mathfrak{L}_{\mathrm{n}}(L,M)$ for which C_T is contained in L_1 and $\mathrm{ran}(T)$ is contained in M_1. From Definition 4.1 it is clear that $\mathfrak{L}_\Phi(L,M) \subset \mathfrak{L}_{\mathrm{n}}(L_1,M_1)$. Hence, it is sufficient to show that $\mathfrak{L}_\Phi(L,M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L_1,M_1)$. For this purpose we define

$$\alpha : \mathfrak{L}_{\mathrm{n}}(L_\Omega,M_\Omega) \to \mathfrak{L}_{\mathrm{n}}(L_1,M_1)$$

by $\alpha(S) = \psi_1 S \varphi_1$ for all $S \in \mathfrak{L}_{\mathrm{n}}(L_\Omega,M_\Omega)$. Since α is bi-positive, it is a Riesz isomorphism from $\mathfrak{L}_{\mathrm{n}}(L_\Omega,M_\Omega)$ onto $\mathfrak{L}_{\mathrm{n}}(L_1,M_1)$. Observe that the relation (4.1) in Definition 4.1 can now be written as $T = \alpha(S_m)$, and so

$$\mathfrak{L}_\Phi(L,M) = \alpha[\mathfrak{L}_{\mathfrak{M}}(L_\Omega,M_\Omega)].$$

By Proposition 3.5, $\mathfrak{L}_{\mathfrak{M}}(L_\Omega,M_\Omega)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L_\Omega,M_\Omega)$. Therefore, $\mathfrak{L}_\Phi(L,M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L_1,M_1)$, and hence a band in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

(ii) By Proposition 3.5, $m \mapsto S_m$ is a Riesz isomorphism from $\mathfrak{M}(L_\Omega,M_\Omega)$ onto $\mathfrak{L}_{\mathfrak{M}}(L_\Omega,M_\Omega)$, and then the statement (ii) is now also clear. ∎

As a result of the preceding theorem and Example 4.2 we obtain the following well known corollary. This result was proved by A.R. Schep in [16].

Corollary 4.5. *Let (X,Σ,μ) and (Y,Λ,ν) be σ-finite measure spaces and let $L \subset L^0(Y,\Lambda,\nu)$ and $M \subset L^0(X,\Sigma,\mu)$ be ideals. The set of all absolute kernel operators from L into M is a band in $\mathfrak{L}_{\mathrm{n}}(L,M)$ which is Riesz isomorphic to the ideal of all kernels $\mathfrak{K}(L,M)$ in $L^0(X\times Y,\Sigma\otimes\lambda,\mu\otimes\nu)$.*

Considering adjoints, we show that the adjoint of an MCE-representable operator is again MCE-representable.

Theorem 4.6. *Let (X,Σ,μ) and (Y,Λ,ν) be σ-finite measure spaces and let $L \subset L^0(Y,\Lambda,\nu)$ and $M \subset L^0(X,\Sigma,\mu)$ be order dense ideals for which L' and M' are order dense ideals as well. Let $T \in \mathfrak{L}_\Phi(L,M)$, where $\Phi = ((\Omega,\mathfrak{F},\mathbb{P}),\varphi_L,\psi_M)$ is a representation triple for L and M, have Φ-kernel m. Then the adjoint T' satisfies $T' \in \mathfrak{L}_{\Phi'}(M',L')$ where $\Phi' = ((\Omega,\mathfrak{F},\mathbb{P}),\psi'_M,\varphi'_L)$ is a representation triple for M' and L' and T' has Φ'-kernel m.*

Proof. Since both φ_L and ψ_M are order continuous interval preserving Riesz homomorphisms, φ'_L and ψ'_M are, by Lemma 2.2, order continuous interval preserving Riesz homomorphisms as well and can be extended uniquely to order continuous interval preserving Riesz homomorphisms $\varphi'_L : L^0(\Omega,\mathfrak{F}_L,\mathbb{P}) \to L^0(Y,\Lambda,\nu)$ and $\psi'_M : L^0(X,\Sigma,\mu) \to L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$. Since $\varphi_L(\mathbb{1}_Y)$ is strictly positive, $\varphi_L(L)$ is order dense in $L^0(\Omega,\mathfrak{F}_L,\mathbb{P})$ and it follows that φ'_L is injective by Lemma 2.2 (i). Also, since ψ_M is injective, $\psi'_M(\mathbb{1}_X)$ is strictly positive (Lemma 2.2 (ii)). By Proposition 2.1 (ii), $\operatorname{ran}(\psi_M)$ is a band in $L^0(X,\Sigma,\mu)$ and so, since $M_\Omega = \psi_M^{-1}(M)$ it easily follows that $\psi_M(M_\Omega) = M \cap \operatorname{ran}(\psi_M)$ which is a band in M. Thus, by Lemma 2.1 (iii), we have that $\psi'_M(M') = M'_\Omega$. Hence, using the notation in Definition 4.1 we have that $(M')_\Omega = \psi'_M(M') = M'_\Omega$. Also, $\widehat{\psi'_M}(X) = \Omega$ and by the fact that ψ'_M is interval preserving, it follows that $\mathfrak{F}_{M'} := \widehat{\psi'_M}(\Sigma) = \mathfrak{F}_M$. Take $\mathfrak{F}_{L'} = \mathfrak{F}_L$ and define, as required, $(L')_\Omega = (\varphi'_L)^{-1}(L') \supset L'_\Omega$.

Let $m \in \mathfrak{M}(L,M)$ and consider the operator $S_m : L_\Omega \to M_\Omega$ defined by $S_m u := \mathbb{E}(mu \,|\, \mathfrak{F}_M)$. We claim that $m \in \mathfrak{M}(M'_\Omega, L'_\Omega) \subset \mathfrak{M}((M')_\Omega,(L')_\Omega)$ and that

$$S'_m f = \mathbb{E}(mf \,|\, \mathfrak{F}_L) \quad \text{for all } f \in M'_\Omega.$$

Indeed, if $f \in M'_\Omega$ and $g \in L_\Omega$, then

$$\begin{aligned}\int_\Omega |g|\mathbb{E}(|mf| \,|\, \mathfrak{F}_L)\,\mathrm{d}\mathbb{P} &= \int_\Omega \mathbb{E}(|mfg| \,|\, \mathfrak{F}_L)\,\mathrm{d}\mathbb{P} \\ &= \int_\Omega |mfg|\,\mathrm{d}\mathbb{P} = \int_\Omega |f|\mathbb{E}(|mg| \,|\, \mathfrak{F}_M)\,\mathrm{d}\mathbb{P} < \infty,\end{aligned}$$

since $\mathbb{E}(|mg| \,|\, \mathfrak{F}_M) \in M_\Omega$. This shows that $\mathbb{E}(|mf| \,|\, \mathfrak{F}_L) \in L'_\Omega$ for all $f \in M'_\Omega$. It follows from this that $m \in \mathfrak{M}(M'_\Omega, L'_\Omega)$ and applying again Proposition 2.5 we get

