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ON ULTRAPOWERS OF NON COMMUTATIVE L_p SPACES

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Abstract. It is well known that for every von Neumann Algebra $\mathcal{a}$, every ultrapower of its predual $\mathcal{a}_*$ is isometric to the predual of a von Neumann Algebra $\mathcal{A}$. We study the modular automorphism groups associated with states of $\mathcal{A}$ in terms of those for $\mathcal{a}$. As an application we show that the ultrapower of the Haagerup $L_p(\mathcal{a})$ spaces are isometrically identifiable with the corresponding $L_p(\mathcal{A})$ spaces (for every $0 < p < \infty$).

Keywords: *Ultrapowers, von Neumann algebras, Haagerup L_p spaces.*

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INTRODUCTION

Since their introduction in Banach Space Theory by [2], ultrapowers (and ultraproducts) proved to be a somewhat useful tool, especially in the study of the qualitative aspects of local theory, due to their close connection to finite representability (see also [9] for an equivalent theory). However, although the definition of the ultrapowers of an *abstract* Banach space X is quite simple, it is generally not a simple task to describe the ultrapowers of *concrete* spaces (i.e. classical Banach spaces), and the class of those spaces for which such a description is known is quite small. Roughly speaking, this class contains essentially Banach lattices, in fact Lebesgue spaces and the spaces obtained from them by simple operations like latticial tensorization (i.e. the operation $(L, X) \mapsto L(X)$, where L is a Banach lattice and X a Banach space) or latticial interpolation. Outside of this frame are essentially the C^*-algebras and the preduals of von Neumann algebras (or W^* algebras), which are the non commutative analogues of, respectively, $C(K)$ spaces and L_1 spaces. While the case of C^*-algebras is simple and its treatment goes back to [2], that of preduals of von Neumann algebras's is a little more involved and is a by-product of [5].

The aim of the present paper is to give a description of the ultrapowers of non-commutative L_p spaces. Since their introduction by Haagerup ([7]), these spaces have been given several other equivalent constructions (see [16], [25], [12]), but we shall follow Haagerup's construction as developed in the first chapters of [24]. An appeal to these generalized L_p spaces is unavoidable even when dealing with the ultrapowers of Schatten classes $S_p(H)$ (in other words, the class of L_p spaces associated with a normal semifinite trace is not closed under ultrapowers). The question of the representation of ultrapowers (or, equivalently, of nonstandard hulls) of $S_p(H)$ was asked as Problem 16 in [10].

Before to do this, we revisit (in Section 1 below) the case $p = 1$, giving another proof of the representation theorem of [5], starting with an appropriate representation of the given von Neumann algebra (VNA) $\mathcal{a}$ as subalgebra of $B(H)$. In this description, the ultrapower of the predual $\mathcal{a}_*$ coincides with the predual of the VNA $\mathcal{A}$ generated by the ultrapower of $\mathcal{a}$, when this last ultrapower is realized as a sub-C^*-algebra of $B(\widetilde{H})$, where $\widetilde{H}$ is the ultrapower of H. The commutant of $\mathcal{A}$ is then generated as VNA by the ultrapower of the commutant of $\mathcal{a}$. These facts (which of course also takes place for ultraproducts) permit us to elucidate (in Section 2) the local modular structure of $\mathcal{A}$ in terms of that of $\mathcal{a}$. More precisely, if (φ_i) is a family of normal states of $\mathcal{a}$ and $\widetilde{\varphi}$ is the corresponding normal state of $\mathcal{A}$, the modular automorphism $\sigma_t^{\tilde{\varphi}}$ (of the reduced $\mathcal{A}_{\tilde{\varphi}}$ obtained by reduction to the support of $\widetilde{\varphi}$) is the "ultrapower map" of the family $\widetilde{\sigma_t^{\varphi_i}}$ of modular automorphisms of the reduced VNA's $\mathcal{a}_{\varphi_i}$. An analogous result takes place for relative modular theory (relative to two normal states $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$ of $\mathcal{A}$).

In Section 3 we pass to the study of ultrapowers of $L_p(\mathcal{a})$ space. Using Haagerup's formalism, we introduce the *Mazur maps* $L_p(\mathcal{a}) \to L_1(\mathcal{a})$: these non-linear locally uniform homeomorphisms are the analogues of the classical Mazur maps $f \mapsto f|f|^{p-1}$ in the commutative case. Then, like in commutative case, the vector space L_p appears as L_1 equipped with new vector space operations and the philosophy of the Section 3 is that these operations "pass to ultrapower". Finally, we recover the ultrapowers of $L_p(\mathcal{a})$ as $L_p(\mathcal{A})$ spaces, isometrically and as bimodules (relative to the action of the ultrapower of $\mathcal{a}$); by the construction itself, the Mazur map for $L_p(\mathcal{A})$ is the ultrapower of the corresponding Mazur map of $L_p(\mathcal{a})$. These results are valid in the case $0 < p < 1$ too, but this requiers a special argument developed in Section 4. Finally, in Section 5 we show that the identifications of the ultrapowers of $L_p(\mathcal{a})$ with spaces $L_p(\mathcal{A})$ for various values of p are compatible with the multiplication maps $L_p(\mathcal{a}) \times L_q(\mathcal{a}) \to L_r(\mathcal{a})$ given by the noncommutative Hölder theorem. We remark also that these identifications preserve the natural operator space structures in the sense of Effros-Ruan.

It should be emphasized that all the results of this paper are valid for ultraproducts of a family of L_p spaces (for a fixed p) as well. The proofs are the same as for ultrapowers, and it is only for convenience of writing that the results are stated and the proofs are given only for ultrapowers. (In facts, the results of Section 2 rely on the ultraproduct version of certain results of Section 1.)

The results concerning the cases $p = 1$ and $p = 2$ were announced in the seminar notes [20].

DEFINITIONS AND NOTATION

If X is a Banach space and $\mathcal{U}$ an ultrafilter over some set of indices I, let $\ell_\infty(I;X)$ be the space of bounded X-valued families indexed by I, equipped with the sup norm; the ultrapower $X^I/\mathcal{U}$ (also frequently denoted by $X_\mathcal{U}$ in the literature), is the quotient space $\ell_\infty(I;X)/\mathcal{N}$, where $\mathcal{N}$ is the (closed) subspace of $\ell_\infty(I;X)$ consisting of those families $(x_i) \in \ell_\infty(I;X)$ such that $\lim_{i,\mathcal{U}} \|x_i\| = 0$. The element of $X^I/\mathcal{U}$ represented by (x_i) will be denoted by $(x_i)^\bullet$. Recall that the quotient norm is simply given by $\|(x_i)^\bullet\| = \lim_{i,\mathcal{U}} \|x_i\|$. If we consider a family of Banach spaces $(X_i)_{i\in I}$ and the space $\left(\bigoplus_{i\in I} X_i\right)_\infty$ (of bounded families $(x_i) \in \prod_{i\in I} X_i$) in place of $\ell_\infty(I;X)$, we obtain the ultraproduct $\prod_{i\in I} X_i/\mathcal{U}$. We refer to [8], [22] for basic facts about ultrapowers and ultraproducts of Banach spaces. In this paper the set I and the ultrafilter $\mathcal{U}$ are fixed once for all (except for the end of Section 1: see Lemmas 1.13 and 1.16) and we shall generally denote by $\widetilde{X}$ the ultrapower $X^I/\mathcal{U}$. A bounded linear map T from the Banach space X into the Banach space Y induces naturally a bounded linear map $\widetilde{T} : \widetilde{X} \to \widetilde{Y}$ defined for $\widetilde{x} = (x_i)^\bullet_{i\in I} \in \widetilde{X}$ as $\widetilde{T}(\widetilde{x}) = (Tx_i)^\bullet_{i\in I}$ (note that changing of representing family for $\widetilde{x}$ does not change $\widetilde{T}(\widetilde{x})$); we shall speak of $\widetilde{T}$ as the *ultrapower map* of T. More generally, we can consider a bounded family (T_i) in the space $B(X,Y)$ of bounded linear operators from X to Y and define the *ultraproduct map* $\widetilde{T} : (x_i)^\bullet_{i\in I} \mapsto (T_i x_i)^\bullet_{i\in I}$; this provides a natural (linear, isometric) embedding from $B(X,Y)^I/\mathcal{U}$ into $B(\widetilde{X},\widetilde{Y})$. The linearity of T is in fact not required by this construction, and, more generally, we can define the ultrapower map $\widetilde{F}$ of a map $F : X \to Y$ which is locally uniformly continuous (i.e. uniformly continuous on every ball of X).

In the following, H will be a Hilbert space and $B(H)$ the space of bounded linear operators on H. We shall denote by $\mathcal{A}$ a von Neumann algebra (shortly, VNA) over H and $\mathcal{A}_*$ its predual (the space of normal bounded linear forms over $\mathcal{A}$). We refer to [3] and [21] for the basic facts on VNA and to [15] and [23] for the facts on modular theory. The basic facts about $L_p(\mathcal{A})$ spaces will be recalled in due time (Section 3).

We denote by $\mathcal{A}_*^+$ the set of positive elements of $\mathcal{A}_*$; if $\xi \in H$, we denote as usual by ω_ξ the positive normal linear form defined by:

$$\forall\, x \in \mathcal{A}, \quad \omega_\xi(x) = (x\xi,\xi)_H.$$

1. ULTRAPOWERS OF PREDUALS OF VON NEUMANN ALGEBRAS REVISITED

Since by Sakai's theorem ([21], Corollary 1.13.3) the predual $\mathcal{a}_*$ of the VNA $\mathcal{a}$ depends only on the isometry class of $\mathcal{a}$, we have the choice of an appropriate representation for identifying the ultrapowers $\widetilde{\mathcal{a}_*}$.

We shall consider a representation of $\mathcal{a}$ over the Hilbert space H verifying the condition:

(R) For every $\varphi \in \mathcal{a}_*^+$, there exists a vector ξ in H such that $\varphi = \omega_\xi$.

For a general representation (for example if $\mathcal{a} = B(H)$), we only have that there exists a sequence (ξ_n) in H such that $\varphi = \sum_n \omega_{\xi_n}$ ([3], I, Section 4, Theorem 1). But given a representation $\mathcal{a}$ (over H) there is an *amplification* of $\mathcal{a}$ verifying property (R) (this is simply the algebra $\mathcal{a} \otimes \mathbb{C}I$ acting on the Hilbert tensor product $H \otimes \ell_2$). Another important case is when there exists a separating vector in H for $\mathcal{a}$ (see [15], Theorem 7.2.3).

From now on, we consider a VNA $\mathcal{a}$ satisfying property (R).

The inclusion $\mathcal{a} \subset B(H)$ induces a natural inclusion $\widetilde{\mathcal{a}} \subset \widetilde{B(H)}$ (i.e. $\widetilde{\mathcal{a}}$ is a sub-C^*-algebra of the C^*-algebra $\widetilde{B(H)}$). On the other hand, we have an isometric embedding $j : \widetilde{B(H)} \hookrightarrow B(\widetilde{H})$ defined by:

$$\forall\, \widetilde{T} = (T_i)^\bullet \in \widetilde{B(H)},\ \forall\, \widetilde{x} = (x_i)^\bullet \in \widetilde{H}, \quad j(\widetilde{T})(\widetilde{x}) = (T_i(x_i))^\bullet_{i\in I}.$$

The map j is a unital injective $*$-homomorphism. Hence, we may consider $\widetilde{B(H)}$ (and so $\widetilde{\mathcal{a}}$) as a sub-C^*-algebra of $B(\widetilde{H})$.

Let $\mathcal{A}$ be the VNA generated by $\widetilde{\mathcal{a}}$ in $B(\widetilde{H})$ ($\mathcal{A}$ is simply the weak operator closure of $\widetilde{\mathcal{a}}$; it coincides also with the bicommutant $\widetilde{\mathcal{a}}''$ of $\widetilde{\mathcal{a}}$).

Theorem 1.1. *The ultrapower* $(\widetilde{\mathcal{a}_*})$ *identifies with the predual* $\mathcal{A}_*$ *of the VNA* $\mathcal{A}$.

Before proving Theorem 1.1, we recall a well known fact of the theory of ultrapowers ([22], Section 11):

Fact 1.2. For every Banach space X there is a natural isometric inclusion $i_X : (\widetilde{X^*}) \to (\widetilde{X})^*$ defined by:

$$\langle i_X((x_i^*)^\bullet), (x_i)^\bullet \rangle = \lim_{i,\mathcal{U}} \langle x_i^*, x_i \rangle.$$

Moreover, the image of i_X is w*-dense in $(\widetilde{X})^*$, and even the image of the unit ball of $\widetilde{X^*}$ is w*-dense in the unit ball of $(\widetilde{X})^*$.

Let us denote by $\mathcal{L}$ the ultrapower $(\widetilde{\mathcal{a}_*})$. By applying Fact 1.2 with $X = \mathcal{a}_*$, we obtain an isometric embedding $i : \widetilde{\mathcal{a}} \to \mathcal{L}^*$.

LEMMA 1.3. *The embedding* $i : \widetilde{\mathcal{A}} \to \mathcal{L}^*$ *is weak-operator to* w*-*continuous.*

Proof. For every $\widetilde{\varphi} \in \mathcal{L}$ we have to prove that the map $\widetilde{x} \mapsto \langle i(\widetilde{x}), \widetilde{\varphi}\rangle$ is weak-operator continuous on $\widetilde{\mathcal{A}}$. We have $\widetilde{\varphi} = (\varphi_i)^\bullet$, and we can suppose that every φ_i belongs to $\mathcal{A}_*^+$ (every element of the predual is a linear combination with coefficients ± 1, $\pm i$ of four positive elements with smaller or equal norms). By (R), for every i we have $\varphi_i = \omega_{\xi_i}$ for some $\xi_i \in H$, and since $\|\xi_i\|^2 = \omega_{\xi_i}(I) = \|\varphi_i\|$, the family (ξ_i) is bounded in H, and represents an element $\widetilde{\xi}$ of $\widetilde{H}$. We have then:

$$\forall\, \widetilde{x} \in \widetilde{\mathcal{A}}, \quad \langle i(\widetilde{x}), \widetilde{\varphi}\rangle = \lim_{i,\mathcal{U}} \langle x_i, \varphi_i\rangle = \lim_{i,\mathcal{U}} \langle x_i \xi_i, \xi_i\rangle = \langle \widetilde{x}\widetilde{\xi}, \widetilde{\xi}\rangle = \omega_{\widetilde{\xi}}(\widetilde{x}).$$

The right member is clearly weak-operator continuous as a function of $\widetilde{x}$. ∎

Proof of Theorem 1.1. Since by Kaplansky's Theorem the unit ball $B_{\tilde{a}}$ of $\widetilde{\mathcal{A}}$ is weak-operator dense in $B_{\mathcal{A}}$ and $i(B_{\tilde{a}}) \subset B_{\mathcal{L}^*}$ is relatively w*-compact, we see that the map i has a unique extension to a weak-operator to w*-continuous map $B_{\mathcal{A}} \to B_{\mathcal{L}^*}$, and then by homogeneity to $\widehat{\imath} : \mathcal{A} \to \mathcal{L}^*$ (for every $x \in B_{\mathcal{A}}$, $\widehat{\imath}(x)$ is the unique w*-clusterpoint of the set $i(\widetilde{x})$ when $\widetilde{x} \in B_{\tilde{a}}$ weak-operator converges to x). It is clear that $\widehat{\imath}$ is linear with norm less than 1.

We prove first that the map $\widehat{\imath}$ is isometric. For, if $x \in \mathcal{A}$ and $\varepsilon > 0$, we can find $\widetilde{\xi}$ and $\widetilde{\eta}$ in $\widetilde{H}$ with $\|\widetilde{\xi}\|, \|\widetilde{\eta}\| \leqslant 1$ and $\|x\| \leqslant (1+\varepsilon)(x\widetilde{\xi}, \widetilde{\eta}) = \omega_{\tilde{\xi},\tilde{\eta}}(x)$. Let (ξ_i), (η_i) be representing families for $\widetilde{\xi}$, respectively $\widetilde{\eta}$, and let $\varphi_i = \omega_{\xi_i,\eta_i}$ (an element of $\mathcal{A}_*$). We have $\|\varphi_i\| \leqslant \|\xi_i\| \cdot \|\eta_i\|$, hence $\sup_i \|\varphi_i\| < \infty$, and the family (φ_i) represents an element $\widetilde{\varphi}$ of $\mathcal{L}$ (with $\|\widetilde{\varphi}\| = \lim_{i,\mathcal{U}} \|\xi_i\|\,\|\eta_i\| \leqslant 1$). We have then:

$$\forall\, \widetilde{a} = (a_i)^\bullet \in \widetilde{\mathcal{A}}, \quad \langle i(\widetilde{a}), \widetilde{\varphi}\rangle = \lim_{i,\mathcal{U}} \langle a_i, \varphi_i\rangle = \lim_{i,\mathcal{U}} \langle a_i \xi_i, \eta_i\rangle = \langle \widetilde{a}\widetilde{\xi}, \widetilde{\eta}\rangle.$$

Letting $\widetilde{a}$ weak-operator converge to x, we deduce (by weak-operator to w*-continuity of $\widehat{\imath}$) that $\langle \widehat{\imath}(x), \widetilde{\varphi}\rangle = \langle x\widetilde{\xi}, \widetilde{\eta}\rangle$, hence $\|\widehat{\imath}(x)\| \geqslant |\langle \widehat{\imath}(x), \widetilde{\varphi}\rangle| \geqslant (1+\varepsilon)^{-1}\|x\|$. Then let $\varepsilon \to 0$.

Now we prove that the map $\widehat{\imath}$ is surjective. For, $\widehat{\imath}(B_{\mathcal{A}})$ is w*-closed (since $B_{\mathcal{A}}$ is weak-operator compact) and contains the set $i(B_{\tilde{a}})$ which is w*-dense in $B_{\mathcal{L}^*}$; thus $\widehat{\imath}(B_{\mathcal{A}})$ contains $B_{\mathcal{L}^*}$. ∎

REMARK 1.4. Let $\mathcal{A}_1$ be another realization of $\mathcal{L}^*$ as VNA. By Fact 1.2 the C^*-algebra $\widetilde{\mathcal{A}}$ is isometrically embedded in $\mathcal{A}_1$ as a w*-dense subspace. We have a linear isometry ρ of $\mathcal{A}$ onto $\mathcal{A}_1$ (preserving the subalgebra $\widetilde{\mathcal{A}}$). This isometry ρ is a priori a Jordan isomorphism ([14]), but if the product in $\mathcal{A}_1$ restricts to that of $\widetilde{\mathcal{A}}$, the isometry ρ (which is w*-continuous) must preserve the products and is thus an isomorphism of VNA.

THE IDENTIFICATION MAP BETWEEN $\widetilde{\mathcal{A}_*}$ AND $\mathcal{A}_*$. From now on we write i in place of $\widehat{\imath}$. Since this map is weak-operator to w*-continuous, and a fortiori w* to w*-continuous (when $\mathcal{A}$ is considered as the dual of $\mathcal{A}_*$), we have $i^*(\mathcal{L}) \subset \mathcal{A}_*$ (where $i^* : \mathcal{L}^{**} \to \mathcal{A}^*$ is the conjugate map). Since i is injective, $i^*(\mathcal{L})$ is dense in $\mathcal{A}_*$, and since i is an onto isometry, so is i^*; in particular, $i^*(\mathcal{L})$ is closed and thus $i^*(\mathcal{L}) = \mathcal{A}_*$. Let i_* be the restriction of i to $\mathcal{L}$; this is the desired identification of $\mathcal{L}$ with $\mathcal{A}_*$.

We list now some "good" properties of the map i_*.

PROPOSITION 1.5. *The identification map i_* preserves the natural $\widetilde{\mathcal{a}}$ bimodule structures of $\widetilde{\mathcal{a}_*}$ and $\mathcal{A}_*$, as well as the natural conjugation maps.*

Proof. (a) i_* preserves the actions of $\widetilde{\mathcal{a}}$.

The natural right and left actions of $\mathcal{a}$ on $\mathcal{a}_*$, i.e. the maps $\mathcal{a} \times \mathcal{a}_* \to \mathcal{a}_*$, $(x, \varphi) \mapsto x \cdot \varphi = \varphi(\,\cdot\, x)$ and $(x, \varphi) \mapsto \varphi \cdot x = \varphi(x\,\cdot\,)$ induce right and left actions of $\widetilde{\mathcal{a}}$ on $\mathcal{L} = \widetilde{\mathcal{a}_*}$ (the ultrapower maps); on the other hand, the right and left actions of $\mathcal{A}$ on $\mathcal{A}_* = i_*(\mathcal{L})$ restrict to actions of $\widetilde{\mathcal{a}}$ on $\mathcal{A}_*$.

The map i_* is compatible with these actions of $\widetilde{\mathcal{a}}$:

$$\forall\, \widetilde{\varphi} \in \mathcal{L}, \forall\, \widetilde{x} \in \widetilde{\mathcal{a}}, \quad i_*(\widetilde{x} \cdot \widetilde{\varphi}) = \widetilde{x} \cdot i_*(\widetilde{\varphi}) \text{ and } i_*(\widetilde{\varphi} \cdot \widetilde{x}) = i_*(\widetilde{\varphi}) \cdot \widetilde{x}.$$

For, if $\widetilde{y}$ is an arbitrary element of $\widetilde{\mathcal{a}}$, we have for example:

$$\begin{aligned}\langle \widetilde{y}, i_*(\widetilde{x} \cdot \widetilde{\varphi})\rangle &= \langle i(\widetilde{y}), \widetilde{x} \cdot \widetilde{\varphi}\rangle = \lim_{i,\mathcal{U}} \langle y_i, x_i \cdot \varphi_i\rangle = \lim_{i,\mathcal{U}} \langle y_i \cdot x_i, \varphi_i\rangle \\ &= \langle i(\widetilde{y} \cdot \widetilde{x}), \widetilde{\varphi}\rangle = \langle \widetilde{y} \cdot \widetilde{x}, i_*(\widetilde{\varphi})\rangle = \langle \widetilde{y}, \widetilde{x} \cdot i_*(\widetilde{\varphi})\rangle.\end{aligned}$$

By w*-density of $\widetilde{\mathcal{a}}$ in $\mathcal{A}$ we deduce the first equality $i_*(\widetilde{x} \cdot \widetilde{\varphi}) = \widetilde{x} \cdot i_*(\widetilde{\varphi})$.

(b) i_* preserves the conjugation map.

On $\mathcal{a}_*$ the conjugation map is defined as usual by

$$\forall\, \varphi \in \mathcal{a}_*, \forall\, x \in \mathcal{a}, \quad \varphi^*(x) = \overline{\varphi(x^*)}.$$

Then $\varphi \mapsto \varphi^*$ is an antilinear isometric involution of $\mathcal{a}_*$. By passing to the ultrapower, we deduce an antilinear isometric involution of $\mathcal{L}$: $\widetilde{\varphi} = (\varphi_i)^\bullet \mapsto (\varphi_i^*)^\bullet =: (\widetilde{\varphi})^*$. On the other hand, $\mathcal{A}_*$ is equipped with the conjugation map induced by that of $\mathcal{A}$. It is straightforward to verify the compatibility of i_* with these two conjugation maps:

$$\forall\, \widetilde{\varphi} \in \mathcal{L}, \quad i_*(\widetilde{\varphi}^*) = i_*(\widetilde{\varphi})^*$$

(again the two members coincide on $\widetilde{\mathcal{a}}$, hence are equal). ∎

The space $\mathcal{L}$ is ordered by the mean of the cone ultrapower of the positive cone $\mathcal{a}_*^+$:

$$\mathcal{L}_+ = \{(\varphi_i)^\bullet \in \mathcal{L} : \varphi_i \geqslant 0, \forall\, i \in I\}.$$

Note that an element of $\mathcal{L}_+$ is characterized by the existence of a positive representing family (not every representing family is positive).

PROPOSITION 1.6. *We have $i_*(\mathcal{L}_+) = \mathcal{A}_*^+$, i.e. the maps i_* and $(i_*)^{-1}$ are positivity preserving.*

Proof. If $\widetilde{\varphi} \in \mathcal{L}_+$ it is clear that $\langle \widetilde{x}, i_*(\widetilde{\varphi})\rangle \geqslant 0$ for every $\widetilde{x} \in \widetilde{\mathcal{a}}_+$ since such an element $\widetilde{x}$ has a representing family (x_i) with $x_i \geqslant 0$ for every $i \in I$ (note that $\widetilde{x} = \widetilde{y}^*\widetilde{y}$ for some $\widetilde{y} = (y_i)^\bullet \in \widetilde{\mathcal{a}}$, then set $x_i = y_i^* \cdot y_i$). But $\widetilde{\mathcal{a}}_+$ is w*-dense in $\mathcal{A}_+$ (this follows easily from the strong-operator density of $\widetilde{\mathcal{a}}$ in $\mathcal{A}$ and the fact that $\mathcal{A}_+ = \{y^*y : y \in \mathcal{A}\}$). Hence $\langle y, i_*(\widetilde{\varphi})\rangle \geqslant 0$ for every $y \in \mathcal{A}_+$, i.e. $i_*(\widetilde{\varphi}) \in \mathcal{A}_*^+$. So i_* is positivity preserving.

Conversely, if $i_*(\widetilde{\varphi}) \geqslant 0$ then in particular $i_*(\widetilde{\varphi})$ is hermitian:

$$i_*(\widetilde{\varphi}) = \frac{i_*(\widetilde{\varphi}) + i_*(\widetilde{\varphi})^*}{2} = i_*\left(\frac{\widetilde{\varphi} + \widetilde{\varphi}^*}{2}\right)$$

hence $\widetilde{\varphi} = \frac{\widetilde{\varphi}+\widetilde{\varphi}^*}{2}$, i.e. $\widetilde{\varphi}$ has an hermitian representing family (φ_i) ($\varphi_i = \varphi_i^*$ for each $i \in I$). Let $\varphi_i = \varphi_i^+ - \varphi_i^-$ be the decomposition of φ_i in positive and negative part, we have $\varphi_i^+, \varphi_i^- \geqslant 0$ and $\|\varphi_i\| = \|\varphi_i^+\| + \|\varphi_i^-\|$ ([21], Theorem 1.14.3). Let $\widetilde{\varphi}^+ = (\varphi_i^+)^\bullet$ and $\widetilde{\varphi}^- = (\varphi_i^-)^\bullet$. Then $\widetilde{\varphi} = \widetilde{\varphi}^+ - \widetilde{\varphi}^-$, $\widetilde{\varphi}^+, \widetilde{\varphi}^- \geqslant 0$ and $\|\widetilde{\varphi}\| = \|\widetilde{\varphi}^+\| + \|\widetilde{\varphi}^-\|$. We have $i_*(\widetilde{\varphi}) = i_*(\widetilde{\varphi}^+) - i_*(\widetilde{\varphi}^-)$ and since $i_*(\widetilde{\varphi}^-) \geqslant 0$ because i_* preserves positivity, we see that $0 \leqslant i_*(\widetilde{\varphi}) \leqslant i_*(\widetilde{\varphi}^+)$, which implies $\|i_*(\widetilde{\varphi})\| \leqslant \|i_*(\widetilde{\varphi}^+)\|$. But $\|i_*(\widetilde{\varphi})\| = \|i_*(\widetilde{\varphi}^+)\| + \|i_*(\widetilde{\varphi}^-)\|$ since i_* is an isometry, hence $\|i_*(\widetilde{\varphi}^-)\| = 0$, and $\widetilde{\varphi}^- = 0$. This means that (φ_i^+) is a representing family for $\widetilde{\varphi}$, which belongs thus to the cone $\mathcal{L}_+$. ∎

Proposition 1.7. *The absolute value map* $V : \mathcal{A}_* \to \mathcal{A}_*^+$, $\varphi \to |\varphi|$ *is locally uniformly continuous and induces an ultrapower map* $\widetilde{V} : \mathcal{L} \to \mathcal{L}_+$, *which is transformed by* i_* *in the absolute value map of* $\mathcal{A}_*$*:*

$$\forall\, \widetilde{\varphi} \in \mathcal{L}, \quad i_*(\widetilde{V}\widetilde{\varphi}) = |i_*(\widetilde{\varphi})|.$$

Proof. If $\varphi \in \mathcal{A}_*$, its absolute value $|\varphi|$ is characterized by the conditions:
(1) $|\varphi| \geqslant 0$;
(2) there exists a partial isometry $v \in \mathcal{A}$ such that $\varphi = v|\varphi|$ and $|\varphi| = v^*\varphi$.

(Note that the polar decomposition $\varphi = u|\varphi|$ is then given by $u = vp_{|\varphi|}$, where $p_{|\varphi|}$ is the support of $|\varphi|$, i.e. the least projection p in $\mathcal{A}$ such that $|\varphi| = p \cdot |\varphi|$.)

These conditions pass to ultrapowers by Proposition 1.5 and 1.6, hence:

$$\widetilde{\varphi} \in \mathcal{L}, \quad \widetilde{\varphi} = (\widetilde{\varphi}_i)^\bullet \Rightarrow |i_*(\widetilde{\varphi})| = i_*[(|\varphi_i|)^\bullet].$$

By a standard reasoning, this implies that the map V is locally uniformly continuous (in fact, it is well known that V is $\frac{1}{2}$-Hölder; see [18]). ∎

The commutant of $\mathcal{A}$. Let $\mathcal{A}'$ be the commutant of $\mathcal{A}$ in $B(H)$. Like for $\mathcal{A}$, the ultrapower $\widetilde{\mathcal{A}'}$ identifies with a unital C^*-algebra over $\widetilde{H}$. Let $\mathcal{B}$ be the VNA generated by $\widetilde{\mathcal{A}'}$ in $B(\widetilde{H})$. We have the following general result (where no assumption is made on the VNA $\mathcal{A}$; so $\mathcal{A}$ is simply the VNA generated by $\widetilde{\mathcal{A}}$ in $B(\widetilde{H})$, which is perhaps not identifiable with the dual of $\widetilde{\mathcal{A}_*}$).

Theorem 1.8. *The* VNA $\mathcal{B}$ *coincides with the commutant* $\mathcal{A}'$ *of* $\mathcal{A}$ *in* $B(\widetilde{H})$.

We shall use the following lemma:

Lemma 1.9. *Let* $\xi, \eta \in H$ *and* $\psi \in \mathcal{A}_*^+$ *such that* $\omega_\eta \leqslant \omega_\xi + \psi$. *Then there exist* $x' \in \mathcal{A}'$ *and* $\zeta \in \mathcal{H}$ *with* $\|x'\| \leqslant 1$ *and* $\|\zeta\| \leqslant \|\psi\|^{1/2}$ *such that* $\eta = x'\xi + \zeta$.

Proof. We can find a sequence $(\xi_n)_n$ in H such that $\psi = \sum_n \omega_{\xi_n}$. We interpret the equation $\omega_\eta \leqslant \omega_\xi + \sum_n \omega_{\xi_n}$ in the predual of the amplification $\mathcal{A} \otimes \mathbb{C} \cdot I$ of $\mathcal{A}$ over $H \otimes \ell_2 = \ell_2(H)$. Let $\widehat{\eta} = (\eta, 0, \ldots, 0, \ldots)$ and $\widehat{\xi} = (\xi, \xi_1, \ldots, \xi_n, \ldots)$. We have $\omega_{\hat{\eta}} \leqslant \omega_{\hat{\xi}}$; hence ([3], I, Section 4, Lemma 1) there exists $\widehat{x}' \in (\mathcal{A} \otimes \mathbb{C} \cdot I)' = \mathcal{A}' \overline{\otimes} B(\ell_2)$ such that $\|\widehat{x}'\| \leqslant 1$ and $\widehat{x}'\widehat{\xi} = \widehat{\eta}$. Writing $\widehat{x}'$ as an infinite matrix $(x'_{mn})_{mn}$ with entries in $\mathcal{a}'$ we obtain $\eta = x'_{00}\xi + \sum_n x'_{0n}\xi_n$ with $\|x'_{00}\| \leqslant 1$ and $\|\sum_n x'_{0n}x'^*_{0n}\| \leqslant 1$. Set $x' = x'_{00}$, $\zeta = \sum_n x'_{0n}\xi_n$, we have $\|\zeta\| \leqslant \sum_n (\|\xi_n\|^2)^{1/2} = \|\psi\|$. ∎

Proof of Theorem 1.8. It is clear that $\widetilde{\mathcal{a}'} \subset (\widetilde{\mathcal{a}})' = \mathcal{A}'$, hence $\mathcal{B} \subset \mathcal{A}'$. Conversely, let $T \in \mathcal{A}'$.

We show that for every finite family $\widetilde{\xi}_1, \dots, \widetilde{\xi}_N \in \widetilde{H}$ there exists $\widetilde{x}' \in \widetilde{\mathcal{a}'}$ such that $\widetilde{x}'\widetilde{\xi}_l = T\widetilde{\xi}_l$, for each $l = 1, \dots, N$. We start with the case $N = 1$.

Let $\widetilde{\eta} = T\widetilde{\xi}$, we have for every $x \in \mathcal{A}_+$

$$\omega_{\widetilde{\eta}}(x) = \|x^{1/2}T\widetilde{\xi}\|^2 = \|Tx^{1/2}\widetilde{\xi}\|^2 \leqslant \|T\|^2\|x^{1/2}\widetilde{\xi}\|^2 = \|T\|^2\omega_{\widetilde{\xi}}(x),$$

i.e. $\omega_{\widetilde{\eta}} \leqslant \|T\|^2\omega_{\widetilde{\xi}}$. Let $(\xi_i)_i$ and $(\eta_i)_i$ be representing families for $\widetilde{\xi}$, respectively $\widetilde{\eta}$. For every $i \in I$, the form $\psi_i = \omega_{\eta_i} - \|T\|^2\omega_{\xi_i}$ is hermitian; it can be decomposed as $\psi_i = \psi_i^+ - \psi_i^-$, where $\psi_i^+, \psi_i^- \in \mathcal{a}_*^+$ have disjoint supports p_i^+, p_i^-. Let $\widetilde{p}^+ = (p_i^+)_i^\bullet$. We have $0 \leqslant \|\psi_i^+\| = \langle \psi_i, p_i^+\rangle$ and $\lim_{i,\mathcal{U}} \langle \psi_i, p_i^+\rangle = \langle \omega_{\widetilde{\eta}} - \|T\|^2\omega_{\widetilde{\xi}}, \widetilde{p}^+\rangle \leqslant 0$, hence $\lim_{i,\mathcal{U}} \|\psi_i^+\| = 0$. Since

$$\forall\, i \in I, \quad \omega_{\eta_i} \leqslant \omega_{\|T\|\xi_i} + \psi_i^+,$$

there exists by Lemma 1.9 an element $a'_i \in \mathcal{a}'$ and a $\zeta_i \in H$ such that $\|a'_i\| \leqslant 1$, $\|\zeta_i\| \leqslant \|\psi_i^+\|^{1/2}$ and

$$\eta_i = \|T\|a'_i\xi_i + \zeta_i.$$

Let $\widetilde{x}' = \|T\|(a'_i)_i^\bullet$; we have $\widetilde{x}' \in \widetilde{\mathcal{a}'}$, and $\widetilde{\eta} = x'\widetilde{\xi}$.