$$\begin{aligned}\int_\Omega g\mathbb{E}(mf \,|\, \mathfrak{F}_L)\,\mathrm{d}\mathbb{P} &= \int_\Omega \mathbb{E}(mfg \,|\, \mathfrak{F}_L)\,\mathrm{d}\mathbb{P} = \int_\Omega mfg\,\mathrm{d}\mathbb{P} \\ &= \int_\Omega \mathbb{E}(mfg \,|\, \mathfrak{F}_M)\,\mathrm{d}\mathbb{P} = \int_\Omega f\mathbb{E}(mg \,|\, \mathfrak{F}_M)\,\mathrm{d}\mathbb{P},\end{aligned}$$

proving the claim. Therefore, for any $T = \psi_M S_m \varphi_L \in \mathfrak{L}_\Phi(L,M)$, we have that $T' = \varphi'_L S'_m \psi'_M$ with $S'_m = \mathbb{E}(m(\cdot)\,|\,\mathfrak{F}_L)$, and $m \in \mathfrak{M}((M')_\Omega, (L')_\Omega)$. Therefore, since ψ'_M and φ'_L satisfy all the conditions required in Definition 4.1, the proof is complete. ∎

5. ABSOLUTE τ-KERNEL OPERATORS

We will discuss in this section the notion of *absolute τ-kernel operators* which is an extension of kernel operators and partial integral operators. We show that these operators are MCE-representable.

Let (X,Σ,μ),(Y,Λ,ν) and (Z,Γ,λ) be σ-finite measure spaces and suppose that $L \subseteq L^0(Y,\Lambda,\nu)$ and $M \subseteq L^0(X,\Sigma,\mu)$ are ideals with carriers Y and X respectively. Let $\tau : X \times Z \to Y$ be a $(\Sigma \otimes \Gamma, \Lambda)$-measurable null-preserving mapping with respect to $\mu \otimes \lambda$ and ν.

Definition 5.1. A function $k \in L^0(X \times Z, \Sigma \otimes \Gamma, \mu \otimes \lambda)$ is called an *absolute τ-kernel for L and M* if:

$$\int_Z |k(\,\cdot\,,z) f(\tau(\,\cdot\,,z))| \,\mathrm{d}\lambda(z) \in M \quad \text{for all } f \in L.$$

The collection of all such τ-kernels will be denoted by $\mathfrak{K}_\tau(L,M)$.

For $k \in \mathfrak{K}_\tau(L,M)$ and $f \in L$ the function

$$Kf(x) = \int_Z k(x,z) f(\tau(x,z)) \,\mathrm{d}\lambda(z)$$

is μ-a.e. well defined on X and $Kf \in M$. This defines a linear, order bounded, order continuous operator K from L into M, i.e., $K \in \mathfrak{L}_{\mathrm{n}}(L,M)$. Such an operator K will be called an *absolute τ-kernel operator with kernel $k(x,z)$*. The collection of all such operators will be denoted by $\mathfrak{L}_{\tau k}(L,M)$. So $\mathfrak{L}_{\tau k}(L,M) \subseteq \mathfrak{L}_{\mathrm{n}}(L,M)$ as a linear subspace. It is clear that $k \geqslant 0$ $\mu \otimes \lambda$-a.e. on $X \times Z$ implies that $K \geqslant 0$.

We note that if $(Z,\Gamma,\lambda) = (Y,\Lambda,\nu)$ and if $\tau(x,z) = z$ then the τ-absolute kernel operator K is an absolute kernel operator as defined in Example 4.2.

Example 5.2. Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be σ-finite measure spaces and suppose that L and M are ideals in $L^0(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$, with carrier $X_1 \times X_2$. Given a function $k_1 \in L^0(X_1 \times X_2 \times X_1, \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_1, \mu_1 \otimes \mu_2 \otimes \mu_1)$ such that

$$\int_{X_1} |k_1(\cdot,\cdot,z) f(z,\cdot)| \,\mathrm{d}\mu_1(z) \in M, \quad \forall\, f \in L,$$

we define the corresponding operator $K_1 : L \to M$ by

$$(K_1 f)(x_1,x_2) = \int_{X_1} k_1(x_1,x_2,z) f(z,x_2) \,\mathrm{d}\mu_1(z), \quad \mu_1 \otimes \mu_2\text{-a.e. on } X_1 \times X_2 \tag{5.1}$$

for all $f \in L$. Such an operator K_1 will be called an absolute partial integral operator (see e.g. [2]). The collection of all operators of this form will be denoted by $\mathfrak{P}_1(L,M)$. Taking in the above definition $X = Y = X_1 \times X_2$, $Z = X_1$ and $\tau_1 : X \times Z \to Y$ defined by $\tau_1(x_1,x_2,z) = (z,x_2)$, we see that such operators are τ_1-kernel operators.

Similarly, if $k_2 \in L^0(X_1 \times X_2 \times X_2, \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_2, \mu_1 \otimes \mu_2 \otimes \mu_2)$ satisfies

$$\int\limits_{X_2} |k_2(\cdot,\cdot,z) f(\cdot,z)| \,\mathrm{d}\mu_2(z) \in M, \quad \forall\, f \in L,$$

the corresponding partial integral operator is defined by

$$(K_2 f)(x_1,x_2) = \int\limits_{X_2} k_1(x_1,x_2,z) f(x_1,z) \,\mathrm{d}\mu_2(z), \quad \mu_1 \otimes \mu_2\text{-a.e. on } X_1 \times X_2$$

for all $f \in L$. The collection of all operators of this form will be denoted by $\mathfrak{P}_2(L,M)$. These operators are examples of τ-kernel operators as well, as is seen by taking $X = Y = X_1 \times X_2$, $Z = X_2$ and $\tau_2 : X \times Z \to Y$ defined by $\tau_2(x_1,x_2,z) = (x_1,z)$.

In the formulation of the next theorem it will be convenient to introduce the following sub-σ-algebras of $\Sigma \otimes \Gamma$:

(i) $\Lambda_0 := \{\tau^{-1}(A) : A \in \Lambda\} = \{\tau_*(A) : A \in \Lambda\}$;

(ii) $\Sigma_0 := \{A \times Z : A \in \Sigma\}$.

Theorem 5.3. *Let $(X,\Sigma,\mu),(Y,\Lambda,\nu)$ and (Z,Γ,λ) be three σ-finite measure spaces and suppose that $L \subseteq L^0(Y,\Lambda,\nu)$ and $M \subseteq L^0(X,\Sigma,\mu)$ are ideals with carriers Y and X respectively. Suppose that $\tau : X \times Z \to Y$ is a $(\Sigma \otimes \Gamma, \Lambda)$-measurable null-preserving mapping. Then the following hold:*

(i) *Every absolute τ-kernel operator $T \in \mathfrak{L}_{\tau k}(L,M)$ is* MCE*-representable via a fixed representation triple Φ and $\mathfrak{L}_{\tau k}(L,M) = \mathfrak{L}_\Phi(L,M)$.*

(ii) *$\mathfrak{L}_{\tau k}(L,M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L,M)$.*

(iii) *If $\sigma(\Lambda_0,\Sigma_0) = \Sigma \otimes \Gamma$ (modulo $\mu \otimes \lambda$-null sets), then for each $T \in \mathfrak{L}_{\tau k}(L,M)$ its corresponding kernel $k \in \mathfrak{K}_\tau(L,M)$ is uniquely determined. Moreover, $T \geqslant 0$ if and only if $k \geqslant 0$; the kernel of $|T|$ is $|k|$.*

(iv) *In general, if $T \in \mathfrak{L}_{\tau k}(L,M)$ and $T \geqslant 0$ then there exists a kernel $k \in \mathfrak{K}_\tau(L,M)$ for T which satisfies $k \geqslant 0$; if $T \in \mathfrak{L}_{\tau k}(L,M)$ there exists a kernel $k \in \mathfrak{K}_\tau(L,M)$ for T such that $|T|$ has kernel $|k|$.*

Proof. Let $0 \leqslant w_1 \in L^1(X,\Sigma,\mu)$ be such that $w_1(x) > 0$ μ-a.e. and $\int\limits_X w_1 \,\mathrm{d}\mu = 1$ and let $0 \leqslant w_2 \in L^1(Z,\Gamma,\lambda)$ be such that $w_2(z) > 0$ λ-a.e. and $\int\limits_Z w_2 \,\mathrm{d}\lambda = 1$. Define $\Omega := X \times Z$, $\mathfrak{F} = \Sigma \otimes \Gamma$ and $\mathbb{P} = (w_1\mu) \otimes (w_2\lambda)$. Then $(\Omega,\mathfrak{F},\mathbb{P})$ is a probability space and $\mathbb{P}$ is equivalent to the product measure $\mu \otimes \lambda$. Define

$$\varphi_L := \varphi_\tau : L^0(Y,\Lambda,\nu) \to L^0(\Omega,\mathfrak{F},\mathbb{P}).$$

Then φ_L is a Riesz homomorphism satisfying $\varphi_L(\mathbb{1}) = \mathbb{1}$ and $\mathfrak{F}_L = \Lambda_0$ in the notation defined above. Let $\mathfrak{F}_M = \Sigma_0$ and note that $f \in L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ if and only $f = g\mathbb{1}_Z$ with $g \in L^0(X,\Sigma,\mu)$. Define $\psi_M : L^0(\Omega,\mathfrak{F}_M,\mathbb{P}) \to L^0(X,\Sigma,\mu)$ by $\psi_M(f) = \psi_M(g\mathbb{1}_Z) = g$. Then ψ_M is a Riesz isomorphism which is interval preserving (in fact it is a surjection). We will use $\Phi = ((\Omega,\mathfrak{F},\mathbb{P}),\varphi_L,\psi_M)$ as a representation triple and show that $\mathfrak{L}_\Phi(L,M) = \mathfrak{L}_{\tau k}(L,M)$ which will prove (i) and (ii).

Since $L_\Omega = \{f\circ\tau : f\in L\}$, it follows from the result in Example 2.4 that $m\in\mathfrak{M}(L_\Omega,M_\Omega)$ if and only if

$$\mathbb{E}(|mf|\,|\,\Sigma_0)(x,z) = \int_Z |m(x,z)f(\tau(x,z))w_2(z)|\,\mathrm{d}\lambda(z) < \infty, \quad \mu\text{-a.e. on } X,$$

and

$$\mathbb{E}(|mf|\,|\,\Sigma_0)(\cdot,z) = \int_Z |m(\cdot,z)f(\tau(\cdot,z))w_2(z)|\,\mathrm{d}\lambda(z) \in M \quad \forall f\in L.$$

Consequently, $m\in\mathfrak{M}(L_\Omega,M_\Omega)$ if and only if $m\cdot w_2\in\mathfrak{K}_\tau(L,M)$. Moreover, if $m\in\mathfrak{M}(L_\Omega,M_\Omega)$ then

$$\psi_M S_m\,\varphi_L f(x) = \int_Z m(x,z)f(\tau(x,z))w_2(z)\,\mathrm{d}\lambda(z),$$

which shows that $\psi_M S_m\,\varphi_L\in\mathfrak{L}_{\tau k}(L,M)$ with kernel $k(x,z) = m(x,z)w_2(z)$.

Conversely, if $k\in\mathfrak{K}_\tau(L,M)$ then $kw_2^{-1}\in\mathfrak{M}(L_\Omega,M_\Omega)$ and so the corresponding τ-kernel operator T satisfies

$$T = \psi_M S_{kw_2^{-1}}\,\varphi_L\in\mathfrak{L}_\Phi(L,M).$$

This shows that $\mathfrak{L}_{\tau k}(L,M) = \mathfrak{L}_\Phi(L,M)$ and that $\mathfrak{K}_\tau(L,M) = \{m\cdot w_2 : m\in\mathfrak{M}(L_\Omega,M_\Omega)\}$. Hence, $\mathfrak{K}_\tau(L,M)$ and $\mathfrak{M}(L_\Omega,M_\Omega)$ are Riesz isomorphic. Since $\mathfrak{L}_\Phi(L,M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L,M)$, it is now clear that $\mathfrak{L}_{\tau k}(L,M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

If Φ is minimal, then the mapping $m\mapsto\psi_M S_m\,\varphi_L$ is a Riesz isomorphism from $\mathfrak{M}(L_\Omega,M_\Omega)$ onto $\mathfrak{L}_\Phi(L,M)$. Since $k\mapsto kw_2^{-1}$ is a Riesz isomorphism from $\mathfrak{K}_\tau(L,M)$ onto $\mathfrak{M}(L_\Omega,M_\Omega)$, the results of (iii) are clear.

In general we replace $\Sigma\otimes\Gamma = \mathfrak{F}$ by $\sigma(\Lambda_0,\Sigma_0)$, to get a minimal representation. In this case we also replace $m\in\mathfrak{M}(L_\Omega,M_\Omega)$ by $\mathbb{E}(m\,|\,\sigma(\Lambda_0,\Sigma_0))$ without changing the corresponding operator. Consequently, if $T\geqslant 0$ in $\mathfrak{L}_{\tau k}(L,M)$ with kernel $k\in\mathfrak{K}_\tau(L,M)$, and we define

$$\widetilde{k} = \mathbb{E}(kw_2^{-1}\,|\,\sigma(\Lambda_0,\Sigma_0))w_2, \tag{5.2}$$

then $\widetilde{k}\in\mathfrak{K}_\tau(L,M)$ is a kernel for T which satisfies $\widetilde{k}\geqslant 0$. If $T\in\mathfrak{L}_{\tau k}(L,M)$ with kernel $k\in\mathfrak{K}_\tau(L,M)$, define $\widetilde{k}$ by (5.2), then $|\widetilde{k}|$ is a kernel for $|T|$. ∎