For the general case $N \geqslant 1$, we consider the spatial tensor product $\mathcal{a}^{(N)} = \mathcal{a} \otimes M_N(\mathbb{C})$, acting on the Hilbert tensor product $H^{(N)} = H \otimes \ell_2^N$. We have $(\mathcal{a}^{(N)})' = \mathcal{a}' \otimes \mathbb{C} \cdot I_N$, $\widetilde{H^{(N)}} = \widetilde{H}^{(N)}$, and $\widetilde{\mathcal{a}^{(N)}} = (\widetilde{\mathcal{a}})^{(N)}$, whose commutant in $B(\widetilde{H}^{(N)})$ is $\mathcal{A}' \otimes \mathbb{C} \cdot I_N$. Applying the preceding result to the element $\widehat{\xi} = (\widetilde{\xi}_1, \dots, \widetilde{\xi}_N)$ of $\widetilde{H}^{(N)}$ and the element $T \otimes I_N$ of $\mathcal{A}' \otimes \mathbb{C} \cdot I_N$ gives an element $\widetilde{x}' \otimes I_N$ of $\widetilde{\mathcal{a}'} \otimes \mathbb{C} \cdot I_N$ (the ultrapower of $(\mathcal{a}^{(N)})'$) such that $(T \otimes I_N)(\widehat{\xi}) = (\widetilde{x}' \otimes I_N)(\widehat{\xi})$, i.e. $T\widetilde{\xi}_l = x'\widetilde{\xi}_l$, $l = 1, \dots, N$. ∎

Remark 1.10. Exchanging the roles of $\mathcal{a}$ and $\mathcal{a}'$, we see that for every $x \in \mathcal{A}$ and every finite family $\widetilde{\xi}_1, \dots, \widetilde{\xi}_N$ in $\widetilde{H}$, there is an element $\widetilde{x} \in \widetilde{\mathcal{a}}$ such that $\|\widetilde{x}\| \leqslant \|x\|$ and $\widetilde{x}\widetilde{\xi}_l = x\widetilde{\xi}_l$, $l = 1, \dots, N$. Let $s(\widetilde{\xi}_1, \dots, \widetilde{\xi}_N)$ be the $\mathcal{A}$-support of the family $\widetilde{\xi}_1, \dots, \widetilde{\xi}_N$ (the least projection in $\mathcal{A}$ preserving the vectors $\widetilde{\xi}_1, \dots, \widetilde{\xi}_N$, which is also the projection on the closure of $\mathcal{A}'\widetilde{\xi}_1 + \dots + \mathcal{A}'\widetilde{\xi}_N$), then x and $\widetilde{x}$ coincide on the range of $s(\widetilde{\xi}_1, \dots, \widetilde{\xi}_N)$.

Corollary 1.11. *Let $\mathcal{A} = \widetilde{\mathcal{a}_*}$. For each $\widetilde{\varphi} \in \mathcal{A}_*^+$ and $x \in \mathcal{A}$ there is an $\widetilde{x} \in \widetilde{\mathcal{a}}$ such that $\|\widetilde{x}\| \leqslant \|x\|$ and $xp_{\widetilde{\varphi}} = \widetilde{x}p_{\widetilde{\varphi}}$, where $p_{\widetilde{\varphi}}$ is the support of the normal functional $\widetilde{\varphi}$.*

Proof. Suppose that the realization of $\mathcal{a}$ in $B(H)$ verifies the condition (R), so $\mathcal{A}$ coincides with the VNA generated by $\widetilde{\mathcal{a}}$ in $B(\widetilde{H})$; moreover, the realization of $\mathcal{A}$ in $B(\widetilde{H})$ verifies the condition (R) too. Hence $\widetilde{\varphi} = \omega_{\widetilde{\xi}}$ for some $\widetilde{\xi} \in \widetilde{H}$, so $p_{\widetilde{\varphi}} = s(\widetilde{\xi})$. Then Remark 1.10 implies the result. ∎

ON THE TYPE OF $\boldsymbol{\mathcal{A}}$. The ultrapower and ultraproduct procedures for VNA-preduals do not preserve semifiniteness for the dual VNA's. We show this for the case of $B(H)$ (ultrapowers) and of finite matrix spaces M_n (ultraproducts). The first lemma is probably a matter of folklore; we give a proof for the reader's convenience.

LEMMA 1.12. *$B(H)^{**}$ is not semifinite.*

Proof. Let $\mathcal{M}$ be a type III hyperfinite factor. Then $\mathcal{M}$ is injective, hence if $\mathcal{M} \subset B(H)$ is a realization of $\mathcal{M}$ there exists a contractive projection $P : B(H) \to \mathcal{M}$. Passing to the biconjugate we obtain a *normal* contractive projection P^{**} from $B(H)^{**}$ onto its sub VNA $\mathcal{M}^{**}$. Then P^{**} is a conditional expectation by Tomiyama's theorem. By a result of Sakai, if $B(H)^{**}$ is semifinite, so is $\mathcal{M}^{**}$ (see the proof of Lemma 2.6.5 of [21]). But there is a central projection z of $\mathcal{M}^{**}$ such that $\mathcal{M} \approx z\mathcal{M}^{**}$ ($*$-isomorphically). If $\mathcal{M}^{**}$ is semifinite, so is $z\mathcal{M}^{**}$ (and $\mathcal{M}$), a contradiction. ∎

LEMMA 1.13. *Let $\mathfrak{a}$ be a* VNA. *There exist a set I, a ultrafilter $\mathcal{U}$ on I and a* w**-continuous isometric $*$-embedding $\rho : \mathfrak{a}^{**} \hookrightarrow \mathcal{A} = [(\mathfrak{a}_*)_\mathcal{U}]^*$ whose range $\rho(\mathfrak{a}^{**})$ is the range of a* w**-continuous contractive projection preserving the identity.*

Proof. Let $j : \mathfrak{a}^* \hookrightarrow (\mathfrak{a}_*)_\mathcal{U}$ be an embedding given by the local reflexivity theorem (for appropriate I and $\mathcal{U}$, see e.g. [8]) and $i : \mathfrak{a} \hookrightarrow \mathfrak{a}_\mathcal{U}$ be the natural isometric embedding. In particular, we have $\langle \varphi, x \rangle_{\mathfrak{a}^* \times \mathfrak{a}} = \langle j(\varphi), i(x) \rangle_{\mathfrak{a}_\mathcal{U}^* \times \mathfrak{a}_\mathcal{U}}$ for every $\varphi \in \mathfrak{a}^*$ and $x \in \mathfrak{a}$. Define a linear contraction $P : (\mathfrak{a}_*)_\mathcal{U} \to \mathfrak{a}^*$ by:

$$\langle P\widetilde{\varphi}, x \rangle = \lim_{\alpha, \mathcal{U}} \langle \varphi_\alpha, x \rangle = \langle \widetilde{\varphi}, i(x) \rangle$$

for every $\widetilde{\varphi} = (\varphi_\alpha)^\bullet \in (\mathfrak{a}_*)_\mathcal{U}$ and $x \in \mathfrak{a}$. In particular, $\langle Pj(\varphi), x \rangle = \langle j(\varphi), i(x) \rangle = \langle \varphi, x \rangle$, so $Pj = \mathrm{id}_{\mathfrak{a}^*}$.

Consider the conjugate maps $\rho = P^* : \mathfrak{a}^{**} \to (\mathfrak{a}_*)_\mathcal{U}^* = \mathcal{A}$ and $Q = j^* : \mathcal{A} \to \mathfrak{a}^{**}$; then ρ, Q are w*-continuous contractions and $Q\rho = \mathrm{id}_{\mathfrak{a}^{**}}$, so ρ is an isometry and ρQ is a projection from $\mathcal{A}$ onto $\rho(\mathfrak{a}^{**})$.

If $x \in \mathfrak{a}$ and $\widetilde{\varphi} = (\varphi_\alpha)^\bullet \in (\mathfrak{a}_*)_\mathcal{U}$, we have:

$$\langle \rho(x), \widetilde{\varphi} \rangle_{\mathcal{A} \times \mathcal{A}_*} = \langle x, P\widetilde{\varphi} \rangle = \langle i(x), \widetilde{\varphi} \rangle_{\mathfrak{a}_\mathcal{U} \times (\mathfrak{a}_*)_\mathcal{U}},$$

hence the restriction $\rho|_\mathfrak{a}$ identifies with the map $i : \mathfrak{a} \to \mathfrak{a}_\mathcal{U}$ (when this last C^*-algebra is considered as embedded in $\mathcal{A}$). In particular, $\rho|_\mathfrak{a}$ preserves the product of $\mathfrak{a}$. Since ρ is w*-w*-continuous and the product is separately w*-continuous, we obtain that $\rho(xy) = \rho(x)\rho(y)$ for every $x, y \in \mathfrak{a}^{**}$. Since P is positive, so is ρ, which is thus a $*$-embedding. ∎

Let $S_1(H)$ be the trace class over H.

Proposition 1.14. *For a suitable ultrafilter $\mathcal{U}$, $[S_1(H)_{\mathcal{U}}]^*$ is not semifinite.*

Proof. By the Lemma 1.13 with $\mathfrak{a} = B(H)$, the VNA $B(H)^{**}$ is $*$-isomorphic to a sub-VNA of $\mathcal{A} = (S_1(H)_{\mathcal{U}})^*$ which is complemented in $\mathcal{A}$ by a normal contractive projection (hence a normal conditional expectation by Tomiyama's theorem). Since $B(H)^{**}$ is not semifinite, $\mathcal{A}$ cannot be semifinite. ∎

For every natural number n, let $S_1^n = S_1(\ell_2^n)$ be the trace class over the n-dimensional Hilbert space ℓ_2^n. Similarly, let $S_1 := S_1(\ell_2)$.

Lemma 1.15. *For every ultrafilter $\mathcal{U}$, the* VNA $(S_1)^*_{\mathcal{U}}$ *is $*$-isomorphic to a contractively* w**-complemented sub-*VNA *of some ultraproduct* $\Big(\prod_k S_1^{n(k)}/\mathcal{W}\Big)^*$ *(for suitable ultrafilter $\mathcal{W}$ and map $k \mapsto n(k)$, $\mathbb{N} \to \mathbb{N}$).*

Proof. Let us identify S_1^n with the subspace $p_n S_1 p_n$ of S_1, where p_n is the natural orthogonal projection $\ell_2 \to \ell_2^n$. Let $\mathcal{V}$ be any free ultrafilter over $\mathbb{N}$; we have a natural isometric embedding $j : S_1 \hookrightarrow \prod_n S_1^n/\mathcal{V}$ defined by $x \mapsto (p_n x p_n)^\bullet$. Let $P : \prod_n S_1^n/\mathcal{V} \to S_1$ be defined by

$$P((x_n)^\bullet) = w^* \lim_{n,\mathcal{U}} x_n$$

(the w*-topology is relative to the duality of $S_1(\ell_2)$ with $K(\ell_2)$). Then P is a contraction and $Pj = \mathrm{id}_{S_1}$, so $j(S_1)$ is 1-complemented in $\prod_n S_1^n/\mathcal{V}$.

Passing to the ultrapowers along any ultrafilter $\mathcal{U}$ we obtain an isometric embedding $j_{\mathcal{U}} : (S_1)_{\mathcal{U}} \hookrightarrow \Big(\prod_n S_1^n/\mathcal{V}\Big)_{\mathcal{U}}$ and a surjective contraction $P_{\mathcal{U}} : \Big(\prod_n S_1^n/\mathcal{V}\Big)_{\mathcal{U}} \to (S_1)_{\mathcal{U}}$ such that $P_{\mathcal{U}} j_{\mathcal{U}}$ is the identity of $(S_1)_{\mathcal{U}}$.

Note that $\Big(\prod_n S_1^n/\mathcal{V}\Big)_{\mathcal{U}} = \prod_{(n,l)} S_1^n/\mathcal{V}\times\mathcal{U} = \prod_k S_1^{n(k)}/\mathcal{W}$ where $\mathcal{V}\times\mathcal{U}$ is the product ultrafilter and $\mathcal{W}$ is its image by some bijection from $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$.

By dualizing, we deduce a w*-continuous isometric embedding $\rho = P^*_{\mathcal{U}}$ of $\mathcal{A} = (S_1)^*_{\mathcal{U}}$ into $\mathcal{B} = \Big(\prod_k S_1^{n(k)}/\mathcal{W}\Big)^*$, and a w*-continuous contractive surjection $Q = j^*_{\mathcal{U}}$ from $\mathcal{B}$ onto $\mathcal{A}$ such that $QP = \mathrm{id}_{\mathcal{A}}$.

Let us show that ρ is multiplicative. We make this map explicit over the sub-C^*-algebra $K(H)_{\mathcal{U}}$ of $\mathcal{A}$. If $\widetilde{x} = (x_k)^\bullet \in K(H)_{\mathcal{U}}$ and $\widetilde{\varphi} = (\varphi_{n,l})^\bullet \in \prod_{n,l} S_1^n/\mathcal{V}\times\mathcal{U}$, we have:

$$\langle\rho(\widetilde{x}),\widetilde{\varphi}\rangle = \langle\widetilde{x}, P_{\mathcal{U}}\widetilde{\varphi}\rangle = \lim_{l,\mathcal{U}}\lim_{n,\mathcal{V}}\langle x_l,\varphi_{n,l}\rangle = \lim_{l,\mathcal{U}}\lim_{n,\mathcal{V}}\langle p_n x_l p_n,\varphi_{n,l}\rangle = \langle i(\widetilde{x}),\widetilde{\varphi}\rangle$$

where $i : (x_l)^\bullet \mapsto (p_n x_l p_n)^\bullet$ is the natural isometric embedding $B(H)_{\mathcal{U}} \to \Big(\prod_n B(l_2^n)/\mathcal{V}\Big)_{\mathcal{U}}$ (which is a subalgebra of $\mathcal{B}$). This last embedding preserve products on $K(H)_{\mathcal{U}}$ (but of course not on the whole of $B(H)_{\mathcal{U}}$). Using the w*-density of $K(H)$ in $B(H)$ and that of $B(H)_{\mathcal{U}}$ in $\mathcal{A}$, one easily sees that $K(H)_{\mathcal{U}}$ is w*-dense in $\mathcal{A}$; so by w*-continuity of ρ and separate w*-continuity of the product, we see that ρ preserves the product. Since ρ is positive, it is a $*$-embedding. Its image is complemented by the contractive w*-continuous projection ρQ. ∎

LEMMA 1.16. *$\Big(\prod_k S_1^{n(k)}/\mathcal{W}\Big)^*$ is isomorphic to a contractively* w**-complemented sub*-VNA *of some* $\Big(\prod_m S_1^m/\mathcal{V}\Big)^*$ *(with same identity).*

Proof. One can find a strictly increasing function $m \mapsto m(k)$, $\mathbb{N} \to \mathbb{N}$ such that $n(k)$ divides $m(k)$ for every k (note that $k \mapsto n(k)$ has no reason to be increasing). We have then $S_1^{m(k)} \approx S_1^{n(k)} \otimes S_1^{p(k)}$ (linearly) for some $p(k) \in \mathbb{N}$. We have isometric embeddings $i_k : S_1^{n(k)} \to S_1^{m(k)}$ and contractive surjections $Q_k : S_1^{m(k)} \to S_1^{n(k)}$ defined by:

$$i_k(x) = \frac{1}{p(k)} x \otimes I_{p(k)}, \quad \forall\, x \in S_1^{n(k)}$$
$$Q_k = \mathrm{id}_{S_1^{n(k)}} \otimes \mathrm{Tr}_{p(k)}$$

where $I_{p(k)}$ is the identity operator on $l_2^{p(k)}$ and $\mathrm{Tr}_{p(k)}$ is the normalized trace $S_1^{p(k)} \to \mathbb{C}$. Passing to ultraproducts, we obtain an isometric embedding $i = (i_k)^\bullet$ from $\prod_k S_1^{n(k)}/\mathcal{W}$ into $\prod_k S_1^{m(k)}/\mathcal{W}$ and a contractive surjection $Q = (Q_k)^\bullet$ from $\prod_k S_1^{m(k)}/\mathcal{W}$ onto $\prod_k S_1^{n(k)}/\mathcal{W}$ with $Qi = \mathrm{id}$. Dualizing, we obtain an isometric embedding $\rho = Q^*$ of $\Big(\prod_k S_1^{n(k)}/\mathcal{W}\Big)^*$ into $\Big(\prod_k S_1^{m(k)}/\mathcal{W}\Big)^*$ and a contractive surjection $P = i^*$ from $\Big(\prod_k S_1^{m(k)}/\mathcal{W}\Big)^*$ onto $\Big(\prod_k S_1^{n(k)}/\mathcal{W}\Big)^*$, both w*-continuous, with $P\rho = \mathrm{id}$. Again ρ coincides with the trivial $*$-embeding $(x_k)^\bullet \mapsto (x_k \otimes I_{p(k)})^\bullet$ on the w*-dense subalgebra $\prod_k B(\ell_2^{n(k)})/\mathcal{W}$, so it is a $*$-embedding of VNA's. Finally, it is immediate that $\prod_k S_1^{m(k)}/\mathcal{W}$ is equal to $\prod_k S_1^n/\mathcal{V}$ for a suitable ultrafilter containing the sequence $\{n(k) : k \in \mathbb{N}\}$. ∎

PROPOSITION 1.17. *For a suitable ultrafilter $\mathcal{V}$ over $\mathbb{N}$, the* VNA $(\prod S_1^n/\mathcal{V})^*$ *is not semifinite.*

Proof. By Lemmas 1.15 and 1.16 this VNA contains $(S_1)^*_{\mathcal{U}}$ as weak*-1-complemented sub-VNA, for some ultrafilter $\mathcal{U}$ for which this last VNA is not semifinite. ∎

2. ELEMENTS OF MODULAR THEORY FOR $\mathcal{A}$

2.A. Local modular automorphism groups. If φ is a normal positive linear form on $\mathcal{a}$, let p_φ be the support of φ. The local modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ will simply be the usual automorphism group of the VNA $\mathcal{a}_\varphi := p_\varphi \mathcal{a} p_\varphi$ with respect to the restriction of φ. We denote by Δ_φ the modular operator associated with φ: it is a positive selfadjoint unbounded operator on the Hilbert space $\mathcal{H}_\varphi = L_2(\varphi)$, completion of $\mathcal{a}_\varphi$ for the scalar product $(\cdot\,,\cdot)_\varphi$ associated with φ (defined as usual by $(x,y)_\varphi = \varphi(y^*x)$).

Let $\Phi = (\varphi_i)_{i\in I}$ be a bounded family in $\mathcal{a}_*^+$ and $\widetilde{\varphi}$ be the element represented by Φ in the ultrapower $\widetilde{\mathcal{a}_*}$, which we identify with the predual of a VNA $\mathcal{A}$ in which $\widetilde{\mathcal{a}}$ embeds as C^*-subalgebra. Let $\widetilde{p}_\Phi = (p_{\varphi_i})^\bullet$ and $\mathcal{A}_\Phi := \widetilde{p}_\Phi \mathcal{A} \widetilde{p}_\Phi$. Since $\widetilde{\varphi}(\widetilde{p}_\Phi^\perp) = \lim_{i,\mathcal{U}} \langle \varphi_i, p_{\varphi_i}^\perp\rangle = 0$, we have $p_{\tilde\varphi} \leqslant \widetilde{p}_\Phi$, hence $\mathcal{A}_{\tilde\varphi} = p_{\tilde\varphi}\mathcal{A}_\Phi p_{\tilde\varphi}$. Note that $(\mathcal{A}_\Phi)_* = \widetilde{p}_\Phi \mathcal{A}_* \widetilde{p}_\Phi$ identifies with $\prod_{i\in I} (\mathcal{a}_{\varphi_i *})/\mathcal{U}$; following Section 1 (which is valid in the case of ultraproducts as well as in the case of ultrapowers) we may identify $\mathcal{A}_\Phi$ with the VNA generated by the C^*-algebra $\widetilde{\mathcal{a}}_\Phi$ in $B(\widetilde{\mathcal{H}}_\Phi)$, where $\widetilde{\mathcal{H}}_\Phi = \prod_{i\in I} \mathcal{H}_{\varphi_i}/\mathcal{U}$. (Note that since each space $\mathcal{H}_{\varphi_i}$ has a vector which is cyclic and separating for $\mathcal{a}_{\varphi_i}$, it verifies condition (R).) We may consider:

– the local modular automorphism group $(\sigma_t^{\tilde\varphi})_{t\in\mathbb{R}}$ acting on the VNA $\mathcal{A}_{\tilde\varphi} = p_{\tilde\varphi}\mathcal{A}p_{\tilde\varphi}$;

– the group of $*$-automorphisms $(\widetilde{\sigma}_t^\Phi)_{t\in\mathbb{R}}$ acting on the C^*-algebra $\widetilde{\mathcal{a}}_\Phi := \widetilde{p}_\Phi \widetilde{\mathcal{a}} \widetilde{p}_\Phi$, where $\widetilde{\sigma}_t^\Phi$ is the ultrapower map associated with the family $(\widetilde{\sigma_t^{\varphi_i}})_i$. This group of automorphisms extends naturally to a group of $*$-automorphisms of $\mathcal{A}_\Phi$ since it is implemented by a group of unitary operators $(U_t)_{t\in\mathbb{R}}$ on $\widetilde{\mathcal{H}}_\Phi$: U_t is simply the ultrapower map associated with the family $(\Delta_{\varphi_j}^{\mathrm{it}})_{j\in I}$. We still denote by $(\widetilde{\sigma}_t^\Phi)$ these extensions to $\mathcal{A}_\Phi$.

Theorem 2.1. *For every element $\widetilde{\varphi} \in \mathcal{A}_*^+$ and every representing family $\Phi = (\varphi_i)_{i\in I} \subset \mathcal{a}_*^+$ of $\widetilde{\varphi}$, the reduced* VNA *$\mathcal{A}_{\tilde\varphi} = p_{\tilde\varphi}\mathcal{A}p_{\tilde\varphi}$ is preserved by the automorphims $\widetilde{\sigma}_t^\Phi$ of $\mathcal{A}_\Phi := \widetilde{p}_\Phi\mathcal{A}\widetilde{p}_\Phi$ and the restrictions to $\mathcal{A}_{\tilde\varphi}$ of these automorphisms coincide with the local modular automorphisms $\sigma_t^{\tilde\varphi}$ associated with $\widetilde{\varphi}$.*

Proof. For every $i \in I$, there is a natural separating cyclic vector $\xi_{\varphi_i} \in \mathcal{H}_{\varphi_i}$ for $\mathcal{a}_{\varphi_i}$; we have $\varphi_i = \omega_{\xi_i}$. Let $\widetilde{\xi}_\Phi$ be the vector of $\widetilde{\mathcal{H}}_\Phi$ represented by $(\xi_{\varphi_i})_{i\in I}$; then $\widetilde{\varphi} = \omega_{\tilde\xi}$. Let $\mathcal{a}'_{\varphi_i}$ denote the commutant of $\mathcal{a}_{\varphi_i}$ in $B(H_{\varphi_i})$. Then $\widetilde{\mathcal{a}'}_\Phi := \prod_{i\in I} \mathcal{a}'_{\varphi_i}/\mathcal{U}$ generates the commutant $\mathcal{A}'_\Phi$ of $\mathcal{A}_\Phi$ in $B(\widetilde{\mathcal{H}}_\Phi)$ (see Theorem 1.8). As a consequence, $p_{\tilde\varphi} \cdot \widetilde{\mathcal{H}}_\Phi$ is the closure of $\widetilde{\mathcal{a}'_\Phi} \cdot \widetilde{\xi}$.

(i) We show first that $\mathcal{A}_{\tilde\varphi}$ is preserved by each $\widetilde{\sigma}_t^\Phi$. If $\widetilde{x}' = (x'_i)^\bullet \in \widetilde{\mathcal{a}'_\Phi}$ we have:

$$U_t \widetilde{x}' \widetilde{\xi}_\Phi = (\Delta_{\varphi_j}^{\mathrm{it}} x'_j \xi_j)_j^\bullet = (\tau_{-t}^{\varphi'_j}(x'_j)\xi_j)_j^\bullet \in \widetilde{\mathcal{a}'_\Phi} \cdot \widetilde{\xi}$$

where $(\tau_t^{\varphi'_j})_t$ is the modular automorphism group of $\mathcal{a}'_{\varphi_j}$ relative to the normal semifinite (n.s.f.) linear form $\varphi'_j = \omega_{\xi_j}$. Hence $U_t\widetilde{x}'\widetilde{\xi}_\Phi \in p_{\tilde\varphi}\widetilde{\mathcal{H}}_\Phi$, and so $U_t(p_{\tilde\varphi}\widetilde{\mathcal{H}}_\Phi) \subset p_{\tilde\varphi}\widetilde{\mathcal{H}}_\Phi$. Since $U_t^{-1} = U_{-t}$ we have in fact $U_t(p_{\tilde\varphi}\widetilde{\mathcal{H}}_\Phi) = p_{\tilde\varphi}\widetilde{\mathcal{H}}_\Phi$, hence $\widetilde{\sigma}_t^\Phi(p_{\tilde\varphi}) = U_t p_{\tilde\varphi} U_t^* = p_{\tilde\varphi}$.

(ii) We show now that the group $(\widetilde{\sigma}_t^\Phi)$ (of $*$-automorphisms of $\mathcal{A}_{\tilde\varphi}$) verifies the KMS modular condition relative to $\widetilde{\varphi}$ ([15], Definition 9.2.10). This will imply that $(\widetilde{\sigma}_t^\Phi)$ coincides with the modular automorphism group $(\sigma_t^{\tilde\varphi})$ ([15], Theorems 9.2.13 and 9.2.16).

Let $x, y \in p_{\tilde\varphi}\widetilde{\mathcal{a}}p_{\tilde\varphi}$ ($= p_{\tilde\varphi}\mathcal{A}p_{\tilde\varphi}$, see Corollary 1.11), i.e. $x = p_{\tilde\varphi}\widetilde{x}p_{\tilde\varphi}$, $y = p_{\tilde\varphi}\widetilde{y}p_{\tilde\varphi}$, with $\widetilde{x} = (x_i)^\bullet$, $\widetilde{y} = (y_i)_i^\bullet$, and $x_i, y_i \in \mathcal{a}_{\varphi_i}$. We have:

$$\begin{aligned}\widetilde{\varphi}(\widetilde{\sigma}_t^\Phi(x)y) &= \widetilde{\varphi}(\widetilde{\sigma}_t^\Phi(p_{\tilde\varphi}\widetilde{x}p_{\tilde\varphi})p_{\tilde\varphi}\widetilde{y}p_{\tilde\varphi}) = \widetilde{\varphi}(p_{\tilde\varphi}\widetilde{\sigma}_t^\Phi(\widetilde{x})p_{\tilde\varphi}\widetilde{y}p_{\tilde\varphi})\\ &= \widetilde{\varphi}(\widetilde{\sigma}_t^\Phi(\widetilde{x})p_{\tilde\varphi}\widetilde{y}) = (p_{\tilde\varphi}\widetilde{y}\widetilde{\xi}, \widetilde{\sigma}_t^\Phi(\widetilde{x})^*\widetilde{\xi})_{\tilde{\mathcal{H}}_\Phi}\end{aligned}$$

and similarly

$$\widetilde{\varphi}(y\widetilde{\sigma}_t^\Phi(x)) = \widetilde{\varphi}(\widetilde{y}p_{\tilde\varphi}\widetilde{\sigma}_t^\Phi(\widetilde{x})) = (\widetilde{\sigma}_t^\Phi(\widetilde{x})\widetilde{\xi}, p_{\tilde\varphi}\widetilde{y}^*\widetilde{\xi})_{\tilde{\mathcal{H}}_\Phi}.$$

Since the unit ball of $\widetilde{\mathcal{a}}_\Phi$ is $*$-strongly dense in that of $\mathcal{A}_\Phi$, there exists a sequence $(\widetilde{z}_n)_n$ in $\widetilde{\mathcal{a}}_\Phi$ such that:

$$\|\widetilde{z}_n\|_{\tilde{a}_\Phi} \leqslant \|p_{\tilde\varphi}\widetilde{y}p_{\tilde\varphi}\|_{\mathcal{A}_\Phi}, \quad \widetilde{z}_n\widetilde{\xi} \underset{n}{\longrightarrow} p_{\tilde\varphi}\widetilde{y}p_{\tilde\varphi}\widetilde{\xi} = p_{\tilde\varphi}\widetilde{y}\widetilde{\xi}, \quad \widetilde{z}_n^*\widetilde{\xi} \underset{n}{\longrightarrow} p_{\tilde\varphi}\widetilde{y}^*p_{\tilde\varphi}\widetilde{\xi} = p_{\tilde\varphi}\widetilde{y}^*\widetilde{\xi}.$$

Then:

$$\begin{aligned}\widetilde{\varphi}(\widetilde{\sigma}_t^\Phi(x)y) &= \lim_n (\widetilde{z}_n\widetilde{\xi}, \widetilde{\sigma}_t^\Phi(\widetilde{x})^*\widetilde{\xi}) = \lim_n \lim_{j,\mathcal{U}} (z_{n,j}\xi_j, \sigma_t^{\varphi_j}(x_j)^*\xi_j)\\ &= \lim_n \lim_{j,\mathcal{U}} \varphi_j(\sigma_t^{\varphi_j}(x_j)z_{n,j})\end{aligned}$$

and similarly:

$$\widetilde{\varphi}(y\widetilde{\sigma}_t^\Phi(x)) = \lim_n \lim_{j,\mathcal{U}} (\sigma_t^{\varphi_j}(x_j)\xi_j, z_{n,j}^*\widetilde{\xi}_j) = \lim_n \lim_{j,\mathcal{U}} \varphi_j(z_{n,j}\sigma_t^{\varphi_j}(x_j)).$$

For every $j \in I$, the KMS condition for the group $(\sigma_t^{\varphi_j})_t$ relatively to φ_j yields a bounded continuous $\mathbb{C}$-valued function $F_{n,j}$ defined on the closed strip $S = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Im} z \leqslant 1\}$, analytic on the open strip $\overset{\circ}{S}$, such that:

$$\forall\, t \in \mathbb{R}, \qquad \begin{cases} F_{n,j}(t) = \varphi_j(\sigma_t^{\varphi_j}(x_j)z_{n,j}) \\ F_{n,j}(t+i) = \varphi_j(z_{n,j}\sigma_t^{\varphi_j}(x_j)). \end{cases}$$

Since

$$\max\{|F_{n,j}(t)|, |F_{n,j}(t+i)|\} \leqslant \|\varphi_j\|\,\|\sigma_t^{\varphi_j}(x_j)\|\,\|z_{n,j}\| = \|\varphi_j\|\,\|x_j\|\,\|z_{n,j}\|,$$

the functions $F_{n,j}$ are actually uniformly bounded on S by the maximum principle (Phragmen-Lindelöf). We shall prove that the limits

$$\begin{cases} \lim_n \lim_{j,\mathcal{U}} F_{n,j}(t) = \widetilde{\varphi}(\widetilde{\sigma}_t^\Phi(x)\widetilde{y}) \\ \lim_n \lim_{j,\mathcal{U}} F_{n,j}(t+\mathrm{i}) = \widetilde{\varphi}(\widetilde{y}\widetilde{\sigma}_t^\Phi(x)) \end{cases}$$

are locally uniform with respect to t (i.e. uniformly for $t \in [-M, +M]$, for every $M > 0$). Since, by the maximum principle and the usual device of multiplication by the analytic fonctions $G_\varepsilon(z) = \mathrm{e}^{-\varepsilon z^2}$, the locally uniform Cauchy condition on the boundary of S for the $F_{n,j}$ implies a locally uniform Cauchy condition on the whole of S, we shall obtain that the $F_{n,j}$ converge uniformly on compact sets of S to a bounded continuous function F, which will be of course analytic on the interior of S, and such that:

$$\forall\, t \in \mathbb{R}, \quad \begin{cases} F(t) = \widetilde{\varphi}(\widetilde{\sigma}^{\Phi}_t(x)y) \\ F(t+i) = \widetilde{\varphi}(y\widetilde{\sigma}^{\Phi}_t(x)). \end{cases}$$

We treat only the convergence of the $F_{n,j}$ on $\mathbb{R}$; the case of the convergence on $\mathbb{R} + \mathrm{i}$ can be done similarly. We have:

$$\forall\, t \in \mathbb{R}, \quad F_{n,j}(t) = (z_{n,j}\xi_j, \Delta^{\mathrm{i}t}_{\varphi_j} x^*_j \xi_j).$$

Consider the following spectral projectors of the selfadjoint operator Δ_{φ_j}:

$$e^{(1)}_j = \mathbb{1}_{[0,1]}(\Delta_{\varphi_j}) \quad \text{and} \quad e^{(2)}_j = \mathbb{1}_{]1,\infty)}(\Delta_{\varphi_j}) = I - e^{(1)}_j;$$

then we obtain a decomposition $F_{n,j}(t) = g^{(1)}_{n,j}(t) + g^{(2)}_{n,j}(t)$ where for $l = 1, 2$:

$$g^{(l)}_{n,j}(t) = (z_{n,j}\xi_j, e^{(l)}_j \Delta^{\mathrm{i}t}_{\varphi_j} x^*_j \xi_j).$$

We show now that the functions $g^{(2)}_{n,j}$, $n \in \mathbb{N}$, $j \in I$ are equicontinuous on $\mathbb{R}$, while the functions $g^{(1)}_{n,j}$ are *asymptotically equicontinuous* in the sense that for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and for every $n \geqslant n_0$ a set $U_n \in \mathcal{U}$ and an equicontinuous family of functions $h_{n,j}$ such that $\sup_{n \geqslant n_0,\, j \in U_n} \|g^{(1)}_{n,j} - h_{n,j}\|_\infty < \varepsilon$. Then the functions $F_{n,j}$ (restricted to $\mathbb{R}$) are also asymptotically equicontinuous, which implies by a variant of Ascoli's theorem that their convergence is uniform on compact sets.