Corollary 5.4. *Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2), L and M be as in Example* 5.2. *Then both collections $\mathfrak{P}_1(L,M)$ and $\mathfrak{P}_2(L,M)$ of partial integral operators are bands in $\mathfrak{L}_{\mathrm{n}}(L,M)$. Moreover, if $K_1\in\mathfrak{P}_1(L,M)$ is given by* (5.1) *then the absolute value of K_1 is given by*

$$|K_1|f(x_1,x_2) = \int_{X_1} |k_1(x_1,x_2,z)|f(z,x_2)\,\mathrm{d}\mu_1(z), \quad \mu_1\otimes\mu_2\text{-a.e. on } X_1\times X_2$$

for all $f \in L$*, and similarly for* $\mathfrak{P}_2(L, M)$*.*

Proof. If we take $X = Y = X_1 \times X_2$, $Z = X_1$ and $\tau : X \times Z \to Y$ given by $\tau_1(x_1, x_2, z) = (z, x_2)$, then $\mathfrak{P}_1(L, M) = \mathfrak{L}_{\tau_1 k}(L, M)$, as has already been observed in Example 5.2. Hence it follows immediately from (ii) in the above theorem that $\mathfrak{P}_1(L, M)$ is a band in $\mathfrak{L}_{\mathrm{n}}(L, M)$. Furthermore, it is easily seen that the condition in (iii) in Theorem 5.3 is satisfied, which yields the second statement of the corollary. The results for $\mathfrak{P}_2(L, M)$ follow analogously. ∎

6. DISJOINTNESS OF MCE-REPRESENTABLE OPERATORS

The main result in this section is Theorem 6.1, which gives sufficient conditions on two representation triples Φ and Φ' implying that the corresponding bands $\mathfrak{L}_\Phi(L, M)$ and $\mathfrak{L}_{\Phi'}(L, M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L, M)$. This result is then illustrated by a number of examples, involving kernel operators, Riesz homomorphisms and partial integral operators.

Let (Y, Λ, ν) and (X, Σ, μ) be σ-finite measure spaces. Suppose that L is an ideal in $L^0(Y, \Lambda, \nu)$ and that M is an ideal in $L^0(X, \Sigma, \mu)$. We assume that the carriers of L and M are equal to Y and X respectively. For $Y_1 \in \Lambda$ we use the notation $\Lambda_{Y_1} := \{A \in \Lambda : A \subseteq Y_1\}$, which is itself a σ-algebra. We recall that the measure ν is called *non-atomic* on the sub-σ-algebra $\Gamma \subset \Lambda_{Y_1}$ if for every $A \in \Gamma$ with $\nu(A) > 0$ there exists some $B \in \Gamma$ such that $B \subseteq A$, $\nu(B) > 0$ and $\nu(A \setminus B) > 0$.

Theorem 6.1. *Let* $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$ *and* $\Phi' = ((\Omega', \mathfrak{F}', \mathbb{P}'), \varphi'_L, \psi'_M)$ *be two representation triples for* L *and* M*. Let* $Y_1 \in \Lambda$ *be the carrier set of* $C_{\varphi_L} \cap C_{\varphi'_L}$*. Assume that there exists a sub-*σ*-algebra* $\Lambda_0 \subseteq \Lambda_{Y_1}$ *such that:*

(i) ν *is non-atomic on* Λ_0*;*

(ii) $\widehat{\varphi}_L(\Lambda_0)$ *and* $\mathfrak{F}_M$ *are independent, i.e.,* $\mathbb{P}(F_1 \cap F_2) = \mathbb{P}(F_1)\mathbb{P}(F_2)$ *for all* $F_1 \in \widehat{\varphi}_L(\Lambda_0)$ *and* $F_2 \in \mathfrak{F}_M$*;*

(iii) $\widehat{\varphi}'_L(\Lambda_0) \subseteq \mathfrak{F}'_M$*.*

Then the bands $\mathfrak{L}_\Phi(L, M)$ *and* $\mathfrak{L}_{\Phi'}(L, M)$ *are disjoint in* $\mathfrak{L}_{\mathrm{n}}(L, M)$*.*

Proof. For convenience we assume, as we may, that $\varphi_L(\mathbb{1}) = \mathbb{1}$ and $\varphi'_L(\mathbb{1}) = \mathbb{1}$. We have to show that $T \wedge S = 0$ for any $0 \leqslant T \in \mathfrak{L}_\Phi(L, M)$ and $0 \leqslant S \in \mathfrak{L}_{\Phi'}(L, M)$. The proof will divided into several steps.

Step 1. Let $0 \leqslant m \in \mathfrak{M}(L_\Omega, M_\Omega)$ be such that $T = \psi_M S_m \varphi_L$, where $S_m : L_\Omega \to M_\Omega$ is as before. For $k = 1, 2 \ldots$ let $T_k = \psi_M S_{m \wedge k\mathbb{1}} \varphi_L$. Now $k(m \wedge \mathbb{1}) \geqslant m \wedge k\mathbb{1} \uparrow m$ in $\mathfrak{M}(L_\Omega, M_\Omega)$ implies that $kT_1 \geqslant T_k \uparrow T$ in $\mathfrak{L}_\Phi(L, M)$, and so it is sufficient to show that $T_1 \wedge S = 0$. Consequently, we may assume without loss of generality that $0 \leqslant m \leqslant 1$ on Ω.

Step 2. In the next step we show that for every $\varepsilon > 0$ there exist disjoint $A_1, \ldots, A_n \in \Lambda_0$ such that $\bigcup_{j=1}^{n} A_j = Y_1$ and

$$\bigvee_{j=1}^{n} T(u\mathbb{1}_{A_j}) \leqslant \varepsilon \psi_M(\mathbb{1})$$

for all $0 \leqslant u \leqslant \mathbb{1}$ in L.

To this end we first observe that $\mathbb{P}$ is non-atomic on $\widehat{\varphi}_L(\Lambda_0)$. Indeed, let $F \in \widehat{\varphi}_L(\Lambda_0)$ with $\mathbb{P}(F) > 0$ be given. Then, $F = \widehat{\varphi}_L(A)$ for some $A \in \Lambda_0$. Since ν is non-atomic on Λ_0 and $\nu(A) > 0$, there exists $B \in \Lambda_0$ such that $B \subseteq A$, $\nu(B) > 0$ and $\nu(A \setminus B) > 0$. Let $G = \widehat{\varphi}_L(B)$. Then $G \in \widehat{\varphi}_L(\Lambda_0)$, $G \subseteq F$, and since $\widehat{\varphi}_L$ is injective on Λ_0 we have $\mathbb{P}(G) > 0$ and $\mathbb{P}(F \setminus G) > 0$.

Given $\varepsilon > 0$ there exist disjoint $F_1, \ldots, F_n$ in $\widehat{\varphi}_L(\Lambda_0)$ such that $\bigcup_{j=1}^n F_j = \widehat{\varphi}_L(Y_1)$ and $\mathbb{P}(F_j) \leqslant \varepsilon$ for all $j = 1, \ldots, n$. Take $A_1, \ldots, A_n \in \Lambda_0$ such that $F_j = \widehat{\varphi}_L(A_j)$. Using once more that $\widehat{\varphi}_L$ is injective on Λ_0 it follows that $A_1, \ldots, A_n$ are mutually disjoint and $\bigcup_{j=1}^n A_j = Y_1$.