For every $t, s \in \mathbb{R}$ and $\lambda > 0$ we have the elementary inequality

$$|\lambda^{\mathrm{i}(t+s)} - \lambda^{\mathrm{i}t}| \leqslant |s||\log \lambda| \leqslant 2|s| \max(\lambda^{1/2}, \lambda^{-1/2}) \tag{2.1}$$

from which we infer that:

$$\|(\Delta^{\mathrm{i}(t+s)}_{\varphi_j} - \Delta^{\mathrm{i}t}_{\varphi_j}) e^{(2)}_j x^*_j \xi_j\| \leqslant 2|s|\, \|\Delta^{1/2}_{\varphi_j} x^*_j \xi_j\| = 2|s|\, \|x_j \xi_j\| \leqslant 2C|s|$$

with $C = \sup_j \|x_j\xi_j\|$, which clearly shows that the family $(g^{(2)}_{n,j})_{n \in \mathbb{N},\, j \in I}$ is equicontinuous, in fact equi-Lipschitz.

For the functions $g^{(1)}_{n,j}$, we note that since $\lim_n \widetilde{z}_n \widetilde{\xi} = p_{\widetilde{\varphi}} \widetilde{y} \widetilde{\xi}$ belongs to the closure of $\widetilde{\mathcal{A}'_\Phi} \cdot \widetilde{\xi}$, there exist for every $\varepsilon > 0$ an integer n_0 and an element $\widetilde{z}'^{\varepsilon} = (z'^{\varepsilon}_j)^\bullet_{j \in I}$ of $\widetilde{\mathcal{A}'_\Phi}$ such that

$$\forall\, n \geqslant n_0,\ \exists U_n \in \mathcal{U} \quad \text{such that} \quad \forall\, j \in U_n,\ \|z_{n,j}\xi_j - z'^{\varepsilon}_j \xi_j\| < \varepsilon.$$

Set

$$g^{\varepsilon}_j(t) = (z'^{\varepsilon}_j \xi_j, e^{(1)}_j \Delta^{\mathrm{i}t}_{\varphi_j} x^*_j \xi_j) = (e^{(1)}_j \Delta^{-\mathrm{i}t}_{\varphi_j} z'^{\varepsilon}_j \xi_j, x^*_j \xi_j);$$

from inequality (2.1) we have:

$$\|(\Delta_{\varphi_j}^{-\mathrm{i}(t+s)} - \Delta_{\varphi_j}^{-\mathrm{i}t})e_j^{(1)} z_j'^{\varepsilon}\xi_j\| \leqslant 2|s|\, \|\Delta_{\varphi_j}^{-1/2} z_j'^{\varepsilon}\xi_j\| = 2|s|\, \|z_j'^{\varepsilon *}\xi_j\| \leqslant 2C'_\varepsilon |s|$$

where $C'_\varepsilon = \sup_j \|z_j'^{\varepsilon}\| \cdot \|\xi_j\|$; so for each $\varepsilon > 0$ the family $(g_j^\varepsilon)_j$ is equi-Lipschitz. Moreover:

$$\forall\, n \geqslant n_0,\, \forall\, j \in U_n, \quad \|g_{n,j}^{(1)} - g_j^\varepsilon\|_\infty \leqslant B\varepsilon$$

where $B = \sup_j \|x_j^* \xi_j\|$. ∎

2.b. Relative modular theory. Given a VNA $\mathcal{a}$ and two normal positive linear forms φ_1 and φ_2 with same support p, we consider (following the classical construction of A. Connes, see Section 1.2 of [1]) the (spatial) tensor product $\mathcal{a}^{(2)} = \mathcal{a} \otimes M_2(\mathbb{C})$ of $\mathcal{a}$ with the algebra of 2×2 complex matrices and the normal positive form ψ on $\mathcal{a}^{(2)}$ defined by $\psi(\sum x_{kl} \otimes e_{kl}) = \varphi_1(x_{11}) + \varphi_2(x_{22})$ (the $e_{k,l}$, $k,l = 1,2$ are the elementary matrices of M_2). Identifying the predual $\mathcal{a}_*^{(2)}$ with the (algebraic) tensor product $\mathcal{a}_* \otimes M_2$, we have $\psi = \varphi_1 \otimes e_{11} + \varphi_2 \otimes e_{22}$. The support of ψ is $p^{(2)} = p \otimes e_{11} + p \otimes e_{22}$. Then the modular automorphisms σ_t^ψ of $p^{(2)}\mathcal{a}^{(2)}p^{(2)} = (p\mathcal{a}p)^{(2)}$ relative to ψ preserve the subspaces $p\mathcal{a}p \otimes e_{k,l}$ of $p^{(2)}\mathcal{a}^{(2)}p^{(2)}$; in particular, for every $x \in p\mathcal{a}p$, $\sigma_t^\psi(x \otimes e_{12}) = \sigma_t^{\varphi_1\varphi_2}(x) \otimes e_{12}$. We have:

$$\sigma_t^{\varphi_1\varphi_2}(x) = \sigma_t^{\varphi_1}(x)(D\varphi_1 : D\varphi_2)_t = (D\varphi_1 : D\varphi_2)_t \sigma_t^{\varphi_2}(x)$$

where the family of elements $(D\varphi_1 : D\varphi_2)_t = \sigma_t^{\varphi_1\varphi_2}(p) \in p\mathcal{a}p$ is the Radon-Nikodym cocycle of φ_1 with respect to φ_2.

Using the KMS condition for the modular automorphism group $(\sigma_t^\psi)_{t\in\mathbb{R}}$, we obtain for each couple $(x,y) \in p\mathcal{a}p$ a (unique) bounded continuous function $F_{x,y} : S \to \mathbb{C}$, analytic on $\mathring{S}$, verifying the boundary conditions:

$$\forall\, t \in \mathbb{R}, \quad \begin{cases} \varphi_1(\sigma_t^{\varphi_1\varphi_2}(x)y) = F_{x,y}(t) \\ \varphi_2(y\sigma_t^{\varphi_1\varphi_2}(x)) = F_{x,y}(t+\mathrm{i}). \end{cases}$$

Set $(\varphi_1^{1-\theta} \cdot \varphi_2^\theta)(x) = F_{p,x}(\mathrm{i}\theta)$. Then $\varphi_1^{1-\theta} \cdot \varphi_2^\theta$ is clearly a bounded linear form on $p\mathcal{a}p$, in general not hermitian; it is in fact w*-continuous, as can easily be seen using the Poisson integral representation formula for $F_{x,p}(z)$, $z \in \mathring{S}$); hence it is an element of $(p\mathcal{a}p)_*$ which extends naturally to an element of $p\mathcal{a}_*p$. It is easy to see (using the three lines theorem) that $\|\varphi_1^{1-\theta} \cdot \varphi_2^\theta\| \leqslant \|\varphi_1\|^{1-\theta}\|\varphi_2\|^\theta$.

We study now the behaviour of these constructions under ultraproducts. If $\mathcal{a}$ is represented as a VNA over the Hilbert space H, then $\mathcal{a}^{(2)}$ is represented as a VNA over the Hilbert space $H \oplus H = H \otimes_2 \ell_2^2$. Identifying $\widetilde{\mathcal{a}}$ with a sub C^*-algebra C of $B(\widetilde{H})$, we can identify $\widetilde{\mathcal{a}^{(2)}}$ with the C^*-algebra $C^{(2)}$ of $B(\widetilde{H} \oplus \widetilde{H}) = B(\widetilde{H}) \otimes M_2$. Let $\mathcal{A}$ be the dual of $\widetilde{\mathcal{a}_*}$. We have:

$$(\mathcal{a}_* \otimes M_2)^\sim = \widetilde{\mathcal{a}_*} \otimes M_2 = \mathcal{A}_* \otimes M_2 = (\mathcal{A} \otimes M_2)_*$$

where the identifications are algebraic ones (i.e. the norms of the first and last member are a priori only equivalent). But the duals of these spaces admit respectively 1-norming subspaces $\widetilde{\mathfrak{a}^{(2)}}$ and $(\widetilde{\mathfrak{a}})^{(2)}$ which isometrically identifies, so the identification $(\mathfrak{a}_* \otimes M_2)^{\sim} = (\mathcal{A} \otimes M_2)_*$ is isometric. Moreover, the C^*-algebra $\widetilde{\mathfrak{a}^{(2)}}$ is a sub-C^*-algebra of $\mathcal{A}^{(2)}$, so $\mathcal{A}^{(2)}$ identifies (as VNA) with the dual VNA of $\widetilde{\mathfrak{a}^{(2)}_*}$ (see Remark 1.4).

Proposition 2.2. *Let $\widetilde{\varphi}_1, \widetilde{\varphi}_2$ be two positive elements of $\mathcal{A}_*$ with the same support p. We can choose representing families $(\varphi_{1,i})_{i\in I}$ and $(\varphi_{2,i})_{i\in I}$ in $\mathfrak{a}_*^+$ such that for every $i \in I$, the normal forms have the same support p_i. Then:*

(i) $p\,((D\varphi_{1,i} : D\varphi_{2,i})_t)_i^\bullet = ((D\varphi_{1,i} : D\varphi_{2,i})_t)_i^\bullet\, p = (D\widetilde{\varphi}_1 : D\widetilde{\varphi}_2)_t$ *(for every $t \in \mathbb{R}$);*

(ii) $(\varphi_{1,i}^{(1-\theta)} \cdot \varphi_{2,i}^{\theta})_i^\bullet = \widetilde{\varphi}_1^{(1-\theta)} \cdot \widetilde{\varphi}_2^{\theta}$ *(for every $\theta \in [0,1]$).*

Proof. Let $(\varphi_{1,i})_{i\in I}$ and $(\varphi_{2,i})_{i\in I}$ be two arbitrary representing families for $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$. If the ultrafilter $\mathcal{U}$ is countably incomplete we can find a family of positive real numbers $(\varepsilon_i)_{i\in I}$ converging to zero along $\mathcal{U}$. Put

$$\begin{cases} \varphi_{1,i}^0 = \varphi_{1,i} + \varepsilon_i \varphi_{2,i} \\ \varphi_{2,i}^0 = \varepsilon_i \varphi_{1,i} + \varphi_{2,i}. \end{cases}$$

Then $\varphi_{1,i}^0, \varphi_{2,i}^0$ have the same support and $(\varphi_{1,i}^0)_{i\in I}$, $(\varphi_{2,i}^0)_{i\in I}$ represent also the elements $\widetilde{\varphi}_1$, respectively $\widetilde{\varphi}_2$. The case where $\mathcal{U}$ is not countably incomplete is essentially the trivial one (in this case the supports $p_{1,i}$ and $p_{2,i}$ of $\varphi_{1,i}$, respectively $\varphi_{2,i}$, coincide for every i belonging to some element U of the ultrafilter, since $\lim_{i,\mathcal{U}} \langle \varphi_{1,i}, p_{2,i}^\perp \rangle = \langle \widetilde{\varphi}_1, (p_{2,i})^{\bullet\perp} \rangle \leqslant \langle \widetilde{\varphi}_1, p^\perp \rangle = 0$ implies the equality $\langle \varphi_{1,i}, p_{2,i}^\perp \rangle = 0$, i.e. $p_{1,i} \leqslant p_{2,i}$, for every i in some $U_1 \in \mathcal{U}$; and similarly for the converse inequality).

Proof of (i). Set $\psi_i = \varphi_{1,i} \otimes e_{11} + \varphi_{2,i} \otimes e_{22}$ and apply Theorem 2.1 to $\widetilde{\psi} = \widetilde{\varphi}_1 \otimes e_{11} + \widetilde{\varphi}_2 \otimes e_{22} = (\psi_i)_i^\bullet$; considering the action of $\sigma_t^{\widetilde{\psi}}$ on $p \otimes e_{1,2}$, we obtain:

$$p\,((D\varphi_{1,i} : D\varphi_{2,i})_t)_i^\bullet\, p = (D\widetilde{\varphi}_1 : D\widetilde{\varphi}_2)_t.$$

Let p_i be the common support of $\varphi_{1,i}$ and $\varphi_{2,i}$; then $p_i^{(2)} = p_i \otimes e_{11} + p_i \otimes e_{22}$ is the support of ψ_i. Let $\widetilde{p} = (p_i)_i^\bullet$, and $\widetilde{p}^{(2)} = \widetilde{p} \otimes e_{11} + \widetilde{p} \otimes e_{22} = (p_i^{(2)})_i^\bullet$. Let $\widetilde{\sigma}_t^\Psi$ be the extension to $(\widetilde{p}\mathcal{A}\widetilde{p})^{(2)}$ of the ultrapower map of the family $(\sigma_t^{\psi_i})_i^\bullet$; by Theorem 2.1 we have $\widetilde{\sigma}_t^\Psi(p^{(2)}) = p^{(2)}$. Since clearly $\widetilde{p} \otimes e_{12}$ commutes with $p^{(2)}$, so does $\widetilde{\sigma}_t^\Psi(\widetilde{p} \otimes e_{12})$ which means exactly that

$$p\,((D\varphi_{1,i} : D\varphi_{2,i})_t)_i^\bullet = ((D\varphi_{1,i} : D\varphi_{2,i})_t)_i^\bullet\, p.$$

To prove the point (ii), note that if $\widetilde{x} \in \widetilde{p}\widetilde{\mathfrak{a}}\widetilde{p}$, we have by the preceding:

$$\begin{aligned} \widetilde{\varphi}_1((D\widetilde{\varphi}_1 : D\widetilde{\varphi}_2)_t p\widetilde{x}p)) &= \langle \widetilde{\varphi}_1, p(D\varphi_{1,i} : D\varphi_{2,i})_t)_i^\bullet (x_i)_i^\bullet \rangle \\ &= \lim_{i,\mathcal{U}} \varphi_{1,i}\,((D\varphi_{1,i} : D\varphi_{2,i})_t x_i) \end{aligned}$$

and similarly

$$\widetilde{\varphi}_2(p\widetilde{x}p(D\widetilde{\varphi}_1 : D\widetilde{\varphi}_2)_t) = \lim_{i,\mathcal{U}} \varphi_{2,i}\,(x_i(D\varphi_{1,i} : D\varphi_{2,i})_t).$$

The proof of Theorem 2.1 shows that these limits are uniform with respect to t varying in compact sets of $\mathbb{R}$. Consequently,

$$F_{p,p\tilde{x}p}(z) = \lim_{i,\mathcal{U}} F_{p_i,x_i}(z)$$

for every $z \in S$. In particular, for $z = \mathrm{i}\theta$ we obtain

$$\widetilde{\varphi}_1^{(1-\theta)} \cdot \widetilde{\varphi}_2^{\theta}(p\widetilde{x}p) = \lim_{i,\mathcal{U}} \varphi_{1,i}^{(1-\theta)} \cdot \varphi_{2,i}^{\theta}(x_i) = \langle (\varphi_{1,i}^{(1-\theta)} \cdot \varphi_{2,i}^{\theta})_i^{\bullet}, \widetilde{x}\rangle$$

which shows that $(\varphi_{1,i}^{(1-\theta)} \cdot \varphi_{2,i}^{\theta})_i^{\bullet}$ has right and left supports equal to p and coincides with $\widetilde{\varphi}_1^{(1-\theta)} \cdot \widetilde{\varphi}_2^{\theta}$. ∎

3. ULTRAPOWERS OF L_p SPACES

The main result of this section is the following

THEOREM 3.1. *Let $\mathcal{a}$ be a von Neumann algebra, $0 < p < \infty$ and $\mathcal{U}$ an ultrafilter on the set I. The ultrapower $L_p(\mathcal{a})^I/\mathcal{U}$ is isometric to the $L_p(\mathcal{A})$ space associated with the* VNA $\mathcal{A} = (L_1(\mathcal{a})^I/\mathcal{U})^*$.

Before proving this theorem we recall a few elements of the construction of Haagerup's $L_p(\mathcal{a})$ spaces associated with a von Neumann algebra $\mathcal{a}$ (see [6], [24]).

There is a semifinite VNA $\mathcal{M}$ containing $\mathcal{a}$ (the crossed product $\mathcal{a} \rtimes \mathbb{R}$ of $\mathcal{a}$ by its modular automorphism group) and a strongly continuous one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ of automorphisms of $\mathcal{M}$, such that $\mathcal{a}$ is the space of fixed points of this automorphism group:

$$\mathcal{a} = \{h \in \mathcal{M} : \theta_s(h) = h,\ \forall\, s \in \mathbb{R}\}$$

where $\mathcal{M}$ is equipped with a n.s.f. trace τ such that $\tau\circ\theta_s = \mathrm{e}^{-s}\tau$ for every $s \in \mathbb{R}$. Let $L_0(\mathcal{M}, \tau)$ be the involutive algebra of τ-measurable (unbounded) operators affiliated with $\mathcal{M}$: $h \in L_0(\mathcal{M})$ iff h is affiliated with $\mathcal{M}$ and for each $\varepsilon > 0$ there exists a projection $p \in \mathcal{M}$ such that hp is bounded and $\tau(p^{\perp}) < \varepsilon$. A basis of neighborhoods of zero in $L_0(\mathcal{M}, \tau)$ consists of the sets

$$N_{\varepsilon,\delta} = \{h \in L_0(\mathcal{M}, \tau) : \exists p \in \mathcal{M} \text{ projection such that } \|hp\| < \delta \text{ and } \tau(p^{\perp}) < \varepsilon\}$$

and provides $L_0(\mathcal{M}, \tau)$ with a structure of linear topological $*$-algebra.