For $u \in L$ such that $0 \leqslant u \leqslant \mathbb{1}$, we find that

$$\bigvee_{j=1}^n T(u\mathbb{1}_{A_j}) = \bigvee_{j=1}^n \psi_M S_m \varphi_L(u\mathbb{1}_{A_j}) = \psi_M \bigvee_{j=1}^n S_m(\varphi_L(u)\mathbb{1}_{F_j}) \leqslant \psi_M \bigvee_{j=1}^n S_m(\mathbb{1}_{F_j}).$$

Since $0 \leqslant m \leqslant \mathbb{1}$, we have

$$S_m(\mathbb{1}_{F_j}) = \mathbb{E}(m\mathbb{1}_{F_j} \,|\, \mathfrak{F}_M) \leqslant \mathbb{E}(\mathbb{1}_{F_j}|\mathfrak{F}_M) = \mathbb{P}(F_j)\mathbb{1},$$

where we use in the last equality that $\mathbb{1}_{F_j}$ is independent of $\mathfrak{F}_M$. Consequently,

$$\bigvee_{j=1}^n T(u\mathbb{1}_{A_j}) \leqslant \psi_M \bigvee_{j=1}^n \mathbb{P}(F_j)\mathbb{1} \leqslant \varepsilon\psi_M(\mathbb{1}).$$

Step 3. For $0 \leqslant u \in L$ and disjoint sets A and B in Λ_0 we have $S(u\mathbb{1}_A) \wedge S(u\mathbb{1}_B) = 0$. Indeed, let $0 \leqslant m' \in \mathfrak{M}(L_{\Omega'}, M_{\Omega'})$ be such that $S = \psi'_M S_{m'} \varphi'_L$. Given $A \in \Lambda_0$, put $F := \widehat{\varphi}'_L(A)$. Since by hypothesis $F \in \mathfrak{F}'_M$, we have

$$S_{m'}\varphi'_L(u\mathbb{1}_A) = S_{m'}(\varphi'_L(u)\mathbb{1}_F) = \mathbb{E}(m'\,\varphi'_L(u)\mathbb{1}_F \,|\, \mathfrak{F}'_M) = \mathbb{1}_F\mathbb{E}(m'\,\varphi'_L(u) \,|\, \mathfrak{F}'_M).$$

The assertion now follows from the fact that $\widehat{\varphi}'_L$ preserves disjointness and from the fact that ψ'_M is a Riesz homomorphism.

Step 4. Take $u \in L$ with $0 \leqslant u \leqslant \mathbb{1}$. First observe that

$$(S \wedge T)(u) = (S \wedge T)(u\mathbb{1}_{Y_1}),$$

since the carrier of $S \wedge T$ is contained in $C_{\varphi_L} \cap C_{\varphi'_L}$. Now let $\varepsilon > 0$ be given and let $A_j \in \Lambda_0$ $(j = 1, \ldots, n)$ be as in Step 2 of the proof. Using that, by Step 3, the $S(u\mathbb{1}_{A_j})$'s are mutually disjoint, we find that

$$\begin{aligned}(S \wedge T)(u\mathbb{1}_{Y_1}) &= \sum_{j=1}^n (S \wedge T)(u\mathbb{1}_{A_j}) \leqslant \sum_{j=1}^n S(u\mathbb{1}_{A_j}) \wedge T(u\mathbb{1}_{A_j}) \\ &= \bigvee_{j=1}^n S(u\mathbb{1}_{A_j}) \wedge T(u\mathbb{1}_{A_j}) \leqslant \bigvee_{j=1}^n S(u\mathbb{1}_{A_j}) \wedge \varepsilon\psi_M(\mathbb{1}) \leqslant (Su) \wedge \varepsilon\psi_M(\mathbb{1}).\end{aligned}$$

This shows that

$$(S \wedge T)u \leqslant (Su) \wedge \varepsilon\psi_M(\mathbb{1})$$

for all $\varepsilon > 0$ and so $(S \wedge T)u = 0$ for all $0 \leqslant u \leqslant \mathbb{1}$ in L. For arbitrary $0 \leqslant u \in L$ note that $0 \leqslant u \wedge k\mathbb{1} \uparrow u$ and $0 \leqslant u \wedge k\mathbb{1} \leqslant k(u \wedge \mathbb{1})$. By the above, $(S \wedge T)(u \wedge \mathbb{1}) = 0$. Hence, $(S \wedge T)(u \wedge k\mathbb{1}) = 0$ for all $k = 1, 2, \ldots$. By the order continuity of $S \wedge T$ this implies $(S \wedge T)u = 0$. We may thus conclude that $S \wedge T = 0$ and the proof is complete. ∎

Remark 6.2. (i) Observe that the above theorem applies in particular if there exists a sub-σ-algebra $\Lambda_1 \subseteq \Lambda$ such that ν is non-atomic on Λ_1, $\widehat{\varphi}_L(\Lambda_1)$ and $\mathfrak{F}_M$ are independent, and $\widehat{\varphi}'_L(\Lambda_1) \subseteq \mathfrak{F}'_M$. Indeed, let $\Lambda_0 = \{A \cap Y_1 : A \in \Lambda_1\}$. Then it remains to show that ν is non-atomic on Λ_0. For this we may assume that ν is finite, replacing ν by an equivalent measure if necessary. Take $A \in \Lambda_1$ such that $\nu(A \cap Y_1) = \beta > 0$. There exists a partition $\{A_j\}_{j=1}^n$ in Λ_1 of A such that $\nu(A_j) < \beta$ for all $j = 1, \dots, n$. Since $\sum \nu(A_j \cap Y_1) = \beta$, this sum has at least two non-zero terms, $\nu(A_k \cap Y_1)$ and $\nu(A_\ell \cap Y_1)$, say. Then $\nu(A_k \cap Y_1) > 0$ and $\nu(Y_1 \setminus (A_k \cap Y_1)) \geqslant \nu(Y_1 \cap A_\ell) > 0$. This shows that ν is non-atomic on Λ_0.

(ii) If, in the situation of the above theorem, the carrier ideals C_{φ_L} and $C_{\varphi'_L}$ are disjoint, then it is obvious $\mathfrak{L}_\Phi(L, M)$ and $\mathfrak{L}_{\Phi'}(L, M)$ are disjoint bands. This case is included in the theorem, since $Y_1 = \emptyset$ in this case.

Now we will discuss some examples which are applications of the above theorem.

Example 6.3. Let (X, Σ, μ) and (Y, Λ, ν) be σ-finite measure spaces with ν non-atomic, and let L and M be order dense ideals in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$ respectively. Then the band $\mathfrak{L}_k(L, M)$ of all absolute kernel operators is disjoint to all Riesz homomorphisms from L into M (cf. Theorem 94.7 from [19], and Example 4.3 from [10]). Indeed, it was shown in Example 4.2 that all operators in $\mathfrak{L}_k(L, M)$ are MCE-representable with respect to $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$. In this case it is clear that $\widehat{\varphi}_L(\Lambda) = \mathfrak{F}_L = \{X \times B : B \in \Lambda\}$ is independent of $\mathfrak{F}_M = \{A \times Y : A \in \Sigma\}$.

Now suppose that $T : L \to M$ is a Riesz homomorphism. To show that T is disjoint to $\mathfrak{L}_k(L, M)$ we may assume that T is order continuous, as $\mathfrak{L}_k(L, M) \subseteq \mathfrak{L}_{\mathrm{n}}(L, M)$. Let $\Phi' = ((\Omega', \mathfrak{F}', \mathbb{P}'), \varphi'_L, \psi'_M)$ be the representation triple of T as constructed in Example 4.3. Then $\widehat{\varphi}'_L(\Lambda) = \mathfrak{F}'_L \subseteq \mathfrak{F}'_M = \mathfrak{F}'$. Hence the conditions of Theorem 6.1 are fulfilled and we may conclude that T is disjoint to $\mathfrak{L}_k(L, M)$.

Example 6.4. Let (X_1, Σ_1, μ_1), (X_2, Σ_2, μ_2), L and M be as in Example 5.2. We will show that if μ_1 is non-atomic, then $\mathfrak{P}_1(L, M)$ is disjoint to all Riesz homomorphisms from L into M. As above, it is suficient to show that $\mathfrak{P}_1(L, M)$ is disjoint to any order continuous Riesz homomorphism $T : L \to M$. Let $\Phi' = ((\Omega', \mathfrak{F}', \mathbb{P}'), \varphi'_L, \psi'_M)$ be the representation triple of T as given in Example 4.3. Then $\widehat{\varphi}'_L(\Sigma_1 \otimes \Sigma_2) \subseteq \mathfrak{F}'_M$, so condition (iii) in Theorem 6.1 is satisfied for any $\Lambda_0 \subseteq \Sigma_1 \otimes \Sigma_2$.

Now let $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$ be the representation triple for $\mathfrak{P}_1(L, M)$, as constructed in the proof of Theorem 5.3. In the present situation we get $\Omega = X_1 \times X_2 \times X_1$, $\mathfrak{F} = \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_1$ and for $\mathbb{P}$ we can take the product measure $w_1\mu_1 \otimes w_2\mu_2 \otimes w_1\mu_1$, where $w_i \geqslant 0$ and $\int_{X_i} w_i \, \mathrm{d}\mu_i = 1$ for $i = 1, 2$. Furthermore, $\mathfrak{F}_M = \{C \times X_1 : C \in \Sigma_1 \otimes \Sigma_2\}$ and $\varphi_L = \varphi_{\tau_1}$, where $\tau_1 : X_1 \times X_2 \times X_1 \to X_1 \times X_2$ is given by $\tau_1(x_1, x_2, z) = (z, x_2)$. Hence,

$$\widehat{\varphi}_L(A \times B) = X_1 \times B \times A \quad \forall\, A \in \Sigma_1,\, B \in \Sigma_2.$$

Define $\Lambda_0 = \{A \times X_2 : A \in \Sigma_1\}$. Since μ_1 is non-atomic it is clear that $\mu_1 \otimes \mu_2$ is non-atomic on Λ_0. Moreover,

$$\widehat{\varphi}_L(\Lambda_0) = \{X_1 \times X_2 \times A : A \in \Sigma_2\},$$

so $\widehat{\varphi}_L(\Lambda_0)$ is independent of $\mathfrak{F}_M$. Hence conditions (i) and (ii) in Theorem 6.1 are satisfied as well and it follows that $\mathfrak{P}_1(L,M)$ and T are disjoint.

Similarly, if μ_2 is non-atomic, then $\mathfrak{P}_2(L,M)$ is disjoint to all Riesz homomorphisms from L into M.

Before we proceed with some more applications of Theorem 6.1 we first formulate a special case of this theorem, dealing with absolute τ-kernel operators. The verification is left to the reader.

Corollary 6.5. *Let (X,Σ,μ), (Y,Λ,ν) and (Z_j,Γ_j,λ_j), for $j=1,2$, be σ-finite measure spaces and let $L\subset L^0(Y,\Lambda,\nu)$ and $M\subset L^0(X,\Sigma,\mu)$ be order dense ideals. Suppose $\tau_j : X\times Z_j\to Y$, for $j=1,2$ are $(\Sigma\otimes\Gamma_j,\Lambda)$ measurable and null preserving with respect to $\mu\otimes\lambda_j$ and ν. If Λ_0 is a sub-σ-algebra of Λ such that*

(i) *ν is non-atomic on Λ_0;*
(ii) *$\tau_1^{-1}(B)\subseteq\{X\times C : C\in\Gamma_1\}$ for all $B\in\Lambda_0$;*
(iii) *$\tau_2^{-1}(B)\subseteq\{A\times Z_2 : A\in\Sigma\}$ for all $B\in\Lambda_0$;*

then the bands $\mathfrak{L}_{\tau_1 k}(L,M)$ and $\mathfrak{L}_{\tau_2 k}(L,M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

Example 6.6. We return to the situation as in Example 6.4. We will show that, if μ_2 is non-atomic, then the bands $\mathfrak{P}_1(L,M)$ and $\mathfrak{L}_k(L,M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L,M)$ and if μ_1 is non-atomic, then the bands $\mathfrak{P}_2(L,M)$ and $\mathfrak{L}_k(L,M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

To this end, take $(X,\Sigma,\mu)=(Y,\Lambda,\nu)=(X_1\times X_2,\Sigma_1\otimes\Sigma_2,\mu_1\otimes\mu_2)$, $(Z_1,\Gamma_1,\lambda_1)=(X_1\times X_2,\Sigma_1\otimes\Sigma_2,\mu_1\otimes\mu_2)$ and $(Z_2,\Gamma_2,\lambda_2)=(X_1,\Sigma_1,\mu_1)$. Furthermore, define $\tau_1:(X_1\times X_2)\times(X_1\times X_2)\to X_1\times X_2$ by $\tau_1(x_1,x_2,z_1,z_2)=(z_1,z_2)$, and define $\tau_2:(X_1\times X_2)\times Z_2\to X_1\times X_2$ by $\tau_2(x_1,x_2,z)=(z,x_2)$. Let $\Lambda_0=\{X_1\times B : B\in\Sigma_2\}$. Since μ_2 is non-atomic, it follows that $\mu_1\otimes\mu_2$ is non-atomic on Λ_0. For $X_1\times B\in\Lambda_0$ we have

$$\begin{aligned}\tau_1^{-1}(X_1\times B)&=(X_1\times X_2)\times(X_1\times B),\\ \tau_2^{-1}(X_1\times B)&=(X_1\times B)\times X_1,\end{aligned}$$

which shows that conditions (ii) and (iii) of Corollary 6.5 are satisfied. Therefore we may conclude that $\mathfrak{L}_{\tau_1 k}(L,M)=\mathfrak{L}_k(L,M)$ and $\mathfrak{L}_{\tau_2 k}(L,M)=\mathfrak{P}_1(L,M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

A similar argument shows that, if μ_1 is non-atomic, then $\mathfrak{P}_2(L,M)$ and $\mathfrak{L}_k(L,M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

Example 6.7. We continue in the same situation as in the previous example and we will show that if μ_1 or μ_2 is non-atomic, then $\mathfrak{P}_1(L,M)$ and $\mathfrak{P}_2(L,M)$ are disjoint in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