There is a linear homeomorphism $\varphi \to h_{\varphi}$ from $\mathcal{a}_*$ onto the closed linear subspace $L_1(\mathcal{a})$ of $L_0(\mathcal{M})$ whose elements are those operators $h \in L_0(\mathcal{M})$ for which

$$\forall\, s \in \mathbb{R}, \quad \theta_s(h) = \mathrm{e}^{-s}h.$$

This homeomorphism preserves the natural structures of $\mathcal{a}$-bimodule of $\mathcal{a}_*$ and $L_1(\mathcal{a})$, as well as the conjugation map and the absolute value map.

For every $p \in (0,\infty]$, the space $L_p(\mathcal{A})$ is defined as the closed subspace of $L_0(\mathcal{M})$ consisting of the operators h for which

$$\forall\, s \in \mathbb{R}, \quad \theta_s(h) = \mathrm{e}^{-s/p} h.$$

Then $L_p(\mathcal{A})$ is an $\mathcal{A}$-bimodule, closed under conjugation and absolute value. If $p = \infty$, it turns out that $L_\infty(\mathcal{A}) = \mathcal{A}$ while if $p < \infty$, $L_p(\mathcal{A})$ is characterized by

$$\text{for } h \in L_0(\mathcal{M}), \quad h \in L_p(\mathcal{A}) \quad \Longleftrightarrow \quad |h|^p \in L_1(\mathcal{A}).$$

The norm on $L_p(\mathcal{A})$ is defined by $\|h\|_p = \|\,|h|^p\|^{1/p}$ (which equals also $\|h^*\|_p$). This is indeed a norm when $p \geqslant 1$, and a p-norm when $0 < p \leqslant 1$ (see [17]). We shall denote by $L_p(\mathcal{A})^+$ the cone $L_0(\mathcal{M})^+ \cap L_p(\mathcal{A})$.

If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, there is a natural bilinear map

$$L_p(\mathcal{A}) \times L_q(\mathcal{A}) \to L_r(\mathcal{A}), \quad (h,k) \mapsto h \cdot k$$

which satisfies Hölder inequality $\|h \cdot k\|_r \leqslant \|h\|_p \|k\|_q$ (see [24] for $r \geqslant 1$ and [17] for the general case). Conversely, we have:

$$\|h\|_p = \sup\{\|h \cdot k\|_r : k \in L_q(\mathcal{A}), \|k\|_q \leqslant 1\}.$$

The case $p = 2$ is special, since the norm on $L_2(\mathcal{A})$ derives from a hermitian scalar product $(h,k) := \mathrm{Tr}(k^* \cdot h)$ where Tr is the distinguished positive linear form on $L_1(\mathcal{A})$ identified with the element $I \in \mathcal{A}$ (when $L_1(\mathcal{A})$ is identified with $\mathcal{A}_*$). There is a natural left action π of $\mathcal{A}$ on $L_2(\mathcal{A})$: $\pi(x) \cdot h = x \cdot h$ for every $x \in \mathcal{A}$ and $h \in L_2(\mathcal{A})$ and a natural antilinear isometric involution J of $L_2(\mathcal{A})$, namely $Jx = x^*$. Then π is an injective normal $*$-representation of $\mathcal{A}$ on $L_2(\mathcal{A})$, and $(\pi(\mathcal{A}), L_2(\mathcal{A}), J, L_2(\mathcal{A})_+)$ a standard form for $\mathcal{A}$ in the sense of [7].

The Mazur maps. Let $0 < p < \infty$. We define a *Mazur map* S_p: $L_0(\mathcal{M}) \to L_0(\mathcal{M})$ in the following way: if $h \in L_0(\mathcal{M})$ has polar decomposition $h = u|h|$ we set $S_p(h) = u|h|^p$. This formula gives also the polar decomposition of $S_p(h)$, since the range projection of $|h|^p$ coincides with that of $|h|$ so $|S_p(h)| = |h|^p$. Note that if $h \in L_p(\mathcal{A})$ then $u \in L_\infty(\mathcal{A})$ and $|h|^p \in L_1(\mathcal{A})$, hence $S_p(h) \in L_1(\mathcal{A})$; we have then $\|S_p(h)\|_1 = \|\,|h|^p\|_1 = \|h\|_p^p$.

Lemma 3.2. *The map S_p is a locally uniform homeomorphism between the spaces $L_p(\mathcal{A})$ and $L_1(\mathcal{A})$.*

Proof. It is clear that the map S_p is bijective, with inverse map $S_p^{-1} = S_{1/p}$.

So it suffices to prove that S_p is locally uniformly continuous for the uniform structure of $L_0(\mathcal{M})$. Note first that the square map $h \mapsto |h|^2 = h^* \cdot h$ is locally uniformly continuous in $L_0(\mathcal{M})$ since the conjugation map $h \mapsto h^*$ and the bilinear map $(h,k) \mapsto h \cdot k$ are. The next step is to prove that for every $\alpha > 0$ the map $h \mapsto h^\alpha$ is locally uniformly continuous from $L_0(\mathcal{M})_+$ into $L_0(\mathcal{M})_+$. This map is indeed locally uniformly continuous from $\mathcal{M}_+$ into $\mathcal{M}_+$ (approximate the function $t \mapsto t^\alpha$ by polynomials, uniformly on compact sets of $\mathbb{R}_+$); let ω_A be its modulus of continuity on the ball of radius A in $\mathcal{M}$. Let B be a bounded set of $L_0(\mathcal{M})_+$. Let δ be a positive real number. For every $h, k \in B$ we can find spectral projections p, q (of h, respectively k) with $\tau(p^\perp) < \delta$, $\tau(q^\perp) < \delta$ and $\|hp\| < C_B(\delta)$, $\|kq\| < C_B(\delta)$; then $\|h^\alpha p - k^\alpha q\| = \|(hp)^\alpha - (kq)^\alpha\| \leqslant \omega_{C_B(\delta)}(\|hp - kq\|)$. If moreover there is a projection $r \in \mathcal{M}$ such that $\tau(r^\perp) < \delta$ and $\|(h-k)r\| < \varepsilon$, then $s = p \wedge q \wedge r$ verifies $\tau(s^\perp) < 3\delta$ and $\|(h^\alpha - k^\alpha)s\| \leqslant \omega_{C_B(\delta)}(\varepsilon) < \delta$ for sufficiently small ε.

Now for every $p > 1$ we may write $S_p h = h|h|^{p-1}$. From the preceding, the map $h \mapsto |h|^{p-1}$ is locally uniformly continuous in $L_0(\mathcal{M})$; so is the product map, hence S_p is. When $p < 1$ consider the maps $S_{p,\varepsilon}$ defined by $S_{p,\varepsilon}(h) = h(\varepsilon + |h|)^{p-1}$. The same reasoning proves that for each $\varepsilon > 0$ the map $S_{p,\varepsilon}$ is locally uniformly continuous in $L_0(\mathcal{M})$. But for every $h \in L_0(\mathcal{M})$ we have $S_{p,\varepsilon}(h) - S_p(h) \in \mathcal{M}$ and $\|S_{p,\varepsilon}(h) - S_p(h)\|_{\mathcal{M}} \leqslant \varepsilon^p$ (for positive autoadjoint h this comes from the spectral calculus and the elementary inequality $|t(\varepsilon + |t|)^{p-1} - t^p| \leqslant \varepsilon^p$ valid for every positive real t). This uniform approximation shows that S_p is also locally uniformly continuous. ∎

Remark 3.3. It is not hard to see that the modulus of continuity of S_p over a ball of radius R depends only on R and p (not on $\mathfrak{a}$); this would be important when dealing with ultraproducts in place of ultrapowers.

Remark 3.4. Several results close to Lemma 3.2 do exist in the literature. Theorem 4.2 of [17] states that the restriction of S_p to the positive cone is an homeomorphism when $p \geqslant 1$. In fact, using a generalized Power-Størmer inequality due to Kosaki too (see the Appendix of [17]) one can easily show that this homeomorphism is locally uniform, and obtain an Hölder estimate for the modulus of continuity, namely:

$$\|a-b\|_p^p \leqslant \|a^p - b^p\|_1 \leqslant M^k \|a-b\|_p^{p-k} + kM^{p-1}\|a-b\|_p$$

for every $a, b \in L_p(\mathfrak{a})_+$ with $\max(\|a\|_p, \|b\|_p) \leqslant M$; here k is the greatest integer strictly less than p. Analogous estimates can be given for the case $p < 1$. Writing $S_p(a) = a|a|^{p-1}$ one can deduce that S_p is locally uniformly continuous for $p \geqslant 1$, but to deduce the local uniform continuity of the inverse S_p^{-1} does not seem so immediate.

Remark 3.5. The map S_p preserves conjugation: $S_p(h^*) = S_p(h)^*$ for every $h \in L_p(\mathfrak{a})$.

Proof. If $h = u|h|$ is the polar decomposition of h then $h^* = u^*|h^*|$ is that of h^*. Then $S_p(h) = u|h|^p$ and $S_p(h^*) = u^*|h^*|^p$. Moreover $|h^*| = u|h|u^*$ implies easily that $|h^*|^p = u|h|^p u^*$, hence $S_p(h^*) = u^*u|h|^p u^* = |h|^p u^* = S_p(h)^*$. ∎

Now we can give a more precise version of Theorem 3.1:

Theorem 3.6. *Let $\mathfrak{a}$ be a* VNA *and identify $\widetilde{\mathfrak{a}_*}$ with $\mathcal{A}_*$ like in Section* 1. *Identifying preduals of* VNA*'s with the corresponding Haagerup L_1 spaces we get an identification map $\Lambda_1 : L_1(\mathfrak{a})_U \to L_1(\mathcal{A})$. Let $S_p^{\mathfrak{a}} : L_p(\mathfrak{a}) \to L_1(\mathfrak{a})$ and $S_p^{\mathcal{A}} : L_p(\mathcal{A}) \to L_1(\mathcal{A})$ be the Mazur maps. Let $\widetilde{S}_p : L_p(\mathfrak{a})_{\mathcal{U}} \to L_1(\mathfrak{a})_{\mathcal{U}}$ be the ultrapower map of $S_p^{\mathfrak{a}}$. Then $\Lambda_p := (S_p^{\mathcal{A}})^{-1} \circ \Lambda_1 \circ \widetilde{S}_p$ is a linear bijective isometry between $L_p(\mathfrak{a})_{\mathcal{U}}$ and $L_p(\mathcal{A})$. Moreover, this map preserves conjugation, positivity and the natural $\widetilde{\mathfrak{a}}$ bimodule stuctures of $L_p(\mathfrak{a})_{\mathcal{U}}$ and $L_p(\mathcal{A})$.*

The non trivial point in this statement is the fact that Λ_p is linear and a bi-module homomorphism. The cases $p > 1$ is treated in this section, the proof of Theorem 3.6 in the case $0 < p < 1$ is postponed to Section 4 (the method for treating the first case does not extends to the second one). In the following we omit generally the map Λ_1 (considered as the identity).

THE $p > 1$ CASE. Let q be the conjugate exponent to p. We shall prove that

$$\forall \widetilde{h} \in L_p(\mathcal{a})_{\mathcal{U}}, \forall \widetilde{k} \in L_q(\mathcal{a})_{\mathcal{U}}, \quad \Lambda_p(\widetilde{h}) \cdot \Lambda_q(\widetilde{k}) = \widetilde{h} \cdot \widetilde{k} \tag{3.1}$$

where in the left member the product is the natural bilinear map $L_p(\mathcal{A}) \times L_q(\mathcal{A}) \to L_1(\mathcal{A})$, while in the right member the product means the ultrapower of the bilinear map $L_p(\mathcal{a}) \times L_q(\mathcal{a}) \to L_1(\mathcal{a})$.

If (3.1) is true, then for every $\widetilde{h}_1, \widetilde{h}_2 \in L_p(\mathcal{a})_{\mathcal{U}}$ and $\widetilde{k} \in L_q(\mathcal{a})_{\mathcal{U}}$:

$$\Lambda_p(\widetilde{h}_1 + \widetilde{h}_2) \cdot \Lambda_q(\widetilde{k}) = (\widetilde{h}_1 + \widetilde{h}_2) \cdot \widetilde{k} = \widetilde{h}_1 \cdot \widetilde{k} + \widetilde{h}_2 \cdot \widetilde{k} = (\Lambda_p(\widetilde{h}_1) + \Lambda_p(\widetilde{h}_2)) \cdot \Lambda_q(\widetilde{k}).$$

But if the elements $\widetilde{f}_1, \widetilde{f}_2$ of $L_q(\mathcal{A})$ verify the equalities $\widetilde{f}_1 \cdot \widetilde{a} = \widetilde{f}_2 \cdot \widetilde{a}$ for every $\widetilde{a} \in L_q(\mathcal{A})$, then $\widetilde{f}_1 = \widetilde{f}_2$ (by the converse of Hölder inequality); so we conclude that $\Lambda_p(\widetilde{h}_1 + \widetilde{h}_2) = \Lambda_p(\widetilde{h}_1) + \Lambda_p(\widetilde{h}_2)$.

Similarly, we have for every $\widetilde{x} \in \widetilde{\mathcal{a}}$, $\widetilde{h} \in \widetilde{L_p(\mathcal{a})}$, $\widetilde{k} \in \widetilde{L_q(\mathcal{a})}$:

$$\Lambda_p(\widetilde{x} \cdot \widetilde{h}) \cdot \Lambda_q(\widetilde{k}) = (\widetilde{x} \cdot \widetilde{h}) \cdot \widetilde{k} = \widetilde{x} \cdot (\widetilde{h} \cdot \widetilde{k}) = \widetilde{x} \cdot (\Lambda_p(\widetilde{h}) \cdot \Lambda_q(\widetilde{k})) = (\widetilde{x} \cdot \Lambda_p(\widetilde{h})) \cdot \Lambda_q(\widetilde{k})$$

were we implicitely used the fact that the left actions of $\widetilde{\mathcal{a}}$ are preserved by the identification of $L_1(\mathcal{a})_{\mathcal{U}}$ with $L_1(\mathcal{A})$. By the same argument we conclude that $\Lambda_p(\widetilde{x} \cdot \widetilde{h}) = \widetilde{x} \cdot \Lambda_p(\widetilde{h})$ and the conclusion of Theorem 3.6 holds.

To prove the relations (3.1), we consider the map $G_p^{\mathcal{a}} : L_1(\mathcal{a}) \times L_1(\mathcal{a}) \to L_1(\mathcal{a})$, defined by $G_p^{\mathcal{a}}(h,k) = S_{1/p}h \cdot S_{1/q}k$. This map is locally uniformly continuous, so we can define its ultrapower map $\widetilde{G_p^{\mathcal{a}}}$. Then (3.1) is equivalent to the relation

$$\widetilde{G_p^{\mathcal{a}}} = G_p^{\mathcal{A}}$$

so we have reduced the proof of the theorem to that of:

PROPOSITION 3.7. *For every* $p \geqslant 1$, $\widetilde{G_p^{\mathcal{a}}} = G_p^{\mathcal{A}}$.

We shall use the following lemma, which is essentially classical (see [16], Section 8).

LEMMA 3.8. *If* $\varphi_1, \varphi_2 \in \mathcal{a}_*^+$ *have the same support then* $G_p^{\mathcal{a}}(h_{\varphi_1}, h_{\varphi_2}) = h_{(\varphi_1^{1-\theta} \cdot \varphi_2^{\theta})}$, *where* $\theta = 1 - 1/p$.

Proof. Note first that if $\varphi \in (\mathcal{a}_*)_+$ and p_φ is the support of φ, the closures in $L_2(\mathcal{M})$ of the subspaces $\mathcal{a}_\varphi h_\varphi^{1/2}$ and $h_\varphi^{1/2}\mathcal{a}_\varphi$ coincide with $p_\varphi \cdot L_2(\mathcal{a}) \cdot p_\varphi$: for, if $k \in p_\varphi \cdot L_2(\mathcal{a}) \cdot p_\varphi$ is orthogonal to $\mathcal{a}_\varphi \cdot h_\varphi^{1/2}$ then $\mathrm{Tr}(xh_\varphi^{1/2}k^*) = \mathrm{Tr}(k^*xh_\varphi^{1/2}) = \langle p_\varphi x h_\varphi^{1/2}, k\rangle = 0$ for every $x \in \mathcal{a}$, hence $h_\varphi^{1/2}k^* = 0$ as an element of $L_1(\mathcal{a})$, so k^* has range included in $\ker h_\varphi^{1/2} = p_\varphi^\perp$; since $k^* = p_\varphi k^*$ this means that $k^* = 0$, i.e. $k = 0$.

For every $x, y \in \mathcal{a}_\varphi$, the equation $F_{x,y}(z) = \mathrm{Tr}(h_\varphi^{1+\mathrm{i}z} x h_\varphi^{-\mathrm{i}z} y)$ defines a continuous function over the strip S and analytic in the interior. The analyticity is due to the fact that the $L_0(\mathcal{M})$-valued map $\zeta \mapsto h_\varphi^\zeta$ is analytic over the open half-plane $\{\mathrm{Re}\,\zeta > 0\}$, see [24]; for the continuity of $F_{x,y}$ at the boundary $\mathbb{R}$ of S, note that approximating $yh_\varphi^{1/2}$ by $h_\varphi^{1/2}y'$ in $L_2(\mathcal{a})$ one gets an approximation of $F_{x,y}$ by functions $G_{x,y'}$ of the form $G_{x,y'}(z) = \mathrm{Tr}(h_\varphi^{1/2+\mathrm{i}z}xh_\varphi^{1/2-\mathrm{i}z}y')$ which is uniform on a neighborhood of $\mathbb{R}$ in S; the treatment of the continuity of $F_{x,y}$ at the boundary $\mathbb{R}+\mathrm{i}$ is analogous (approximating now $xh_\varphi^{1/2}$ by some $h_\varphi^{1/2}x'$).