Assume that μ_1 is non-atomic. In Corollary 6.5 we take $(X,\Sigma,\mu)=(Y,\Lambda,\nu)$ $=(X_1\times X_2,\Sigma_1\otimes\Sigma_2,\mu_1\otimes\mu_2)$ and $(Z_j,\Gamma_j,\lambda_j)=(X_j,\Sigma_j,\mu_j)$ for $j=1,2$. Define $\tau_1:X_1\times X_2\times X_1\to X_1\times X_2$ by $\tau_1(x_1,x_2,z)=(z,x_2)$ and $\tau_2:X_1\times X_2\times X_2\to X_1\times X_2$ by $\tau_2(x_1,x_2,z)=(x_1,z)$. Then $\mathfrak{L}_{\tau_j k}(L,M)=\mathfrak{P}_j(L,M)$ for $j=1,2$. Let $\Lambda_0=\{A\times X_2 : A\in\Sigma_1\}$. For $A\times X_2\in\Lambda_0$ we have

$$\begin{aligned}\tau_1^{-1}(A\times X_2)&=X_1\times X_2\times A,\\ \tau_2^{-1}(A\times X_2)&=A\times X_2\times X_2,\end{aligned}$$

so conditions (ii) and (iii) of Corollary 6.5 are fulfilled. Since $\mu_1 \otimes \mu_2$ is non-atomic on Λ_0, we may now conclude that $\mathfrak{P}_1(L,M)$ and $\mathfrak{P}_2(L,M)$ are disjoint bands in $\mathfrak{L}_{\mathrm{n}}(L,M)$.

If we assume μ_2 is non-atomic, the proof follows the same lines.

For the sake of convenience we collect the above results in the following corollary.

COROLLARY 6.8. *Let (X_i, Σ_i, μ_i), $i = 1, 2$, be non-atomic σ-finite measure spaces and let L and M be order dense ideals in $L^0(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$. Then the band $\mathfrak{L}_k(L,M)$ of absolute kernel operators, the bands $\mathfrak{P}_1(L,M)$ and $\mathfrak{P}_2(L,M)$ of partial integral operators, and the band $\mathfrak{H}(L,M)$ generated in $\mathfrak{L}_{\mathrm{b}}(L,M)$ by all Riesz homomorphisms, are mutually disjoint.*

In particular, if an operator $T \in \mathfrak{L}_{\mathrm{b}}(L,M)$ can be written as $T = K + K_1 + K_2 + S$, with $K \in \mathfrak{L}_k(L,M)$, $K_i \in \mathfrak{P}_i(L,M)$, $i = 1, 2$, and $S \in \mathfrak{H}(L,M)$, then this representation is unique and $|T| = |K| + |K_1| + |K_2| + |S|$.

Before we formulate the next corollary we recall that a Banach function space on the σ-finite measure space (X, Σ, μ) is an order dense ideal $M \subseteq L^0(X, \Sigma, \mu)$ equipped with a norm ρ such that (M, ρ) is a Banach lattice. The restriction of the adjoint norm ρ^* to M' is called the associate norm of ρ and is denoted by ρ' (see Chapter 15 from [18]).

COROLLARY 6.9. *Let (X_i, Σ_i, μ_i), $i = 1, 2$, be non-atomic σ-finite measure spaces. Let $L = L^1(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$ and let (M, ρ) be a Banach function space on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$ such that $(M')' = M$ and ρ' is order continuous on M'. Then $\mathfrak{P}_1(L,M) = \mathfrak{P}_2(L,M) = \mathfrak{H}(L,M) = \{0\}$. This is in particular the case if $M = L^p(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$ with $1 < p \leqslant \infty$.*

Proof. By Theorem 98.3 from [19] every continuous operator from L into M is an absolute kernel operator. Every partial integral operator and every Riesz homomorphism is order bounded and therefore norm continuous. It follows from Corollary 6.8 that each of these operators is disjoint to itself and therefore zero. ∎

EXAMPLE 6.10. Let (X_n, Σ_n, μ_n), $n \in \mathbb{N}$, be non-atomic measure spaces with $\mu_n(X_n) = 1$ for all n. We denote by (X, Σ, μ) the product measure space of $\{(X_n, \Sigma_n, \mu_n) : n \in \mathbb{N}\}$, so

$$X = \prod_{n=1}^{\infty} X_n, \quad \Sigma = \bigotimes_{n=1}^{\infty} \Sigma_n, \quad \mu = \bigotimes_{n=1}^{\infty} \mu_n$$

(for information concerning product measure spaces we refer to e.g. Section 2.2 of [9]).

Let L and M be order dense ideals in $L^0(X, \Sigma, \mu)$. We will need some further notation. For $n \in \mathbb{N}$ we denote by $p_n : X \to X_n$ the projection onto the n-th coordinate. For a non-empty subset Δ of $\mathbb{N}$ we write $X_\Delta = \prod_{n \in \Delta} X_n$ and $(X_\Delta, \Sigma_\Delta, \mu_\Delta)$ denotes the corresponding product measure space. For $n \in \Delta$ let $p_n^\Delta : X_\Delta \to X_n$ be the projection on the n-th coordinate. Now assume in addition that Δ is a proper subset of $\mathbb{N}$. Then (X, Σ, μ) is the product of $(X_\Delta, \Sigma_\Delta, \mu_\Delta)$ and $(X_{\Delta^c}, \Sigma_{\Delta^c}, \mu_{\Delta^c})$, so $x \in X$ can be written as $x = (v, w)$ with $v \in X_\Delta$ and $w \in X_{\Delta^c}$.

Define $\tau_\Delta : X \times X_\Delta \to X$ by

$$\tau_\Delta(v, w, z) = (z, w), \quad \forall\, v, z \in X_\Delta,\ w \in X_{\Delta^c},$$

which is a null preserving mapping for $\mu \otimes \mu_\Delta$ and μ.

By $\mathfrak{P}_\Delta(L, M)$ we denote the corresponding band in $\mathfrak{L}_{\rm n}(L, M)$ of absolute τ-kernel operators (partial integral operators). So every $K \in \mathfrak{P}_\Delta(L, M)$ can be represented as

$$(Kf)(v, w) = \int_{X_\Delta} k(v, w, z) f(z, w)\, \mathrm{d}\mu_\Delta(z) \quad \mu\text{-a.e. on } X,$$

for all $f \in L$ and an appropriate kernel k.