For every $x \in \mathcal{a}_\varphi$ set: $\alpha_t(x) = h_\varphi^{\mathrm{i}t}xh_\varphi^{-\mathrm{i}t}$; then (α_t) is a one parameter group of $*$-isomorphisms of $\mathcal{a}_\varphi$ which by the preceding verifies the KMS condition relative to φ; so (α_t) coincides with the modular automorphism group $(\sigma_t^\varphi)_t$ of φ.

Similarly, if φ_1, φ_2 are elements of $(\mathcal{a}_*)_+$ with same support p, and $\widehat{\varphi} = \varphi_1 \otimes e_{11} + \varphi_2 \otimes e_{22}$ is the associated normal form on $\mathcal{a}^{(2)} = \mathcal{a} \otimes M_2$, consider in $L_0(\mathcal{M}) \otimes M_2 = L_0(\mathcal{M} \otimes M_2)$ the operator $h_{\widehat{\varphi}} = h_{\varphi_1} \otimes e_{11} + h_{\varphi_2} \otimes e_{22}$. The same reasoning proves that $\sigma_t^{\hat{\varphi}}(\widehat{x}) = h_{\hat{\varphi}}^{\mathrm{i}t}\widehat{x}h_{\hat{\varphi}}^{-\mathrm{i}t}$ for every $\widehat{x} \in (p\mathcal{a}p)^{(2)}$, from which one deduces easily that $\sigma_t^{\varphi_1\varphi_2}(x) = h_{\varphi_1}^{\mathrm{i}t} \cdot x \cdot h_{\varphi_2}^{-\mathrm{i}t}$ for every $x \in p\mathcal{a}p$.

Let $F(z) = \mathrm{Tr}(h_{\varphi_1}^{1+\mathrm{i}z} \cdot h_{\varphi_2}^{-\mathrm{i}z}x)$ for $z \in S$: this function is bounded continuous on S, analytic on $\overset{\circ}{S}$, and $F(t) = \varphi_1((D\varphi_1 : D\varphi_2)_t\, x)$, $F(t+\mathrm{i}) = \varphi_2(x(D\varphi_1 : D\varphi_2)_t)$, so we obtain $\langle \varphi_1^{1-\theta} \cdot \varphi_2^\theta, x\rangle = F(\mathrm{i}\theta) = \mathrm{Tr}(h_{\varphi_1}^{1-\theta}h_{\varphi_2}^\theta \cdot x)$. Hence $h_{(\varphi_1^{1-\theta} \cdot \varphi_2^\theta)} = h_{\varphi_1}^{1-\theta} \cdot h_{\varphi_2}^\theta = G_p^{\mathcal{a}}(h_{\varphi_1}, h_{\varphi_2})$. ∎

Proof of Proposition 3.7. By the preceding Lemma 3.8 and Proposition 2.2, if $\widetilde{\varphi}_1, \widetilde{\varphi}_2 \in \mathcal{A}_*^+$ have the same supports and have representing families $(\varphi_{1,i})_i$ and $(\varphi_{2,i})_i$ such that for each $i \in I$, $\varphi_{1,i}$ and $\varphi_{2,i}$ have the same supports, then $(G_p^{\mathrm{a}}(\varphi_{1,i}, \varphi_{2,i}))_i^\bullet = G_p^{\mathcal{A}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2)$, which means $\widetilde{G}_p^{\mathrm{a}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2) = G_p^{\mathcal{A}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2)$. One can easily get rid of the support conditions by approximating $\widetilde{\varphi}_1$, $\widetilde{\varphi}_2$ by $\widetilde{\varphi}_1^\varepsilon = \widetilde{\varphi}_1 + \varepsilon\widetilde{\varphi}_2$, $\widetilde{\varphi}_2^\varepsilon = \varepsilon\widetilde{\varphi}_1 + \widetilde{\varphi}_2$, $\varepsilon > 0$, and letting $\varepsilon \to 0$ using the locally uniform continuity of G_p^{a} and $G_p^{\mathcal{A}}$ (note that $\widetilde{\varphi}_1^\varepsilon$, $\widetilde{\varphi}_2^\varepsilon$ have the same support $p_{\tilde{\varphi}_1} \vee p_{\tilde{\varphi}_2}$).

If now the elements $\widetilde{\varphi}_1 = (\varphi_{1,i})^\bullet$, $\widetilde{\varphi}_2 = (\varphi_{2,i})^\bullet$ are no more supposed to be positive, consider the polar decompositions $\varphi_{1,i} = u_{1,i}|\varphi_{1,i}|$, $\varphi_{2,i}^* = v_{2,i}|\varphi_{2,i}^*|$ and set $\widetilde{u}_1 = (u_{1,i})_i^\bullet$, $\widetilde{v}_2 = (v_{2,i})_i^\bullet$; we know that $|\widetilde{\varphi}_1| = (|\varphi_{1,i}|)_i^\bullet$, $|\widetilde{\varphi}_2^*| = (|\varphi_{2,i}^*|)_i^\bullet$ and that, setting $u_1 = \widetilde{u}_1 p_{|\tilde{\varphi}_1|}$, $v_2 = \widetilde{v}_2 p_{|\tilde{\varphi}_2^*|}$, then $\widetilde{\varphi}_1 = u_1|\widetilde{\varphi}_1|$, $\widetilde{\varphi}_2^* = v_2|\widetilde{\varphi}_2^*|$ are the polar decompositions of $\widetilde{\varphi}_1$, respectively $\widetilde{\varphi}_2^*$. Then

$$G_p^{\mathcal{A}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2) = u_1 G_p^{\mathcal{A}}(|\widetilde{\varphi}_1|, |\widetilde{\varphi}_2^*|)v_2^* = \widetilde{u}_1 G_p^{\mathcal{A}}(|\widetilde{\varphi}_1|, |\widetilde{\varphi}_2^*|)\widetilde{v}_2^*$$

since it is clear from the definition that $p_{|\tilde{\varphi}_1|}G_p^{\mathcal{A}}(|\widetilde{\varphi}_1|, |\widetilde{\varphi}_2^*|)p_{|\tilde{\varphi}_2^*|} = G_p^{\mathcal{a}}(|\widetilde{\varphi}_1|, |\widetilde{\varphi}_2^*|)$. So by the preceding:

$$G_p^{\mathcal{A}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2) = (u_{1,i}G_p^{\mathcal{a}}(|\varphi_{1,i}|, |\varphi_{2,i}^*|)v_{2,i}^*)_i^\bullet = (G_p^{\mathcal{a}}(\varphi_{1,i}, \varphi_{2,i}))_i^\bullet = \widetilde{G_p^{\mathcal{a}}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2). \quad ∎$$

Standard forms. If $\mathcal{P}$ is a cone in H, we denote by $\widetilde{\mathcal{P}}$ the ultrapower cone, i.e.:

$$\widetilde{\mathcal{P}} = \{(x_i)_i^\bullet \in \widetilde{H} : x_i \in \mathcal{P}, \forall\, i \in I\}.$$

The following corollary was announced in [20]; it has now a straightforward proof, but it could also be easily deduced directly from Theorem 1.8:

COROLLARY 3.9. *If $(\mathcal{a}, H, J, \mathcal{P})$ is a standard form for $\mathcal{a}$ then $(\mathcal{A}, \widetilde{H}, \widetilde{J}, \widetilde{\mathcal{P}})$ is a standard form for $\mathcal{A}$.*

Proof. Since all standard forms are spatially equivalent, we can assume that $(\mathcal{a}, H, J, \mathcal{P}) = (\mathcal{a}, L_2(\mathcal{a}), J, L_2(\mathcal{a})_+)$, where J is the conjugation map on $L_2(\mathcal{a})$. The map Λ_2 permits to identify $(L_2(\mathcal{a})_{\mathcal{U}}, \widetilde{J}, L_2(\mathcal{a})_{\mathcal{U}+})$ with $(L_2(\mathcal{A}), J, L_2(\mathcal{A})_+)$ and the action of $\widetilde{\mathcal{a}}$ on $\widetilde{L_2(\mathcal{a})}$ with the restriction of the action of $\mathcal{A}$ on $L_2(\mathcal{A})$. Finally, $(\mathcal{A}, \widetilde{H}, \widetilde{J}, \widetilde{\mathcal{P}})$ is spatially equivalent to the standard form $(\mathcal{A}, L_2(\mathcal{A}), J, L_2(\mathcal{A})_+)$. ∎

4. THE CASE $0 < p < 1$

In this section we prove the case $0 < p < 1$ of Theorem 3.6. This is done by induction, by proving that if Theorem 3.6 is true for p and every VNA $\mathcal{a}$, it is true for $p/2$ and every VNA. We shall make repeatedly use of the following remark.

Consider the VNA's $\mathcal{a}^{(2)} = \mathcal{a} \otimes M_2$ and $\mathcal{A}^{(2)} = \mathcal{A} \otimes M_2$. Then $(\mathcal{a}^{(2)})_* = \mathcal{a}_* \otimes (M_2)_*$ and $(\mathcal{A}_{(2)})_* = \mathcal{A}_* \otimes (M_2)_*$ identifies with $(\mathcal{a}_{(2)*})_{\mathcal{U}} = (\mathcal{a}_*)_{\mathcal{U}} \otimes (M_2)_*$ (see Section 1). On the other hand, $L_p(\mathcal{a}^{(2)})$ identifies linearly with $(L_p(\mathcal{a}))_{(2)} = M_2(L_p(\mathcal{a}))$ (see [24]). This proceeds from an identification of the crossed product $\mathcal{a}^{(2)} \rtimes \mathbb{R}$ with $M_2(\mathcal{a}^{(2)} \rtimes \mathbb{R}) = \mathcal{M}^{(2)}$ when the weight over $\mathcal{a}^{(2)}$ is correctly choosen; then $L_0(\mathcal{M}^{(2)})$ identifies with $M_2(L_0(\mathcal{M}))$ and the dual automorphism group over $\mathcal{M}^{(2)}$ takes simply the form $(\theta_s \otimes \mathrm{Id}_{M_2})$. In these identifications the bilinear multiplication map $L_p(\mathcal{a}^{(2)}) \times L_q(\mathcal{a}^{(2)}) \to L_r(\mathcal{a}^{(2)})$ (where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$) corresponds to the matricial multiplication $M_2(L_p(\mathcal{a})) \times M_2(L_q(\mathcal{a})) \to M_2(L_r(\mathcal{a}))$.

LEMMA 4.1. *Let $0 < p < \infty$ and $\widetilde{h} \in L_p(\mathcal{a})_{\mathcal{U}}$. Then:*
(i) $\Lambda_p(\lambda \widetilde{h}) = \lambda \Lambda_p(\widetilde{h})$ *for every* $\lambda \in \mathbb{C}$;
(ii) $\Lambda_p(\widetilde{v}\widetilde{h}) = \widetilde{v}\Lambda_p(\widetilde{h})$ *for every unitary* $\widetilde{v} \in \mathcal{a}_{\mathcal{U}}$;
(iii) $\Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{h}) = (\Lambda_p \widetilde{h})^* \cdot (\Lambda_p \widetilde{h})$.

Proof. The point (i) is trivial. For the point (ii), note simply that if $h = u|h|$ is the polar decomposition of $h \in L_p(\mathcal{a}))$ and v is an unitary of $\mathcal{a}$ then $vh = (vu)|h|$ is the polar decomposition of vh; hence $S_p(v \cdot h) = v \cdot S_p h$. For the point (iii) we have:

$$\begin{aligned}\Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{h}) = \Lambda_{p/2}(|\widetilde{h}|^2) &= S_{2/p}\Lambda_1 \widetilde{S}_{p/2}(|\widetilde{h}|^2) = S_{2/p}\Lambda_1 \widetilde{S}_p(|\widetilde{h}|)\\ &= S_2 \Lambda_p(|\widetilde{h}|) = \Lambda_p(|\widetilde{h}|)^2.\end{aligned}$$

By the definition of Λ_p and the properties of S_p, we have $\Lambda_p(|\widetilde{h}|) = |\Lambda_p(\widetilde{h})|$, and the lemma follows.

LEMMA 4.2. *For every $0 < p < \infty$ and every self-adjoint element $\widetilde{h}$ of $L_p(\mathcal{a})_{\mathcal{U}}$, we have*

$$\Lambda_p(\widetilde{h}_+) = (\Lambda_p \widetilde{h})_+, \quad \Lambda_p(\widetilde{h}_-) = (\Lambda_p \widetilde{h})_-$$

where $\widetilde{h}_+$, respectively $-\widetilde{h}_-$ denotes the positive (respectively negative) part of the self-adjoint element $\widetilde{h}$.

Proof. We have $S_p(h_+) = (S_p h)_+$ and similarly $S_p(h_-) = (S_p h)_-$for every self-adjoint element h of $L_p(\mathcal{A})$ or $L_p(\mathcal{A})$. For, if $h = (e_+ - e_-)|h|$ is the polar decomposition of h, then $S_p h = (e_+ - e_-)S_p|h|$ is that of $S_p h$, so

$$(S_p h)_+ = e_+ S_p|h| = e_+|h|^p = (e_+ h)^p = S_p h_+$$

and similarly for the negative parts. ∎

Lemma 4.3. *If the map Λ_p is linear for $\mathcal{A}^{(2)}$ then $\Lambda_{p/2}$ is positively additive, i.e.:*

$$\forall \widetilde{h}, \widetilde{k} \in L_{p/2}(\mathcal{A})^+_{\mathcal{U}}, \quad \Lambda_{p/2}(\widetilde{h} + \widetilde{k}) = \Lambda_{p/2}(\widetilde{h}) + \Lambda_{p/2}(\widetilde{k}).$$

Proof. Note that for every $0 < p < \infty$ and $x \in L_p(\mathcal{A}))$ ve have $S_p(x \otimes e_{ij}) = (S_p(x) \otimes e_{ij})$, for every $i, j \in \{1, 2\}$, since $x \otimes e_{ij} = (u \otimes e_{ij})(|x| \otimes e_{jj})$ is the polar decomposition of $x \otimes e_{ij}$ if $x = u|x|$ is that of x. So we obtain

$$\Lambda_p(\widetilde{x} \otimes e_{ij}) = \Lambda_p(\widetilde{x}) \otimes e_{ij}$$

for every $0 < p < \infty$, $\widetilde{x} \in L_p(\mathcal{A})_{\mathcal{U}}$ and $i, j \in \{1, 2\}$. If moreover Λ_p is linear on $L_p(\mathcal{A}^{(2)})_{\mathcal{U}}$, we obtain $\Lambda_p([\widetilde{x}_{ij}]) = [\Lambda_p(x_{ij})]$ for every $\widetilde{X} = [\widetilde{x}_{ij}] \in M_2(L_p(\mathcal{A}))_{\mathcal{U}}$.

Now we apply Lemma 4.1 to $L_p(\mathcal{A}^{(2)})$. Let $\widetilde{h}, \widetilde{k} \in L_p(\mathcal{A})_{\mathcal{U}}$ and $\widetilde{X} = \begin{bmatrix} \widetilde{h} & 0 \\ \widetilde{k} & 0 \end{bmatrix} \in M_2(L_p(\mathcal{A})_{\mathcal{U}})$. Then $\widetilde{X}^* \cdot \widetilde{X} = \begin{bmatrix} \widetilde{h}^* \cdot \widetilde{h} + \widetilde{k}^* \cdot \widetilde{k} & 0 \\ 0 & 0 \end{bmatrix}$, so $\Lambda_{p/2}(\widetilde{X}^* \cdot \widetilde{X}) = \begin{bmatrix} \Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{h} + \widetilde{k}^* \cdot \widetilde{k}) & 0 \\ 0 & 0 \end{bmatrix}$.