We claim that, if Δ and Δ' are two different non-empty proper subsets of $\mathbb{N}$, then the bands $\mathfrak{P}_\Delta(L, M)$ and $\mathfrak{P}_{\Delta'}(L, M)$ are disjoint. For this purpose we define for $n \in \mathbb{N}$ the sub-σ-algebra

$$\Lambda_n = \{p_n^{-1}(A) : A \in \Sigma_n\}.$$

Since μ_n is non-atomic, it follows that μ is non-atomic on Λ_n. Now observe that:

(i) if $n \in \Delta$ and $A \in \Sigma_n$, then

$$\tau_\Delta^{-1}(p_n^{-1}(A)) = X_\Delta \times X_{\Delta^c} \times (p_n^\Delta)^{-1}(A) = X \times (p_n^\Delta)^{-1}(A),$$

so $\tau_\Delta^{-1}(B) \in \{X \times C : C \in \Sigma_\Delta\}$ for all $B \in \Lambda_n$;

(ii) if $n \notin \Delta$ and $A \in \Sigma_n$, then

$$\tau_\Delta^{-1}(p_n^{-1}(A)) = X_\Delta \times (p_n^{\Delta^c})^{-1}(A) \times X_\Delta,$$

so $\tau_\Delta^{-1}(B) \in \{D \times X_\Delta : D \in \Sigma\}$ for all $B \in \Lambda_n$.

From these observations, in combination with Corollary 6.5, our claim follows immediately.

Consequently, $\{\mathfrak{P}_\Delta(L, M) : \emptyset \neq \Delta \subsetneqq \mathbb{N}\}$ is a disjoint collection of bands in $\mathfrak{L}_{\rm n}(L, M)$, and it follows from Examples 6.4 and 6.6 that all these bands are disjoint to $\mathfrak{L}_k(L, M)$ and to all Riesz homomorphisms as well. In general, it may happen that all these bands are equal to $\{0\}$ (cf. Corollary 6.9). However, if for example $L = M = L_p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, then it is easily seen that all these bands are non-zero. Therefore, in this situation, $\{\mathfrak{P}_\Delta(L, M) : \emptyset \neq \Delta \subsetneqq \mathbb{N}\}$ is an uncountable disjoint collection of bands, which is disjoint to $\mathfrak{L}_k(L, M)$ and $\mathfrak{H}(L, M)$.

We return to the case in which $L \subset L^0(Y, \Lambda, \nu)$ and $M \subset L^0(X, \Sigma, \mu)$ are Banach function spaces. Let $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$ be a representation triple for L and M. Then $L_1 = C_{\varphi_L} \cap L$ is a band in L and consequently norm closed. Therefore, L_1 is a Banach lattice which is Riesz isomorphic to L_Ω via the Riesz isomorphism φ_1 as in the proof of Proposition 4.4. We equip L_Ω with a Riesz norm by defining $\|\varphi_1(u)\| = \|u\|$, $u \in L_1$. With this norm L_Ω is a Banach function space.

Similarly, M_Ω is Riesz isomorphic to the band $\operatorname{ran}(\psi_M) \cap M$ via the Riesz isomorphism ψ_1 as in the proof of Proposition 4.4 and can be equipped with a Riesz norm by defining $\|u\| := \|\psi_1(u)\|$ for all $u \in M_\Omega$. With this definition ψ_1 is an isometry onto the Banach lattice $\operatorname{ran}(\psi_M) \cap M$ and M_Ω is also a Banach function space.

If $T \in \mathfrak{L}_\Phi(L, M)$ is represented as $T = \psi_M S_m \varphi_L$ then $S_m = \psi_1^{-1} T \varphi_1^{-1}$ and so $T : L \to M$ is compact if and only if $S_m : L_\Omega \to M_\Omega$ is compact. We now prove:

Theorem 6.11. *Let $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \varphi_L, \psi_M)$ be a representation triple for the Banach function spaces L and M. Suppose there exists a sub-σ-algebra $\Lambda_0 \subset \Lambda$ such that $\nu|\Lambda_0$ is non-atomic and such that $\widehat{\varphi}_L(\Lambda_0) := \mathfrak{F}_{\Lambda_0} \subset \mathfrak{F}_M$. Then $T \in \mathfrak{L}_\Phi(L, M)$ is compact if and only if $T = 0$.*

Proof. Let $T \in \mathfrak{L}_\Phi(L, M)$ be compact. As remarked above, the corresponding operator $S_m : L_\Omega \to M_\Omega$ is compact and from the second step of the proof of Theorem 6.1 we have that $\mathbb{P}|\mathfrak{F}_{\Lambda_0}$ is non-atomic. Let $u \in L_\Omega$, $\|u\| \leqslant 1$ and assume there exists $\varepsilon > 0$ such that $U_\varepsilon := \{\omega \in \Omega : |S_m u(\omega)| \geqslant \varepsilon\}$ have positive measure $\alpha > 0$. Put $\mathfrak{F}_\varepsilon := \{A \cap U_\varepsilon : A \in \mathfrak{F}_{\Lambda_0}\}$ and note that by the proof given in Remark 6.2, $\mathbb{P}|\mathfrak{F}_\varepsilon$ is non-atomic. Also, since $U_\varepsilon \in \mathfrak{F}_M$ and $\mathfrak{F}_{\Lambda_0} \subseteq \mathfrak{F}_M$ by hypothesis, $\mathfrak{F}_\varepsilon \subseteq \mathfrak{F}_M$. Let (r_n) be a sequence of Rademacher functions, i.e., a sequence of independent, symmetric, $\{-1, 1\}$-valued random variables on the probability space $(U_\varepsilon, \mathfrak{F}_\varepsilon, \alpha^{-1}\mathbb{P})$. Extend each r_n to Ω by defining it to be zero on $\Omega \setminus U_\varepsilon$. Let $u_n := r_n u$. Then (u_n) is a sequence in the unit ball of L_Ω and $g_n := S_m u_n = r_n S_m u$ since r_n is $\mathfrak{F}_M$-measureable for each $n \in \mathbb{N}$. By the compactness of S_m, using Theorem 100.6 in [19], the sequence (g_n) has a subsequence which converges pointwise $\mathbb{P}$-a.e. on Ω. The inequality

$$|r_n - r_m| \leqslant \varepsilon^{-1}|g_n - g_m| \quad \text{for all } n, m \in \mathbb{N}$$

then shows that the sequence of Rademacher functions (r_n) has an almost everywhere convergent subsequence which is a contradiction. It follows that $\mathbb{P}(U_\varepsilon) = 0$ for all $\varepsilon > 0$ and so $S_m u = 0$ for all u in the unit ball. Hence, $S_m = 0$, i.e., $T = 0$. ∎

Corollary 6.12. *Let L, M and Φ be as in the preceding theorem. Assume in addition that L^* and M have order continuous norms. Then every compact operator $0 \leqslant T : L \to M$ is disjoint to every $S \in L_\Phi(L, M)$. In particular, if (X_i, Σ_i, μ_i), $i = 1, 2$, are non-atomic σ-finite measure spaces and if L and M are Banach function spaces on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$ such that L^* and M have order continuous norms, then every partial integral operator in $\mathfrak{P}_i(L, M)$, $i = 1, 2$, is disjoint to every positive compact operator.*

Proof. Let $T \geqslant 0$ be compact. By the Dodds-Fremlin theorem ([6]; see also [14]) one has for every $S \in L_\Phi(L, M)$ that $|S| \wedge T$ is compact and belongs to $L_\Phi(L, M)$. Hence it is zero. ∎

Acknowledgements. The authors would like to express their gratitude to the Departments of Mathematics at the Delft University of Technology and at Potchefstroom University for hospitality offered during reciprocal visits.

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Received October 12, 1999.