If Λ_p is linear, we have $\Lambda_p(\widetilde{X}) = \begin{bmatrix} \Lambda_p(\widetilde{h}) & 0 \\ \Lambda_p(\widetilde{k}) & 0 \end{bmatrix}$, so the equality $\Lambda_{p/2}(\widetilde{X}^* \cdot \widetilde{X}) = \Lambda_p(\widetilde{X})^* \cdot \Lambda_p(\widetilde{X})$ reads $\Lambda_{p/2}(\widetilde{h}^*\widetilde{h} + \widetilde{k}^*\widetilde{k}) = \Lambda_p(\widetilde{h})^* \cdot \Lambda_p(\widetilde{h}) + \Lambda_p(\widetilde{k})^* \cdot \Lambda_p(\widetilde{k})$, which equals $\Lambda_{p/2}(\widetilde{h}^*\widetilde{h}) + \Lambda_{p/2}(\widetilde{k}^*\widetilde{k})$ by Lemma 4.1 again. ∎

Lemma 4.4. *If Λ_p is linear for $\mathcal{A}^{(2)}$, then $\Lambda_{p/2}$ is additive (hence real linear) on the selfadjoint part of $L_p(\mathcal{A})_{\mathcal{U}}$.*

Proof. Assume that $\widetilde{h}, \widetilde{k} \in L_p(\mathcal{A})^+_{\mathcal{U}}$. From

$$\widetilde{h} + (\widetilde{h} - \widetilde{k})_- = \widetilde{k} + (\widetilde{h} - \widetilde{k})_+$$

we deduce using Lemma 4.3 that

$$\Lambda_{p/2}\widetilde{h} + \Lambda_{p/2}[(\widetilde{h} - \widetilde{k})_-] = \Lambda_{p/2}\widetilde{k} + \Lambda_{p/2}[(\widetilde{h} - \widetilde{k})_+].$$

Hence, using Lemma 4.2:

$$\Lambda_{p/2}\widetilde{h} - \Lambda_{p/2}\widetilde{k} = (\Lambda_{p/2}(\widetilde{h} - \widetilde{k}))_+ - (\Lambda_{p/2}(\widetilde{h} - \widetilde{k}))_- = \Lambda_{p/2}(\widetilde{h} - \widetilde{k}).$$

So $\Lambda_{p/2}$ preserves differences as well as sums of positive elements. Since every selfadjoint element of $L_p(\mathcal{A})_{\mathcal{U}}$ is a difference of two positive elements, we are done. ∎

LEMMA 4.5. *If Λ_p is linear in the case of $\mathcal{A}^{(4)}$, then for every $\widetilde{h}, \widetilde{k} \in L_p(\mathcal{A})_{\mathcal{U}}$ we have*

$$\Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{k}) = (\Lambda_p \widetilde{h})^* \cdot (\Lambda_p \widetilde{k}).$$

Proof. (a) From the equality

$$\widetilde{h}^* \cdot \widetilde{k} + \widetilde{k}^* \cdot \widetilde{h} = \frac{1}{2}[(\widetilde{h} + \widetilde{k})^* \cdot (\widetilde{h} + \widetilde{k}) - (\widetilde{h} - \widetilde{k})^* \cdot (\widetilde{h} - \widetilde{k})]$$

we easily deduce using Lemma 4.4, Lemma 4.1 and the linearity of Λ_p that

$$\begin{aligned}
&2\Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{k} + \widetilde{k}^* \cdot \widetilde{h}) \\
&\quad = \Lambda_{p/2}[(\widetilde{h} + \widetilde{k})^* \cdot (\widetilde{h} + \widetilde{k})] - \Lambda_{p/2}[(\widetilde{h} - \widetilde{k})^* \cdot (\widetilde{h} - \widetilde{k})] \\
&\quad = (\Lambda_p(\widetilde{h} + \widetilde{k}))^* \cdot \Lambda_p(\widetilde{h} + \widetilde{k}) - (\Lambda_p(\widetilde{h} - \widetilde{k}))^* \cdot \Lambda_p(\widetilde{h} - \widetilde{k}) \\
&\quad = (\Lambda_p \widetilde{h} + \Lambda_p \widetilde{k}))^* \cdot (\Lambda_p \widetilde{h} + \Lambda_p \widetilde{k})) - (\Lambda_p \widetilde{h} - \Lambda_p \widetilde{k}))^* \cdot (\Lambda_p \widetilde{h} - \Lambda_p \widetilde{k})) \\
&\quad = 2[(\Lambda_p \widetilde{h})^* \cdot (\Lambda_p \widetilde{k}) + (\Lambda_p \widetilde{k})^* \cdot (\Lambda_p \widetilde{h})].
\end{aligned}$$

(b) We shall use the following observation: for every $0 < r < \infty$, and $h, k \in L_r(\mathcal{A})$

$$S_r\left(\begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}\right) = \begin{bmatrix} S_r h & 0 \\ 0 & S_r k \end{bmatrix}$$

from which we infer that

$$\Lambda_r\left(\begin{bmatrix} \widetilde{h} & \underset{\sim}{0} \\ 0 & \widetilde{k} \end{bmatrix}\right) = \begin{bmatrix} \Lambda_r \widetilde{h} & 0 \\ 0 & \Lambda_r \widetilde{k} \end{bmatrix}$$

for every $0 < r < \infty$, and $\widetilde{h}, \widetilde{k} \in L_r(\mathcal{A})_{\mathcal{U}}$. We apply now the point (a) above to the following elements of $L_p(\mathcal{A}^{(2)})_{\mathcal{U}}$:

$$\widetilde{A} = \begin{bmatrix} \widetilde{h} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{B} = \begin{bmatrix} \underset{\sim}{0} & \widetilde{k} \\ k & 0 \end{bmatrix}.$$

We have

$$\widetilde{A}^* \cdot \widetilde{B} + \widetilde{B}^* \cdot \widetilde{A} = \begin{bmatrix} 0 & \widetilde{h}^* \cdot \widetilde{k} \\ \widetilde{k}^* \cdot \widetilde{h} & 0 \end{bmatrix}.$$

Since $\begin{bmatrix} 0 & \widetilde{h}^* \cdot \widetilde{k} \\ \widetilde{k}^* \cdot \widetilde{h} & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \widetilde{h}^* \cdot \widetilde{k} & 0 \\ 0 & \widetilde{k}^* \cdot \widetilde{h} \end{bmatrix}$ and the matrix $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is unitary, we have

$$\begin{aligned}
\Lambda_{p/2}\left(\begin{bmatrix} 0 & \widetilde{h}^* \cdot \widetilde{k} \\ \widetilde{k}^* \cdot \widetilde{h} & 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \Lambda_{p/2}\left(\begin{bmatrix} \widetilde{h}^* \cdot \widetilde{k} & 0 \\ 0 & \widetilde{k}^* \cdot \widetilde{h} \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{k}) & 0 \\ 0 & \Lambda_{p/2}(\widetilde{k}^* \cdot \widetilde{h}) \end{bmatrix} \qquad (4.1) \\
&= \begin{bmatrix} 0 & \Lambda_{p/2}(\widetilde{h}^* \cdot \widetilde{k}) \\ \Lambda_{p/2}(\widetilde{k}^* \cdot \widetilde{h}) & 0 \end{bmatrix}.
\end{aligned}$$

On the other hand,

$$\Lambda_p\widetilde{A} = \begin{bmatrix} \Lambda_p\widetilde{h} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda_p\widetilde{B} = \begin{bmatrix} 0 & \Lambda_p\widetilde{k} \\ \Lambda_p\widetilde{k} & 0 \end{bmatrix}$$

hence

$$\Lambda_p\widetilde{A}^* \cdot \Lambda_p\widetilde{B} + \Lambda_p\widetilde{B}^* \cdot \Lambda_p\widetilde{A} = \begin{bmatrix} 0 & \Lambda_p\widetilde{h}^* \cdot \Lambda_p\widetilde{k} \\ \Lambda_p\widetilde{k}^* \cdot \Lambda_p\widetilde{h} & 0 \end{bmatrix}. \tag{4.2}$$

By the point (a) above and the comparison of formulas (4.1) and (4.2) we are done. ∎

Lemma 4.6. *If Λ_p is linear and a bimodule homomorphism ("modular" in brief) in the case of $\mathcal{A}^{(4)}$, so is $\Lambda_{p/2}$ (in the case of $\mathcal{A}$).*

Proof. (a) We show first the $\mathcal{A}_{\mathcal{U}}$-modularity.

Let $\widetilde{h} \in L_{p/2}(\mathcal{A})_{\mathcal{U}}$ and $\widetilde{x} \in \mathcal{A}_{\mathcal{U}}$. We decompose $\widetilde{h} = \widetilde{S}_{1/2}\widetilde{h} \cdot \widetilde{S}_{1/2}|\widetilde{h}|$, where $\widetilde{S}_{1/2} = (S_{1/2})_{\mathcal{U}} : L_{p/2}(\mathcal{A})_{\mathcal{U}} \to L_p(\mathcal{A})_{\mathcal{U}}$. Using Lemma 4.5, the left modularity of Λ_p and Lemma 4.5 again, we have:

$$\Lambda_{p/2}(\widetilde{x} \cdot \widetilde{h}) = \Lambda_p(\widetilde{x} \cdot \widetilde{S}_{1/2}\widetilde{h}) \cdot \Lambda_p(\widetilde{S}_{1/2}|\widetilde{h}|) = \widetilde{x} \cdot \Lambda_p(\widetilde{S}_{1/2}\widetilde{h}) \cdot \Lambda_p(\widetilde{S}_{1/2}|\widetilde{h}|) = \widetilde{x} \cdot \Lambda_{p/2}(\widetilde{h})$$

and similarly for the right modularity.

(b) We deduce the linearity.

If $\widetilde{h}, \widetilde{k} \in L_{p/2}(\mathcal{A})_{\mathcal{U}}$, with $\widetilde{h} = (h_i)^\bullet$, $\widetilde{k} = (k_i)^\bullet$, set $l_i = (h_i^* \cdot h_i + k_i^* \cdot k_i)^{1/2}$. For $p \leqslant 4$, we have $\|l_i\|_{p/2} \leqslant (\|h_i\|_{p/2}^{p/2} + \|k_i\|_{p/2}^{p/2})^{2/p}$, so the family (l_i) represents an element $(l_i)^\bullet$ of $L_{p/2}(\mathcal{A})_{\mathcal{U}}$. Since $h_i^* \cdot h_i \leqslant l_i^* \cdot l_i$ and $k_i^* \cdot k_i \leqslant l_i^* \cdot l_i$, there exist u_i, v_i in the unit ball of $\mathcal{M}$ such that

$$h_i = u_i l_i \quad \text{and} \quad k_i = v_i l_i;$$

u_i and v_i are uniquely determined if we ask that their right support is included in the support of l_i. Since the support of l_i is θ_s-invariant, it is easy to see that $\theta_s(u_i) = u_i$, $\theta_s(v_i) = v_i$ for every $s \in \mathbb{R}$, so in fact $u_i, v_i \in \mathcal{A}$.

Set $\widetilde{u} = (u_i)^\bullet$ and $\widetilde{v} = (v_i)^\bullet$, we have $\widetilde{h} = \widetilde{u} \cdot \widetilde{l}$ and $\widetilde{k} = \widetilde{v} \cdot \widetilde{l}$, whence by the point (a):

$$\begin{aligned} \Lambda_{p/2}(\widetilde{h} + \widetilde{k}) &= \Lambda_{p/2}((\widetilde{u} + \widetilde{v}) \cdot \widetilde{l}) = (\widetilde{u} + \widetilde{v}) \cdot \Lambda_{p/2}(\widetilde{l}) \\ &= \widetilde{u} \cdot \Lambda_{p/2}(\widetilde{l}) + \widetilde{v} \cdot \Lambda_{p/2}(\widetilde{l}) = \Lambda_{p/2}(\widetilde{u} \cdot \widetilde{l}) + \Lambda_{p/2}(\widetilde{v} \cdot \widetilde{l}) \\ &= \Lambda_{p/2}(\widetilde{h}) + \Lambda_{p/2}(\widetilde{k}). \end{aligned}$$ ∎

5. FINAL REMARKS

5.A. THE PRODUCT MAP $L_p \times L_q \to L_r$ AND ITS ULTRAPOWERS. We prove now (following [13]) that the identification of $L_p(\mathcal{a})_{\mathcal{U}}$ with $L_p(\mathcal{A})$ for various p is compatible with the natural product from $L_p \times L_q$ into L_r.

THEOREM 5.1. *Let $0 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then for every $\widetilde{h} \in L_p(\mathcal{a})_{\mathcal{U}}$, $\widetilde{k} \in L_q(\mathcal{a})_{\mathcal{U}}$ we have:*

$$\Lambda_p(\widetilde{h}) \cdot \Lambda_q(\widetilde{k}) = \Lambda_r(\widetilde{h} \cdot \widetilde{k})$$

(where in the left-hand side the dot denotes the natural product $L_p(\mathcal{A}) \times L_q(\mathcal{A}) \to L_r(\mathcal{A})$ while in the right-hand side it refers to the ultrapower map of the operation $L_p(\mathcal{a}) \times L_q(\mathcal{a}) \to L_r(\mathcal{a})$).

Proof. Let $\widetilde{h} = (h_i)^\bullet$, $\widetilde{k} = (k_i)^\bullet$. For every $i \in I$ set

$$a_i = S_{1/2}(S_{2p/r}|h_i| + S_{2q/r}|k_i^*|) \in L_r(\mathcal{a})_+$$

(in other terms $a_i = (|h_i|^{2p/r} + |k_i^*|^{2q/r})^{1/2}$). Since $\frac{p}{r} \geqslant 1$ we have

$$|h_i|^{2p/r} \leqslant a_i^2 \Longrightarrow |h_i|^2 \leqslant a_i^{2r/p} \Longleftrightarrow h_i = u_i a_i^{r/p} \text{with } u_i \in \mathcal{a},\ \|u_i\| \leqslant 1$$

and similarly $k_i = a_i^{r/q} v_i$ with $v_i \in \mathcal{a}$, $\|v_i\| \leqslant 1$. Setting now $\widetilde{a} = (a_i)^\bullet$, $\widetilde{u} = (u_i)^\bullet$, $\widetilde{v} = (v_i)^\bullet$, we have

$$\widetilde{h} \cdot \widetilde{k} = [\widetilde{u} \cdot (\widetilde{S}_{r/p}\widetilde{a})] \cdot [(\widetilde{S}_{r/q}\widetilde{a}) \cdot \widetilde{v}] = \widetilde{u} \cdot [(\widetilde{S}_{r/p}\widetilde{a}) \cdot (\widetilde{S}_{r/q}\widetilde{a})] \cdot \widetilde{v} = \widetilde{u} \cdot \widetilde{a} \cdot \widetilde{v}.$$

Hence, since Λ_r preserves right and left actions of $\mathcal{a}_{\mathcal{U}}$:

$$\Lambda_r(\widetilde{h} \cdot \widetilde{k}) = \Lambda_r(\widetilde{u} \cdot \widetilde{a} \cdot \widetilde{v}) = \widetilde{u} \cdot \Lambda_r \widetilde{a} \cdot \widetilde{v};$$

on the other hand

$$\Lambda_p \widetilde{h} = \Lambda_p(\widetilde{u} \cdot (\widetilde{S}_{r/p}\widetilde{a})) = \widetilde{u} \cdot \Lambda_p(\widetilde{S}_{r/p}\widetilde{a}) = \widetilde{u} \cdot S_{r/p}(\Lambda_r \widetilde{a})$$

and similarly

$$\Lambda_q \widetilde{k} = S_{r/q}(\Lambda_r \widetilde{a}) \cdot \widetilde{v}$$

so finally:

$$\Lambda_p(\widetilde{h}) \cdot \Lambda_q(\widetilde{k}) = \widetilde{u} \cdot S_{r/p}(\Lambda_r \widetilde{a}) \cdot S_{r/q}(\Lambda_r \widetilde{a}) \cdot \widetilde{v} = \widetilde{u} \cdot \Lambda_r \widetilde{a} \cdot \widetilde{v} = \Lambda_r(\widetilde{h} \cdot \widetilde{k}). \qquad \blacksquare$$

5.B. THE OPERATOR SPACE STRUCTURE OF $L_p(\mathcal{a})_{\mathcal{U}}$. If E is an operator space, its ultrapower $E_{\mathcal{U}}$ is equipped with a natural structure of operator space given by the equality:

$$M_n(E_{\mathcal{U}}) = (M_n(E))_{\mathcal{U}}$$

(this definition verifies Ruan's axioms, see Section 3 of [19]).

The spaces $L_p(\mathcal{a})$ are equipped with a natural operator space structure given by interpolation between the operator spaces structure of $\mathcal{a}$ and that of $\mathcal{a}_*$ (the dual operator space structure) (see [19]).

In fact, the norms on the spaces $M_n(L_p(\mathcal{a}))$ can be given an intrinsic definition, see Theorem 9 of [4]:

$$\|x\|_{M_n(L_p(\mathcal{a}))} = \sup_{\substack{a,b \in S_{2p}^n \\ \|a\|_p \leqslant 1,\ \|b\|_p \leqslant 1}} \begin{cases} \|axb\|_{L_p(M_n(\mathcal{a}))} & \text{if } 2 \leqslant p < \infty, \\ \|a\,{}^{\mathrm{t}}xb\|_{L_p(M_n(\mathcal{a}))} & \text{if } 1 \leqslant p \leqslant 2; \end{cases}$$

where S_{2p}^n denotes the Schatten class of exponent $2p$ over the space l_2^n while ${}^{\mathrm{t}}x = [x_{ji}]$ is the transposed matrix of $x = [x_{ij}]$.

PROPOSITION 5.2. *For every $1 \leqslant p < \infty$ the identification map $\Lambda_p : L_p(\mathfrak{A})_{\mathcal{U}} \to L_p(\mathcal{A})$ constructed in Section 3 is a complete isometry.*

Proof. As for $n = 2$, we have for any n:

$$M_n(\mathfrak{A})_{*\mathcal{U}} = M_n(\mathcal{A})_*.$$

It results (as in the case $n = 2$, see the proof of Lemma 4.3) that $\Lambda_p^{(n)} = \mathrm{Id}_n \otimes \Lambda_p$ is the identification map $L_p(M_n(\mathfrak{A}))_{\mathcal{U}} \to L_p(M_n(\mathcal{A}))$. Since clearly $\Lambda_p^{(n)}(a\widetilde{x}b) = a\Lambda_p^{(n)}(\widetilde{x})b$ for every $\widetilde{x}$ in $L_p(M_n(A))_{\mathcal{U}}$ and $a, b \in S_{2p}^n$, and $\Lambda_p^{(n)}$ commutes with transposition, we obtain easily that:

$$\|\widetilde{x}\|_{M_n(L_p(\mathfrak{A})_{\mathcal{U}})} = \|\Lambda_p^{(n)}(\widetilde{x})\|_{M_n(L_p(\mathcal{A}))}.$$

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